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Reformulation of the Nash Social Welfare Function for a Continuum of Individuals

by

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Abstract: This paper provides a new formulation of the Nash social welfare function for a continuum of individuals. The new formulation removes redundant assumptions of Kaneko (1981)'s formulation and in particular shows that the continuity axiom on social orderings is unnecessary. The proof of the derivation of the Nash social welfare function is also much directer and shorter than Kaneko's original proof.

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1. Introduction

Models with a continuum of individuals are quite useful in considering problems in large economies or societies and indispensable for some economic or social problems. The Nash social welfare function is formulated for societies with a continuum of individuals in Kaneko (1981) and is applied to the optimal income taxation problem in Kaneko (1982). However, the formulation of the Nash social welfare fucntion in Kaneko (1981) is not fully elaborated. For example, it includes redundant assumptions, unnecessary restrictions and generalities. The purpose of this paper is to remove them and make the theory transparent as much as possible.

The format of this paper is as follows. The next section provides a new formulation of the Nash social welfare function and compares it with the original one of Kaneko (1981). The new set of axioms for the Nash social welfare function does not include the continuity of social orderings, which played an important role in Kaneko's original proof of the derivation of the Nash social welfare function. Section 3 proves that the set of axioms derives the Nash social welfare fuction. The proof is much shorter and directer than the original proof.

2. The Axioms for the Nash Social Welfare Function

This section provides preliminaries and the axioms for the Nash social welfare function for a continuum of individuals.

Let (N,B_N,λ) be a measure space of individuals, where N is the interval $[0,1]$, B_N the set of Borel subsets of N and λ the Lebesgue measure on N.

This assumption seems to be strong and Kaneko (1981) treats a more general measure space. However, the measure space (N,B_N,λ) is an approximation of a large finite population and every individual should be counted with the same weights. If some structure has a special distribution on N, e.g., labor productivities, then it could be described as a measurable function on N. Therefore it is unnecessary to consider such a general measure space.

Let (X^*, B^*_X) be a pair of the set of all <u>pure alternatives</u> and a σ-field of subsets of X^*. We associate X^* with the <u>origin</u> x_o. Let $X = X^* \cup \{x_o\}$ and let B_X be the minimal σ-field which includes B^*_X and $\{x_o\}$. It is assumed that $\{x\} \in B_X$ for all $x \in X$.

The origin x_o should be interpreted as the "hell" where every member of the society must die immediately. In this sense, the origin x_o is the worst state for every individual. The full description of the necessity of this assumption will be given from a contractarian viewpoint in a forthcoming paper.

Let P be the set of all probability measures p on (X, B_X). The set of all real-valued functions on B_X is a linear space on reals and includes P. It is easily verified that P is also a convex subset of this linear space. Therefore for $(\alpha_1, \ldots, \alpha_n)$ with $\sum_{t=1}^{n} \alpha_t = 1$ and $\alpha_t \geqq 0$ for all t, $p = (\alpha_1 p_1 + \ldots + \alpha_n p_n)$ is well-defined and p is given as $p(E) = \sum_{t=1}^{n} \alpha_t p_t(E)$ for all $E \in B_X$. Regarding a pure alternative $x \in X$ as the measure p such that $p(\{x\}) = 1$, X can be considered to be a subset of P. In particular, $(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n)$ means the measure which has n possible outcomes x_t (pure alternatives) with probabilities α_t, respectively. A measure $p \in P$ is also called a <u>(mixed) alternative</u>.

An <u>individual preference ordering</u> R_i is a complete preordering on P. Here $p_1 R_i p_2$ means that individual i prefers p_1 to p_2 or is indifferent between p_1 and p_2. The nonsymmetric part of R_i is denoted by S_i and the symmetric part of R_i is denoted by I_i. The relation S_i is called the <u>strict preference</u>

<u>ordering</u> and I_i the <u>indifference</u> <u>relation</u> of R_i . It is assumed that every individual preference ordering R_i satisfies

(P.1) $x \; R_i \; x_o$ for all $x \in X^*$.

This axiom corresponds to the above interpretation of the origin x_o, i.e., the origin is the worst state for every individual.

A function f from N to the set of all individual preference orderings is called a <u>profile</u> if there is a real-valued function u(i,p) on $N \times P$ such that

(P.2) for each $i \in N$, $u(i,p_1) \geqq u(i,p_2)$ if and only if $p_1 \; f(i) \; p_2$;

(P.3) for each $i \in N$, u(i,x) is a B_X-measurable function of x and $u(i,p) = \int_X u(i,x) \, dp$ for all $p \in P$;

(P.4) for each $p \in P$, u(i,p) is a B_N-measurable function of i and the integral $\int_N \log(u(i,p)-u(i,x_o)) \, d\lambda$ is determinate with $\int_N \log(u(i,p)-u(i,x_o)) \, d\lambda < +\infty$, more precisely, $\int_N \max(0,\log(u(i,p)-u(i,x_o)) \, d\lambda < +\infty$;

(P.5) for some $p^* \in P$, $\log(u(i,p^*)-u(i,x_o))$ is a real-valued integrable function of i .

The set of all profiles is denoted by F. Note that axiom (P.3) implies that u(i,p) is linear with respect to mixture operation, i.e.,

$$u(i,\alpha p_1+(1-\alpha)p_2) = \alpha u(i,p_1) + (1-\alpha)u(i,p_2) \quad \text{for all } \alpha \in [0,1] \text{ and } p_1,p_2 \in P.$$

Axiom (P.2) means that each individual preference ordering is represented by u(i,.) and (P.3) means that u(i,.) satisfies the expected utility hypothesis.

In fact, this axiom can be derived from a certain set of axioms on individual preference orderings, that is, it can be a consequence of a certain type of von Neumann-Morgenstern utility theory. See DeGroot (1970) and Kaneko (1978). Axioms (P.4) and (P.5) are rather technical conditions, which are quite natural when we work on a model with a continuum of individuals. If $u(i,p)-u(i,x_o)$ is integrable, then (P.4) is satisfied, but it may be the case that $\int_N \log(u(i,p)-u(i,x_o))\,d\lambda = -\infty$. Axiom (P.5) requires the existence of p^* for which $\log(u(i,p^*)-u(i,x_o))$ is integrable. We will discuss axioms (P.4) and (P.5) again after the main theorem will be stated.

Lemma 1. Let f be a profile and let u satisfy (P.2) - (P.5). Then v satisfies (P.2) - (P.5) if and only if there are B_N-measurable real-valued functions a(i) and b(i) such that (i) $a(i) > 0$ for all i, (ii) $v(i,p) = a(i)u(i,p)+b(i)$ for all i and $p \in P$ and (iii) log a(i) is integrable.

Proof. Sufficiency is clear. It can be proved in the same way as Lemma 1 of Kaneko (1981) that if u and v satisfy (P.2)-(P.5), then there exist B_N-measurable functions a(i) and b(i) on N with (i) and (ii). Note that $\log(u(i,p^*)-u(i,x_o))$ and $\log(v(i,p^{**})-v(i,x_o))$ are integrable for some p^* and p^{**} in P. Since $a(i) = (v(i,p^*)-v(i,x_o))/(u(i,p^*)-u(i,x_o))$ by (ii), i.e., $\log a(i) = \log(v(i,p^*)-v(i,x_o))-\log(u(i,p^*)-u(i,x_o))$, the integral $\int_N \log a(i)\,d\lambda$ is determinate with $\int_N \log a(i)\,d\lambda < +\infty$ by axiom (P.4) and the above note. Since $\log(v(i,p^{**})-v(i,x_o))$ is integrable, $\int_N \log a(i)\,d\lambda = \int_N \log(v(i,p^{**})-v(i,x_o))d\lambda - \int_N \log(u(i,p^{**})-u(i,x_o))d\lambda > -\infty$ by axiom (P.4). Hence log a(i) is integrable. Q.E.D.

Let U(f) be the set of all utility functions u(.,.) representing profile f, i.e., satisfying (P.2)-(P.5). An alternative $p \in P$ is said to be non-singular for profile f if

$$\log(u(i,p)-u(i,x_o)) \text{ is an integrable real-valued function of } i \text{ for some } u \in U(f). \quad (1)$$

For each $f \in F$, the set of all non-singular alternatives for f is denoted by P_f. Note that P_f is always nonempty by axiom (P.5).

A <u>social welfare function</u> W(f) is a function on F which assigns a complete preordering $R_f = W(f)$ on P_f to each f in F. More precisely, W(f) assigns to each profile $f \in F$ the subset P_f of P and a complete preordering R_f on the set P_f. R_f is called a <u>social ordering</u>. The expression $p_1 \; R_f \; p_2$ means that alternative p_1 is socially better than p_2 or is indifferent to p_2 when the profile is f. The nonsymmetric part of R_f is denoted by S_f and the symmetric part of R_f is denoted by I_f. We call S_f the <u>strict social ordering</u> and I_f the <u>social indifference relation</u> of R_f.

Now we are in a position to state the axioms which determine the Nash social welfare function.

(S.1) <u>Pareto Condition</u>: Let f be an arbitrary profile in F with $f(i) = R_i$. Let p_1 and p_2 be any alternatives in P_f such that $p_1 \; S_i \; p_2$ for almost all $i \in N$. Then $p_1 \; S_f \; p_2$.

(S.2) <u>Anonymity</u>: Let θ be a measure-preserving automorphism on N, i.e., is a one-one onto measurable function from N to N such that the inverse function of f is also measurable and $\lambda(S) = \lambda(\theta(S))$ for all $S \in B_N$. Then $W(f) = W(\theta f)$ for all $f \in F$, where $\theta f(i) = f(\theta(i))$ for all $i \in N$.[1)]

(S.3) <u>Independence of Irrelevant Alternatives other than the Origin with Neutral Property</u>: Let f and g be any two alternatives in F. Let p_1, p_2 be any alternatives in P_f and let p_3, p_4 be any alternatives in P_g.

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1) Note that $P_f = P_{\theta f}$.

Suppose that for all $i \varepsilon N$,

$$(\alpha_1 p_1+\alpha_2 p_2+\alpha_3 x_o)\ f(i)\ (\beta_1 p_1+\beta_2 p_2+\beta_3 x_o)$$

if and only if

$$(\alpha_1 p_3+\alpha_2 p_4+\alpha_3 x_o)\ g(i)\ (\beta_1 p_3+\beta_2 p_4+\beta_3 x_o)$$

for all probability vectors $(\alpha_1,\alpha_2,\alpha_3)$ and $(\beta_1,\beta_2,\beta_3)$. Then $p_1\ R_f\ p_2$ if and only if $p_3\ R_g\ p_4$.

Although the anonymity axiom (S.2) is written in a different form from that in Kaneko (1981), they are almost equivalent. The Pareto axiom (S.1) and the independence axiom (S.3) are exactly the same as those in Kaneko (1981). However, the continuity axiom for social welfare functions needs to be written in a different form.

A sequence $\{f_n\}$ of profiles in F is said to <u>converge to</u> profile g if $\{i\varepsilon N : f_n(i) = g(i)\}\ \varepsilon\ B_N$ for all n and $\lim_{n \to +\infty} \lambda(\{i\varepsilon N : f_n(i) = g(i)\}) = \lambda(N)$. Let $T_n = \{i \varepsilon N: f_n(i) \neq g(i)\}$ for all n. A converging sequence $\{f_n\}$ is said to be <u>cannonical for alternative</u> $p\ \varepsilon\ P$ if for all n, there is a $u_n\ \varepsilon\ U(f_n)$ such that (i) $\log(u_n(i,p)-u_n(i,x_o))$ is integrable and (ii) $\int_{T_n} \log(u_n(i,p)-u_n(i,x_o))\, d\lambda \to 0\ (n \to \infty)$.[2)]

(S.4) <u>Continuity</u>: Let $\{f_n\}$ be a converging sequence to g. If $\{f_n\}$ is cannonical for p_1,p_2 with $p_1\ R_{f_n}\ p_2$ for all n and $p_1,p_2\ \varepsilon\ P_g$, then $p_1\ R_g\ p_2$.

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2) The reader might feel that the requirement $\lim \int_{T_n} \log(u_n(i,p)-u_n(i,x_o))d\lambda = 0$ is strange. However, for example, Lebesgue's dominated convergence theorem needs integrable boundedness. The above condition is weaker than integrable boundedness ; that is, if $\log(u_n(i,p)-u_n(i,x_o))$ is integrably bounded, then the above condition holds. Thus the process of taking limit in measure theory needs this kind of condition.

To state the main theorem, we need to define the Nash social welfare function. The Nash social welfare function W_o is defined by

$$\text{for any } f \in F \text{ and } p_1, p_2 \in P_f\ ,\ p_1\ W_o(f)\ p_2 \text{ if and only if for some } u \in U(f),\ \int_N \log(u(i,p_1)-u(i,x_o))\,d\lambda \geq \int_N \log(u(i,p_2)-u(i,x_o))\,d\lambda > +\infty\ . \tag{1}$$

The well-definedness of (1) can be ensured by Lemma 1 and the following lemma.

Lemma 2. For any $p_1, p_2 \in P_f$ and $u, v \in U(f)$, if $\log(u(i,p_1)-u(i,x_o))$ and $\log(v(i,p_2)-v(i,x_o))$ are integrable, then $\log(u(i,p_2)-u(i,x_o))$ and $\log(v(i,p_1)-v(i,x_o))$ are also integrable.

Proof. By Lemma 1, there are $a(i)$ and $b(i)$ such that $v(i,p) = a(i)u(i,p)+b(i)$ for all i and $p \in P$. Note that $\log a(i)$ is integrable. Since $\log(v(i,p_t)-v(i,x_o)) = \log(u(i,p_t)-u(i,x_o)) + \log a(i)$ for all i and $t = 1,2$, the integrablity of $\log(v(i,p_t)-v(i,x_o))$ is equvalent to that of $\log(u(i,p_t)-u(i,x_o))$. Q.E.D.

Theorem. Assume that X^* has at least three alternatives. Then a social welfare function W satisfies axioms (S.1)-(S.4) if and only if W coincides with the Nash social welfare function W_o .

There are two important differences between the present formulation and the original one of Kaneko (1981). The first one is in the axiomization of utility functions (profiles). This paper allows (log-) integrable utility functions (axiom (P.4)) but Kaneko (1981) did uniformly bounded utility functions. In this sense, the present formulation is more general.

However, the boundedness of each individual's utility function could be justified by the St.Petersburg paradox. See Aumann (1977). Note that this does not yet imply uniform boundedness. In applications, the boundedness of utility functions could be quite important, e.g., the uniform boundedness of $u(i,.)-u(i,x_o)$ played a substantial role in the limit tax rate theorem (Kaneko (1982,Theorem 3.1)). In fact, if there is an integrable upper bound of $u(i,.)-u(i,x_o)$, then u can be transformed into a uniformly bounded utility function without loss of generality. Let M(i) bean upper bound of $\{u(i,p)-u(i,x_o): p\in P\}$. Let $M(i) \geqq 1$ for all $i \in N$ and M(i) be integrable, which imply that log M(i) is also integrable. Difine v by

$$v(i,p) - v(i,x_o) = (u(i,p)-u(i,x_o))/M(i) \quad \text{for all } i \text{ and } p \in P.$$

Then v is uniformly bounded and satisfies the property

$\log(u(i,p)-u(i,x_o))$ is integrable if and only if $\log(v(i,p)-v(i,x_o))$ is integrable.

Thus if there is an integrable upper bound, then uniform boundedness can be assumed without loss of generality in effect.

Another important point is to remove continuity on social orderings, which played an essential role in Kaneko's original proof. The new proof given in the next section does not use the continuity axiom on social orderings at all but the Nash social welfare function satisfies it.[3] Therefore the continuity axiom is not independent from the others (S.1)-(S.4).

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3) In fact, d'Aspremont and Gevers (1977) derived the utilitarian social welfare function without using the continuity axiom in a finite society.

Then it is a natural question whether or not axioms (S.1) - (S.4) are really independent. Unfortunately the present author can not give a complete answer to this question. The following three examples designate that (S.1) - (S.3) are independent axioms. However the author has not succeeded in finding any counter example for axiom (S.4). Of course, axiom (S.4) plays an essential role in the proof of Section 3. Therefore it is an open problem whether or not axiom (S.4) is really necessary in our axiomization of the Nash social welfare function. The author feels that although a counter example would be found, it would be proved that axioms (S.1)-(S.3) imply

$$\text{if } \int_N \log(u(i,p_1)-u(i,x_o))d\lambda > \int_N \log(u(i,p_2)-u(i,x_o))d\lambda, \text{ then } p_1\ S_f\ p_2 .$$

These are remaining open problems of importance in this topic.

Example 1. Define a social welfare function W_P by

$p_1\ W_P(f)\ p_2$ if and only if $\int_N \log(u(i,p_1)-u(i,x_o))d\lambda \leqq \int_N \log(u(i,p_2)-u(i,x_o))d\lambda$ for some $u \in U(f)$.

This W_P satisfies (P.2)-(P.4) but not (P.1).

Example 2. Let $\alpha(i)$ be a measurable function on N with $m \leqq \alpha(i) \leqq M$ for some positive $m < M$ and all i. Define W_A by

$p_1\ W_A(f)\ p_2$ if and only if $\int_N \alpha(i)\log(u(i,p_1)-u(i,x_o))d\lambda \geqq \int_N \alpha(i)\log(u(i,p_2)-u(i,x_o))d\lambda$ for some $u \in U(f)$.

This W_A satisfies (P.1),(P.3),(P.4) but not (P.2).

Example 3. For any $f \in F$, choose an arbitrary u_f from U(f). Then define W_I by

$$p_1\ W_I(f)\ p_2 \text{ if and only if } \int_N (1-e^{-(u_f(i,p_1)-u_f(i,x_o))})d\lambda \geqq \int_N (1-e^{-(u_f(i,p_2)-u_f(i,x_o))})d\lambda .$$

This W_I satisfies (S.1),(S.2),(S.4) but not (S.3).

3. Proof of the Theorem

Sufficiency can be proved in the same way as Kaneko (1981,p.183-186). Here we prove only necessity and assume throughout this section that W(f) is an arbitrary social welfare function which satisfies axioms (S.1)-(S.4).

Let L = {k: k is a real-valued integrable function on N}. A binary relation Q on L is introduced as follows. Let k_1 and k_2 be arbitrary functions in L. For some pure alternatives $x_1, x_2 \in X^*$, define a function u on X by

$$u(i,x_1) = 10^{k_1(i)},\ u(i,x_2) = 10^{k_2(i)} \quad \text{for all } i \in N \text{ and}$$
$$u(i,x) = 0 \quad \text{for all } i \in N \text{ and } x \in X - \{x_1,x_2\}. \tag{2}$$

Since $\{x\} \in B_X$ for all $x \in X$, u(i,x) is a B_X-measurable function for each fixed i. Hence we can define u(i,p) for any p by

$$u(i,p) = \int_X u(i,x)\ dp\ . \tag{3}$$

It is easy to see that this u satisfies axioms (P.4) and (P.5). Therefore for each $i \in N$, we can define a binary relation f(i) by

$$p_1\ f(i)\ p_2 \text{ if and only if } u(i,p_1) \geqq u(i,p_2)\ . \tag{4}$$

Then this f is a profile. Using this profile f, define a binary relation Q on L by

$$k_1\ Q\ k_2 \text{ if and only if } x_1\ W(f)\ x_2\ . \tag{5}$$

Axiom (S.3) ensures the well-definedness of (5), i.e., Q does not depend upon the choice of x_1, x_2 and the remaining part of the profile.

The following lemma can be proved in the same way as Kaneko and Nakamura (1979,Lemma 3.1).

Lemma 3. The binary relation Q is a complete preordering.

Remark. We need the assumption that the number of (pure) alternatives in X^* is at least three, only to prove the transitivity of Q . If $|X^*| = 1$, then the theorem is trivial, and so it does not cover only the case where $|X^*| = 2$.

The nonsymmetric part and symmetric part of Q are denoted by Q_S and Q_I .

Lemma 4. If $k_1(i) > k_2(i)$ for almost all i , then $k_1 \; Q_S \; k_2$.

Proof. It can be proved in the same way as Kaneko and Nakamura (1979,Lemma 3.2).

Lemma 5. For any measure-preserving automorphism θ on N with $\theta\theta(i) = i$ for all i, $\theta k \; Q_I \; k$ for all $k \varepsilon L$.

Proof. Let x_1, x_2 be arbitrary alternatives in X^*. Define u_1 and u_2 by

$$\begin{aligned} &u_1(i,x_1) = 10^{k(i)},\ u_1(i,x_2) = 10^{\theta k(i)} \quad \text{for all } i \text{ and} \\ &u_1(i,x) = 0 \quad \text{for all } i \text{ and } x \;\varepsilon\; X-\{x_1,x_2\}; \text{ and} \\ &u_2(i,x_1) = 10^{\theta k(i)},u_2(i,x_2) = 10^{k(i)} \quad \text{for all } i \text{ and} \\ &u_2(i,x) = 0 \quad \text{for all } i \text{ and } x \;\varepsilon\; X-\{x_1,x_2\}. \end{aligned} \tag{6}$$

Similarly to (2) and (3), u_1 and u_2 can be extended on P. Then there exist profiles f_1 and f_2 such that $u_1 \;\varepsilon\; U(f_1)$ and $u_2 \;\varepsilon\; U(f_2)$. It follows from (6) and axiom (S.3) that $x_1 \; W(f_1) \; x_2$ if and only if $x_2 \; W(f_2) \; x_1$. On the other hand, clearly $f_1(i) = \theta f_2(i) = f_2(\theta(i))$ for all i. This and axiom (S.2) imply that $x_1 \; W(f_1) \; x_2$ if and only if $x_1 \; W(f_2) \; x_2$. Therefore we have $x_1 \; W(f_1) \; x_2$ and $x_2 \; W(f_1) \; x_1$. This implies $k \; Q_I \; \theta k$. Q.E.D.

Lemma 6. For any k_1, k_2 and k_3 in L , $k_1\ Q_I\ k_2$ if and only if $(k_1+k_3)\ Q_I\ (k_2+k_3)$.

Proof. Similarly to the proof of Lemma 5, choose a function u on P such that

$$u(i,x_1) = 10^{k_1(i)+k_2(i)}\ ,\ u(i,x_2) = 10^{k_2(i)+k_3(i)} \text{ for all } i \text{ and } \quad u(i,x) = 0 \text{ for all } i \text{ and } x \in X-\{x_1,x_2\}\ . \tag{7}$$

For some $f \in F$, u belongs to U(f). Consider $v(i,.) = (1/10^{k_3(i)})u(i,.)$ for all i. Then v also belongs to U(f). Therefore $k_1\ Q_I\ k_2$ if and only if $x_1\ I_f\ x_2$ if and only if $(k_1+k_3)\ Q_I\ (k_2+k_3)$. Q.E.D.

Lemma 7. Let $\{k_n\}$ be a sequence in L such that for some $k_\infty \in L$, $\int_N k_n(i)\, d\lambda = \int_N k_\infty(i)\, d\lambda$ for all n and $\lim_{n \to +\infty} \lambda(\{i\in N: k_n(i) = k_\infty(i)\}) = \lambda(N)$. If $k_n\ Q_I\ k^*$ for all n, then $k_\infty\ Q_I\ k^*$.

Proof. Similarly to the proof of Lemma 5, define u_n ($n = 1,\ldots,\infty$) by

$$u_n(i,x_1) = 10^{k_n(i)}\ ,u_n(i,x_2) = 10^{k^*(i)} \text{ for all } i \text{ and } \quad u_n(i,x) = 0 \text{ for all } i \text{ and } x \in X-\{x_1,x_2\}\ . \tag{8}$$

For some $f_n \in F$, u_n belongs to $U(f_n)$ for all $n = 1,\ldots,\infty$. Then it follows from (8) that $k_n\ Q_I\ k^*$ implies $x_1\ I_{f_n}\ x_2$ for all n. It is easy to see that $\lambda(\{i\in N: f_n(i) = f_\infty(i)\}) = \lambda(\{i\in N: k_n(i) = k_\infty(i)\})$ for all n. Let $T_n = \{i\in N: f_n(i) \neq f_\infty(i)\}$ for all n. Then $\lambda(N-T_n) \to \lambda(N)$ $(n \to +\infty)$ and

$$\int_{T_n} \log(u_n(i,x_1)-u_n(i,x_o))\, d\lambda = \int_{T_n} k_n(i)\, d\lambda = \int_{T_n} k_\infty(i)\, d\lambda \to 0 \ (n \to +\infty),$$

because $\int_N k_n(i)\, d\lambda = \int_N k_\infty(i)\, d\lambda$ and $\int_{N-T_n} k_n(i)\, d\lambda = \int_{N-T_n} k_\infty(i)\, d\lambda$ for all n. It is also clear that $\int_{T_n} \log(u_n(i,x_2)-u_n(i,x_o))\, d\lambda = \int_{T_n} k^*(i)\, d\lambda \to 0$ $(n \to +\infty)$. Therefore $\{f_n\}$ is a converging sequence to f_∞ which is cannonical for x_1 and x_2. Hence we have, by axiom (S.4), $x_1\ I_{f_\infty}\ x_2$. This implies $k_\infty\ Q_I\ k^*$. Q.E.D.

Lemma 8. For any k_o and $k_\infty \varepsilon L$, if $\int_N k_o(i)\, d\lambda = \int_N k_\infty(i)\, d\lambda$, then $k_o\ Q_I\ k_\infty$.

Proof. Define a sequence $\{k_n\}$ inductively by

$$k_0(i) = k_o(i) \quad \text{for all } i\varepsilon N; \text{and for all } n \geq 1,$$

$$\begin{aligned} k_n(i) &= k_{n-1}(i) + (k_\infty(i) - k_{n-1}(i)) && \text{if } i \,\varepsilon\, \left(\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}\right] \\ &= k_{n-1}(i) - (k_\infty(i-\frac{1}{2^n}) - k_{n-1}(i-\frac{1}{2^n})) && \text{if } i \,\varepsilon\, \left(\frac{2^n-1}{2^n}, 1\right] \\ &= k_{n-1}(i) && \text{otherwise.} \end{aligned}$$

For any n, let Δ_n be the function such that

$$\begin{aligned} \Delta_n(i) &= (k_\infty(i) - k_n(i))/2 && \text{if } i \,\varepsilon\, \left(\frac{2^{n+1}-2}{2^{n+1}}, \frac{2^{n+1}-1}{2^{n+1}}\right] \\ &= -\,(k_\infty(i-\frac{1}{2^{n+1}}) - k_n(i-\frac{1}{2^{n+1}}))/2 && \text{if } i \,\varepsilon\, \left(\frac{2^{n+1}-1}{2^{n+1}}, 1\right] \\ &= 0 && \text{otherwise;} \end{aligned}$$

and let θ_n be the measure-preserving automorphism on N such that

$$\begin{aligned} \theta_n(i) &= i + \frac{1}{2^{n+1}} && \text{if } i \,\varepsilon\, \left(\frac{2^{n+1}-2}{2^{n+1}}, \frac{2^{n+1}-1}{2^{n+1}}\right] \\ &= i - \frac{1}{2^{n+1}} && \text{if } i \,\varepsilon\, \left(\frac{2^{n+1}-1}{2^{n+1}}, 1\right] \\ &= i && \text{otherwise.} \end{aligned}$$

Since $\Delta_n(\theta_n(\theta_n(i))) = \Delta_n(i)$, $\Delta_n(\theta_n(i)) + \Delta_n(i) = 0$ for all n and i,

$$k_n = \theta_n(\theta_n(k_n+\Delta_n)+\Delta_n)\ .^{4)}$$

It is also easy to see that $k_{n+1} = k_n + 2\Delta_n$ for all n. Using Lemmas 5 and 6 repeatedly, we have

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4) Remember the notation $(\theta_n k_n)(i) = k_n(\theta_n(i))$.

$\theta_n(k_n+\Delta_n)\ Q_I\ (k_n+\Delta_n)$, $(\theta_n(k_n+\Delta_n)+\Delta_n)\ Q_I\ (k_n+2\Delta_n)$

and $k_n = \theta_n(\theta_n(k_n+\Delta_n)+\Delta_n)\ Q_I\ (k_n+2\Delta_n) = k_{n+1}$.

Therefore $k_o\ Q_I\ k_n$ for all n. Since $\int_N \Delta_n(i)\ d\lambda = 0$, we have, for all n,

$$\int_N k_{n+1}(i)\ d\lambda = \int_N k_n(i)\ d\lambda + 2\int_N \Delta_n(i)\ d\lambda = \int_N k_n(i)\ d\lambda .$$

Furthermore since $\{i\varepsilon N: k_n(i) \neq k_\infty(i)\} \subset (\frac{2^n-1}{2^n}, 1]$ for all n, $\lambda(\{i\varepsilon N: k_n(i) = k_\infty(i)\}) \to \lambda(N)$ ($n \to +\infty$). Therefore the sequence $\{k_n\}$ satisfies the supposition of Lemma 7. This implies $k_o\ Q_I\ k_\infty$. Q.E.D.

Lemma 9. For all $k_1, k_2 \varepsilon L, \int_N k_1(i)\ d\lambda \geqq \int_N k_2(i)\ d\lambda$ if and only if $k_1\ Q\ k_2$.

Proof. It suffices to show that $\int_N k_1(i)\ d\lambda > \int_N k_2(i)\ d\lambda$ implies $k_1\ Q_S\ k_2$. Suppose $\int_N k_1(i)\ d\lambda > \int_N k_2(i)\ d\lambda$ and let $\Delta = \int_N k_1(i)\ d\lambda - \int_N k_2(i)\ d\lambda$. Define k_3 by

$$k_3(i) = k_2(i) + \Delta/\lambda(N) \qquad \text{for all i.}$$

Then $k_3\ Q_S\ k_2$, by Lemma 4. Since $\int_N k_1(i)\ d\lambda = \int_N k_3(i)\ d\lambda$, we have, by Lemma 8, $k_1\ Q_I\ k_3$. Therefore $k_1\ Q_I\ k_3\ Q_S\ k_2$, which implies $k_1\ Q_S\ k_2$. Q.E.D.

We are now in a position to complete the proof of the theorem. Let f be a profile. If P_f consists of a unique alternative, then the theorem is trivial. Let us suppose that P_f has at least two alternatives p_1 and p_2 . Then $\log(u(i,p_1) - u(i,x_o))$ and $\log(u(i,p_2)-u(i,x_o))$ are functions in L for some $u\ \varepsilon\ U(f)$ by Lemma 2. Another profile g can be constructed in the same way as (2)-(4): Some $v\ \varepsilon\ U(g)$ satisfies

$v(i,x_1) = 10^{\log(u(i,p_1)-u(i,x_o))}$, $v(i,x_2) = 10^{\log(u(i,p_2)-u(i,x_o))}$ for all i and $v(i,x) = 0$ for all $i\varepsilon N$ and $x\varepsilon X-\{x_1,x_2\}$.

Then it follows from axiom (S.3) that

$$p_1 \; W(f) \; p_2 \text{ if and only if } x_1 \; W(g) \; x_2 \; . \qquad (9)$$

Hence we have, by (5) and Lemma 9,

$x_1 \; W(g) \; x_2$ if and only if $\log(u(i,p_1)-u(i,x_o)) \; Q \; \log(u(i,p_2)-u(i,x_o))$ if and only if $\int_N \log(u(i,p_1)-u(i,x_o)) \, d\lambda \geqq \int_N \log(u(i,p_2)-u(i,x_o)) \, d\lambda$.

This and (9) imply

$$p_1 \; W(f) \; p_2 \text{ if and only if } \int_N \log(u(i,p_1)-u(i,x_o)) \, d\lambda \geqq \int_N \log(u(i,p_2)-u(i,x_o)) \, d\lambda \; .$$

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