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Reversibility of Tandem Blocking Queueing Systems

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Abstract

This paper is concerned with queueing systems of several service stations in series in which each station may consist of multi-servers. Infinite customers are always waiting in front of the first station, and each customer passes through all of the stations in sequence. There are only finite waiting positions between any two adjacent stations. The service time for a customer at any station is assumed to be a random variable. In this mode of operation the servers at any station will at any time be either busy, or idle, or blocked. This blocking system is said to be C-reversible if the capacity remains invariant under the reversed system, which is obtained by reversing the original stations' order, that is, every customer in the reversed system passes through the original stations in the reverse order. It has been already proved that C-reversibility holds for any blocking system in which each station consists of either a single server of nondeterministic service times or multi-servers of deterministic service times, and that the blocking system has a stronger property than C-reversibility. In this paper we show that two-station blocking systems with multi-server stations of nondeterministic service times are C-reversible, but this property can be no longer extended to three- or more station blocking systems with multi-server stations of nondeterministic service times. We also show that the stronger property does not hold even for two-station blocking systems in the case of involving multi-server station of nondeterministic service times.

1. Introduction

Many production lines can be viewed as a series of work stations where the operation times are variable rather than to be constant. This is especially so for automatic transfer production lines and mixed-item assembly lines to produce many kinds of product-items, which have come to attract attention lately. Such lines can be modeled effectively by tandem blocking queueing systems, that is, a series of finite queues.

Since the production rate of the line, which is called "the capacity" in queueing theory, becomes an important measurement standard to evaluate its system performance, the capacity of the tandem blocking queueing system has received an increasing attention. See, for example, Freeman[2], Fujii, Tanioka and Narutaki[3], Hildebrand [4], [5], Hillier and Boling [6], [7], Hunt [8], Knott [9], Makino [10], Suzuki [12] and Tumura and Ishikawa [13].

One of important questions, from the viewpoint of optimization of production rate, is how the capacity is affected by the rearrangement of the service stations' order. A special case of the rearrangement is that of the reversed system, to be explained in Section 2. Makino [10] showed analytically that the capacity does not change under the reversed system in some simple systems. This property of reversibility, C-reversibility (to be defined in Section 2), has drawn many researchers' attention, and some of them conjectured that C-reversibility holds for more general systems consisting of a single server stations on the basis of a series of numerical experiments (see, for example, Fujii et al. [3] and Hillier and Boling [6]).

This conjecture has been proved by Yamazaki and Sakasegawa [14], and they have proved a stronger property, "D-reversibility" (to be defined in Section 2), and also this has been proved by Muth [11] separately. Its theoretical extension and application have been done by Yamazaki, Sakasegawa and Kawashima [15]. Thereafter this subject has been discussed by Dattatreya [1].

The problem whether or not such reversibility can be extended to the blocking systems with multi-server stations of nondeterministic service times, however, remains to be solved. It is the purpose of this paper to seek solutions on this subject. We show that two-station blocking systems with multi-server stations of nondeterministic service times are C-reversible but are not D-reversible. We also show that C-reversibility cannot be extended to three- or more station blocking systems with multi-server stations of nondeterministic service times.

2. Model and Definitions

The beginning part of this section is to formalize the model. After this is done, some terms are defined with reference to the model.

This paper is concerned with the following tandem queueing system: There are K service stations, numbered 1 through K, arranged in tandem, in which each station may consist of multi-servers. Infinite customers are always waiting in front of the first station (station 1), and each customer passes through all of the stations in sequence. There are only finite waiting positions between any two adjacent stations. If the next waiting positions are full when a service to a certain customer is completed by one of servers at a station, this customer stays at the station and blocks the server until one of the waiting positions becomes vacant. The service times of customers at each station are independently and identically distributed nonnegative random variables. The service times at different stations are mutually independent.

Given a blocking system described above (to be defined as "the original system" thereafter), its reversed system is obtained by reversing its stations' order, that is, each customer in the reversed system passes through the stations in the order of K, K-1,..., 2, 1.

The capacity of a tandem queueing system is defined as the departure rate from the last station when infinite customers are always waiting in front of the first station. If the system is stable, of course, the departure rate from each station always remains unchanged.

<u>Definition</u> (Dattatreya [1]). The tandem queueing system is said to be C-reversible

if the original system has the same capacity as its reversed system.

Definition ([1]). Consider a blocking system and its reversed system which start the service processes at time t=0 with both systems vacant. Then, the blocking system is said to be D-reversible if the distributions of times until n-th departure epochs from both systems are identical for every n.

We note that D-reversibility of a blocking system implies C-reversibility when the capacity is well defined, but its contrary does not necessarily hold.

3. C-reversibility

Consider a two-station blocking system where station 1 and station 2 consist of m and n servers, respectively, and where there are N waiting positions between the two stations. Let B, Q and I be the numbers of servers blocked at station 1, customers occupying the waiting positions and idling servers at station 2 at any time in equilibrium, respectively. We similarly define B' (the number of servers blocked at station 2), Q' and I' (the number of idling servers at station 1) for its reversed system.

Then, we have the following:

Theorem.

(i) This blocking system is C-reversible.

(ii) The joint distributions of (B,Q,I) and (I',N-Q',B') are identical in the steady state.

This theorem is proved by constructing supplementary-variables Markov processes for both systems. Details are proved in Appendix.

Remarks.

(i) It is clear that the theorem holds not only under FCFS service discipline but also under any other discipline which selects the customer to be served next among waiting or blocking customers when a customer leaves the system.

(ii) Because of the technique of proof of the theorem, it remains valid when each station consists of heterogeneous servers under suitable disciplines; for example, under discipline which always selects the fastest among the free servers at each station.

This C-reversibility can be no longer extended to three- or more station blocking systems where certain station consists of multi-servers of nondeterministic service times, as shown in the following example:

<u>Counterexample</u>. Consider a three-station blocking system where station 3 consists of 2 servers and others a single server. There are no intermediate waiting positions. The service time of each station is exponentially distributed and the rates of stations 1, 2 and 3 are 2, 2 and 1, respectively. This original and its reversed systems are shown in Figure. The capacity of the original system can be obtained by solving the equilibrium equations and it becomes $86/145 \simeq 0.5931$. Similarly the capacity of its reversed system becomes $22/37 \simeq 0.5946$, and these two values are not idetical.

Blocking systems with multi-server stations of deterministic service times are discussed in the next section.

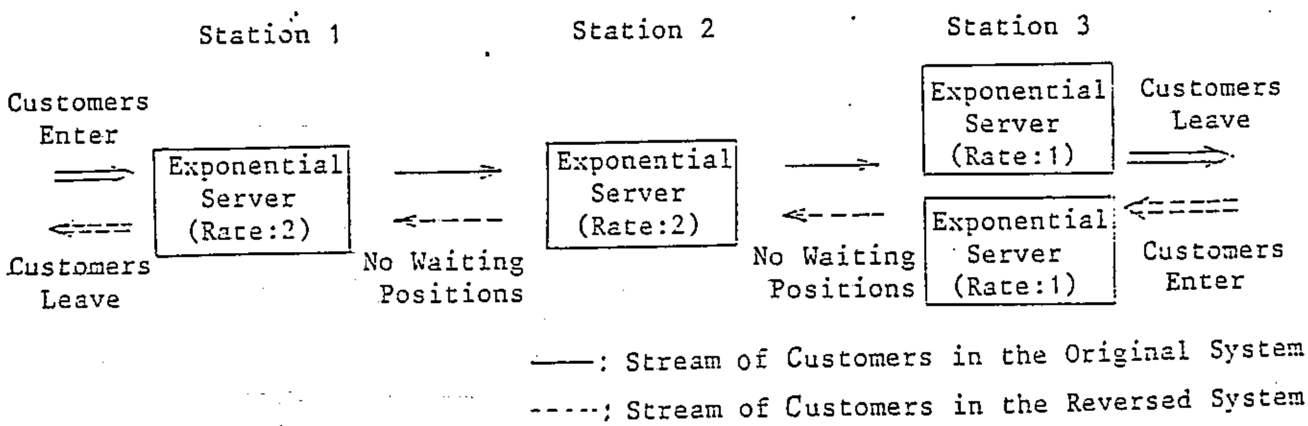

Figure. Three-Station Blocking System Not to Be C-Reversible

4. D-reversibility

D-reversibility of blocking systems proved in the authors' preceding paper [15] is summarized as follows:

<u>Theorem.</u> K-station blocking systems are D-reversible when each station consists of either a single server of nondeterministic service times or multi-servers of deterministic service times.

<u>Remarks.</u> The theorem holds even if the service times of the same customer at the different stations are dependent (see, for example, [1]).

It seems that the blocking systems with this property are featured by the fact that nondeterministic service times are permitted only at the single server stations. This prevents the passing of customers throughout the system according to the variation of service times at each station. In fact, in case where passing of customers occurs, this property cannot be extended even to two-station blocking systems, as shown in the following example:

<u>Counterexample.</u> Consider a two-station blocking system with no intermediate waiting positions, where stations 1 and 2 consist of 2 servers and a single server, respectively. Let S_1 and S_2 be the service times at stations 1 and 2, respectively, and let Prob.$(S_1 = 1)$ = Prob.$(S_1 = 2) = 1/2$, and $S_2 = 2$ for all customers. Denote by D_2 the time until the second customer comes out of the original system when the service process is started at time t=0 with system vacant. We similarly define D_2' for its reversed system. Then, it is easily seen by a little calculation that the distributions of D_2 and D_2' become

$$\text{Prob.}(D_2 = 5) = 3/4,\ \text{Prob.}(D_2 = 6) = 1/4$$

and

$$\text{Prob.}(D_2' = 5) = \text{Prob.}(D_2' = 6) = 1/2.$$

Hence D_2 is not stochastically equal to D_2'.

Appendix

Proof of Theorem in Section 3.

Consider the two-station blocking system described in Section 3. Observing the numbers of servers blocked at station 1 b, customers occupying the waiting positions q and idling servers at station 2 i, and ordered residual service times at station 1 $x_1, x_2, \ldots, x_{m-b}$ and at station 2 $y_1, y_2, \ldots, y_{n-i}$ $(0 \le x_1 \le x_2 \le \ldots \le x_{m-b}\ ;\ 0 \le y_1 \le y_2 \le \ldots \le y_{n-i})$, the behavior of the original system can be obviously by a Markov process X(t) with state space S, where

$$S = \{ (b,\ q,\ i;\ x_1, x_2, \ldots, x_{m-b};\ y_1, y_2, \ldots, y_{n-i}) : 0 \le b \le m,\ 0 \le q \le N,\ 0 \le i \le n,\ 0 \le x_1 \le x_2 \le \ldots \le x_{m-b},\ 0 \le y_1 \le y_2 \le \ldots \le y_{n-i} \} .$$

Note that servers at station 1 never become idle since infinite customers are always waiting in front of the station, and that servers at station 2 are never blocked. Also note that when $q < N$, blocking never occur, that is, b = 0, and that when $q > 0$, idling servers do not exist, that is, i = 0. The transition of X(t) can be described as follows:

Assuming $X(t) = (b,\ q,\ i;\ x_1, \ldots, x_{m-b};\ y_1, \ldots, y_{n-i})$,

$X(t+h) = (b,\ q,\ i;\ x_1-h, \ldots, x_{m-b}-h;\ y_1-h, \ldots, y_{n-i}-h)$ for $0 \le h \le \min(x_1, y_1)$,

and X jumps at $t + \min(x_1, y_1)$ to

$$(0,\ 0,\ i-1;\ R(S_1,\ x_2-x_1, \ldots, x_m-x_1);\ R(S_2,\ y_1-x_1, \ldots, y_{n-i}-x_1)) \text{ if } b=0,\ q=0,\ 0 < i < n \text{ and } x_1 \le y_1 \quad (A.1),$$

$$(0,\ q+1,\ 0;\ R(S_1,\ x_2-x_1, \ldots, x_m-x_1);\ y_1-x_1, \ldots, y_n-x_1) \text{ if } b=0,\ 0 \le q < N,\ i=0 \text{ and } x_1 \le y_1 \quad (A.2),$$

$$(b+1,\ N,\ 0;\ x_2-x_1, \ldots, x_{m-b}-x_1;\ y_1-x_1, \ldots, y_n-x_1) \text{ if } 0 \le b < m,\ q=N,\ i=0 \text{ and } x_1 \le y_1 \quad (A.3),$$

$$(0,\ 0,\ i+1;\ x_1-y_1, \ldots, x_m-y_1;\ y_2-y_1, \ldots, y_{n-i}-y_1) \text{ if } b=0,\ q=0,\ 0 \le i < n, \text{ and } x_1 > y_1 \quad (A.4),$$

$$(0,\ q-1,\ 0;\ x_1-y_1, \ldots, x_m-y_1;\ R(S_2,\ y_2-y_1, \ldots, y_n-y_1)) \text{ if } b=0,\ 0 < q \le N,\ i=0 \text{ and } x_1 > y_1 \quad (A.5),$$

$$(b-1,\ N,\ 0;\ R(S_1,\ x_1-y_1, \ldots, x_{m-b}-y_1);\ R(S_2,\ y_2-y_1, \ldots, y_n-y_1)) \text{ if } 0 < b < m,\ q=N,\ i=0 \text{ and } x_1 > y_1 \quad (A.6),$$

and X jumps at $t + x_1$ to

$$(0, 0, n-1; R(S_1, x_2-x_1,\dots, x_m-x_1); S_2) \text{ if } b=0, q=0 \text{ and } i=n \qquad (A.7),$$

and X jumps at $t + y_1$ to

$$(m-1, N, 0; S_1; R(S_2, y_2-y_1,\dots, y_n-y_1)) \text{ if } b=m, q=N \text{ and } i=0 \qquad (A.8),$$

where S_1 and S_2 are new (residual) service times at stations 1 and 2, respectively, they distribute independently of others, and $R(x_1, x_2,\dots, x_j)$ means reordering x_i's in ascending size.

Similarly to X(t), we can define a Markov process X'(t) with state space S' corresponding to its reversed system, where

$$S' = \{(b', q', i'; y_1', y_2',\dots, y'_{n-b'}; x_1', x_2',\dots, x'_{m-i'}); 0\le b'\le n, 0\le q'\le N,$$
$$0\le i'\le m, 0\le y_1'\le \dots\le y'_{m-b'}, 0\le x_1'\le \dots\le x'_{m-i'}\},$$

b' is the number of servers blocked at station 2 and i' is the number of idling servers at station 1.

Now, let f be an one-to-one mapping from S' to S such that

$$f(b, q, i; y_1, y_2,\dots, y_{n-b}; x_1, x_2,\dots, x_{m-i}) = (i, N-q, b; x_1, x_2,\dots, x_{m-i}; y_1, y_2,\dots, y_{n-b}).$$

Then, we have the following:

<u>Lemma</u>. X(t) and f(X'(t)) have the same transition densities.

Proof. In the original system, when a service to a customer is completed by one of servers at station 1 possible events are the following:

(1) If $i\ge 1$, then i decreases by 1 and new services simultaneously begin at both stations (cf. (A.1) and (A.7)).

(2) If i=0 and $q<N$, then q increases by 1 and a new service begins only at station 1 (cf. (A.2)).

(3) If q=N and i=0, then b increases by 1 (cf. (A.3)).

While in the reversed system, when a service to a customer is completed by one of servers at station 1 which is located in the second, possible events are the following:

(1') If $b'\ge 1$, then b' decreases by 1 and new services simultaneously begin at both stations.

(2') If $b'=0$ and $q'>0$, then q' decreases by 1, that is, $N-q'$ increases by 1, and a new service begins only at station 1.

(3') If $q'=0$ and $b'=0$, then i' increases by 1.

Obviously, (1'), (2') and (3') correspond to (1), (2) and (3), respectively. For the case where a service to a customer is completed by one of servers at station 2, the similar inter-acted correspondence is obtained. $X(t)$ and $f(X'(t))$ have, therefore, the same transition densities.

This lemma implies that it is possible to construct a probability space in which $X(t) = f(X'(t))$ for all t with probability one if $X(0) = s$, $X'(0) = f^{-1}(s)$ for any $s \in S$. Hence a service completion at station 1 in the original system corresponds to that at station 1 in the reversed system. The theorem in Section 3 is induced on the basis of the above fact.

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