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Exact formula for ASN function etc. when log of likelihood-ratio takes only two integral multiples of a constant

by

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1. Introduction

Consider the i.i.d. dichotomous random variables $X_1, X_2, \cdots$ with two values, 0 and 1, whose distribution depends on θ. Let us assume that θ takes on only two values θ_1 and θ_2 and hence that, when $\theta = \theta_1$,

$$P(X_i = 0) = p_{10}, \qquad P(X_i = 1) = p_{11} = 1 - p_{10}$$

and when $\theta = \theta_2$

$$P(X_i = 0) = p_{20} (< p_{10}), \qquad P(X_i = 1) = p_{21} = 1 - p_{20}$$

To discriminate sequentially between θ_1 and θ_2 with specified loss and given priors on θ_1 and θ_2, Bayes sequential test is the sequential probability ratio test based on

$$\lambda_n = \left(\frac{p_{21}}{p_{11}}\right)^{T_n} \left(\frac{p_{20}}{p_{10}}\right)^{n-T_n}, \qquad n = 0, 1, \cdots \tag{1.1}$$

where $T_n = \sum_{i=1}^{n} X_i$, or equivalently on

$$\log \lambda_n = S_n \equiv \sum_{i=1}^{n} Z_i ,$$

$$Z_i = \begin{cases} \log(p_{20}/p_{10}) & \text{if } X_i = 0 \\ \log(p_{21}/p_{11}) & \text{if } X_i = 1 . \end{cases} \tag{1.2}$$

(See Chernoff-Moses [1].)

With the appropriate loss, Bayes sequential test is to accept $H_2 : \theta = \theta_2$ if $\lambda_n \geqq A$, to accept $H_1 : \theta = \theta_1$ if $\lambda_n \leqq B$ and to continue if $B < \lambda_n < A$, where A and B $(A > B)$ are constants depending on the loss, priors and the cost of sampling.

Let N be the first number n such that $\lambda_n \geqq A$ or $\lambda_n \leqq B$, or the stopping time of λ_n process. This paper is related to the exact distribution of λ_N (or equivalently of S_N), ASN function $E(N|\theta_i)$ and the OC function $P(\lambda_N \geqq A|\theta_i)$, $P(\lambda_N \leqq B|\theta_i)$, based on the distribution. Wald's fundamental identity of sequential analysis generally serves this problem, usually with approximation. He dealt, however, with the exact formula when Z_i takes only a finite number of integral multiple of a constant, ω, since in this case the sample path S_n is confined to the set of equi-distanced lattice points on the real line (Wald [2], p181). He argued also that still then the fundamental identity could not be solved explicitly, even by the use of Girshick's method (Girshick [3]).

This paper is to consider the case where

$$\begin{aligned} \log(p_{20}/p_{10}) &= -g_1\omega \\ \log(p_{21}/p_{11}) &= g_2\omega \end{aligned} \qquad (1.4)$$

for some positive integer g_1, g_2 and $\omega > 0$, and to apply the Girshick's method to solve the fundamental identity. Some example problems, which can be generalized easily, are shown to be actually solved explicitly. Various modes of solving the identity explicitly is illustrated together.

One should also note that the condition (1.4) is not restrictive one. On the contrary, the author (Matsubara [4], [5]) proved that this condition is even necessary and sufficient for Z_i to take only a finite number of integral multiples of a constant. It is up to saying that the log-likelihood vectors are dependent over rationals and this number-theoretic property determines the set-theoretic property of the sample path of λ_n process.

Other cases for which exact formula can be obtained are when we assume the special family of distribution. For example, H. B. Kemperman proposed

$$f(x|\theta) = \frac{1-\theta^2}{2} \exp\,(-|x| + \theta x)$$

as such family of distribution (Kemperman [6], or Ferguson [7]).

2. Fundamental Identity

Let a be the smallest integer $\geqq \log A/\omega$ and b be the largest integer $\leqq \log B/\omega$. That is

$$a = [\log A/\omega], \qquad b = -([-\log B/\omega] + 1) \tag{2.1}$$

Then

$$\{\lambda_n \geqq A \quad \text{or} \quad \lambda_n \leqq B\} = \{S_n/\omega \geqq a \quad \text{or} \quad S_n/\omega \leqq b\} \tag{2.2}$$

and

$$N = \min\ \{n;\quad S_n/\omega \geqq a \quad \text{or} \quad S_n/\omega \leqq b\} \tag{2.3}$$

Wald's fundamental identity

$$E\left(\frac{e^{tS_n/\omega}}{(M(t))^n}\right) = 1\ , \qquad M(t) = E(e^{tZ_i/\omega}) \tag{2.4}$$

evaluated at the non-zero solution $t = \tau$ of

$$M(t) = 1 \tag{2.5}$$

is reduced to

$$E(e^{\tau S_N/\omega}) = 1 \tag{2.6}$$

or, putting $u = e^{\tau}$, to

$$E(u^{S_N/\omega}) = 1 \tag{2.7}$$

But since $Z_i = -g_1\omega\ ,\ g_2\omega$ for each i, we have

$$S_N/\omega = (b - g_1 + 1)\ ,\quad (b - g_1 + 2)\ ,\ \cdot\ \cdot\ \cdot\ ,\ b\ ,$$
$$a\ ,\quad (a+1)\ ,\ \cdot\ \cdot\ \cdot,\ (a + g_1 - 1)\ , \qquad (2.8)$$

and the fundamental identity (2.7) becomes

$$\sum_{i=1}^{g} \xi_i \cdot u^{c_i} = 1\ , \qquad (2.9)$$

where $\xi_i = P(S_N/\omega = c_i)$ and c_i $(i = 1, 2, \cdot \cdot \cdot, g\text{-}2;\ g = g_1 + g_2)$ are values listed in (2.8). Thus the distribution of S_N/ω, $\xi_i = P(S_N/\omega = c_i)$, is the solution to (2.9) for all possible non-unity values $u = u_2, \cdot \cdot \cdot, u_g$ $(u_1 = 1)$ satisfying the rational equation (2.5), or in u,

$$p\ u^{-g_1\omega} + (1 - p)u^{g_2\omega} = 1 \qquad (2.10)$$

($p = p_{10}$, p_{20} and $1 - p = p_{11}$, p_{21} respectively, corresponding to θ_1 and θ_2.)

3. Illustrative Examples

There are increasing difficulties of solving (2.10) as g_1, g_2 get large. If $g_1 = g_2$, the problem becomes the classical random walk, which has been investigated in thorough detail. But if $g_1 \neq g_2$, the solution of (2.10) is almost intractable even for g_i being something around 2 and 3. We cannot solve explicity, or the solutions can be complex. In either case, the fundamental equation (2.9) becomes virtually impossible to solve.

This section illustrates modes of difficulty to solve the fundamental identity, which will finally necessitate Girshick's method.

[Example 1] (Simple random walk)

Let

$$p_{10} = 2/3\ ,\qquad p_{11} = 1/3\ ,\qquad p_{20} = 1/3\ ,\qquad p_{21} = 2/3\ . \qquad (3.1)$$

Then we have

$$\log(p_{20}/p_{10}) = -\log 2 \ , \quad \log(p_{21}/p_{11}) = \log 2 \ , \tag{3.2}$$

and this corresponds with

$$\omega = \log 2 \ , \quad g_1 = g_2 = 1 \ . \tag{3.3}$$

Thus we have the classical problem of the simple random walk (Feller [8]);

$$Z_i/\omega = \begin{cases} -1 & \text{with } p \\ 1 & \text{with } 1-p \ , \end{cases} \tag{3.4}$$

where

$$p = 2/3 \quad (\theta = \theta_1) \ , \quad p = 1/3 \quad (\theta = \theta_2) \ . \tag{3.5}$$

The rational equations for u are

$$\text{(i)} \quad \frac{2}{3}\cdot\frac{1}{u} + \frac{1}{3}u = 1 \qquad (\theta = \theta_1) \ , \tag{3.6}$$

and

$$\text{(ii)} \quad \frac{1}{3}\cdot\frac{1}{u} + \frac{2}{3}u = 1 \qquad (\theta = \theta_2) \ . \tag{3.7}$$

(i) yields the solution $u = 2$ and (ii) the solution $u = 1/2$. Fundamental identity for $\theta=\theta_1$ is then

$$\xi_1\cdot 2^b + \xi_2\cdot 2^a = 1 \ , \quad \xi_1 + \xi_2 = 1 \qquad (\theta = \theta_1) \ , \tag{3.8}$$

yielding the solution of ξ_i to be

$$\xi_1 = \frac{2^a - 1}{2^a - 2^b} \ , \quad \xi_2 = \frac{1 - 2^b}{2^a - 2^b} \tag{3.9}$$

and

$$E(S_N/\omega|\theta_1) = b\xi_1 + a\xi_2$$

$$= \frac{b\cdot 2^a - a\cdot 2^b + (a-b)}{2^a - 2^b} \tag{3.10}$$

$$E(N|\theta_1) = \frac{E(S_N/\omega|\theta_1)}{E(Z_i/\omega|\theta_1)}$$

$$= \frac{a\cdot 2^b - b\cdot 2^a + (b-a)}{3(2^a - 2^b)} \tag{3.11}$$

This is illustrated as the case where the fundamental identity is most straightforwardly solved.

[Example 2]

Let

$$p_{10} = 6/7\ , \quad p_{11} = 1/7\ ; \quad p_{20} = 3/7\ , \quad p_{21} = 4/7 \tag{3.12}$$

This case leads to

$$\log(p_{20}/p_{10}) = -\log 2\ , \quad \log(p_{21}/p_{11}) = 2\log 2\ , \tag{3.13}$$

that is,

$$\omega = \log 2\ , \quad g_1 = 1\ , \quad g_2 = 2. \tag{3.14}$$

Thus we have

$$Z_i/\omega = \begin{cases} -1 & \text{with } p \\ 2 & \text{with } 1-p\ , \end{cases} \tag{3.15}$$

where

$$p = 6/7 \quad (\theta = \theta_1)\ , \quad p = 3/7 \quad (\theta = \theta_2)\ . \tag{3.16}$$

In this case rational equations are still explicitly solved, since it is the cubic equation with $u = 1$ as a solution;

(i) $\frac{6}{7}\cdot\frac{1}{u} + \frac{1}{7}u^2 = 1 \qquad (\theta = \theta_1)\ ,$ (3.17)

with solutions $\quad u = -3,\ 2$.

(ii) $\frac{3}{7}\cdot\frac{1}{u} + \frac{4}{7}u^2 = 1 \qquad (\theta = \theta_2)\ ,$ (3.18)

with solutions $\quad u = -3/2\ ,\ 1/2$.

Let us in this case assume the values of a and b; e.g. $a = 3$, $b = -3$. Then the stopped sum of S_N/ω is

$$S_N/\omega = -3,\ 3,\ 4 \tag{3.19}$$

with each probability

$$\xi_1 = P(S_N/\omega = -3)\ ,\quad \xi_2 = P(S_N/\omega = 3)\ ,\quad \xi_3 = P(S_N/\omega = 4)\ , \tag{3.20}$$

$\theta = \theta_1, \theta_2$ being omitted here. The fundamental identity for ξ_1, ξ_2, ξ_3 is, for $\theta = \theta_1$,

$$\begin{cases} \xi_1 + \xi_2 + \xi_3 = 1 \\ (-3)^{-3}\xi_1 + (-3)^3\xi_2 + (-3)^4\cdot\xi_3 = 1 \\ 2^{-3}\cdot\xi_1 + 2^3\cdot\xi_2 + 2^4\cdot\xi_3 = 1\ , \end{cases} \tag{3.21}$$

and in terms of the solution to this we are to obtain

$$E(S_N/\omega) = -3\xi_1 + 3\xi_2 + 4\xi_3 \tag{3.22}$$

$$E(Z_i/\omega) = -4/7 \tag{3.23}$$

$$E(N) = E(S_N)/E(Z_i) \tag{3.24}$$

$$P\ (\theta_1 \text{ accepted } |\theta_1) = \xi_1 \tag{3.25}$$

$$P\ (\theta_2 \text{ accepted } |\theta_1) = \xi_2 + \xi_3 \tag{3.26}$$

In fact, computing (3.21) ~ (3.25) we have

$$\xi_1 = .9191\ ,\quad \xi_2 = .0511\ ,\quad \xi_3 = .0298$$

$$E(S_N/\omega) = -2.4852\ ,\quad E(N) = 4.3492$$

This example is illustrated to show the degree beyond which the equation for u, and therefore the fundamental identity would be hard to handle for explicit solutions.

[Example 3]

Let

$$p_{10} = 14/15 \ , \quad p_{11} = 1/15 \ ; \quad p_{20} = 7/15 \ , \quad p_{21} = 8/15 \ . \tag{3.27}$$

This time we have

$$\omega = \log 2 \ , \quad g_1 = 1 \ , \quad g_2 = 3 \tag{3.28}$$

and the rational equation becomes the algebraic equation of 4-th degree;

$$\text{(i)} \ \frac{14}{15} \cdot \frac{1}{u} + \frac{1}{15} u^3 = 1 \qquad (\theta = \theta_1) \tag{3.29}$$

with solutions

$$u = 1/2 \ , \quad \frac{-3 \pm \sqrt{19}i}{4} \tag{3.30}$$

$$\text{(ii)} \ \frac{7}{15} \cdot \frac{1}{u} + \frac{8}{15} u^3 = 1 \tag{3.31}$$

with solutions

$$u = 2 \quad , \quad \frac{-3 \pm \sqrt{19}i}{4} \tag{3.32}$$

Though complex solutions are worked out they lead to no difficulty, since ultimately ξ's are real. Still the explicit computation of ξ's through the fundamental identity is barely possible if not impossible. The computation would be messy and not systematic.

4. Exact Formula by Girshick's Method

Up to Example 3 in Chapter 3, we can manage to solve the fundamental identities. For $g = g_1 + g_2 = 5$, however, to solve for u is not only prohibitive but in general algebracically impossible in view of the fundamental theorem of algebra.

Consider for example the following case;

[Example 4]

Let

$$p_{10} = 28/31\ ,\quad p_{11} = 3/31\ ;\quad p_{20} = 7/31\ ,\quad p_{21} = 24/31\ . \tag{4.1}$$

This gives

$$\omega = \log 2\ ,\quad g_1 = 2\ ,\quad g_2 = 3 \tag{4.2}$$

and thus

$$Z_i/\omega = \begin{cases} -2 & \text{with } p \\ 3 & \text{with } 1-p\ , \end{cases} \tag{4.3}$$

where

$$p = 28/31 \qquad (\theta = \theta_1)\ ,\quad p = 7/31 \qquad (\theta = \theta_2)\ . \tag{4.4}$$

The equations to solve are

$$\text{(i)}\quad \frac{28}{31}\cdot\frac{1}{u^2} + \frac{3}{31}u^3 = 1 \qquad (\theta = \theta_1) \tag{4.5}$$

and

$$\text{(ii)}\quad \frac{7}{31}\cdot\frac{1}{u^2} + \frac{24}{31}u^3 = 1 \qquad (\theta = \theta_2) \tag{4.6}$$

Let us further assume for example that $a = 7$, $b = -6$ and hence

$$S_N/\omega = -7,\ -6,\ 7,\ 8,\ 9\ . \tag{4.7}$$

If all solutions $u_1 = 1$, $u_2 \cdot\cdot\cdot u_5$ were obtained to (4.5) or (4.6), we could solve the simultaneous equation

$$\xi_1 u_j^{-7} + \xi_2 u_j^{-6} + \xi_3 u_j^{\,7} + \xi_4 u_j^{\,8} + \xi_5 u_j^{\,9} = 1 \tag{4.8}$$

$$(j = 1,\ 2,\ \cdots,\ 5)$$

finally to obtain

$$\xi_i = P(S_N/\omega = c_i) \tag{4.9}$$

The only solutions to (i) or to (ii) easily found are, however, just $u = 2$ for (i) and $u = 1/2$ for (ii). They could not be solved completely.

M. A. Girshick noted the fact that the polynomial equation $f(u)= 0$ for u and the fundamental identity $F(u) = 0$ (in the polybomial form with ξ's being coefficients, as to be determined later) are to have common roots (solutions). Since $f(u)$ is of lesser degree than $F(u)$, the problem for us is therefore to determine such ξ's as to guarantee that $F(u)$ is factored to $f(u)$ and, say, $f^*(u)$. So $f(u)$ is not actually to be solved and instead the problem is to deal with the linear relationships of coefficients on both sides of

$$F(u) = f(u) \cdot f^*(u) \ . \tag{4.10}$$

A. Wald put a limit on the advantage of this method on the ground that $f^*(u)$ now introduces in addition to ξ's new coefficients k's (say) to be obtained and hence new equations to be solved, whereas the original fundamental identity involves only ξ's.

We are to show that, although Wald's comment is generally true, Girshick's method is still valid in our case by demonstrating that the equation $F(u) = f(u) \cdot f^*(u)$ is reduced to a rather simple type of a difference equation in related coefficients, in spite of apparent increase of the number of equations in k's.

So let us in our case consider

$$f(u) \equiv \frac{1-p}{p} u^5 - \frac{1}{p} u^2 + 1 = 0 \tag{4.11}$$

and

$$F(u) \equiv \xi_5 u^{16} + \xi_4 u^{15} + \xi_3 u^{14} - u^7 + \xi_2 u + \xi_1 = 0 \ . \tag{4.12}$$

(4.12) has the solution of (4.11) as its solutions and $f(u)$ divides (i.e. is the factor of) F(u), that is

$$F(u) = f(u)f^*(u) \tag{4.13}$$

where $f^*(u)$ is the proper polynomial:

$$f^*(u) = \sum_{i=0}^{11} k_i u^i \, . \tag{4.14}$$

Comparing the 17 coefficients of powers in both sides of (4.13), we get simultaneous equations in $\xi_1, \cdot\cdot\cdot, \xi_5; k_0, k_1, \cdot\cdot\cdot, k_{11}$.

Wald's criticism is, in the context of this example here, to the effect that we have to solve for 17 unknowns in order actually to get only $\xi_1, \cdot\cdot\cdot, \xi_5$. It is not so. We only have to solve the simultaneous equation only in ξ_1 (= k_0), and ξ_2 (= k_1), only 2 variables. (One should incidentally note that min $(g_1, g_2) = 2$).

Comparing the i-th power of u on both sides, we get,

$$k_i = \xi_{i+1} \qquad (i = 0, 1) \, , \tag{4.15}$$

$$k_i - \frac{1}{p} k_{i-2} = 0 \qquad (i = 2, 3, 4) \, , \tag{4.16}$$

$$k_i - \frac{1}{p} k_{i-2} + \frac{1-p}{p} k_{i-5} = -\delta_{i,7} \qquad (i = 5, \cdots, 11) \, , \tag{4.17}$$

$$- \frac{1}{p} k_{i-2} + \frac{1-p}{p} k_{i-5} = 0 \qquad (i = 12, 13) \, , \tag{4.18}$$

$$\frac{1-p}{p} k_{i-5} = \xi_{i-11} \qquad (i = 14, 15, 16) \, , \tag{4.19}$$

where δ_{ij} means 0 $(i \neq j)$, $1 (i = j)$.

Combining (4.16), (4.17), we have the difference euqation for k_i

$$k_i = \frac{1}{p} k_{i-2} - \frac{1-p}{p} k_{i-5} - \delta_{i,7} \qquad (i = 2, \cdots, 11) \, , \tag{4.20}$$

with the convention

$$k_{-1} = k_{-2} = k_{-3} = 0 .$$

Repeated use of (4.20) yields the expression of $k_7, \dots, k_{11}$ in terms of k_0, k_1 only.

$$k_7 = -2\,\frac{1-p}{p^2}\,k_0 + \frac{1}{p^3}\,k_1 - 1 \tag{4.21}$$

$$k_8 = \frac{1}{p^4}\,k_0 - 2\,\frac{1-p}{p^2}\,k_1 \tag{4.22}$$

$$k_9 = -3\,\frac{1-p}{p^3}\,k_0 + \frac{1}{p^4}\,k_1 - \frac{1}{p} \tag{4.23}$$

$$k_{10} = \left\{\frac{1}{p^5} + \frac{(1-p)^2}{p^2}\right\} k_0 - 3\,\frac{1-p}{p^3}\,k_1 \tag{4.24}$$

$$k_{11} = -4\,\frac{1-p}{p^4}\,k_0 + \left\{\frac{1}{p^5} + \frac{(1-p)^2}{p^2}\right\} k_1 - \frac{1}{p^2} \tag{4.25}$$

By (4.18), $(1-p)k_7$ and k_{10}, and $(1-p)k_8$ and k_{11} are put equal, giving the simultaneous equation in k_0 and k_1 only (instead of all 17 unknowns)

$$\begin{pmatrix} -5p(1-p) & 1 + 3p^3(1-p)^2 \\ 1 + 3p^3(1-p)^2 & -4p^2(1-p) \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \begin{pmatrix} p^3 \\ -p^5(1-p) \end{pmatrix}, \tag{4.26}$$

the solution of which is

$$\begin{aligned} k_0 &= 3p^5(1-p)(1 - Q(p))/D\,, \\ k_1 &= p^3(1 - 2Q(p))/D\,, \end{aligned} \tag{4.27}$$

where*

$$D = 1 - 14Q(p) + 9(Q(p))^2\,,$$

$$Q(p) = p^3(1-p)^2\,.$$

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*) $D = 0$ has no solutions in $p \in [0,1]$.

As for $\xi_1, \cdots, \xi_5$, we note by (4.15), (4.19) the change of variables

$$\begin{aligned} &\xi_1 = k_0 , \qquad \xi_2 = k_1 ; \\ &\xi_3 = \frac{1-p}{p} k_9 , \qquad \xi_4 = \frac{1-p}{p} k_{10} , \qquad \xi_5 = \frac{1-p}{p} k_{11} \end{aligned} \tag{4.28}$$

and hence we get exact final formula through (4.27);

$$\begin{cases} \xi_1 = 3p^5(1-p)(1 - Q(p))/D \\ \xi_2 = p^3(1 - 2Q(p))/D \end{cases} \tag{4.29}$$

and through (4.23), (4.24), (4.25)

$$\begin{cases} \xi_3 = -\frac{3(1-p)^2}{p^4} \xi_1 + \frac{1-p}{p^5} \xi_2 - \frac{1-p}{p^2} \\ \xi_4 = \frac{(1-p)(1 + Q(p))}{p^6} \xi_1 - \frac{3(1-p)^2}{p^4} \xi_2 \\ \xi_5 = \frac{5(1-p)^2}{p^5} \xi_1 + \frac{(1-p)(1 + Q(p))}{p^6} \xi_2 - \frac{1-p}{p^3} \end{cases} \tag{4.30}$$

which were to be obtained.

Incidentally, the expression of ξ_5 might better be

$$\begin{aligned} \xi_5 &= (1-p)k_{11} = (1-p)^2 k_8 \\ &= \frac{(1-p)^2}{p^5} \xi_1 - \frac{2(1-p)^3}{p^3} \xi_2 \\ &= (1-p)^3(1 + Q(p))/D . \end{aligned} \tag{4.31}$$

Figure 1 shows the computational result of;

i) the distribution of S_N

$$\xi_1 = P(S_N/\omega = -7) , \quad \xi_2 = P(S_N/\omega = -6)$$
$$\xi_3 = P(S_N/\omega = 7) , \quad \xi_4 = P(S_N/\omega = 8) ,$$
$$\xi_5 = P(S_N/\omega = 9) , \quad (\omega = \log 2)$$

ii) $E(S_N/\omega) = \xi_1 \cdot (-7) + \cdot \cdot \cdot + \xi_5 \cdot 9$,

iii) $E(Z_i/\omega) = (-2)p + 3(1-p)$,

iv) $E(N) = E(S_N)/E(Z_i)$ (ASN)

for $p = \frac{28}{31}$ $(\theta = \theta_1)$ and $p = \frac{7}{31}$ $(\theta = \theta_2)$.

	$\theta = \theta_1$		$\theta = \theta_2$	
ξ_1	.1918	.9958	.0015	.0141
ξ_2	.8040		.0126	
ξ_3	.0027	.0042	.3478	.9859
ξ_4	.0005		.1212	
ξ_5	.0010		.5170	
$E(S_N/\omega)$	-6.1347		7.9708	
$E(Z_i/\omega)$	-1.5161		1.8710	
$E(N)$	4.0463		4.2603	

Fig. 1 Exact quantities for Example 4

One should note

a) the SPRT test has sufficiently good operating characteristic, .9958 and .9859 under each state of nature,

b) two types of errors are threfore considerably low, with .0042 and .0141 each, and

c) the ASN are roughly the same.

It seems that E(N) is not very much affected by θ.

One also may conjecture that the less time one <u>need</u> for decision the more precise it would be, since both represent at least seemingly the high discriminatory power of observations. We look at 4.0463 > 4.2603 and .9958 < .9859.

The following example may point to the same effect, though we could not still rule out on this ground only the opposite reasoning that slow decision will be the accurate one.

[Example 5]

Let the probability structure be the same with Example 4 but this time $a = 2$, $b = -4$. Then

$$S_N/\omega = -5\ ,\ -4\ ,\ 2\ ,\ 3\ ,\ 4$$

and ξ_i be the probability for each.

Girshick's method again applied to this case yields the exact formula

$$\xi_1 = p^4(1-p)/D\ , \quad \xi_2 = p^2/D\ ,$$

$$\xi_3 = p^2(1-p)^2/D\ , \quad \xi_4 = (1-p)/D\ , \quad \xi_5 = p(1-p)^2/D\ ,$$

$$D = 1 - p^3(1-p)^2$$

Fig. 2 shows the computational result.

	$\theta = \theta_1$		$\theta = \theta_2$	
ξ_1	.0649	.8863	.0020	.0534
ξ_2	.8215		.0513	
ξ_3	.0077	.1137	.0308	.9466
ξ_4	.0974		.7796	
ξ_5	.0085		.1363	
$E(S_N/\omega)$	-3.2684		2.7299	
$E(Z_i/\omega)$	-1.5161		1.8710	
$E(N)$	2.1588		1.4591	

Fig. 2 Exact quantities for Example 5

5. Conclusions

We could get the exact formula of the distribution ξ_i's of S_N, E(N) and the operating characteristic. Girshick's method turned out to be useful in this case. We only had to solve the simultaneous equations in just min (g_1, g_2) unknowns, thus having the decrease of the number of unknowns instead of increase. This is a great advantage.

It is not the purpose of this paper but will be left to other occasions to evaluate the Wald's approximation compared with the exact result we have obtained.

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