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URBAN LAND USE PATTERNS WITH LAND OWNERSHIP; TOWARDS A GAME-THEORETIC FRAMEWORK FOR URBAN LAND USE THEORY+

by

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CONTENTS

ABSTRACT

The purpose of the present paper is to provide a theory of urban land use patterns with land ownership and to build a game-theoretic framework for urban land use theory. In recent land use theories, income of a household is given exogeneously and there are no initial endowments of land. Using the basic model in which income of the household is determined endogeneously by the change of land price, we present a residential location and a competitive urban land use pattern. Next, we build an urban land use model in which there are both types of household with and without land ownership. We obtain a location replacement theorem, that is, the household with initial land ownership closest to the CBD locates at the edge of the city and vice versa. Next, we build an optimum land use model, and we develop the correspondence theorem between the optimum and the market. We make a framework for an urban land use game, and present the equivalence theorem between stable and competitive land use patterns, that is, the core of the urban land use game is equivalent with the competitive equilibrium.

1. INTRODUCTION

Recently we have had many contributions to urban land use theories, in particular, residential land use has been analyzed by many authors. However, in recent papers on urban land use theories, usually in the formulation of the residential behavior of households, it is assumed that households only have incomes at the initial state. That is, the income of the household is exogeneously predetermined, and there are no initial endowments of land.

Generally speaking, in the real world initial endowments of land ownership play an important role for the formation of urban land use patterns. Therefore, in the present paper we provide urban land use patterns with land ownership. We first formulate the basic model of residential choice, with land ownership, where the income of a household is determined endogeneously by the change of land prices. Using this basic model of residential location with land ownership, we present the equilibrium spatial structure of urban land use. The concepts of bid prices and bid price curves play the fundamental role for these analysis. The proof of the existence and the uniqueness of the equilibrium is suggested by the three kinds of methods used. One is the constructive method. A second is the method using the correspondence between the optimum model and the market model. A third is the method using by the equivalence theorem between the core of the urban land game and the competitive equilibrium of the market model.

We present the location replacement theorem at equilibrium in the section three. That is, the equilibrium location of households with land ownership is ranked according to the initial location of land ownership. In other words, the farther the initial location of the land owned by a household is from the CBD, the closer to the CBD its optimal location. The replacement of location occures in this model.

Next, we show the relative advantage of land ownership in urban land use patterns in a model with and without land ownership. If the change of land prices is relatively larger, households with land ownership get the relative advantage in equilibrium. As a result, households with land ownership close to the CBD gains much money, and they reside farther from the CBD than the richer households without land ownership.

An optimum urban land use model, with land ownership, is formulated similarly to the standard optimum model which does not consider land ownership. Showing the correspondence between the optimum model and the market model, with subsidies, the existence and the uniqueness of the equilibrium are indirectly verified. This concept of equilibrium is a kind of compensated equilibrium, because it needs income subsidies for the equilibrium location of the households. Last of all we present a framework for an urban land use game. A stable spatial structure (Core) of the urban land game is defined, and presented the equivalènce theorem between the core of the urban game and the equilibrium spatial structure of the market mechanism.

Here, we review the literatures on urban land use theories with land ownership. The analysis of the effect of land ownership on urban spatial structure is carried out by methods similar to the comparative statics of income change. Hartwick, et. al., (1976) developed comparative statics of a residential city with a finite number of income classes. They obtained that "under the assumptions, in equilibrium, people in different classes reside in characteristic concentric rings around the CBD with poorer classes living closer to the CBD." (page 55). This is very similar to the findings of the present paper. However, they gave the income of households exogeneously.

Wheaton (1977) discussed the relationship between income and urban residence, and analyzed consumer demand for location. Fujita (1982b) defined the concept of the relative steepness of bid price curves and gave the relative location of households by income classes. Our discussion in this paper has been done using the concept of the relative steepness of bid price curves.

The existence and the uniqueness of a general competitive equilibrium of economies with land (or, urban spatial economies) have long been of interest to economists and regional scientists.

Koopmans and Beckmann (1957) showed that there does not necessarily exist a set of decentralized prices for a class of location-assignment problems. This is due to the indivisibility or discreteness of agents, that is, one and only one agent can be assigned to a location.

Beckmann (1969) extended the discrete model to the continuous model, with the assumption that there is a continuum of identical consumers.

Mills (1972) showed that in a simple urban setting with a continuum of agents, there exists a competitive equilibrium. Note that a continuum of traders means to allow fractional allocations of agents.

Scotchmer (1981) gave a continuum model with land using a measure-theoretic approach and proved the existence of a competitive equilibrium. However, in this model the income of a household is exogenously given and there are no initial endowments. Moreover, she used the concept of a compensated equilibrium, that is, lump sum income transfers were allowed and needed to prove the existence of an equilibrium.

Schweizer, et. al., (1976) tried to prove the existence of an equilibrium for a large number of consumers, parallel to the Arrow-Debreu type general equilibrium model. They also used the concept of compensated equilibrium.

However, as Fujita (1982b) criticized them, their model gave only a trivial solution, i.e., uniform distribution of all activities among all locations. This is due to the assumption of a homogeneous space, which means that the utility functions of households and technologies of firms are independent of their loction, and the initial endowment of resources is equal over the space.

Starret (1978) showed that under this assumption, there is no competitive equilibrium with a positive total transport cost. This theorem is named the Spatial Impossibility Theorm by Fujita (1982b). Note that our model of this paper does not satisfy the homogeneous space assumption.

One of the best methods to prove the equilibrium of an economy with land is the constructive method suggested by Fujita (1982b), which is like the constructive computational method for an economy without land by Scarf and Hansen (1973).

Another method was developed by Ando (1981) and Fujita (1982a) using the correspondence between the optimum model and the market model, because the existence of the optimum for urban land use is not difficult to prove by the maximum principle of the optimal control theory. This method was initially applied by Fujta (1976a) and (1976b) for a dynamic theory of urban land use. The optimum model of this approach is a continuous extension of the Herbert and Stevens Model (1960). At the same time Wheaton (1974) succeeded in formulating the continuous Herbert and Stevens Model independent of Fujtia. An equilibrium by the correspondence between the optimum and the market is a compensated equilibrium, because it needs income subsidies to verify an equilibrium.

A third method is to use the equivalence theorem between the stable spatial structure (core) of an urban land game and the equilibrium of the market model. This is a game-theoretic approach to urban land use theoreis. This approach needs a continuum of agents to verify the existence of an equilibrium for an economy with land, because usually there does not exist an equilibrium for an economy with land of a discrete number of agents.

One of the efforts has been done by McLean and Muench (1979) to eliminate the discontinuity between the discrete model and the continuum model. They considered a continuum city as a limit of a sequence of cities with a finite number of locations and agents. This method has seemed to be a forth method for verifying the existence of an equilibrium of an economy with land. This method used is similar to the relationship between the limit theorem for a discrete number of agents of an economy by Debreu and Scarf (1963) and the equivalence theorem of the core and competitive equilibrium using a measure-theoretic approach by Aumann (1964)(1966), and Hildenbrand (1974).

In this paper we develop a simple Lagrangian approach to the proof of the equivalence of the core of the urban game and the equilibrium of the market using a method similar to the method used by the Schweizer (1982) for the proof of the limit theorem on the core of an economy.

2. THE BASIC MODEL OF RESIDENTIAL LOCATION WITH LAND OWNERSHIP

First, we build the basic model of residential location of households with land ownership. That is, in this paper we focus mainly on the residential land use as in papers on recent urban land use theories.

To illustrate the basic theoretical structure, models in this paper are as simple as possible. The common assumptions employed in the residential land models here are as follows.

Assumption 2.1

(i) The city is monocentric and all the residents are assumed to commute to the CBD which can be seen to be an origin point.

(ii) The city is located on a featureless uniform plain. The transportation system in this city is radial and dense everywhere. There is no traffic congestion, hence transport cost is solely a function of distance from the CBD. The commuting cost from r to CBD, $T(r) \geqq 0$, is an exogeneously given, continuously differentiable function of distance $r \geqq 0$, with $T(0)=0$, where r is the distance from the CBD.

(iii) Households are classified $m \geqq 1$ types. Every household in a particular type i ($i = 1,2, \ldots, m$) has the same initial endowments of composite good a_i and land l_i at the location i which has the distance d_i from the CBD, $d_1 > d_2 > \cdots > d_{m-1} > d_m = 0$ ($\equiv$ CBD).

(iv) Every household of each type resides at only one location. If the number of households in type i is given by the continuous number N_i, and $n_i(r)$ denotes the number of households in type i locating at the distance r, then

$$\int_0^{\bar{r}} n_i\,(r)\,d_r = N_i \qquad (i = 1, 2, \ldots, m)$$

where $\bar{r}$ is the distance to the urban fringe.

(v) There exist no externalities such as environmental damage or neighborhood externalities and there are no public goods.

(vi) Assume that all the households have the same utility function,

$$U = u(z,q) \tag{2.1}$$

where z denotes the quantity of composit good which is considered as the numeraire, and q is the amount of land owned.[1]

And, the budget constraint of a household of type i is given by

$$z + P(r)q + T(r) = a_i + P(d_i)l_i \tag{2.2}$$

where P(r) is the land price at distance r from the CBD.[2]

Please note that income,[3] $y_i \equiv a_i + P(d_i)l_i$, of a household of type i is not given exogeneously, but is determined endogeneously by the market price of land.

The behavior of the residential choice of the household is described by

$$\left\{\begin{array}{l} \max\limits_{z,q,r} \; u\,(z,\,q) \\ \\ \text{subject to:} \\ z + P(r)\,q + T(r) = a_i + P(d_i)l_i \end{array}\right. \tag{2.3}$$

where by definition, $z,q,r,d_i \geq 0$.

We call (2.3) the basic model of residential choice of household with land ownership.

Moreover, for simplicity we assume that every household has the same amount of the composite good and land, so that the difference among households is only the initial loction of holding land. That is, we assume

$$(2.4) \quad \begin{cases} a_i = a & \text{for every } i \quad (i = 1,2, \ldots, m) \\ l_i = l & \text{for every } i \quad (i = 1,2, \ldots, m) \end{cases}$$

Then, the basic model (2.3) of the residential behavior of a household of type i is described as follows.

$$(2.5) \quad \begin{cases} \max\limits_{z,q,r} \; u\,(z,\, q) \\ \text{subject to;} \\ z + P(r)q + T(r) = a + P(d_i)\ell \end{cases}$$

The land available to urban residential use at distance d_i is given by $L(d_i) = N_i\ell \geqq 0$.

And the land available to urban residential use at distance r except d_i is given exogeneously by L(r) and owned by absentee landlords. The number of absentee landlords is the continuous number Na, and they have equal ownership for absentee land.[4] That is,

$$(2.6) \quad L(x) = \begin{cases} L(d_i) = N_i\ell \quad (i = 1,2,\ldots,m) \text{ --- private owners (households)} \\ L(r) \quad (r \neq d_i) \text{ ------------------ absentee landlords} \end{cases}$$

Next, we introduce the concept of <u>bid-price curves</u> of the household, and examine the fundamental characteristics of these curves. Bid price curves of the household are defined as follows.

<u>Definition 2.1</u>

Given a household of type i, specify its utility level to an arbitrary constant, U_i. Then, the bid price, $\psi_i\ (r,U_i)$, of the household at distance r corresponding to utility level U_i is defined as the maximum price (per unit of land) which the household could pay at that distance while enjoying that level of utility. When we emphasize that $\psi_i(r,U_i)$ is a function of distance r, it is called the bid price curve of the household corresponding to utility level U_i.

(For simplicity, we omit, without notice, the suffix i hereafter in this section).

Mathematical definition of bid price $\psi(r,U)$ is given as follows.

$$\psi(r,U\ ) = \max_{z,q} \left\{ \frac{a + P(d_i)\ell - T(r) - z}{q} \;\middle|\; u(z,q) = U,\ z \geqq 0,\ q \geqq 0 \right\} \tag{2.7}$$

If the market is competitive, then, all households must act as price-takers, and the income of a household of type i is assumed to be as if exogeneously predetermined, that is, $y_i \equiv a + P(d_i)\ell$. Hence, we can discuss the properties of bid prices similarly with the standard bid price theory with the case of exogeneously predetermined income which was, for example, presented in Fujita (1982a).

Let us examine the basic properties of bid prices and bid price curves defined by (2.7). For this purpose, we assume that utility function $u(z,q)$ is well behaved in the following sense.

Assumption 2.2

Utility function $u(z,q)$ satisfies the following set of conditions on its domain $(z,q) \geqq 0$.

(i) It is strictly quasi-concave and twice continuously differentiable

(ii) $\partial u/\partial z > 0$ and $\partial u/\partial q > 0$

(iii) Each difference curve has no intersection with z-axis or q-axis.

The assumption (iii) means that at any set of prices for z and q, some positive amount of each good will be purchased, namely each good is indispensable.

We first examine how to obtain bid rent $\psi(r, U)$ for a given combination of (r, U). Given utility level U, the corresponding indifference curve, $u(z,q) = U$, can be drawn in Figure 2.1. Given distance r, if land price there is $P(r)$, then the budget constraint of the household at this distance is given by $z + P(r)q + T(r) = a + P(d_i)l$, or equivalently by

$$z = (a + P(d_i)l - T(r)) - P(r)q \tag{2.8}$$

In Figure 2.1, equation (2.8) defines a straight line passing through point A, of which (negative) slope is $P(r)$. In this figure, the height of point A, $a + P(d_i)l - T(r) = y_i - T(r) \equiv I(r)$, represents the disposable income of the household at distance r.

First, suppose that land price P(r) was equal to P_1, in Figure 2.1. Then, the budget line (2.8) would be given by straight live AE, which has no common point with the indifference curve U. This implies that, if land price P(r) was as high as P_1, then the household could not derive the specified utility level U at distance r. Next, suppose that land price P(r) was P_2 (< P_1) in Figure 2.1. Then, the budget line (2.8) would be given by line AD, which intercepts with indifference curve U. Hence, if the land price at r was as low as P_2, the household could enjoy the required utility level U. However, the budget line (2.8) intercepts with indifference curve U even if P(r) is slightly higher than P_2, which implies that the household could derive utility level U at a higher land price than P_2. Therefore, we can conclude that in Figure 2.1 the highest land price which the household could pay at distance r while attaining the specific utility level U is given by the slope of the budget line, $\overline{ABC}$. Namely, the bid land price $\ell(r,U)$ is given by the slope of the budget line at loction r which is just tangent to the indifference curve.

Next, to obtain the mathematical derivation of the bid land price $\ell(r,U)$ we solve equation $u(z,q) = U$ for z, and obtain the equation of the corresponding indifference curve, $z(q,U)$, which gives the amount of composite good z necessary to achieve utility level U where the lot size is specified at each value q.

Because of assumption A2.2, function $z(q, U)$ has the following properties.

<u>Property 2.1.</u>

(2.9) $z(q,U)$ is strictly convex with respect to q, and it is twice continuously differentiable with respect to q and U.

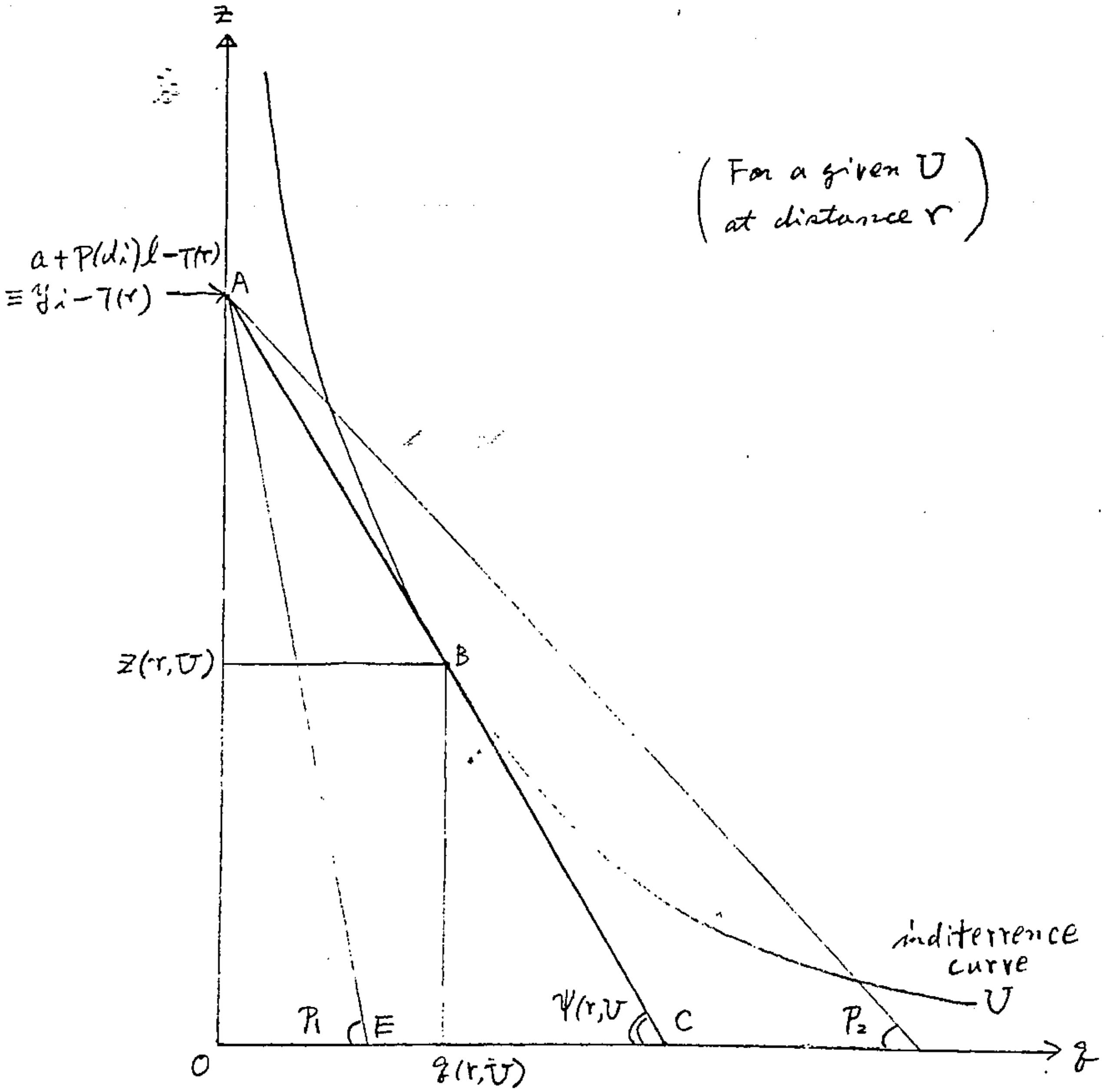

Figure 2.1: Geometric Derivation of Bid Price $\psi(r,\overline{U})$

$$(2.10) \qquad z(q,U) > 0,\ \partial z / \partial q < 0,\ \partial z / \partial U > 0$$

$$(2.11) \qquad \lim_{q \to 0} \partial z/\partial q = -\infty,\ \lim_{q \to \infty} \partial z/\partial q = 0$$

Condition (2.11) implies that the indifference curve never intercepts with the z-axis or q-axis. Hence, both z and q are positive on each indifference curve. Therefore, using function z(q,U), the equation of bid land price (2.7) can be reformulated as follows.

$$(2.12) \qquad \psi(r,U) = \max_{q} \frac{a + P(d_i)l - T(r) - z(q,\ U)}{q}$$

The first order condition for the solution of the maximization problem in the (2.12) is given by

$$- \frac{\partial z(q,U)}{\partial q} = \frac{a + P(di)l - T(r) - z(q,U)}{q}$$

or, equivalently, since at the solution of (2.12)

$$(2.13a) \qquad \psi(r,U) = \frac{a + P(di)l - T(r) - z(q,U)}{q}$$

the first order condition above can be restated as follows.

$$(2.13b) \qquad - \frac{\partial z(q,U)}{\partial q} = \psi(r,U)$$

Solving the system of equations (2.13) for q and ψ, we obtain the bid land price function.

Note that when bid price $\psi(r,U)$ is determined by (2.13), the optimal lot size, q (r,U), and the optimal amount of composite good, z(r,U) = z(q(r,U), U) corresponding to $\psi(r,U)$ are also determined. Both of the optimal solutions q(r,U) and z(r,U) are denoted in Figure 2.1. Geometrically, condition (2.13b) expresses the requirement tht budget line $\overline{AC}$ [given by (2.13a)] is to be tangent to indifference curve U at the optimal consumption bundle B in Figure 2.1. Substituting q(r,U) and z(r,U) into (2.13a), we have,

$$(2.14) \qquad \psi(r,U) = \frac{a + P(di)l - T(r) - z(r,U)}{q(r,U)}$$

Next, we examine the basic properties of the bid price function defined by (2.7), namely, by (2.12). For this purpose, we assume the following condition

Assumption 2.3

$T'(r) = dT(r) / dr > 0$. Namely, transportation cost T(r) is continuously increasing with distance r.

We first examine how bid price $\psi(r,U)$ and the corresponding lot size q(r,U) change with distance r. For this purpose, take two distances, $r_1 < r_2$. Then, since $a + P(d_i)l - T(r_1) < a + P(d_i)l - T(r_2)$, we see from Figure 2.2, that $\psi(r_1,U) > \psi(r_2,U)$ and $q(r_1,U) < q(r_2,U)$.
Hence, under assumptions A2.1-A2.3, we have

Property 2.2

(i) Bid price $\psi(r,U)$ continuously decreases with the increase in distance r. That is,

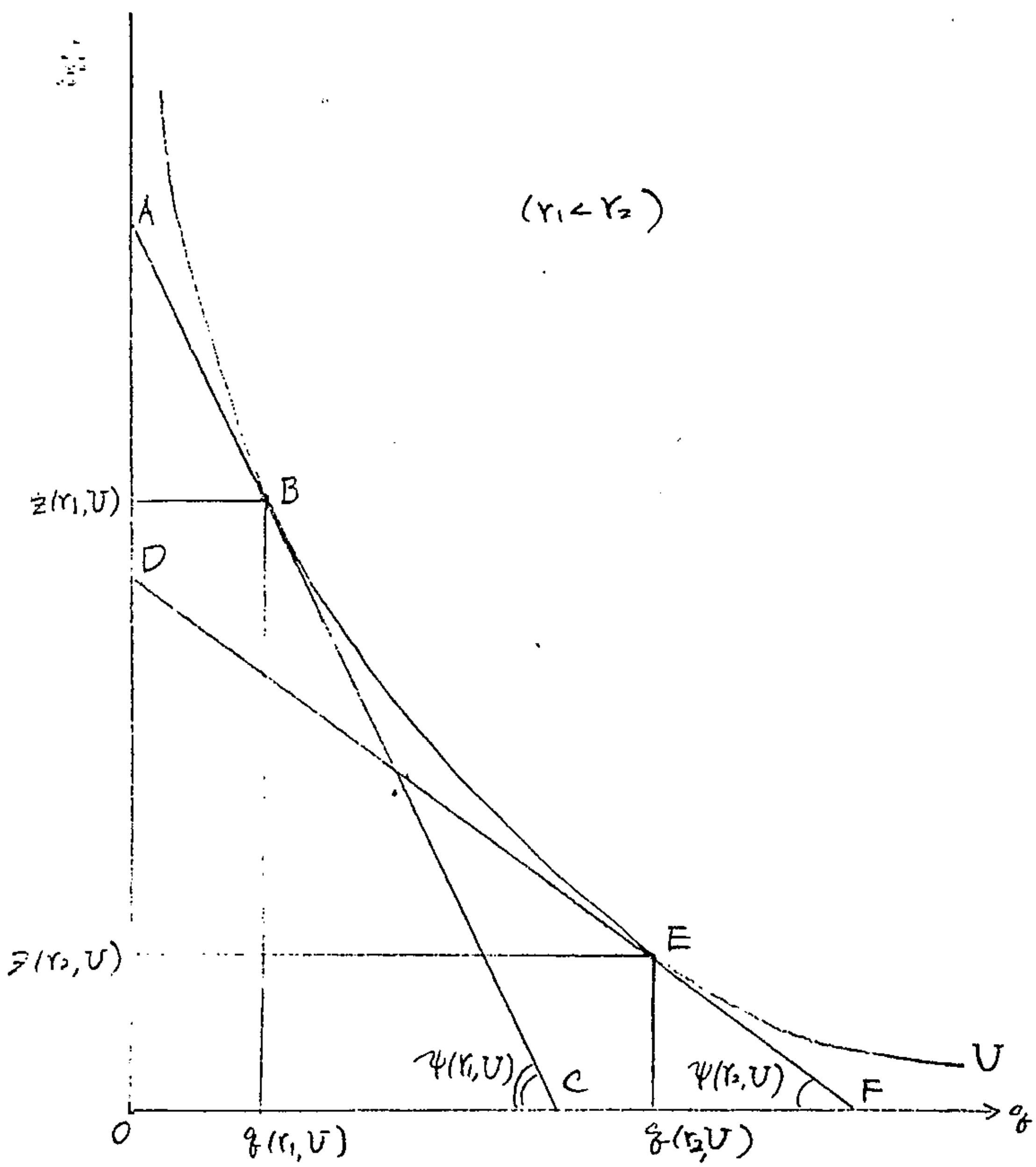

Figure 2.2: Variation of Bid Price ψ(r,U) with Respect to Distance r

$$\frac{\partial\psi(r,U)}{\partial r} = - \frac{T'(r)}{q(r,U)} < 0$$

(ii) Lot size $q(r,U)$ continuously increases with the increase in distance r. That is,

$$\frac{\partial q(r,U)}{\partial r} = \frac{\partial q(r,U)}{\partial\psi(r,U)} \cdot \frac{\partial\psi(r,U)}{\partial r} > 0$$

The economic interpretation of this property is given as follows. The household can achieve the same utility level U under a smaller disposable income $I(r) \equiv a + P(d_i)l - T(r)$ at location r only when the land price $\psi(r,U)$ is lower so that a greater amount of land $q(r,U)$ can be purchased. Mathematically applying the Envelope Theorem [eg., cf. Variain (1978)], we have

$$\frac{\partial\psi(r,U)}{\partial r} = - \frac{T'(r)}{q(r,U)} < 0 \tag{2.15}$$

Next, let us examine how bid price $\psi(r,U)$ and lot size $q(r,U)$ change with the increase in utility level U. For this purpose, take two utility levels, $U^1 < U^2$. Then, from (2.10) (i.e., $\partial z/\partial U > 0$), indifference curve U^2 must lie above indifference curve U^1 in Figure 2.3. Therefore, from Figure 2.3 we see that $\psi(r,U^1) > \psi(r,U^2)$.

The direction of change of $q(r,U)$ with respect to U_i is not always unique. However, we assume the following condition.

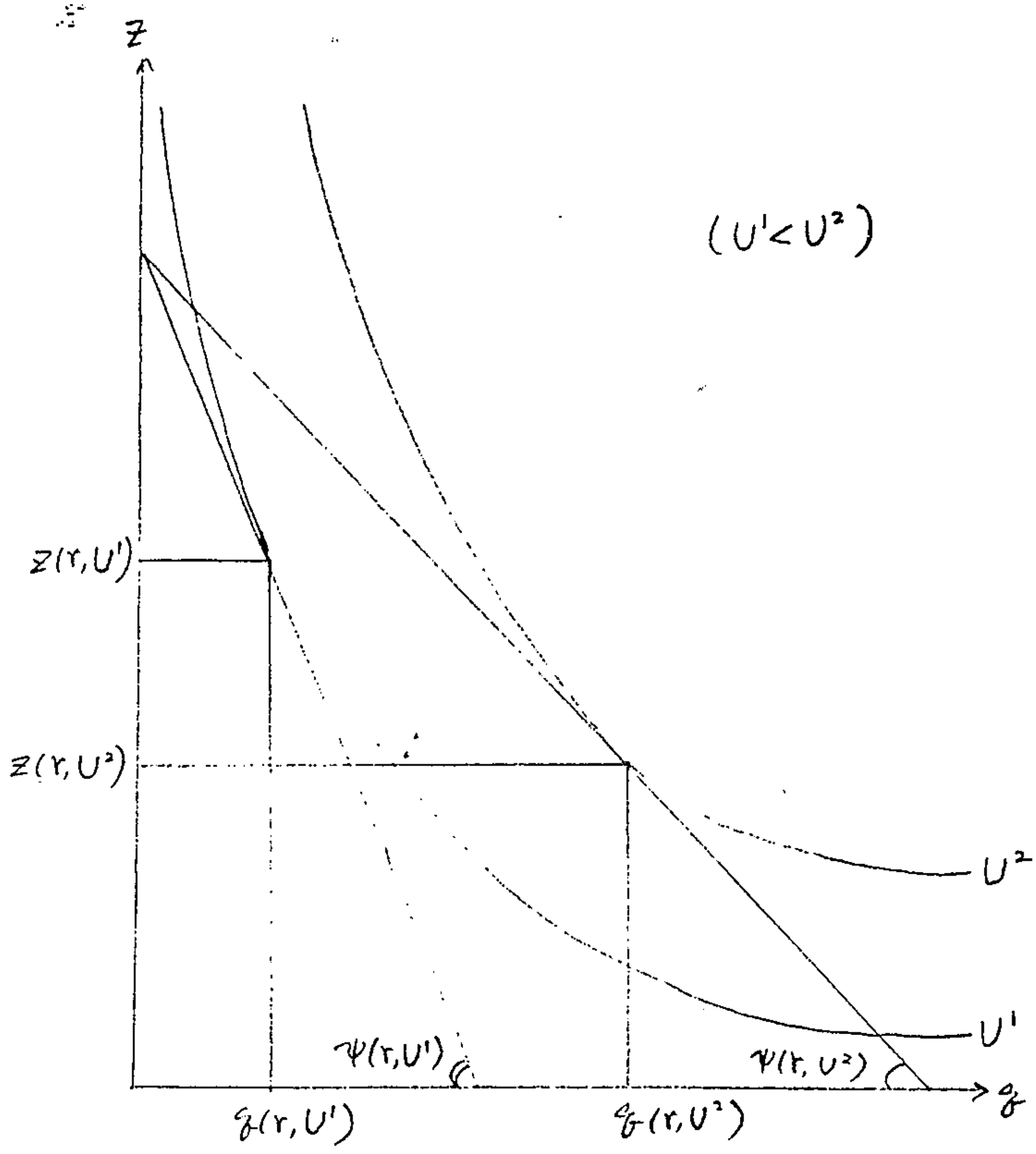

Figure 2.3: Variation of Bid Price ψ(r,U) with Change of Utility Level

Assumption 2.4

Land q is a normal good, that is, $\partial^2 z\ (q,U)/\partial q\ \partial U > 0$.

This assumption means that for each fixed q, indifference curve z(q,U) becomes steeper with the increase in U. In this case, as we can see from Figure 2.3, it always holds that $q(r,U^1) < q(r,U^2)$. Hence obtain (i) (under A.2.2) and (ii) (under A.2.2 and A.2.3) in the following property.

Property 2.3

(i) Bid price ψ(r,U) continuously decreases with the increase in utility level U. That is, $\frac{\partial\ \psi(r,U)}{\partial U} < 0$

(ii) Lot size q(r,U) increases with the increase in utility level U. That is, $\frac{\partial q(r,U)}{\partial\ U} = \frac{\partial q(r,u)}{\partial \psi(r,U)} \cdot \frac{\partial \psi(r,U)}{\partial\ U} > 0$

The first proposition (i) is clear since the household could obtain a higher utility level under a fixed disposal income $I(r) \equiv a + P(d_i)l - T(r) \equiv y_i - T(r)$ only when land price ψ(r,U) was lower. Mathematically, applying the Envelope Theorem as before, we have

$$(2.16) \qquad \frac{\partial \psi(r,U)}{\partial U} = -\frac{1}{q(r,U)} \cdot \left.\frac{\partial z(q,U)}{\partial U}\right|_{q\ =\ q(r,U)}$$

which is negative since $\partial z/\partial u > 0$ from (2.10).

Combining Properties 2.2 and 2.3, the general shape of bid price curves ψ(r,U) can be depicted as in Figure 2.4. From property 2.2 (i), for each fixed U bid price ψ(r,U) continuously increases with the increase in utility level U. Next, from properties 2.1 and 2.2, the general shape of lot size curves q(r,U) can be depicted as in Figure 2.4. From property 2.2 (i), for each fixed U bid price ψ(r,U) continuously increases with the increase in utility level U. Next, from properties 2.1 and 2.2, the general shape of lot

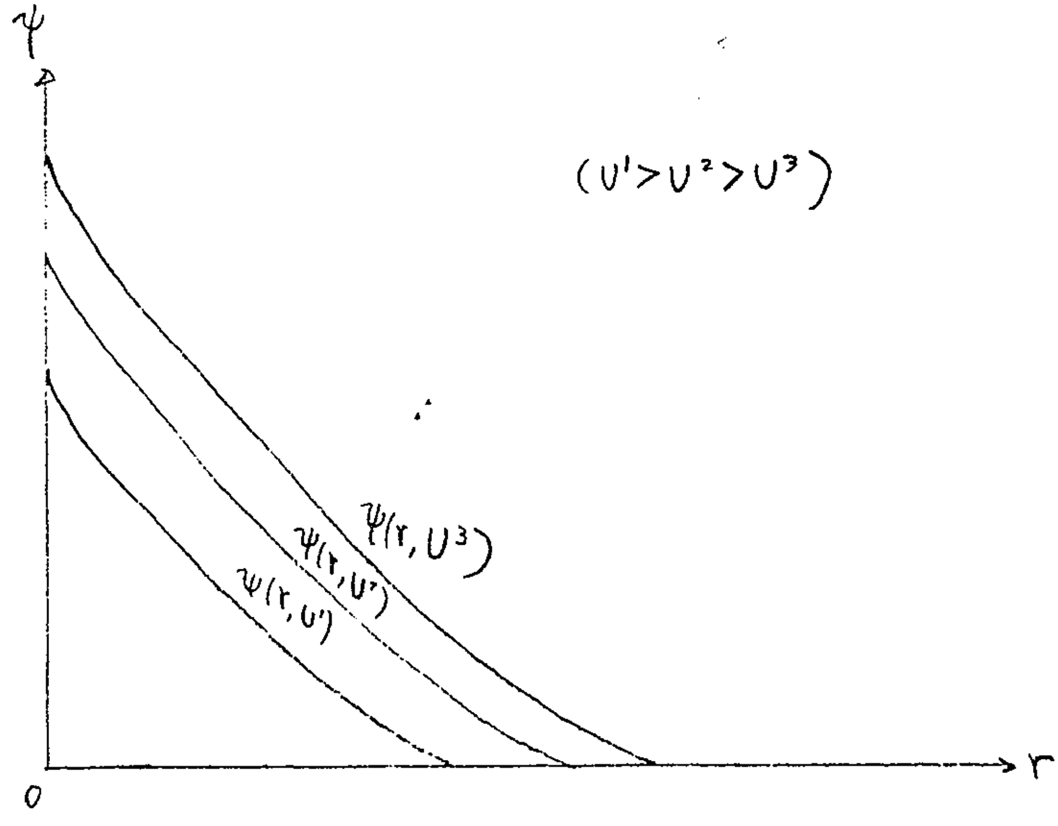

Figure 2.4: General Shape of Bid Price Curves ψ(r,U)

size curves $q(r,U)$ can be depicted as in Figure 2.5. From Property 2.2 (i), for each fixed U, lot size $q(r,U)$ continuously increases with the increase in distance r. And, from Property 2.3 (ii), at each fixed distance r, lot size $q(r,U)$ continuously increases with the increase in utility level U.

Note that bid price curves are not always convex as depicted in Figure 2.4. However, suppose that we have

Assumption 2.5

Transportation cost $T(r)$ is increasing at a nonincreasing rate with the increase in distance r. That is, $T''(r) = \frac{d^2T(r)}{dr^2} \leqq 0$.

Then, from (2.15) and recalling Property 2.2 (ii), we have

$$\frac{\partial^2 \psi_i(r,U_i)}{\partial r^2} = - \frac{T''(r)}{q(r,U_i)} + \frac{T'(r)}{q(r,U_i)^2} \frac{\partial q(r,U_i)}{\partial r} > 0 \tag{2.17}$$

Namely, under A.2.2 and A.2.5, we have

Property 2.4

Bid price curves are strictly convex with respect to distance r.

In concluding the above discussions, notice that each bid price curve defines an indifference curve on an urban space. Hence, in determining the optimal location among all possible locations in the urban space, bid price curves play the role of indifference curves on commodity space in standard microeconomics. For this reason the concept of bid price curves plays the basic role in developing the urban land use theory.

We next study, by using bid price curves, how the optimal location of a household is determined in the city. If the market is competitive, then all households must behave as price-takers. Hence, we can assume that the actual

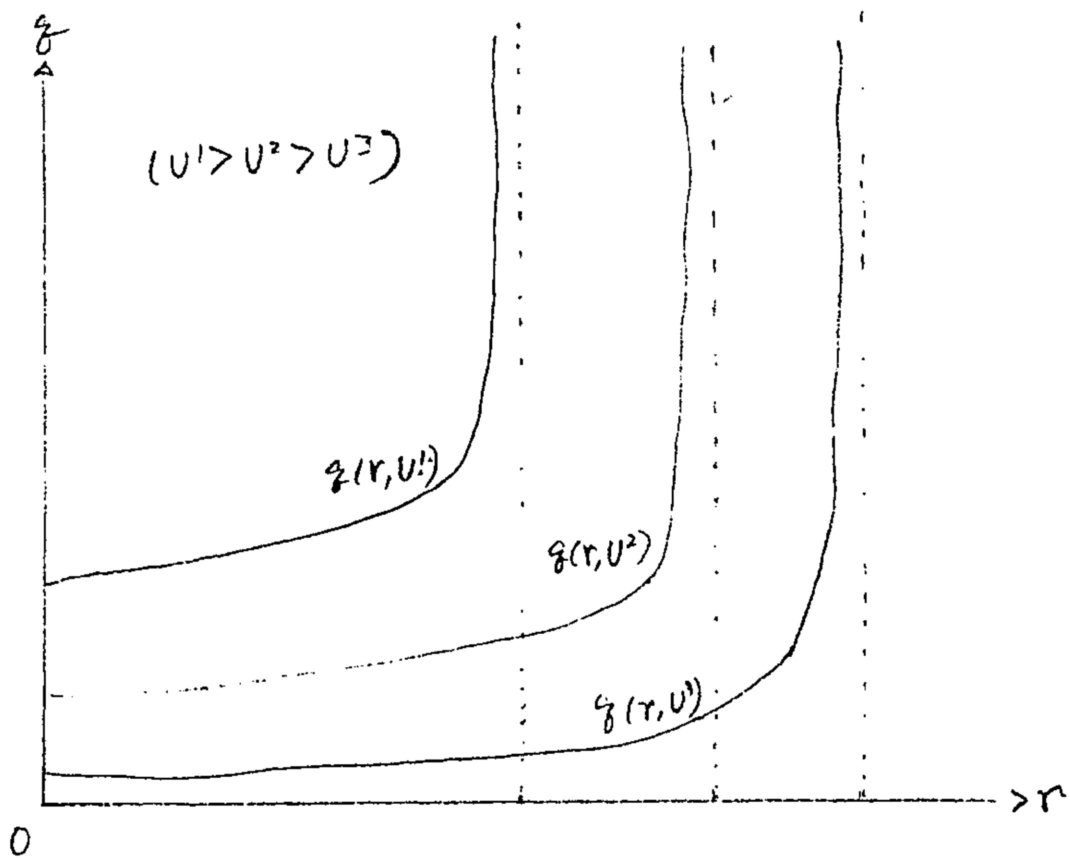

Figure 2.5: General Shape of Lot Size Curves $q(r,\overline{U})$

market land price curve of the city is given exogeneously by a curve P(r).

Suppose that we superimpose the actual market land price curve P(r) on Figure 2.4, and obtain Figure 2.6. Then, the optimal location of the household is given at distance r* at which a bid price curve $\psi(r,U^2)$ is tangent to the market land price curve P(r) from the bottom. That is, we obtain the following theorem

Theorem 2.1 (Optimal Location of A Household)

The optimal location of the household in the city is given at the location where one of its bid price curves is tangent to the actual market land price curve of the city from the bottom.

Let us call the maximum utility level which households can obtain in the city the equilibrium utility level, and denote it by U*; in the case of Figure 2.6, $U^* = U^2$. Then the tangency condition in Theorem 2.1 is generally stated as follows.

(2.18) $$P(r^*) = \psi(r^*,U^*) \text{ and } P(r) \geq \psi(r,U^*) \text{ for all } r$$

where, r* is the optimal location of the household. Note that theorem 2.1 combined with condition (2.18) can apply whether P(r) is smooth or not, or convex or not.

When P(r) and $\psi(r,U^*)$ are differentiable at the optimal location r*, (2.18) can be restated as follows.

(2.19) $$\frac{\partial\psi(r,U^*)}{\partial r} = P'(r) \quad \text{at optimal location } r^*$$

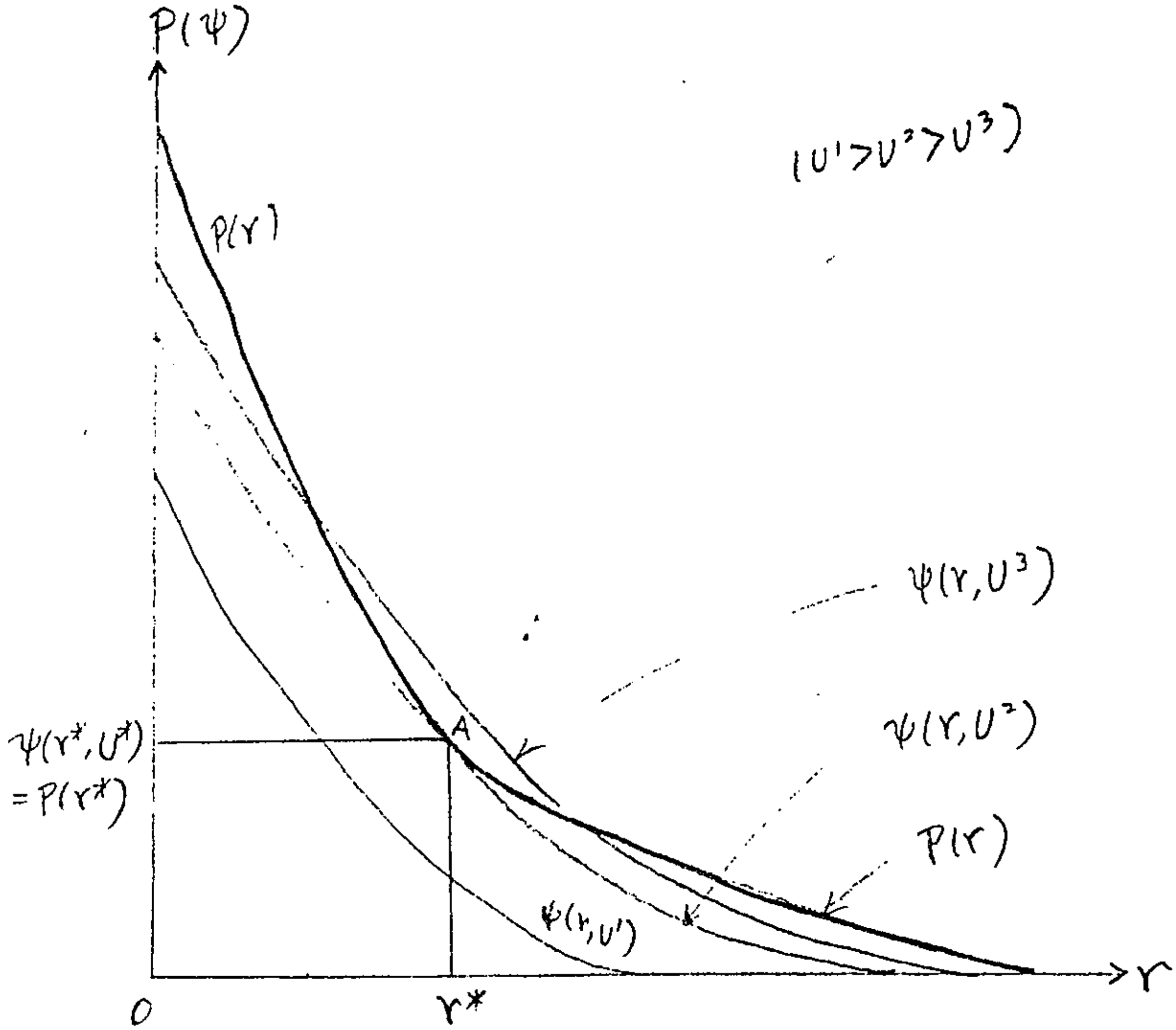

Figure 2.6: Determination of the Optimal Location

Hence, when bid price function is given by (2.14), using (2.15), the tangency condition is restated as follows.

$$(2.20) \qquad T'(r) = -P'(r)q(r,U^*) \quad \text{at optimal location } r^*$$

Namely, at the optimal location, the marginal travel cost $T'(r)$ is just equal to the marginal land cost saving, $-P'(r)q(r,U^*)$.

Next, to study the relative locations of different households in the city, we define the concept of the <u>relative steepness</u> of bid price curves which was first introduced by Fujita (1982a) as follows. Suppose that $\psi_j(r,U_j)$ represents the bid price function for household j, and $\psi_i(r,U_i)$ for household $i(i > j)$. We say that the bid price curves of household j are <u>steeper</u> than the bid price curves of household i if and only if

$$(2.21) \qquad \left| -\frac{\partial\psi_j(r,U_j)}{\partial r} \right| > \left| -\frac{\partial\psi_i(r,U_i)}{\partial r} \right| \text{ i.e., } |\psi_j'(r)| > |\psi_i'(r)|$$

whenever $\psi_j(r,U_j) = \psi_i(r,U_i)$ at each r.
Namely, as shown in Figure 2.7, at each intersection of the bid price curves for household j and household i, the former is always steeper than the latter (in absolute slopes).

From the definition of relative steepness of bid price curves and from Theorem 2.1, we immediately obtain

<u>Theorem 2.2.</u> (Relative Locations of Households)

If bid price curves $\psi_j(r,U_j)$ of household j are steeper than bid price curves $\psi_i(r,U_i)$ of household i, then household j locates closer to the CBD than household i.

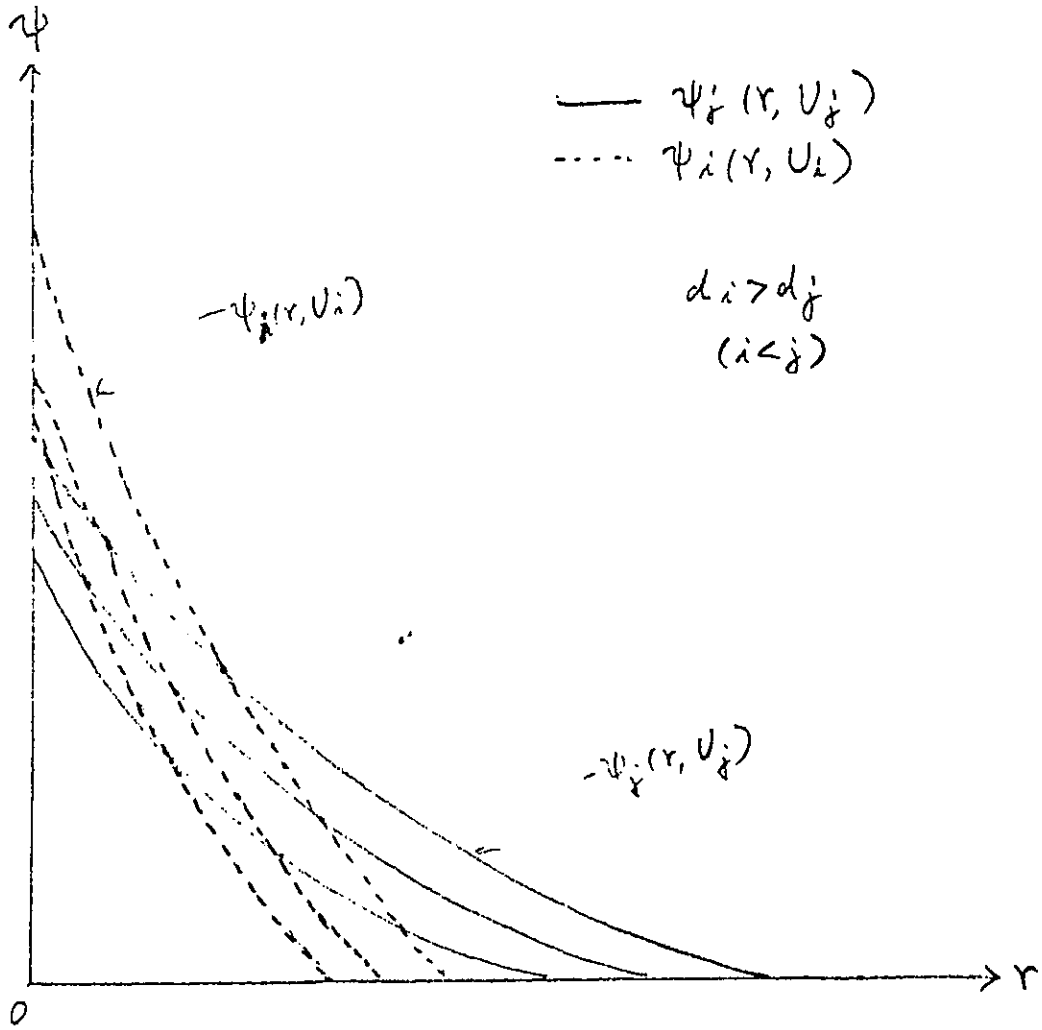

Figure 2.7: Bid Price Curves for Different Income Levels

Suppose that we have two types of household j and i, $(i < j)$ which have the same utility function, and the same transport cost function $T(r)$. However they have different incomes y_j, y_i, $(y_i > y_j)$. If we assume that the actual market land price function $P(r)$ is continuously decreasing with the increase of distance, that is, $P(r_i) > P(r_j)$ $(i < j)$, then $y_i \equiv a + P(d_i)\ell > y_j \equiv a + P(d_j)\ell$.

Then, for each household bid price curves $\psi_j(r,U_j)$, $\psi_i(r,U_i)$ can be depicted as in Figure 2.8. Hence, the bid price curves of household j (lower income) are steeper than those of household i (higher income).

Theorem 2.3.

Other things being equal, the higher the income level of a household, the farther is its optimal location from the CBD.

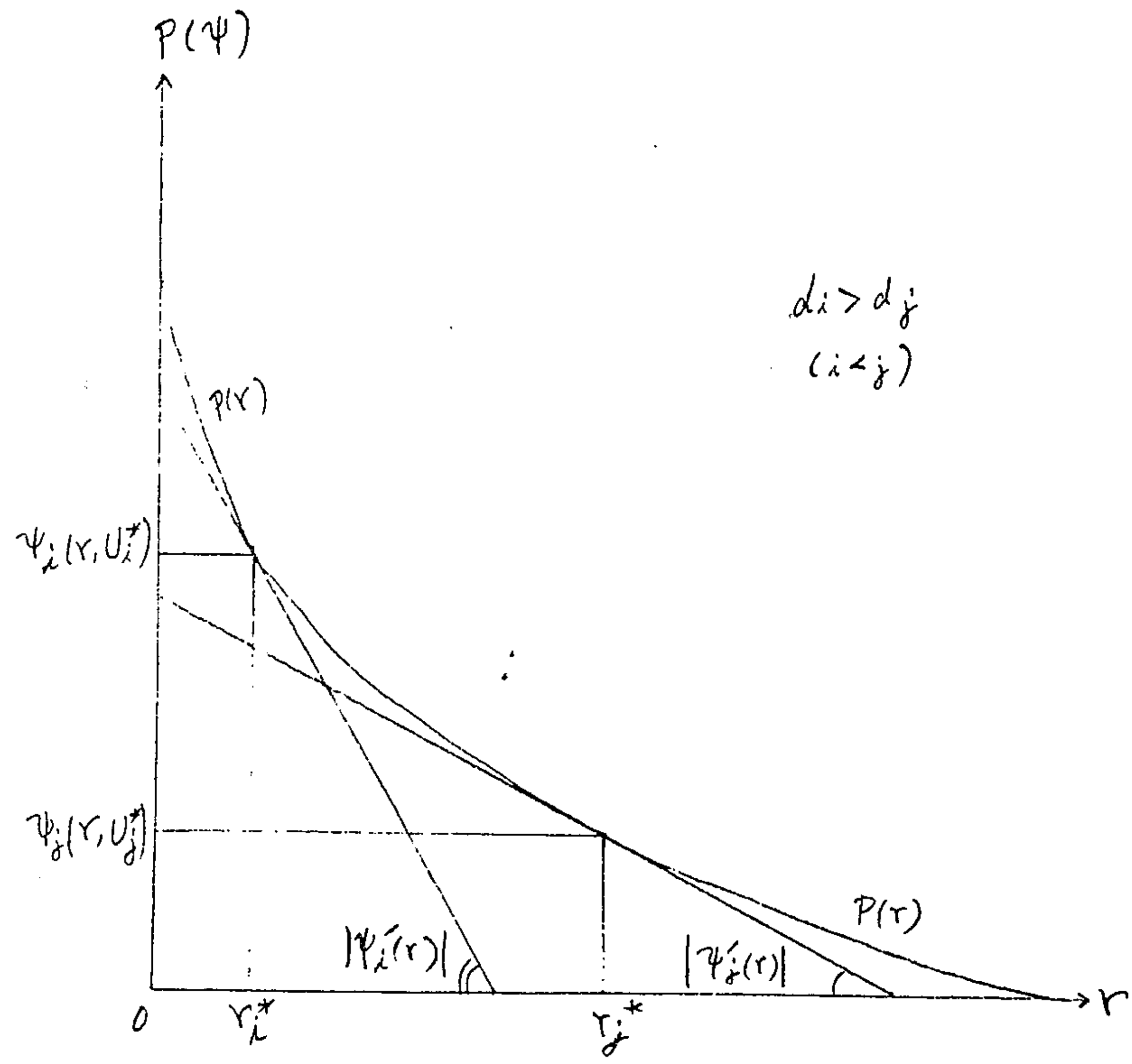

Figure 2.8: Ordering of Relative Locations by Steepness of Bid Price Curves

3. EQUILIBRIUM SPATIAL STRUCTURE OF URBAN LAND USE

Next, under the same basic framework of a monocentric city with land ownership, we discuss how the market equilibrium spatial structure of urban residential land use is determined.

Here, we assume the closed city model, that is, the total population of the city is exogenously given. In this closed city model, the equilibrium utility level of households (of each type) must be determined endogeneously.

It is also assumed that land which is not used for urban residential purposes is used for agricultural activities at given fixed land price, $P_A \geqq 0$. The conversion of agricultural land to residential use is assumed to be costless. Then, since the land market in the city is to be assumed perfectly competitive everywhere, land owners (absentee landlords and households as private land owners) play no active role in our models; they simply sell their land to households or farmers at market prices.

Under the above assumptions, the conditions for the equilibrium land use of the city can be summarized as follows. In the closed city model, an equilibrium spatial structure of the city consists of a market land price curve P(r) and a distribution of households in the city such that:

(i) Equilibrium of all households.

Under the equilibrium market land price curve P(r), each household has optimally determined its residential choice; and hence, no household has motivation to change its optimal choice.

(ii) Market Clearance Condition.

$P(r) \geqq P_A$ at every distance r. And, at each distance r where

$P(r) > P_A$, the total demand for the land by the households residing there is equal to the amount of land existing there.

(iii) Physical Condition (Closed City Model)

Every household resides somehwere in the city; namely, the total number of households of each type in the city is equal to the exogeneously given population.

First, let us formulate the Bid Price Theorem.

Assuming that the households in the city consist of m different types, where all households of the same type are identical in terms of residential choice behavior. Then, it follows immediately that;

Property 3.1 (equal utility property)

In equlibrium, all the households of the same type must achieve the same utility level independent of their locations.

Otherwise, a household could increase its utility by imitating the residential choice of a household in the same type with a higher utility. Let us denate by U_i^* the equilibrium utility level of type i (i = 1, ---,m).

Next, let us denote by $\psi_i(r,U)$ the bid price function of households in type i, and by P(r) the equilibrium land price curve of the city.
Then, the following conditions must be satisfied.

(3.1) $$P(r) \geqq \psi_i(r,U_i^*) \qquad \text{for all i and all r.}$$

(3.2) $$P(r) \geqq P_A \qquad \text{for all r}$$

Because, if $P(r) < \psi_i(r,U_i^*)$, we can see from the definition of bid price function that the household of type i can achieve a higher utility level than U_i^* under $P(r)$. This contradicts the definition of equlibrium. Similarly, (3.2) follows from the non-profitability condition for agriculture activities. In short, we obtain

$$(3.3) \qquad P(r) \geqq \max \{\psi_1(r,U_1^*),\ \psi_2(r,U_2^*), \ldots, \psi_m(r,U_m^*),\ P_A\}$$

at each distance r.

Suppose that inequality could hold in relation (3.3) at each distance r. This implies that any household cannot achieve its equilibrium utility level under P(r) at r, and that agricultural activities get negative profit under P(r) at r. Hence, the land at distance r would be left vacant. On the other hand, the inequality in (3.3) implies $P(r) > 0$ since $P_A > 0$. This is a contradiction since unused land cannot command a positive land price in equilibrium.

Therefore, we conclude that, in equilibrium

$$(3.4) \qquad P(r) = \max \{ \max_i \psi_i(r,U_i^*),\ P_A \} \qquad \text{at each } r$$

Namely, the equilibrium land price curve is the <u>upper envelope of the equilibrium bid land price curves of all households and the agricultural price curve.</u> Moreover, we can conclude that:

(3.5) a household of type i possibly resides at distance r only when $P(r) = \psi_i(r,U_i^*)$

(3.6) land at distance r is possibly used for agriculture only when $P(r) = P_A$

Summing up (3.4), (3.5) and (3.6), we have;

Theorem 3.1 (Bid Price Theorem)

In equilibrium, market land price curve P(r) is the upper envelope of all equilibrium bid price curves of households and agricultural activities. Namely,

(3.7) $$P(r) = \max\{\psi_1(r,U_1^*),\ \psi_2(r,U_2^*),\ldots,\psi_m(r,U_m^*),\ P_A\}$$

at each distance r

when U_i^* is the equilibrium utility level for a household of type i. And each land is occupied by the activity(or, one of activites) which has the highest bid price for that land.

From property 2.2(ii), bid price $\psi_i(r,U_I^*)$ continuously decreases with r. Hence we define the boundary of the city, $\bar{r}$, by

(3.8) $$\max_i \psi_i(\bar{r},U_i^*) = P_A$$

Hence, (3.7) can be rewritten as follows. Combining this property with property 2.2(i), we obtain that;

Property 3.2

Equilibrium market land price curve P(r) continuously decreases with the increse in distance r up to the urban fringe $\bar{r}$, and $P(\bar{r}) = P_A$ (constant) excess to the urban fringe $\bar{r}$. That is,

(3.9) $$P'(r) = \begin{cases} \dfrac{dP(r)}{dr} < 0 & \text{for } 0 < r < \bar{r} \\ 0 & \text{for } r > \bar{r} \end{cases}$$

Once the equilibrium market land price curve is determined, then income of a household of type i is determined as follows.

(3.10) $$y_i \equiv a + P(d_i)q \qquad \text{for all } i\ (i = 1,2, ---, m)$$

And, if we assume that the edge of the city $\bar{r}$ is a distance from d_1 with $\bar{r} > d_1$, then, the income of a household of type is is ranked as follows.

(3.11) $$y_1 < y_2 < ----- < y_{m-1} < y_m$$

That is, we can consider that there are m income classes for m types of households according to the initial location of land ownership.

From Property 2.4, we know that bid price curves become steeper (in the sense of the definition in Section 2) as the income becomes lower. Hence, from Theorem 2.2, we see that, at equilibrium all households of type 1 reside closer to the CBD than households of type 2, and household of type 2 reside closer to the CDB than household of type 3, so on. Hence, households of each type form a concentric ring, on zone, around the CBD. Zones for different types are ranked by the distance from the CBD in the order of income level, i.e., in the order of initial location of land ownership

Let us denote the zone for type i by $r_{i-1} \leqq r \leqq r_i$, where r_{i-1} and r_i are the inner radius and the outer radius of zone i, respectively. Here, $r_0 \equiv 0$ is the CBD, and r_m is the edge of the city, $\bar{r}$. Then, recalling that each bid price curve is continuous with r, from (3.7), we have

(3.12) $$P(r) = \begin{cases} \psi_i(r, U_i^*) & \text{for } r_{i-1} \leqq r \leqq r_i \qquad (i = 1,2,...m) \\ P_A & \text{for } r \geqq r_m \equiv \bar{r} \end{cases}$$

which implies that

$$\text{(3.13)}\qquad \begin{aligned} &\psi_i(r_i, U_i^*) = \psi_{i+1}(r_i, U_{i+1}^*), \qquad \text{for } i = 1,2,\ldots, m-1 \\ &\psi_m(r_m, U_m^*) = P_A \end{aligned}$$

Therefore, the equilibrium land price curve can be depicted as in Figure 3.1. Since each bid price curve is strictly convex (from property 2.3), the equilibrium land price curve is also strictly convex up to the urban fringe.

Next, let us denote by $q_i(r,U_i^*)$ the optimal lot size corresponding to $\psi_i(r,U_i^*)$ which is obtained from (2.13b). Then, from equilibrium conditions (iii), it must hold that

$$\text{(3.14)}\qquad \int_{r_{i-1}}^{r_i} \frac{L(r)}{q(r,U_i^*)}\, dr = N_i \qquad (i = 1,2,\ldots,m)$$

where, $L(r)$ is the land available at distance r. Solving (3.13) and (3.14), we can obtain the equilibrium utility levels $U_i^*(i=1,2,\ldots,m)$ and boundary distances $r_i(i=1,2,\ldots,m)$. Then, the equilibrium land price curve $P(r)$ is obtained from (3.12), and optimal lot size curves $q_i(r,U_i^*)$ can also be obtained. Because of the normality of lot size, at each boundary between two types (income classes), each household of a higher income class (closer initial location type) consumers more land than a household of a lower income class (farther initial location type). Hence,

$$\text{(3.15)}\qquad q_i(r_i,U_i^*) < q_{i+1}(r_i,U_{i+1}^*) \qquad (i=1,2,\ldots,m)$$

And, from Property 2.2, for the same type (same income class), $q_i(r,U_i^*)$ increases continuously with r. Hence, the lot size curve can be depicted as in Figure 3.2.

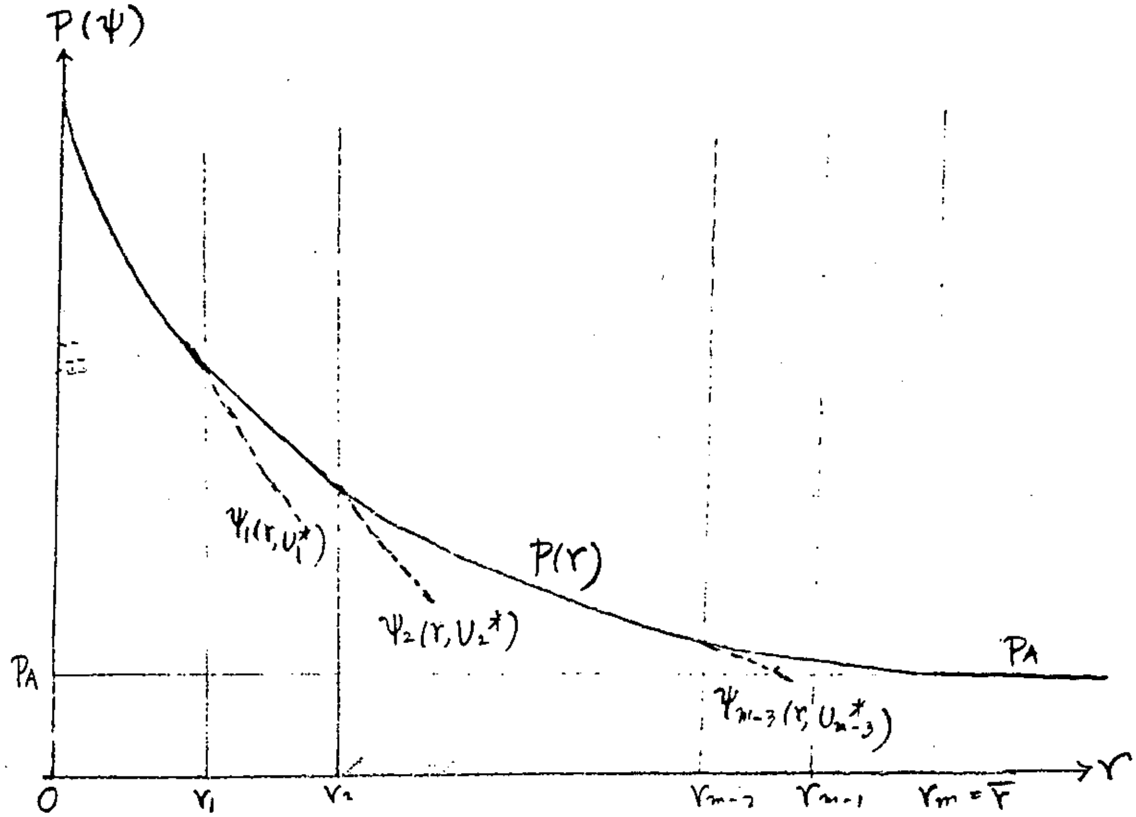

Figure 3.1: Equilibrium Spatial Structure with Land Price Curves

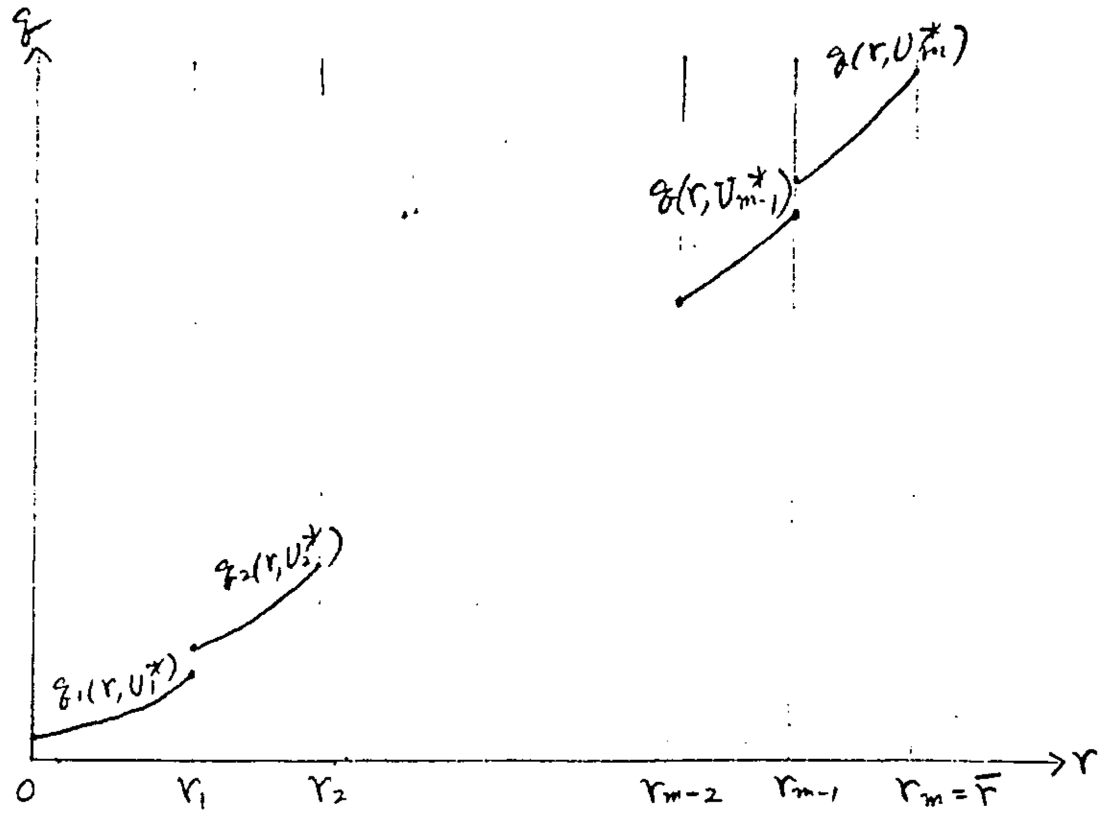

Figure 3.2: Equilibrium Spatial Structure with Lot Size Curves

Concluding the above discussions, we obtain the following location replacement theorem.

Theorem 3.2 (Replacement Theorem).

In equilibrium, the location of households (with land ownership) is ranked according to the initial location of land ownership. That is, the fartherfrom the CBD the initial location of land owned by a household is, the closer its optimal location to the CBD. The replacement of the location occurs in this model.

It is not difficult to prove the existence[5] and uniqueness of the equilibrium spatial structure of urban land use. There are three kinds of methods for the proof of the competitive equilibrium. One is to apply similarly the constructive method which was shown by Fujita (1982a) for the proof of the standard case with exogeneously predetermined finite income classes. A second is to use the corresponding theorem, between the optimum model and the market model, which is shown later in the next section. A third is to use the equibalence theorem between the core of the urban game and the competitive equilibrium of the urban land market.

4. URBAN LAND USE PATTERNS WITH AND WITHOUT LAND OWNERSHIP

Let us analyze the situation where there are two kinds of household with and without land ownership. According to the initial holding of a composite good and land, we divide households into the following class, I, II, and III.

Each household of the class I has the initial holding of only composite good a^1. Each household of type i in the class II has the initial holding of composite good a^2 and the same amount of land $l(> 0)$ at distance d_i $(i=1,2,...,m)$. Each household of class III has the initial holding of only composite good a^3. The number of class I and III are M_1 and M_3, respectively, and the number of household of type i in class II is N_i $(i=1,2,...,m)$.

Assumption 4.1

For simplicity, we assume that the initial level of composite good is ranked as follows.

$$0 < a^1 < a^2 < a^3 \tag{4.1}$$

That is, according to the initial level of composite good, classes I, II and III are the poorest, the middle and the richest, respectively, but only class II has an initial endowment of land.

And, except for the above assumptions, we assume the same conditions as in the previous sections. The behavior determining the residential choice for a household of each class is described as follows.

Class I:

$$(4.2)\quad \begin{cases} \max_{z,q,r} u(z,q) \\ \text{subject to:} \\ z + p(r)\, q + T(r) = a^1 \end{cases}$$

Class II (type i):

$$(4.3)\quad \begin{cases} \max_{z,q,r} U(z,q) \\ \text{subject to:} \\ z + P(r)\, q + T(r) = a^2 + P(d_i)\, \ell \qquad \text{for each } i\ (i=1,2,\ldots,m) \end{cases}$$

Class III:

$$(4.4)\quad \begin{cases} \max_{z,q,r} U(z,q,) \\ \text{subject to:} \\ z + P(r)\, q + T(r) = a^3 \end{cases}$$

Note that income of a household in each class y^1, y_i^2 and y^3 is $y^1 \equiv \equiv a^1$, $y_i^2 \equiv a^2 + P(d_i)l$ and $y^3 \equiv a^3$, respectively. It is clear that;

$$(4.5)\qquad y^1 < y^2 \text{ and } y^1 < y^3$$

However, land prices at distance d_i $(i=1,2,\ldots,m)$ are not predetermined, we cannot previously identify which is higher y^2 or y^3. Moreover, we assume as follows.

Assumption 4.2

The income of class III is larger than that of any type in class II when every land of this city is used for agriculture. That is, we assume:

$$(4.6) \qquad y^3 \equiv a^3 > y_i^2\,(P_A) \equiv a^2 + P_A\,\ell \qquad \text{for every } i \ (i=1,2,\ldots,m)$$

where, $y_i^2(P(d_i))$ denotes the income level of type i when the land price at distance d_i is $P(d_i)$. This assumption means that the initial income of each household in class III is larger than the initial income of every type of household in class II, since we can assume that at the initial state every land in this city is used for agriculture. Hence, at the initial state income level of each class is ranked I, II, III, i.e.,

$$(4.7) \qquad y^1 \equiv a^1 < y^2\,(P_A) \equiv a^2 + P_A\ell < y^3 \equiv a^3$$

In order to obtain several kinds of urban land use patterns, we first examine how much the income level of each class will be at the competitive equilibrium. For this urban land market model, note that since the bid price theorem holds, the relative location by the income level also holds.

We have to identify two cases of equilibrium spatial structure according to the equilibrium land price level at the CBD, P(0).

Case 1: The Case with $P(0) > (a_3 - a_2)/l$. (Figure 4.1)

Since $P(0) \geqq P(r)$ for all r, we obtain,

$$a^3 > a^2 + P(0)\,\ell \geqq a^2 + P(r)\,\ell \quad \text{for all } r$$

Hence,

$$P(0) < (a^3 - a^2)/\ell \Longrightarrow y^3 \equiv a^3 > y_i^2 \equiv a^2 + P(d_i)\ell \text{ for all } i$$

In this case, the income level of each class is lanked as follows.

$$y^1 \equiv a^1 < y_i^2 \equiv a^2 + P(d_i)\,\ell < y^3 \equiv a^3 \qquad \text{for all } i$$

Hence, as shown in Figure 4.1, the market land price curve and the assignment of zones to each class and each type is done according to the final income level. That is, the poorest class resides closest to the CBD. The middle class with land ownership resides in the middle zone. The types within this class have moved within the zone. The type originally closest to the CBD is now the farthest and the type initially the farthest is now the closest to the CBD. The other types within the zone have made corresponding changes. The richest class resides farthest from the CBD.

Case 2; The case with $P(0) > (a_3 - a_2)/\ell$ (Figure 4.2.)

For this case there exists some r_m $(0 < r_m < \bar{r})$ such that

$$P(r_m) = \frac{a_3 - a_2}{\ell}$$

And assume that;

$$d_{k+1} < r_m < d_k \qquad \text{for some } k \; (k=1,2,\ldots,m)$$

Then, in class II

$$y_j^2 \equiv a^2 + P(d_j)\,\ell < a^3 \qquad \text{for type } j \; (j \leqq k)$$

$$y_\ell^2 \equiv a^2 + P(d_\ell)\,\ell > a^3 \qquad \text{for type } l \; (l \geqq k+1)$$

and define

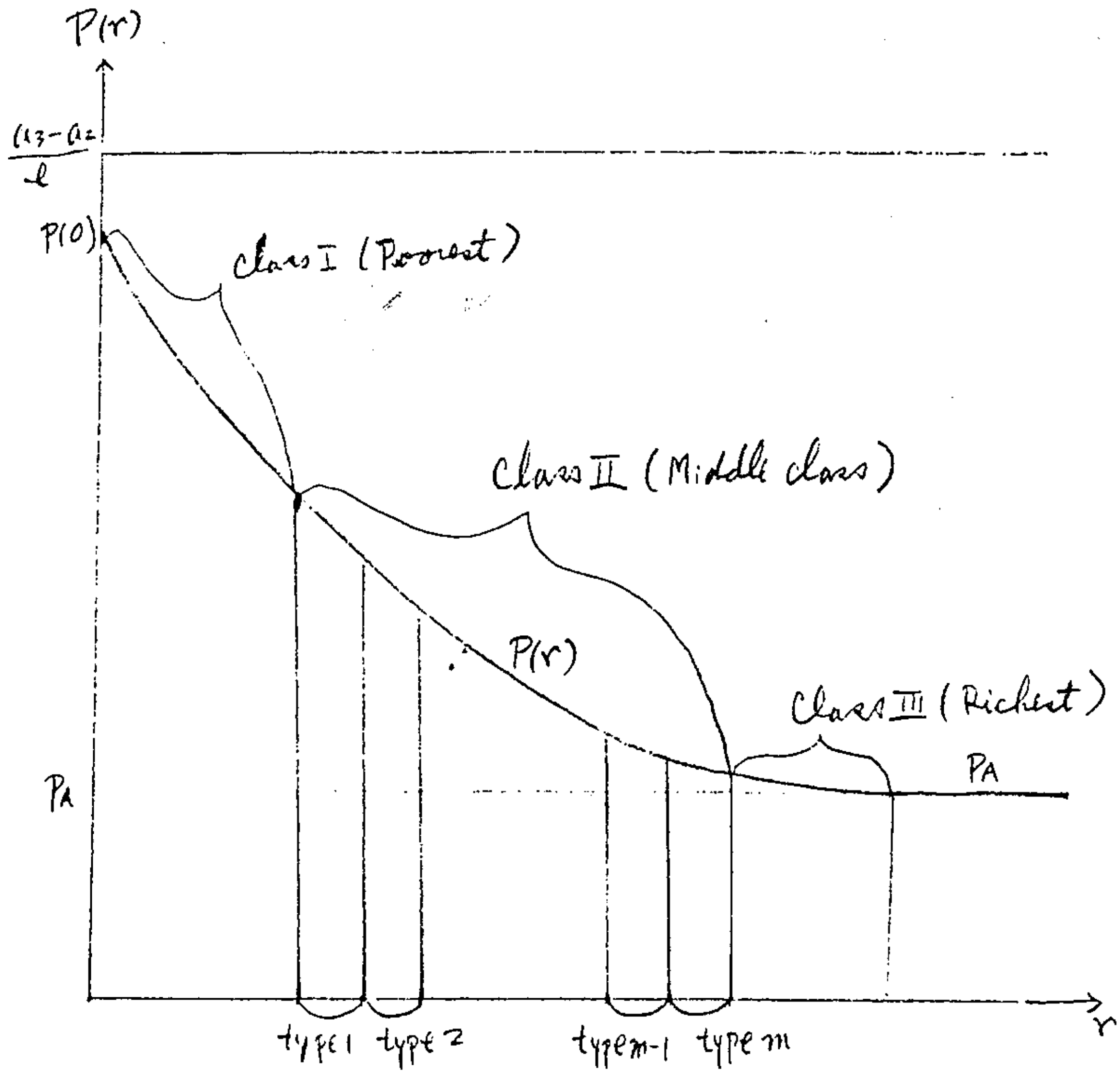

Figure 4.1 (Case 1): Equilibrium Spatial Structure with Land Ownership: $P(0) > (a_3 - a_2)/\ell$

Class II A is a set of households of type j ($j \leq k$), and call it the outer land owner.

Class II B is a set of households of type l ($l \geq k + 1$), and call it the inner land owner.

For this case, the income level of each class is ranked as follows.

$$y^1 \equiv a^1 < y_j^2 \equiv a^2 + P(d_j)\ell < y^3 \equiv a^3 < y_\ell^2 \equiv a^2 + P(d_\ell)\ell$$

Hence, as shown in Figure 4.2, the market land price curve and the assignment of zones to each class and each type is in accordance with the final income level. That is,

The poorest class resides closest to the CBD

The outer land owner class resides next to the poorest class slightly farther from the CBD.

The richest class resides next to the outer land owner class somewhat farther from the CBD.

The inner land owner class resides next to the richest class and is the class farthest from the CBD.

Concluding the above discussions, we say that:

Proposition 4.1 (Relative Advantages of Land Owner)

If the change of the land price in a city is larger, more households with land ownership close to the CBD gain money, and as a result, they reside farther from the CBD than the richer households without land ownership.

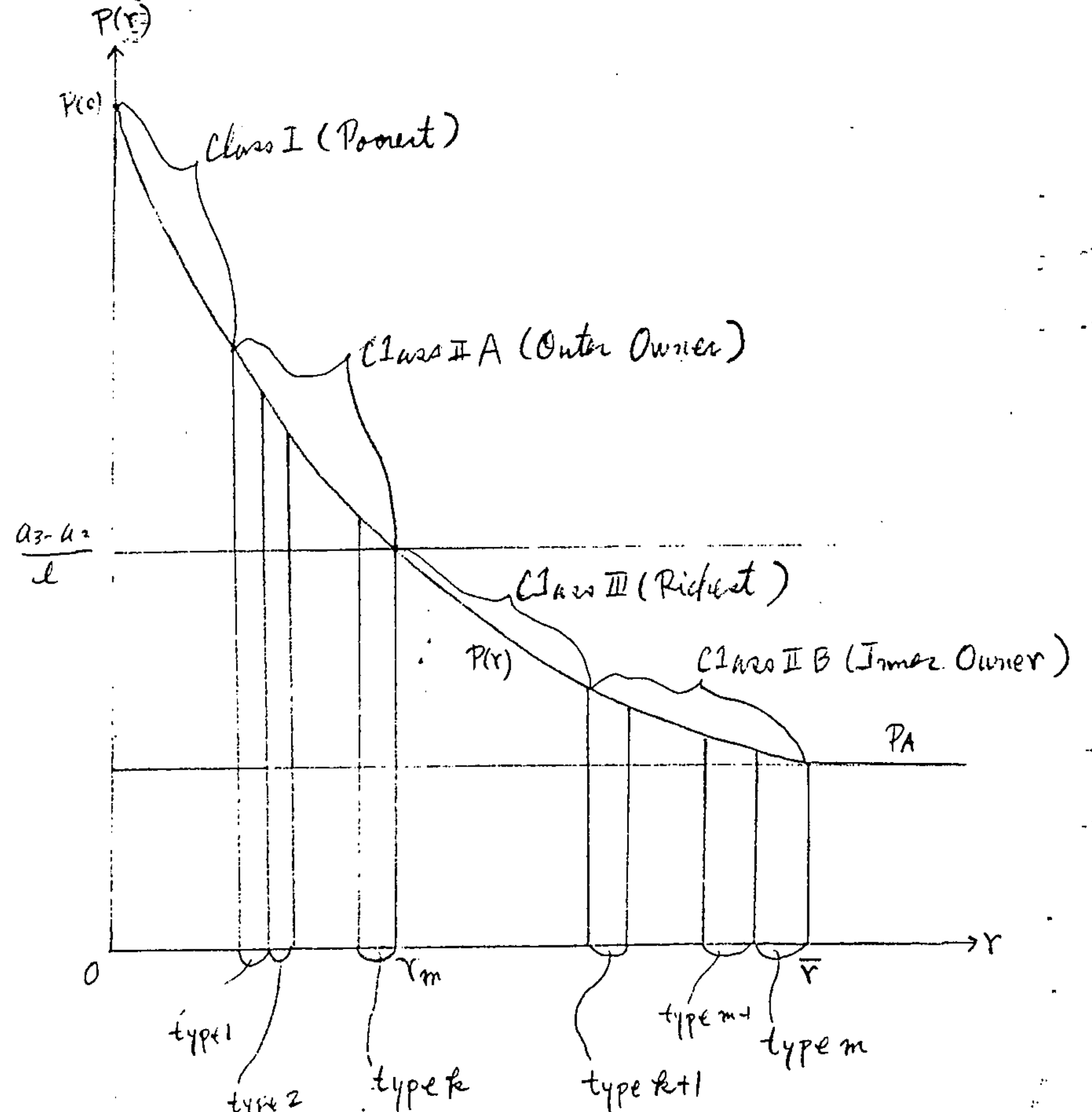

Figure 4.2 (Case 2): Equilibrium Spatial Structure with

Land Ownership: $P(0) < (a_3 - a_2)/\ell$

5. OPTIMUM MODEL OF URBAN LAND USE WITH LAND OWNERSHIP

Here, instead of the determination of land use by competitive markets we consider the optimal allocation of households and land. Though the definition of optimum allocation depends on the criterion used and many different criteria are conceivable, we are concerned with Pareto optimal allocations. In order to define precisely (Pareto) optimum land use pattern, we first introduce the utility equity constraint which requires that all households of the same type should enjoy the same utility level independent of their locations. This constraint is natural since all households of the same type are assumed to be identical. Moreover, if the planning authority should wish to discriminate among some households within the same type, it could achieve this simply by subdividing the households in the same type into a number of subgroups and considering each subgroup as one type of households. Considering the definition of household types in this way, we assume that there are m types of households which are the same as those in section 2. That is, each household of type i has utility function $u_i(z,q)$, initial endowment of composite good a and initial endowment of land (ownership) ℓ at distance d_i (from the CBD), and commuting cost function $T_i(r) = T(r)$ for all i; and the total number of households of type i is given by the continuous number N_i. Let us denote that, $n_i(r)$ is the number of in type i households at distance r, $(z_i(r),q_i(z))$ is the consumption bundle for each household of type i at r , $\bar{r}$ is the urban fringe distance. All of these variables are assumed to be nonnegative. It is not difficult to prove by using the assumption of strictly concave utility functions that all households of the same type located at the same distance which are enjoying the same utility level must consume a same consumption bundle at the Pareto optimum. Therefore, we can consider only the same consumption bundle for all households in the same type at each location.

Let us define an allocation $(\{n_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \bar{r})$ is called feasible if and only if the following set of conditions (5.1)---(5.4) are satisfied.

(i) composite good constraint;

(5.1) $$\int_0^{\bar{r}} \sum_i (z_i(r) + T(r))\, n_i(r)dr \leq \sum_i N_i a \equiv Na \quad (N = \Sigma N)$$

(ii) land constraint;

(5.2) $$\sum_i q_i(r)\, N_i(r) \leq L\,(r) \text{ for each } R$$

(iii) population constraint;

(5.3) $$\int_0^{\bar{r}} n_i(r)\, dr = N_i \text{ for every } i \; (i=1,2,\ldots,m)$$

(iv) utility equity constraint;

There exists a continuous number U_i such that:

(5.4) $$U_i(z_i(r), q_i(r)) = U_i \text{ for all } r \text{ where } n_i(r) > 0 \; (i=1,2,\ldots,m)$$

For each allocations, the total cost of development of residential city is defined by

(5.5) $$TC = \sum_i \int_0^{\bar{r}} n_i(r)(z_i(r) + T(r) + P_A q_i(r))dr$$

where, the term $P_A q_i(r)$, represents the agricultural opportunity cost. A feasible land use pattern is called a Pareto optimal land use pattern if and only if no feasible reallocation could increase the utility level of any household without lowering the utility level of some other type or increasing the total cost, and no feasible reallocation could decrease the total cost without lowering the utility level of some household type.

In order to restate the definition of optimum land use pattern above in a more convenient way, let us define bid price with fixed lot size $q(> 0)$ for household type i (per unit of land) by

$$\psi_i(r,U,q) = \frac{y_i - T(r) - z_i(q,U)}{q} \tag{5.6}$$

where $y_i = a + P(d_i)1$ is income of household type i, and $P(d_i)$ is a price of land (per unit of land) at distance d_i which is set by the planning agency and not necessarily a competitive market price.

Then, given a feasible allocation $(\{n_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \overline{Y})$ we have, by using population constraint (5.3) and utility equity constraint (5.4), that

$$TR \equiv \sum_i y_i N_i - TC \equiv \int_0^{\bar{r}} \sum_i \{\psi_i(r,U_i,q_i(r)) - P_A\} \, q_i(r) n_i(r) dr \tag{5.7}$$

Since the right hand side of (5.7) represents the net land selling revenue from the development of the residential city, we call TR the total net revenue. Then since $\sum_i y_i N_i$ is a constant by assumption, we can redefine the Pareto optimal land use pattern as follows.

Definition 5.1 (Pareto optimal urban land use pattern)

A feasible allocation, $(\{n_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \bar{r})$ a Pareto optimum land use pattern if and only if no feasible reallocation could increase the utility level of any household type without lowering the utility level of some other type or the total revenue, or could increase the total revenue without lowering the utility level of some household type.

Let us denote EF the set of all (m + 1) dimensional vectors consist of a set of utility levels of all households types and total net revenue, $(U_1,U_2,\ldots,U_m,TR)$, each of which corresponds to an optimum land use pattern, and call EF the efficient frontier. A feasible allocation is Pareto optimum if an donly if the corresponding vector $(U_1,U_2,\ldots,U_m,TR)$ is on the efficient frontier.

Our next task is to formulate a mathematical programming model which can generate all possible optimum land use configurations as its solutions. This can be achieved as follows. Let us define the feasible set of utilities $\{U_i\}_{i=1}^m$ which corresponds to the feasible allocation by

$$(5.8)\quad G = \{(U_1,U_2,\ldots,U_m) \,|\, U_i = u_i(z,q) \text{ for some } (z,q) > 0 \quad (i=1,2,\ldots,m)\}$$

Then, given a feasible set of utilities, $\{\bar{U}_i\} \in G$, let us consider the following optimization problem HS($\{\bar{U}_i\}$): (Herbert-Stevens Problem). Choose household distributions $n_i(r)$, composite good curves $z_i(r)$, lot size curves $q_i(r)$ $(0 \leqq r \leqq \bar{r},\ i=1,2,\ldots,m)$ and urban fringe distance $\bar{r}$ so as to maximize

$$(5.9)\qquad TR \equiv \int_0^{\bar{r}} \sum_i [\psi_i(r,\bar{U}_i,q_i(r)) - P_A]\, q_i(r) n_i(r) dr$$

subject to

a) composite good constraint

$$\int_0^{\bar{r}} \sum_i (Z_i(r) + T(r))\, n_i(r)\, dr \leqq \sum_i N_i\, a \equiv Na \tag{5.10}$$

b) land constraint;

$$\sum_i q_i(r) n_i(r) \leqq L(r) \quad \text{at each distance } r\ (0 \leqq r \leqq \bar{r}) \tag{5.11}$$

c) population constraint;

$$\int_0^{\bar{r}} n_i(r)\, dr = N_i \quad \text{for all } i\ (i=1,2,\ldots,m) \tag{5.12}$$

where, $n_i(r) \geqq 0$ and $z_i(r) > 0$, $q_i(r) > 0$, and bid price function ψ_i is defined by (5.6).

Namely, this problem is to determine the allocations of household, composite good and land so as to maximize the total net revenue subject to composite good constraint, land constraint and population constraint while satisfying the target utility levels. Since this problem is a continuous version of the Herbert-Stevens problem (Herbert and Stevens (1960)), we call it <u>HS problem</u>. Each HS problem is characterized by a set of target utility levels, $\{\bar{U}_i\}$. And, by changing $\{\bar{U}_i\}$ all over the feasible set G, we obtain a set of HS problems. We call this set of problems the <u>HS model</u>.

The solution for an HS problem may not be unique. However, it is not difficult to show that each solution of an HS problem gives an optimum land use in the sense of Definition 5.1 (Pareto optimal urban land use pattern). And by changing parameters $\{\bar{U}_i\}$ all over the set G, we can obtain all the optimal land use configurations.

Suppose that we have assumptons A2.1 and A2.2, then by using the Maximum Principle of optimal control theory (see e.g., Pontryagin et al. (1962)), we get the optimality conditions, $OC(\{\overline{U}_i\})$ for problem $HS(\{\overline{U}_i\})$ as follows. (For derivation of the condition, see Appendix 1)

$OC(\{\overline{U}_i\})$; (Optimality Condition)

A land use pattern which consists of household distributions $n_i(r)$, composite good curves $z_i(r)$, lot size curves $q_i(r)$ $(0 \leqq r \leqq r_i,\ i=1,2,\ldots,m)$ and urban fringe distance $\overline{r}$ is optimal for problem HS $(\{\overline{U}\})$ if and only if there exists a set of multipliers $P(r)$, P_z, Q_i $(0 < r < \overline{r},\ i = 1, 2, \ldots, m)$ such that

(i) Land market condition

(5.13) $$P(r) = \max \{\max_i \Phi_i (r,\overline{U}_i,Q_i),P_A\}$$

(5.14) $$P(r) = \Phi_i(r,\overline{U}_i,Q_i) = \psi_i(r,\overline{U}_i,q_i(r), Q_i)$$
$$\text{if } n_i(r) > 0$$

(5.15) $$\sum_i q_i (r)\, n_i(r) \leqq L(r) \text{ at each } r$$

(5.16) $$\sum_i q_i(r)\, n_i(r) = L(r) \text{ if } P(r) > P_A$$

(5.17) $$P(\overline{r}) = P_A$$

(ii) composite good market condition;

(5.18) $$\int_0^{\overline{r}} \sum_i (z_i(r) + T(r))\, n_i(r)\, dr \leqq \sum_i N_i a \equiv N_a$$

(iii) Population constraints;

$$\int_0^{\bar{r}} n_i(r)\, dr = N_i \quad (i = 1, 2, \ldots, m) \tag{5.19}$$

where, $n_i(r) > 0$, $q_i(r) > 0$, and bid price functions $\Phi_i(r, U_i, Q_i)$ and $\psi_i(r, U_i, q_i(r), Q_i)$ are defined by (5.20) and (5.21), respectively.

$$\Phi_i(r, U_i, Q_i) = \max_q \frac{y_i + Q_i - T(r) - z_i(q, U_i)}{q} \tag{5.20}$$

$$\psi_i(r, U_i, q_i, Q_i) = \frac{y_i + Q_i - T(r) - z_i(q, U_i)}{q} \tag{5.21}$$

where $y_i \equiv a + P(d_i)l \quad (i = 1, 2, \ldots, m)$

$\Phi_i(r, U_i, Q_i)$ is called the subsidized bid price, because it represents the bid price which a household of type i can pay after subsidization with income subsidy Q_i. And $\psi_i(r, U_i, q_i, Q_i)$ is called a subsidized bid price function with lot size q_i.

By definition, note the following relation.

$$\begin{aligned} \Phi_i(r, U_i, Q_i) &= \max_q \frac{y_i + Q_i - T(r) - Z_i(q, U_i)}{q} \\ &= \psi_i(r, q_i(r), U_i) + \frac{Q_i}{q_i(r)} \end{aligned} \tag{5.22}$$

Next, we formulate a market model of a residential city which corresponds to the optimum model formulated above. We consider three kinds of agents, viz., households, absentee landlords and the government, and assume that each of them seeks its objective in a decentralized maner through market mechanism.

The structure of the model is given by the basic assumptions except that, equilibrium utility level, $\{U_i^*\}_{i=1}^{m}$, is obtained endogenously in the market. Instead, we consider the income subsidy (or negative tax) $\bar{Q}_i$ for each household of type i as the policy variable to be set by the government.

The model is essentially the same as the Alonso-Muth type market model discussed in section 2 and 3 except for income subsidy $\{\bar{Q}_i\}_{i=1}^{m}$, and thus, it is denoted as AM ($\{\bar{Q}_i\}$)

AM ($\{\bar{Q}_i\}$) (Alonso-Muth Type Market Problem)

(i) Households Behavior;

With a given amount of income subsidy $\bar{Q}_i$, the residential choice behavior of each household type i is formulated as follows

$$\max_{z,q,r} u(z, q)$$

(5.23) subject to

$$z + P(r)q + T(r) = y_i + \bar{Q}_i$$

where $y_i \equiv a + P(d_i)l \quad (i = 1,2,\ldots,m)$

Note that, except $\bar{Q}_i$, all variables and functions above are the same as those in the basic model in section 2.

The meaning of model (5.23) is the same as the basic model except that each household of type i is given an income subsidy (or tax), $\bar{Q}_i$, in addition to its initial holding of composite good a and land l at distance d_i.

The market equilibrium conditions of the residential city is given similarly to those as in section 3.

MC ($\{\bar{Q}_i\}$): (Market Equilibrium Condition)

An urban spatial structure which consists of household distribution $n_i(r)$, composite good curves $z_i(r)$, lot size curves $q_i(r)$, urban fringe distance, $\bar{r}$, market land price curve, $P(r)$, and equilibrium utility levels $\{U_i\}_{i=1}^{m}$ is competitive equilibrium if and only if the following set of conditions is satisfied.

(i) Land market;

(5.24) $$P(r) = \max \left(\max_i \ \psi_i(r, U_i, \bar{Q}_i), P_A \right)$$

(5.25) $$P(r) = \Phi_i(r, U_i, \bar{Q}_i) = \psi_i(r, U_i, q_i(r), \bar{Q}_i) \quad \text{if } n_i(r) > 0$$
$$\text{for } r : o \leqq r \leqq \bar{r}, \quad (i = 1, 2, \ldots, m)$$

(5.26) $$\sum_i q_i(r) \, r_i(r) \leqq L(r)$$

(5.27) $$\sum_i q_i(r) \, n_i(r) = L(r) \quad \text{if } P(r) > P_A$$

(ii) composite good market;

(5.28) $$\int_o^{\bar{r}} \sum_i (z_i(r) + T(r)) \, n_i(r) \, dr \leqq \sum_i N_i a = N_a$$

(iii) population constraints;

(5.29) $$\int_o^{\bar{r}} n_i(r) \, dr = N_i \quad (i = 1, 2, \ldots, m)$$

where, $n_i(r) > 0$, $q_i(r) > 0$ and bid price functions $\Phi_i(r, U_i, Q_i)$ and $\psi_i(r, U_i, q_i(r), Q_i)$ are defined by (5.20) and (5.21), respectively.

The government sets the income subsidies $\{\bar{Q}_i\}$.

Let us examine the property of the correspondence between optimum and market models. First, we find that optimality and market equilibrium conditions, OC ($\{\bar{U}_i\}$) and MC ($\{Q_i\}$) , are equivalent except for the following point.

(a) Utility levels are exogenously given in the optimum model while they are endogenously determined in the market model. And in return,

(b) Income subsidies $\{\bar{Q}_i\}$ are exogenously given in the market model while they are endogenously determined in the optimum model. Hence, we can find the following relation between solutions complying with OC ($\{\bar{U}_i\}$) and MC ($\{\bar{Q}_i\}$)

<u>Property 5.1</u> (correspondence from optimum to market)

Suppose that $\{(n_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \bar{r}, P(r), \{Q_i\})$ is a solution for HS problem with utility levels $\{\bar{U}_i\}$. And, if the government gives income subsidies $\{\bar{Q}_i\}$ to households, then the AM problem with $\{\bar{Q}_i\}$ can generate

$$(\{n_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \bar{r}, P(r), \{\bar{U}_i\})$$

as the market equilibrium solution. Hence, the solution of any HS problem can be achieved through an AM problem using appropriate income subsidies.

Practically, this property is particularly meaningful in the sense that once the government establishes the goal for utility levels $\{\bar{U}_i\}$, this goal can be achieved through the market mechanisms by giving income subsidies $\{\bar{Q}_i\}$ calculated from HS ($\{\bar{U}_i\}$) . The converse of this property is also true.

Property 5.2. (correspondence from market to optimum)

Suppose that $(\{n_i^*(r)\}, \{z_i^*(r)\}, \{q_i^*(r)\}, \bar{r}^*, P^*(r), \{U_i^*\})$ is a solution for $AM(\{\bar{Q}_i\})$. Then $(\{n_i^*(r)\}, \{Z_i^*(r)\}, \{q_i^*(r)\}, \bar{r}^*, P^*(r), \{\bar{Q}_i\})$ is a solution for $HS(\{U_i^*\})$. Thus the solution of any AM problem can be achieved through the HS optimum model associated with appropriate utility levels.

In other words, we can observe a one-to-one correspondence between $HS(\{\bar{U}_i\})$ and $AM(\{\bar{Q}_i\})$, elements of solutions for the optimum and market problems. That is, we can always find an AM problem which gives the same solution as a HS problem, and vice versa. Thus considering Property 3.2, we can conclude the existence of equilibrium solutions.

Property 5.3

Under assumptions 2.2--2.4, there exists an equilibrium solution for $AM(\{\bar{Q}_i\})$.

We have already known that the optimum solution of an HS problem is Pareto optimal with respect to households and absentee landlords. And we know from Property 5.2, that for any AM problem, there exist an HS problem which gives the same solution. Therefore, the following property is straightforward obtained.

Property 5.5

Given income subsidies $\{\bar{Q}_i\}$, there exists a Pareto optimum allocation of households, composite good and land which can be achieved through the competitive market mechanism.

6. A FRAMEWORK FOR AN URBAN LAND USE GAME

Next, we build a framework for an urban land use game with land ownership. Here, we consider two kinds of players, viz., households (with land ownership) and absentee landlords. Assume that each of players seeks its objective by a cooperative game.

Suppose that there are m types of households, $i = 1,2,...m$. The number of households of each type i is exogeneously given N_i (that is, we assume a closed city model).

Each household of type i has initial endowments $e_i = (a, l(d_i))$, where a denotes the initial holding of composite good and $l(d_i)$ denotes the initial holding of land at distance d_i from the CBD. The preference ordering of a household i is described by its utility function u_i such that

$$U_i = u_i\ (z_i,\ q_i(r))$$

where z_i denotes the amount of the composite good and $q_i(r)$ denotes the amount of land at distance r from the CBD. For simplicity, each household of every type has the same utility function, $U_i = u\ (z,q(r))$ for all $i(i=1,2,...,m)$. Hence, each household of the same type has the same utility function and the same initial holdings of the composite good and land at the same distance from the CBD. The only difference among households of different types is the initial holding of land at different distances from the CBD. The utility function satisfies the Assumption 2.2. All land $L(r)$, except at the distance $d_i(i=1,2,...,m)$ is owned by absentee landlords which have an equal land ownership at each distance from the CBD. And the number of absentee landlords is assumed to be Na (continuous).

The preference ordering of an absentee landlord a is described by his utility function v_a such that

$$V_a = v_a\ (z_a)$$

where z_a denotes the amount of the composite good.[6] For simplicity, each absentee landlord has the same utility function, $V_a = v(z_a)$ for all a. We assume that utility function $v(z_a)$ is well behaved in the following sense.

Assumption 6.2

Utility function $v(z_a)$ satisfies the following set of conditions in its domain $z_a \geqq 0$.

(i) It is strictly quasi-concave and twice continuously differentiable

(ii) $\partial U/\partial z > 0$

(iii) The composite good is indispensable for absentee landlords, that is, for z, some amount of good will be purchased.

Households and absentee landlords cooperate in order to develop a residential city. This cooperative urban development can be regarded as a cooperative game with a continuum of players. And we call it the game $(\{N_i\}), N_a)$ consists of N_i members of households of type i and N_a members of absentee landlords. Hence, we can analyze this cooperative urban development game using the mathematical theory of games.

In order to give the simplified definition of an allocation of the game, we first introduce the utility equity constraint which requires that all households of the same type should enjoy the same utility level independent of their locations. If some households of the same type were discriminated against, then these households could be considered other types of households. Moreover, all households of the same type located at the same distance from the CBD which are enjoying the same utility level must consume the same consumption bundle of composite good and land.

Definition 6.1

An allocation of the urban game $(\{N_i\}, N_a)$ is defined as a set $(\{n_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \bar{r}, z_a)$ which consists of,

$n_i(r)$: the number of households of type i which locate at distance r

$x^i(r) = (z_i(r), q_i(r))$: the consumption bundle of a household of type i which locates at distance r, where,

$z_i(r)$: the final holding of composite good

$q_i(r)$: the final holding of land

$\bar{r}$: the edge of the city

z_a: the final holding of composite good for a absentee landlord

such that:

(i) composite good constrants;

$$\int_0^{\bar{r}} \sum_i z_i(r) n_i(r) dr + N_a z_a + \int_o^{\bar{r}} \sum_i T(r) n_i(r) dr \tag{6.1}$$

$$\leqq \sum_i N_i\, a = Na \quad (N \equiv \sum_i N_i)$$

(ii) land constraints;

$$\sum_i q_i(r) n_i(r) \leqq L(r) \quad \text{for each } r \tag{6.2}$$

(iii) population constraints;

$$\int_0^{\bar{r}} n_i(r)\, dr = N_i \quad (i=1,2,\ldots,m) \tag{6.3}$$

Definition 6.2 (Feasible Allocation)

An allocation $(\{n_i(r)\}, \{z_i(r)\}, \{q_i(r)\{, \bar{r}, za)$ is called feasible if and only if the following condition is satisfied;

(iv) utility equity constraint;

(6.4) There exists a set of a continuous number U_i and V_a such that

$u_i(z_i(r), q_i(r)) = U_i$ for all r where $n_i(r) > 0 \quad (i=1,2,\ldots,m)$

$u\ (z_a) = V_a$

In order to define the core of the game, let us define the concept of blockness by a coaliton. A subset $S = (S_1, S_2, \ldots, S_m, S_a)$ $M = (N_1, N_2, \ldots N_m, N_a)$ is called a coalition in this game, where M is the total number of players which consist of all households and all absentee landlords. Since all households of the same type located at the same distance from the CBD which are enjoying the same utility level must consume the same consumption bundle of the composite good and land, they are assumed to be identical. Hence, we may consider only the number of households of each type at each distance from the CBD. And, $s_i = |S_i|$ and $s_a = |S_a|$ denote the number of households of type i and the number of absentee landlords which are members of the coaltion, respectively.

Definition 6.3

We say that the feasible allocation is blocked by a coaliton $S = (S_1, S_2, \ldots, S_m, S_a)$, if it is possible to find an allocation $(\{s_i(r), \{z_i'(r)\}, \{q_i'(r)\}, \overline{r}, z_a')$ with

(i) composite good constraint;

(6.4) $$\int_0^{\overline{r}} \sum_i z_i'(r)\, s_i(r) dr + s_a z_a' + \int_0^{\overline{r}} \sum_i T(r)\, s_i(r) dr \leqq \sum_i s_i a$$

(ii) land constraint;

(6.5) $$\sum_i q_i'(dj)\, s_i(dj) \leqq s_j \ell \quad (j = 1,2,\ldots,m) \quad s_j \equiv |S_j|$$

(6.6) $$\sum_i q_i'(r)\, s_i(r) \leqq \frac{1}{s_a} L(r) \quad (r \neq d_j)$$

(iii) population constraints;

(6.7) $$\int_0^{\overline{r}} s_i(r)\, dr = s_i \quad (i = 1,2,\ldots,m)$$

And both of the following conditions are satisfied:

(iv) utility constraint of households;

$$u(z_i'(r), q_i'(r)) \geqq u(z_i(r), q_i(r))$$

for all i (i=1,2,---,m) at each r, with strict preference for at least one member of $(S_1, S_2, \text{---}, S_m)$ at some r $(0 \leqq r \leqq \overline{r})$

(v) utility constraint of absentee landlords;

$$v(z'_a) \geqq v(s_a)$$

for all $a \in S_a \subseteqq N_a$.

Definition 6.4

The core of the game $(\{N_i\}, N_a)$ is defined as a collection of all allocations

$$(\{n_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \bar{r}, z_a)$$

which cannot be blocked by any coalition

$$S = (S_1, S_z, \text{—}, S_m, S_a) \subseteqq M = (N_1, N_2, \text{—}, N_m, N_a)$$

One can immediately obtain the following property.

Property 6.1

The core of the urban game is Pareto optimal.

Next, we consider the relationship between the core of the game and the market equilibrium. We obtain the following properties which are well known in the field of core theory of a pure exchange economy.

Property 6.2

In the core of the urban game, each household of the same type has the same utility level.

Property 6.3

A market equilibrium spatial structure is included in the core of the game.

Proof). Let $(\{n_i^*(r)\}, \{z_i^*(r)\}, \{q_i^*(r)\}, \bar{r})$ be an equilibrium spatial structure of urban land use but this is not included in the core of the urban game. Let $S = (S_1, S_2, \ldots, S_m, S_a)$ $M = (N_1, N_2, \ldots, N_m, N_a)$ be a possible blocking coalition, so that

$$(6.8) \qquad \int_0^{\bar{r}} \sum_i z_i'(r) s_i(r) dr + s_a z_a' + \int_0^{\bar{r}} \sum_i T(r) s_i(r) dr \leqq \sum_i s_i a$$

$$(6.9) \qquad \sum_i q_i'(d_j) s_i(d_j) \leqq s_j \ell \quad (j=1,2,\ldots,m)$$

$$(6.10) \qquad \sum_i q_i'(r)\, s_i(r) \leqq \frac{1}{s_a} L(r) \quad (r \neq d_j)$$

with $u(z_i'(r), q_i'(r)) \geqq u(z_i^*(r), q_i^*(r))$ for all i in S at each r, and with strict inequality for at least one i at least one r, where $s_i(r)$ denotes the number of households of type i which reside at this distance r. And $v(z_a') \geqq v(z_a^*)$ for all a ε S_a. These implies

$$z'_i(r) + P(r) q'_i(r) + T(r) \geqq a + P(d_i) 1$$

for all i in S_i at each r, with strict inequality for at least one i at some r. And

$$z_a' \geqq \frac{1}{s_a} \int_0^{\bar{r}} P(r)\, L(r) dr, \ (r \neq d_j)$$

for all a in S_a.

Therefore,

$$\int_0^{\bar{r}} \sum_i [z_i'(r) + P(r)q_i'(r) + T(r)]\ s_i(r)dr + s_a z_a' >$$

$$\int_0^{\bar{r}} \sum_i (a + P(d_i)\ell)s_i(r)dr + \int_{0\ (r \neq d_j)}^{\bar{r}} P(r)\ L(r)\ dr$$

Hence,

$$(6.11) \qquad \int_0^{\bar{r}} \sum_i z_i'(r)s_i(r)dr + s_a z_a' + \int_0^{\bar{r}} \sum_i T(r)\ s_i(r)dr +$$

$$\int_0^{\bar{r}} \sum_i P(r)q_i'(r)s_i(r)dr > \sum_i s_i a + \sum_i P(d_i)\ \ell + \int_{0\ (r \neq d_j)}^{\bar{r}} P(r)\ L(r)dr$$

(6.11) contradicts (6.8), (6.9) and (6.10). Q.E.D.

Moreover, we obtain a stronger proposition on the equivalence between the core of the urban game and the equilibrium spatial pattern.

Theorem 6.1 (Equivalence Theorem)

The equilibrium spatial structure of the urban land use market is equivalent with the core of the urban game.

The proof of this theorem will be carried out by a method similar to the equibalence theorem of an atomless economy with a continuum of traders shown in Aumann (1964) or Hildernbrand (1974, Theorem 1, pp. 133-135). In this paper we will give a simple proof using a method similar to the Lagrangian approach for proof of the limit theorem on the core of an economy as recently shown by Schweizer (1982). We have already proved that an equilibrium spatial structure of the land market is in the core of the urban land game $(\{N_i\}, N_a)$ by Property 6.3. Hence, this time we must verify that for any allocation in the core of the urban land game $(\{N_i\}, N_a)$ which has a continuum of households and landlords, an equilibrium is obtained via the corresponding urban land market. This proposition is restated as the following Property 6.4.

Property 6.4

If the allocation $\overline{X} = (\{\overline{n}_i(r)\}, \{\overline{z}_i(r)\}, \{\overline{q}_i(r)\}, \overline{\overline{r}}, \overline{z}_a\}$ is in the core of game $(\{N_i\}, N_a)$ and it satisfies the following strictly positive consumption level condition of the consumption level of the composite good and land,

$$z_i(r) > 0 \quad \text{at each } r\ (0 \leqq r \leqq r) \quad (i = 1,2,\ldots,m)$$

$$q_i(r) > 0 \quad \text{at each } r\ (0 \leqq r \leqq r) \quad (i = 1,2,\ldots,m)$$

then, there exists strictly positive prices of the composite good and land, respectively,

$$P_z > 0$$

$$P(r) > 0 \text{ at each } r\ (0 \leqq r \leqq \overline{r})$$

which sustain the allocation $\overline{X}$ as an equilibrium spatial structure of the urban land market.

If we use a Lagrangian approach like Schweitzer (1982), it is not difficult to prove Property 6.4. As he did, we consider the Lagrangian associated with the maximization problem which emerges from the notion of Pareto efficiency for a variable number of traders. In the game $(\{N_i\}, N_a)$, since we can only consider the number of members of any coalition, any allocation of the core is obtained from the solutions to the maximization problem using the notion of Pareto efficiency for a variable number. That is, differentiating this Lagrangian with respect to the composite good and land at each distance leads to the corresponding prices at which expenditures are minimized for the given utility level. Next, we differentiate this Lagrangian with respect to the number of households at each distance. If we evaluate this Lagrangean at $\overline{X}$ and $\{N_i\}$, N_a, then the corresponding derivatives must vanish. It turns out that this is only the case when $\{\overline{z}_i(r), \overline{q}_i(r)\}$ belongs to the budget constraint of each household of any type i. Hence, the allocation $\overline{X}$ is sustainable as an equilibrium of the market $(\{N_i\}, N_a)$. (For a precise proof, see Appendix 2.) At equilibrium, we need the income subsidies from the government. Hence, this equilibrium is a compensated equilibrium.

7. CONCLUDING REMARKS

In the present paper we have discussed a theorey of urban land use patterns with land ownership and given a game-theoretic framework for urban land use theory. We first formulated the basic market model of urban land use with land ownership. In this model we assumed a finite number of types of households which own the same amount of land at different distances from the CBD. The residential behavior of the individual household, of each type, was analyzed by a method similar to the standard method for the urban land use model with given income classes. As always each household behaves as a price-taker in the competitive market. We obtained a equilibrium spatial structure for this urban land use market with land ownership. The main theorem in the equilibrium land use pattern with land onwership is the location replacement theorem. This states that the farther the initial location of a household's land is from the CBD, the closer its optimal location is to the CBD. The replacement of the location occurs by means of this market structure. Next, we analyzed the market with and without land ownership. We obtained the interesting result of the relative advantage of the land-owner compared to non-landowners for the case of a great change in land prices.

In this paper we suggested three methods for proving the existence of the equilibrium spatial structure. They are the constructive method, the method using the correspondence between the optimum model and the market model, and the method using the equivalence theorem between the core of the urban land game and the equilibrium spatial structure of the urban land market. In section five we built an optimum model with land ownership and presented the correspondence between this optimum and the market equilibrium spatial structure.

Next, we gave a game-theoretic approach to urban land use theory. We first proceeded to formulate an urban land use game with land ownership. We obtained the Equivalence Theorem between the core of the urban game and the competitive spatial structure of the land market. We gave a simple Lagrangian approach to the proof of the equivalence theorem.

We formulated a model with land ownership for a finite number of type of household types with a continuum of households in each type. However, it is not difficult to develop a model for land ownership with a continuum of household types and a finite number of households in each type. Similarly, we will be able to give the proof for the existence of the equilibrium in this model. In following papers we will give a more general framework for urban land games. By using a game-theoretic approach we can study non-competitive situations of urban land use with land ownership, that is, we can give theories of urban spatial structures with the monopolistic supply of land.

APPENDIX 1: THE DERIVATION OF THE OPTIMALITY CONDITION

We can easily derive the optimality condition for our H-S problem by a method similar to that used for the standard H-S problem, as shown by Ando (1981) or Fujita (1982).

First, we define the number of households of type i which reside beyond distance r, $K_i(r)$, as follows.

(A.1.0) $$K_i(r) = \int_r^{\bar{r}} n_i(r)\, dr \qquad (i=1,2,\ldots,m)$$

The population constraint c) is rewritten to the following terminal conditions

c') terminal conditions:

(5.12)' $$K_i(0) = N_i, \; K_i(\bar{r}) = 0$$

Corresponding to the optimal control problem, our problem has $K(r) = (K_1(r),\ldots,K_m(r))$ as the state variable, and $n(r) = (n_1(r),\ldots,n_m(r))$, $q(r) = (q_1(r),\ldots,q_m(r))$ and $z(r) = (z_1(r),\ldots,z_m(r))$ as the control variables. It is non-autonomous in the sense that it involves the parameter r explicitly, while having no constraints due to the control variables. The state variables at the end distances, $r = 0$ and $r = \bar{r}$, are fixed by (5.12'), while $\bar{r}$ is to be determined endogenously in the model.

In this case, the maximum principle requires the existence of a nonzero continuous vector $\bar{\lambda}(r) = (\lambda_0,\lambda(r)) = (\lambda_0,\lambda_1(r),\ldots,\lambda_m(r))$, $(0 \leqq r \leqq \bar{r})$, satisfying the following conditions for the optimality of the solution, $(K(r), n(r), q(r), z(r))$

1) $\lambda_0 \geqq 0$

2) $\lambda_i(r)$ $(i=1,2,\ldots,m)$ is given as a solution of

(A.1.1) $$\dot{\lambda}_i(r) = -\frac{\partial\mathcal{H}}{\partial K_i}$$

3) $$\mathcal{H}(n(r),\ q(r),\ z(r),\ \lambda(r),\ r) = \max_{n,q,z} \mathcal{H}(n,q,z,\ \lambda(r),r)$$

(A.1.2) subject to:

$$\sum_i q_i(r)n_i(r) \leqq L(r) \qquad (5.11)$$

$$\int_0^{\bar{r}} \sum_i (z_i(r) + T(r))\ n_i(r)\ dr \leqq Na \qquad (5.10)$$

and

$$n_i(r) \geqq 0 \text{ and } q_i(r) \geqq 0 \quad (i=1,\ldots,m)$$

for each $r(0 \leqq r \leqq \bar{r})$

4)

(A.1.3) $$\mathcal{H}(n(\bar{r}),\ q(\bar{r}),\ \lambda(\bar{r}),\ \bar{r}) = 0$$

where $\mathcal{H}$ is the Hamiltonian function given by

(A.1.4) $$\mathcal{H}(n,q,z,\lambda(r),r) = \lambda_0 \sum_i \ [\psi_i(q_i,r,\bar{U}_i) - P_A]q_i n_i - \sum_i \lambda_i(r)n_i$$

It is not difficult to see $\lambda_0 > 0$, and thus, we can set $\lambda_0 \equiv 1$. The Hamiltonian function $\mathcal{H}$ does not depend on the state variable K, and from (A.1.1) we get $\dot{\lambda}_i(r) = 0$ $(i=1,2,\ldots,m)$. This implies that each of $\lambda_i(r)$ becomes some constant, and we denote it by $-Q_i$, viz., (A.1.5) $$\lambda_i(r) = - Q_i \quad \text{for } r \ (0 \leq r \leq \overline{r})$$

For each r, the problem (A.1.2) becomes a non-linear programming problem. In order to obtain the optimality condition for the problem (A.1.2), let us define the Lagrangean function by

(A.1.6)
$$\mathcal{L}(n,q,z,Q,u(r),r) = \mathcal{H}(n,q,z,Q,r) + \mu(r)(L(r) - \sum_i q_i n_i)$$
$$+ \mu_z(Na - \int_0^{\overline{r}} \sum_i (z_i(r) + T(r))\ n_i(r)\ dr)$$

where μ (r), μ_z is the Lagrangean multiplier associated with (5.11), and (5.10), respectively.

The Kuhn-Tucker Theorem gives us the following set of first-order condition for optimality of $n_i(r)$, $q_i(r)$ and $z_i(r)$ (i=1,2,...,m).

(A.1.7) $$\frac{\partial}{\partial n_i} = [\psi_i(q_i(r),r,\overline{U}_i) - P_A]q_i(r) + Q_i - \mu(r)q_i(r) \leq 0$$

(A.1.8) $$\{[\psi_i(q_i(r),r,\overline{U}_i) - P_A]q_i(r) + Q_i - \mu(r)q_i(r)\}n_i(r) = 0$$

(A.1.9) $$\frac{\partial\mathcal{L}}{\partial q_i} = [\frac{\partial\psi_i}{\partial q_i} q_i(r) + \psi_i(q_i(r),r,\overline{U}_i) - P_A]n_i(r) - \mu(r)n_i(r)$$

$$= - [\frac{\partial z_i}{\partial q_i} + P_A + \mu(r)]\ n_i(r) \leq 0$$

(A.1.10) $$[\frac{\partial z_i}{\partial q_i} + P_A + \mu(r)]\ n_i(r)\ q_i(r) = 0$$

(A.1.11) $$\frac{\partial\mathcal{L}}{\partial z_i} = - (1 + P_A \frac{\partial q_i}{\partial z_i} + \mu_z)u_i(r) \leq 0$$

(A.1.12) $$\frac{\partial \mathcal{L}}{\partial Z_i} = -(1 + P_A \frac{\partial q}{\partial z_i} + \mu_z) n_i(r) z_i(r) = 0$$

(A.1.13) $$\mu(r)(L(r) - \sum_i q_i(r) n_i(r)) = 0$$

(A.1.14) $$\mu(r) \geqq 0$$

Then (A.1.9) and (A.1.11) can be replaced respectively by

(A.1.15) $$[\frac{\partial z_i}{\partial q_i} + P_A + \mu(r)]\ n_i(r) = 0$$

and

(A.1.16) $$[1 + P_A \frac{\partial q_i}{\partial z_i} + \mu_z] n_i(r) = 0$$

and (A.1.10) and (A.1.12) are to be discorded. By defining

(A.1.17) $$P(r) \equiv \mu(r) + P_A$$

(A.1.18) $$P_z \equiv -\frac{u_z}{\mu(r)} P(r)$$

(A.1.7) and (A.1.14) together imply

(A.1.19) $$P(r) = \max \{\max_i \psi_i(q_i(r), \overline{U}_i) + \frac{Q_i}{q_i(r)}, P_A\}$$

Using $\psi_i(q_i(r), r, \overline{U}_i, Q_i)$ defined in (5.21), this can be rewritten as

(A.1.20) $$P(r) = \max \{\max_i \psi_i(q_i(r), r, \overline{U}_i, Q_i), P_A\}$$

Similarly, (A.1.8) becomes

(A.1.21) $$[P(r) - \psi_i(q_i(r),r,(\overline{U}_i),Q_i)]n_i(r) = 0$$

and (A.1.13) becomes

(A.1.22) $$(P(r) - P_A)(L(r) - \sum_i q_i(r)n_i(r)) = 0$$

On the other hand, since (A.1.15) can be written as

(A.1.23) $$[\frac{\partial z}{\partial q_i} + P(r)]n_i(r) = 0$$

We can observe that

(A.1.24) $$P(r) = -\frac{\partial z_i}{\partial q_i} \text{ whenever } n_i(r) > 0$$

And whenever $n_i(r) > 0$, then we obtain $1 + P_A \frac{\partial q}{\partial z_i} + u_z = 0$. Therefore,

(A.1.25) $$\mu_z \doteq - \mu(r)/P(r)$$

and we get

(A.1.26) $$P_z = 1$$

This states that we can set the composite good as the numeraire. And, (A.1.24) states that whenever households of type i reside at distance r, their marginal rate of substitution between composite good and land must be equal to the price ratio, $P(r)/P_z = P(r)/1 = P(r)$.

In other words, households of type i can live only at places where their maximized utility level, under the land price P(r), coincides with the prespecified utility level $\overline{U}_i$.

APPENDIX 2. THE PROOF OF THE EQUIVALENCE THEOREM

In order to give the precise proof for Property 6.4, we first established the following Lemma which states that the notion of Pareto efficiency for a variable number of households corresponds to the concept of the core.

Lemma:

A set of allocations $X = (\{s_i(r)\}, \{z_i(r)\}, \{q_i(r)\}, \bar{r}, z_a)$ of the sub-games $(\{S_i\}, S_a)$ of the game $(\{N_i\}, N_a)$ are assumed to be the variables of the following maximization problem.

For $\bar{U}_i = u_i(\bar{z}_i(r), \bar{q}_i(r))$,

(A.2.1) $$\bar{U}_1 = \max_{s,z,q,r} u_1(z_1(r), q_1(r))$$

subject to:

(A.2.2) $$u_i(z_i(r), q_i(r)) = \bar{U}_i \quad (i=2,\ldots,m)$$

(A.2.3) $$v(z_a) = \bar{V}_a$$

(A.2.4) $$\int_0^{\bar{r}} \sum_i z_i(r)s_i(r)dr + s_a z_a + \int_0^{\bar{r}} \sum_i T(r)s_i(r)dr = \sum_i s_i a$$

(A.2.5) $$\sum_i q_i(d_j)s_i(d_j) = s_j \ell \quad (j \in S_j)$$

(A.2.6) $$\sum_i q_i(r)s_i(r) = L(r)/s_a \quad (r \neq d_j)$$

(A.2.7) $$\int_0^{\bar{r}} s_i(r)dr = s_i \quad (i=1,2,\ldots m)$$

The proof of this lemma is not difficult using the property of the core, hence we omit the proof here.

The proof of the Property 6.4:

We define the Lagrangian function for this maximization problem as follows.

(A.2.8) $$L = u_1(z_1(r),q_1(r) + \sum_{i=2}^{m} \lambda_i(u_i(z_i(r),q_i(r)) - \overline{U}_i) + \lambda_a(v(z_a) - \overline{V}_a)$$

$$- \mu_z[\int_0^{\overline{r}} \sum_i z_i(r)s_i(r)dr + s_a z_a + \int_0^{\overline{r}} T(r)s_i(r)dr - \sum_i s_i a]$$

$$- \sum_j \mu_j[\sum_i q_i(d_j)s_i(d_j) - s_j \ell]dr$$

$$- \int_{0(r \neq d_i)}^{\overline{r}} \mu(r)[\sum_i q_i(r)s_i(r) - L(r)/s_a]dr$$

$$+ \sum_i \gamma_i \ [\int_0^{\overline{r}} s_i(r)dr - s_i]$$

If we evaluated the first derivatives of the Lagrangian of this maximization problem at the allocation $\overline{X}$ in the core of the game $(\{N_i\}, N_a)$, then it must be zero.

If we first differentiate the Lagrangian L with respect to the composite good and land at each r and evaluate at $\overline{X}$ and $(\{N_i\}, N_a)$, then we obtain the following first order conditions.

(A.2.9) $$\frac{\partial L}{\partial z_1} = \frac{\partial u_1}{\partial z_1} - \mu_z n_1(r) = 0 \text{ at each } r$$

(A.2.10) $$\frac{\partial L}{\partial z_i} = \frac{\partial u_i}{\partial z_i} - \mu_z n_i(r) = 0 \text{ at each } r \ (i=2,\ldots,m)$$

(A.2.11) $$\frac{\partial L}{\partial z_a} = \lambda_a \frac{\partial v}{\partial z_a} - \mu_z n_a = 0$$

(A.2.12) $$\frac{\partial L}{\partial q_1(d_j)} = \frac{\partial v}{\partial z_a} - \mu(d_j)n_1(d_j) = 0 \text{ at each } d_j \ (j=1,2,\ldots,m)$$

(A.2.13) $$\frac{\partial L}{\partial q_i(d_j)} = \lambda_i \frac{\partial u}{\partial q_i} - \mu(d_j)n_i(d_j) = 0 \text{ at each } r (r \neq d_j) \ (i=2,3,\ldots,m)$$

(A.2.14) $$\frac{\partial L}{\partial q_1(r)} = \frac{\partial u_1}{\partial q_1} - \mu(r)n_1(r) = 0 \text{ at each } r \ (r \neq d_j)$$

(A.2.15) $$\frac{\partial L}{\partial q_i(r)} = \lambda_i \frac{\partial u_i}{\partial q_i} - \mu(r)n_i(r) = 0 \text{ at each } r \ (r \neq d_j) \ (i=2,\ldots,m)$$

If we set prices of the composite good and land at each distance as follows,

$$P_z = \mu_z$$
$$P(d_j) = \mu_j \text{ at each } d_j \ (j= 1,2,\ldots,m)$$
$$P(r) = \mu(r) \text{ at each } r \ (r \neq d_j)$$

then, (A.2.9), (A.2.12) and (A.2.13) imply that the prices are strictly positive. Moreover, it follows that the marginal rates of substitution equal the price ratios for all households. Hence, every household of each type i at the location r maximizes its utility at $\overline{X}$ over the following region.

(A.2.16) $$P_z z_i(r) + P(r)q_i(r) + T(r) \leqq P_z \overline{z}_i(r) + P(r)\overline{q}_i(r) + \overline{T}(r)$$

Second, differentiating the Lagrangian L with respect to $s_i(r)$ and evaluating at $\overline{X}$, and $(\{N_i\}, N_a)$ leads to

$$-\mu_z(\overline{z}_i(r) + T(r) - a) - \mu(d_i)(-\ell) - \mu(r)\overline{q}_i(r) + \gamma_i = 0$$

Rearranging the equation, we obtain

$$P_z\overline{z}_i(r) + P(r)\overline{q}_i(r) + P_z\ T(r) = P_z a + P(d_i)\ell + \gamma_i \tag{A.2.17}$$

Since the composite good can be assumed to be the numeraire, one can set $P_z = 1$.

Hence, (A.2.17) means that the consumption bundle $[\overline{z}_i(r), \overline{q}_i(r)]$ of the composite good and land at each distance r, belongs to the budget constraint of a household of type i. Therefore, we obtain that a set of prices $(P_z(=1),\{P(r)\})$ sustains the allocation $\overline{X}$ as a competitive equilibrium of the market.

Here, the multiplier $\gamma_i (i=1,2,\ldots,m)$ is regarded as the income subsidy to a household of type i. Note that the multiplier γ_i corresponds to the $\lambda_i(r) = - Q_i(u \leqq r \leqq \overline{r})$ in the optimal model. The income subsidy γ_i $(i = 1, 2, \ldots, m)$ will be set by the government. Hence, this market equilibrium is a compensated equilibrium. Note that this proof gives a proof for the existence of equilibrium at the same time.

FOOTNOTES

[1]Here, we might regard z,q as stocks instead of flows.

[2]We set the price of composite good $p_z = 1$, since it is considered as the numeraire.

[3]Here, y_i should be regarded as wealth, since it is derived from the stocks. However, in order to discuss the theory parallel to the standard urban land use theory, I will use "income" instead of "wealth" as the terminology in the present paper.

[4]We can consider that the absentee landlords supply the transportation systems in this city. Hence, we can regard the absentee landlords as the government. Then, we can set the number of absentee landlords Na = 1.

[5]For the existence of the equilibrium, we need the condition, $p_z a \geqq T(d_i)$ for all i (i=1,2,...m), which means that every household of any type can pay the transportation cost at its initial location in the city by its initial holding of composite good. Since the transport cost function is increasing with respect to distance, this condition becomes the simple condition, $a \geqq T(d_1)$ (if $p_z \equiv 1$). We will discuss precisely the existence and uniqueness of the competitive equilibrium in the following paper.

[6]Note that the absentee landlords, which supply the transportation systems in the city, receive the amount of composite good Z_t corresponds to the total transportation cost pazed by all households. That is,

$$Z_t = \frac{1}{p_z} \int_0^{\bar{r}} \sum_i T(r)\, n_i(r) dr$$

However, this is used to provide the transportation systems. As a result, an absentee landlord receives the amount of composite good, z_a, from households in exchange for land.

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