No. 169 (82-36)

The Internal Rate of Return and the Selection of Investment

by

Hiroshi Atsumi

October 1982

The Internal Rate of Return and the Selection of Investment

By

Hiroshi Atsumi†

1. It now seems to be widely recognized that the internal rate of return of an income-earning investment may have multiple values if some of the prospective net yields are negative and it does not serve, in general, as a selection criterion among possible investment project, nor can the Keynesian schedule of the marginal efficiency of capital give an unequivocal measure of the level of aggregate investment at a given rate of interest (see Pitchford and Hagger [7]). It has been argued so far, to overcome this difficulty, that if it is practicable to terminate, without any essential cost, an investment project at any stage during its lifetime and if the investor chooses the truncation period so as to maximize the internal rate of return (see Soper [10] and Karmel [4]), or if the investor chooses the truncation period so as to maximize the present value of the project (see Arrow and Levhari [1] and Flemming and Wright [3]), then the internal rate of return is unique. Moreover, Soper's maximal rate of return is equal to the one found by Arrow and Levhari (see Norstrøm [6]). Therefore, as long as an investment project is costlessly terminable, the present value is greater or smaller than the replacement cost according as the rate of interest is smaller or greater than the internal rate of return, respectively. That is to say, the two selection criteria are consistent for such a single investment project.

The main purpose of this paper is to investigate the significance of the internal rate of return of the investment project --- not necessarily terminable, and to examine the role of the internal rate of return in the selection among various investment projects. The assumption of terminable investment means, to be more precise, that it is possible to truncate the cash flow of the project in any period leaving the cash flow up to the point of truncation unaffected.[1] At the first glance, this assumption is similar to the assumption of free disposal in production theory, but these two are quite different.[2] It is plausible, I believe, to assume that we can freely dispose of positive net yields in any period of time but not negative net yields. The latter may be disposed of only if we pay some 'winding-up cost', and this means that the cash flow of the project up to the point of truncation will necessarily be affected.[3] At any rate, when the truncation of a project involves cost substantially or a project can not be truncated we have the possibility of multi-valued internal rate of return. In this paper, we wish to work out a solution of this problem -- the problem which the economists could not help but puzzle over thus far.

Some economists, notably Samuelson [8], adopt the maximization of present value with market rates of discount as a guide in making the selection, since in a perfect capital market the investment project must have a market value equal to the capitalized value, which is uniquely determined. But when a project has multiple values of the internal rate of return, it is possible to face with the situation that the present value is less than the supply price at a given market rate of interest, whilst an internal rate is greater than the market rate (see, for example, Karmel [4]). Can we conclude by the present value alone that the project is not worth undertaking? Or can such a situation really occur in the perfect capital market? Is it not possible to adopt the project to earn the positive difference between the maximal internal rate and the market rate as an additional premium? We are unable to answer these questions unless we clarify the true profit rate of the project, which, whenever the project is undertaken, is expected to equal the market rate of interest under ideal conditions (i.e., the conditions that everyone can borrow or lend in unlimited amounts at the market rate and everyone has free access to the project). It is also well known that a similar dilemma occurs when we compare two projects even if the present value of each project is a monotonic decreasing function of the interest rate. This is so, because the present value of one project exceeds that of another at a market rate of interest but the internal rate of return of the latter may exceed that of the former (see also Karmel [4]). Can this situation also occur in a perfect capital market? It is true that the assumption of terminable investment gives a unique discount rate or a well-behaved present value function, but when one compares two different projects, each being optimally truncated, the same dilemma still remains. Indeed, the assumption of terminable investment is not only unacceptable but also unhelpful. Besides, an n-period project, if terminable in any period, contains n-1 truncated projects, and the search for an optimal truncation period is the selection of an optimal project among these special n projects according to one criterion, either the present value or the internal rate of return. Strictly speaking, not only the above dilemma may be embraced in the process of selection, but the selection itself is logically incomplete without an appropriate theory of investment selection.[4]

The problem of making the selection, or the maximum and the minimum problems subject to constraints on the decision variables, occur very frequently in many branches of economics. It is strange that no inquiry has systematically been made so far in the field of investment theory. We wish to show that once we apply the theory of efficient program of capital accumulation (see Dorfman, Samuelson and Solow [2]) to our problem, it is a fairly simple matter now to develope the theory of investment program, making it clear how the internal rate of return is to be defined or how the discount rates of various projects play a role in the selection of investment. Section 2 will discuss the simplest possible linear model of investment over time with a single project, the case corresponding with a single dynamic process of production, a so-called Ramsey model (see [2], section 11-2). Section 3 will introduce various investment alternatives, and the ordering of projects will be our main concern. We shall see that the efficiency prices of cash flows generated from investment will be cast as the leading actors in the both sections.

2. It is our purpose in this section to deal with a single project, describe our method of analysis and derive the internal rate of return in a precise manner. Let us consider an investment project having a finite life of n periods. Let S be the replacement cost (supply price) of the asset and Q_t be the expected net yield, positive or negative, from the asset in period t. If we are not concerned about how concretely the net yields arise, a complete history of the project is characterized only by a cash flow $(-S, Q_1, \ldots, Q_n)$. Traditionally, the internal rate of return, r, is defined as a solution of the equation

$$S = \frac{Q_1}{1+r} + \frac{Q_2}{(1+r)^2} + \ldots + \frac{Q_n}{(1+r)^n}, \qquad \ldots (1)$$

where Q_n is assumed to be nonzero, otherwise the project essentially reduces to an n-1 period project. However, this definition of the internal rate of return, which is regarded as a measure of profitability, has been accepted as a matter of course and, as far as I know, no one has ever discussed its exact meaning or validity. It is no wonder that one is bewildered when (1) gives multiple solutions.

Multiply the both sides of (1) by $(1+r)^t$, obtaining

$$S(1+r)^t = Q_1(1+r)^{t-1} + \ldots + Q_t + W_t, \qquad \ldots (2)$$

where $0 < t \leqq n$, and W_t is the present value of the project at the end of the t^{th} period from its inception, i.e.,

$$W_t = \frac{Q_{t+1}}{1+r} + \ldots + \frac{Q_n}{(1+r)^{n-t}} . \qquad \ldots (3)$$

W_t is an internal value of the asset, since all future yields are discounted at the internal rate, or since the t year old capital-asset is not always marketable. We will see from (2) that the rate of earnings on the initial outlay S will be r, or what amounts to the same thing, S can grow potentially at the rate r, provided that the net yield Q_τ for $1 \leqq \tau < t$, if positive, is not drawn but is expected to earn at the rate r, and that the cost of capital is r when Q_τ is negative or when discounting the future yields to compute W_t. In order that the rate of earnings on Q_τ is r or the opportunity cost of capital is r, such an investment opportunity must be open to the investor. If only one kind of project is available, then there is no alternative but to let the accumulated funds in the course of the project be reinvested in the same kind of project. Again these funds earn at the rate r if their net yields do the same, and so on. This circular argument does not clarify the significance of the internal rate, and of course, equation (2) holds not only for r but for any other solutions of equation (1), whether positive, negative or even complex, although most of them are usually and traditionally disregarded without any justifiable reason. Still, we see at least that the reinvestment of the net yields seems to be a key to our problem,[5] and we will make the following assumption:

Assumption 1. If $(-S, Q_1, \ldots, Q_n)$ is a project, $(-uS, uQ_1, \ldots, uQ_n)$ is also a feasible project for $u \geq 0$, where u stands for the activity level or intensity of the original project.

This means that if there is a feasible project installing one hundred new machines and making a series of net yields over n periods, then it is also feasible to install, say, two hundred (fifty) new machines of the same kind, doubling (halving) the net yield in each period. The characteristics of this assumption is twofold. One is the divisibility of the project. If this is not possible, u must be an integer. But the assumption of divisibility is an idealization often made by economists in the theory of production or consumer demand. Another is that the project enjoys the constant returns to scale. This is a basic feature adopted in linear economic models, especially in the fields of modern capital theory and of production theory.[6]

Let us now imagine a firm endowed with a single project only and acting in the investors' interests. The investors are interested in profits over time in some sense. Let u_t be the activity level of the project in period t. The program of the firm, i.e., its complete schedule of cash flows between the enterprise and the investors, is completely specified by the list of activity levels, $u_0, u_1, u_2, \ldots$.

In fact, Su_0 is an amount of money invested in period 0, and $Q_1u_0 - Su_1$ shows a cash flow in period 1, and so on. Let us denote by c_t a cash flow, positive or negative, in period t. Generally, the relation between a sequence of activity levels and a feasible sequence of cash flows will be represented by the following inequalities:[7]

$$\begin{aligned} c_0 &\leqq - Su_0, \\ c_1 &\leqq Q_1u_0 - Su_1, \\ c_2 &\leqq Q_2u_0 + Q_1u_1 - Su_2, \\ &\quad . \quad . \quad . \quad . \quad . \quad . \;, \end{aligned} \qquad \ldots \; (4)$$

and $$c_t \leqq Q_nu_{t-n} + \ldots + Q_1u_{t-1} - Su_t, \qquad t = n, n+1, \ldots .$$

A positive c_t means an outflow of money, and we draw an amount of c_t in period t out of the project. If c_t is negative, it means an inflow and we put money in the enterprise. If a project is profitable in the sense that one has at least $Q_1 + Q_2 + \ldots + Q_n > S$, then there exists a sequence of activity levels for which c_t is positive for all $t \geqq n$. For example $u_t = 1$ for all t. For a given feasible sequence of cash flows, if one cannot increase c_τ in period τ without decreasing some other c_t in period t, then the sequence of cash flows is said to be <u>efficient</u>. It is obvious that all the inequalities in (4) must hold with strict equality for any efficient programs. Otherwise, one can at once increase some c_τ without affecting any other c_t's. We will assume that the firm, maximizing its some objective function, chooses an optimal program among the efficient programs of cash flows. In fact, if an optimal program is chosen, it must be efficient, since, other things being equal, a smaller inflow or a greater outflow is always preferred by investors.[8]

Let us now consider a set of feasible cash flows and the efficiency frontier of the set. To make matters simple, we consider, to start with, $t = 0, 1$ only and a feasible set of (c_0, c_1) in two dimensions, and draw the possibilities open to us. We have from (4) that

$$c_0 \leqq - Su_0 \quad \text{and} \quad c_1 \leqq Q_1 u_0 - Su_1 .$$

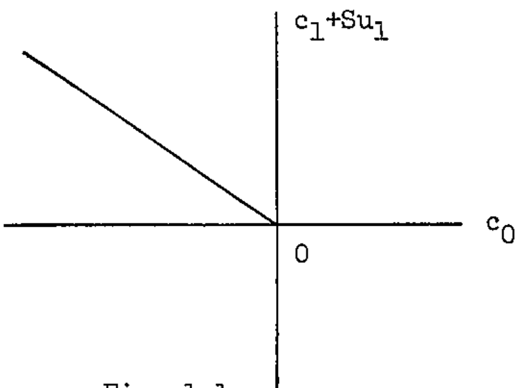

Fig. 1.1

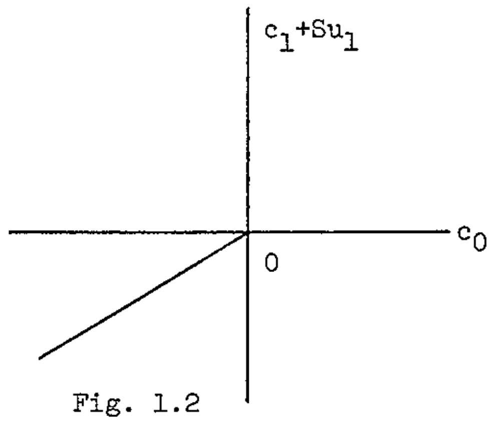

Fig. 1.2

Figure 1.1 is drawn on the assumption that $Q_1 > 0$, while in Figure 1.2 we assume $Q_1 < 0$. The efficient point is $(0, 0)$ only in the latter. If we consider $t = 0, 1, 2$ only and a feasible set of (c_1, c_2), specifying $u_0 = 1$ or $c_0 = - S$, then we have from (4) that

$$c_1 \leqq Q_1 - Su_1 \quad \text{and} \quad c_2 \leqq Q_2 + Q_1 u_1 - Su_2 .$$

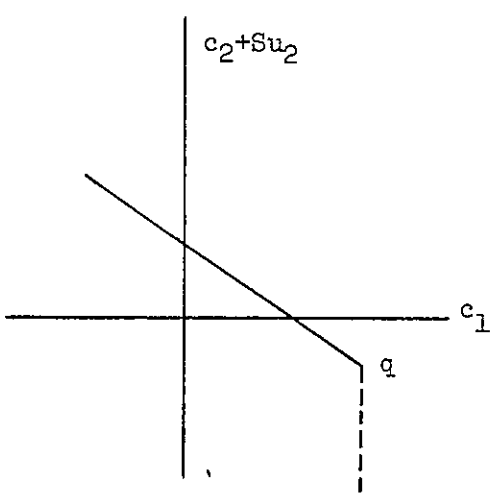

Fig. 2.1

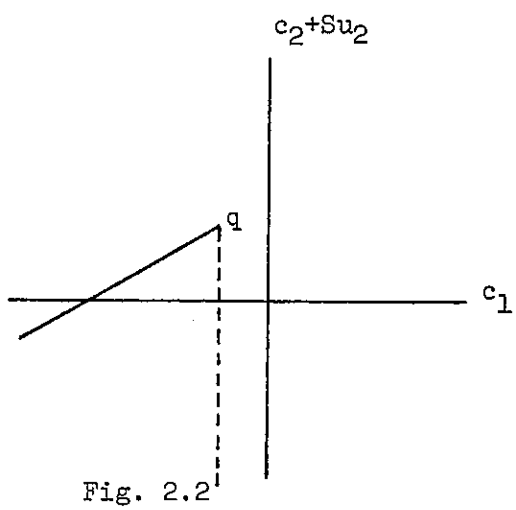

Fig. 2.2

Figure 2.1 is drawn on the assumption that $Q_1 > 0$ and $Q_2 < 0$. Point q represents the coordinate (Q_1, Q_2). In Figure 2.2 we assume $Q_1 < 0$ and $Q_2 > 0$. In Figure 1.1 or 2.1, the slope of the efficiency frontier, namely, the marginal rate of substitution between c_0 and c_1, or between c_1 and c_2 will respectively be given by

$$- \partial c_1/\partial c_0 = Q_1/S \quad \text{for} \quad u_0 > 0 \quad \text{or} \quad - \partial c_2/\partial c_1 = Q_1/S \quad \text{for} \quad u_1 > 0.$$

At the origin of Figure 1.1 or 1.2, where $u_0 = 0$, or at point q in Figure 2.1 or 2.2, where $u_1 = 0$, the marginal rate of substitution is not well defined. But we can draw a price line passing through the origin or q. Let (v_0, v_1) be a nonnegative price vector, which 'supports' the feasible set of (c_0, c_1) at point 0. We must have $v_0/v_1 \geqq Q_1/S$ in Figure 1.1, but we need not consider such restriction in Figure 1.2, since this is automatically satisfied by $v_0/v_1 \geqq 0$ and $Q_1 < 0$.

When we consider $t = 0, 1, 2,$ we can actually draw a feasible set of (c_0, c_1, c_2) or the efficiency surface in three dimensions, or a feasible set of (c_0, c_1) for some constant $c_2 + Su_2$. It is important to notice that the marginal rate of substitution between c_0 and c_1 differs from Q_1/S or the range of v_0/v_1 differs from the previous one. In fact, for any efficient point of (c_0, c_1, c_2), the first three inequalities in (4) must hold with strict equality, and whenever $u_0 > 0$ and $u_1 > 0$ hold, it is possible to differentiate them totally. Noting that $c_2 + Su_2$ is constant, we obtain

$$dc_0 = - Sdu_0, \quad dc_1 = Q_1du_0 - Sdu_1, \quad \text{and} \quad 0 = Q_2du_0 + Q_1du_1,$$

which in turn imply $- \partial c_1/\partial c_0 = (Q_1 + SQ_2/Q_1)/S$. It is easy to see that if we consider $t = 0, 1, 2, 3$ then $- \partial c_2/\partial c_1$ is also $(Q_1 + SQ_2/Q_1)/S$. In general, we can find the feasible and the efficient cash flow programs extending over any number of periods. The marginal rate of substitution between any two c_t and c_τ continues to vary and we must keep revising, say, Figure 1.1 accordingly. The correct value of the marginal rate of substitution between c_0 and c_1 is not obtained unless we consider an infinite number of periods. To illustrate this situation, suppose that we are given a three period project, $(-S, Q_1, Q_2, Q_3)$, and we consider $t = 0, 1, 2$ only. We then have from (4) that

$$c_0 \leqq - Su_0, \quad c_1 \leqq Q_1u_0 - Su_1 \quad \text{and} \quad c_2 \leqq Q_2u_0 + Q_1u_1.$$

Since we do not consider $t = 3, 4, \ldots,$ we cannot take into account Q_3u_0, Q_2u_1 and Q_3u_1 even if u_0 and u_1 are positive. A feasible set of (c_0, c_1, c_2) does not reflect what we really can do or the true structure of feasible cash flows generated from the project. Thus, we must consider $t = 0, 1, 2, 3, 4$ in order to allow for what we have neglected before. We will now add two inequalities:

$$c_3 \leqq Q_3u_0 + Q_2u_1 + Q_1u_2 - Su_3 \quad \text{and} \quad c_4 \leqq Q_3u_1 + Q_2u_2 + Q_1u_3.$$

Since activity levels u_2 and u_3 appear in these inequalities, and this time Q_3u_2, Q_2u_3 and Q_3u_3 are not taken into account, we have to extend the number of periods again. But lengthening our time period turns out always to add new activity levels, and we never seem to get enough inequalities. Only one exception is the case of one period project, where Q_1/S determines the rate of return at once. This argument is closely akin to the one discussed by Samuelson [9] in his overlapping generation model where one must have an infinite time horizon for the determination of the rate of interest.

To make our analysis more systematic, considering $t = 0, 1, \ldots, T$, let us seek to maximize Su_T with $c_0, c_1, \ldots, c_T$ all being feasible and prescribed. We wish to maximize

$$Su_T$$

subject to (4) for $t = 0, 1, \ldots, T$, namely,

$$
\begin{aligned}
& - Su_0 \geqq c_0, \\
& Q_1 u_0 - Su_1 \geqq c_1, \qquad \ldots (4.1) \\
& \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& Q_n u_{T-n} + \cdots + Q_1 u_{T-1} - Su_T \geqq c_T, \\
& \qquad u_t \geqq 0, \qquad t = 0, 1, \ldots, T. \qquad \ldots (4.2)
\end{aligned}
$$

There corresponds a dual minimum problem of the form

$$- v_0 c_0 - v_1 c_1 - \cdots - v_T c_T$$

to be a minimum subject to

$$
\begin{aligned}
& - v_t S + v_{t+1} Q_1 + \cdots + v_{t+n} Q_n \leqq 0, \\
& \qquad\qquad t = 0, 1, \ldots, T-n, \\
& - v_{T-n+1} S + \cdots + v_T Q_{n-1} \leqq 0, \qquad \ldots (5.1) \\
& \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& - v_{T-1} S + v_T Q_1 \leqq 0, \\
& - v_T S \leqq - S, \\
& \qquad v_t \geqq 0, \qquad t = 0, 1, \ldots, T. \qquad \ldots (5.2)
\end{aligned}
$$

Since we prescribe a feasible cash flow $(c_0, \ldots, c_T)$, the maximum problem, of course, has a feasible list of activity levels. (For example, let us choose $u_t = 1$ for $t = 0, 1, \ldots, T$, and consider a cash flow satisfying (4) as equations.) The first inequality in (4.1) shows that any feasible u_0 is bounded. This together with the second inequality shows that u_1 must be bounded. Repeating the same arguments, we see that any feasible u_t is bounded. Thus, Su_T being bounded, the maximum problem has an optimal solution. From the duality theorem of linear programming, the dual problem also has an optimal solution and the values of the two problems are the same:

$$- v_0 c_0 - v_1 c_1 - \dots - v_T c_T = Su_T. \qquad \dots (6)$$

The solution of the dual problem $v = (v_0, \dots, v_T)$ is regarded as a price vector; v_t is a shadow price (an internal price) of money in period t within the enterprise. We see from the last inequality of (5.1) that v_T is always positive, and we may assume without loss of generality that $v_T = 1$. (6) may be rewritten as

$$v_0(-c_0) = v_1 c_1 + \dots + v_T(c_T + Su_T),$$

where v_t is understood as a discounting price; the value of the initial outlay, $v_0(-c_0)$, is equal to a sum of discounted values of future cash flows, $(c_1, \dots, c_T+Su_T)$. Specifically, the internal exchange ratio between the moneies of two periods, v_t/v_{t+1}, now corresponds with the marginal rate of substitution between c_t and c_{t+1}. As will be seen in the appendix, there exists a positive price vector associated with the efficiency frontier of the set of cash flows, the price vector 'normal' to the frontier or 'supporting' the feasible set. This positive optimal price vector v in the minimum problem, however, may not satisfy all inequalities in (5.1) with strict equality. For example, suppose that $Q_1 < 0$. Then $- v_{T-1}S + Q_1 < 0$ holds for any nonnegative v_{T-1}, and we have $u_{T-1} = 0$ for any optimal solution of the maximum problem as is the case with Figure 1.2, where $T = 1$ and we must have $u_0 = 0$ (since the origin $(0, 0)$ is the only one efficient point).

Suppose that we can find a positive price vector v for which the inequalities in (5.1) are satisfied as equations except possibly for the last k inequalities and that these k inequalities do not depend on T.

A necessary and sufficient condition for this to be true is that equation (1) has a root $r > -1$. (See Theorem 1 in the appendix.) In this case we have the difference equation of order n for the unknown v_t:

$$- v_t S + v_{t+1} Q_1 + \dots + v_{t+n} Q_n = 0, \qquad \dots (7)$$

$$t = 0, 1, \dots, \nu, \quad \nu = \min (T-n, T-k),$$

where $v_t > 0$, and the last n inequalities in (5.1) give the conditions which must be met by the initial condition $(v_{T-n+1}, \dots, v_{T-1}, 1)$ of this difference equation. Notice that, as long as (7) holds, efficient is any cash flow $(c_0, \dots, c_T)$ satisfying (4.1) as equations for any non-negative $u = (u_0, \dots, u_T)$ such that $u_t = 0$ for $t > \nu$, including $u = 0$, $u = (1, 0, \dots, 0)$, etc. This means that we may freely choose $u_t \geqq 0$ for $0 \leqq t \leqq \nu$ so that the net yields of a project need not be reinvested, although potentially it is always possible to do so. As we mentioned before, v_0/v_1 will vary as we increase T, and its 'correct' value is not obtained until T goes to an infinity. It is interesting to realize that the characteristic equation of (7) is given by

$$- S + \mu Q_1 + \dots + \mu^n Q_n = 0, \qquad \dots (8)$$

which, upon putting $\mu = (1+r)^{-1}$, is identified with our familiar equation (1). It is natural to conjecture that as T goes to an infinity, the marginal rate of substitution between c_0 and c_1, or v_0/v_1, converges to a unique positive $(1+r)$. Whence this r is to be explained as the genuine internal rate of return, since v_0/v_1 is an internal exchange ratio between the moneies of periods 0 and 1. As we will give a rigorous proof in the appendix, v_0/v_1 approaches a maximal $(1+r)$ satisfying equation (1). Since we may choose u_t freely, the internal rate of return is determined independently of the time pattern of investment.

The following example illustrates that even if (1) produces multiple values of discount rate, the internal rate of return is uniquely determined from (7).

Example. We will consider the same example as in footnote 3.

$$S = \$4,000, \quad Q_1 = \$13,200, \quad Q_2 = -\$14,510, \quad Q_3 = \$5,313.$$

This gives three discount rates, 5, 10, and 15 per cent. It can be showed that there is a positive price vector, $(v_0, \ldots, v_T)$, satisfying all the inequalities in (5.1) with strict equality for arbitrary T (see proposition (D) in the appendix), so that efficient is any cash flow, meeting (4.1) with strict equality for some activity list. In fact, if $T = 3$, we consider three equations to calculate the internal prices:

$$-4,000v_2 + 13,200 = 0, \quad -4,000v_1 + 13,200v_2 - 14,510 = 0,$$

$$\text{and} \quad -4,000v_0 + 13,200v_1 - 14,510v_2 + 5,313 = 0.$$

We obtain $(v_0, v_1, v_2) = (13.32375, 7.2625, 3.3)$. Generally, when T is large, it is convenient to use the following method to calculate v_t's. Let us put $a_i = Q_i/S$ and consider the following matrix:

$$A = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{bmatrix}.$$

Premultiplication of A by $(v_T, 0, 0)$ gives $(v_{T-1}, v_T, 0)$. Repeating the same procedure, it will be seen that premultiplication of A^T by $(v_T, 0, 0)$ gives (v_0, v_1, v_2). When $T = 64$ or $T = 128$, we obtain, normalizing $v_2 = 1$, that (v_0, v_1, v_2) is almost equal to

$$(1.156^2, 1.156, 1) \quad \text{or} \quad (1.150^2, 1.150, 1),$$

respectively. Hence, the internal rate of return is 15 per cent. Also let us put $u_0 = v_T$, $u_1 = v_{T-1}$, $\ldots$, $u_T = v_0$. Then, since $(v_0, \ldots, v_T)$

satisfies (5.1) with strict equality, $(u_0, \ldots, u_{T-1}, 0)$ satisfies (4.1) with strict equality for a cash flow $(-Sv_T, 0, .., 0, Su_T)$. This means that the initial outlay Sv_T, if the net yields are not withdrawn, becomes Su_T in period T, and can grow at the rate 15% ultimately, as v_0/v_1 approaches 1+0.15. The 15% is a maximal growth rate, or an own-rate of interest of money for the enterprise. Consequently, it is regarded as the opportunity cost of capital for the firm. Even if the project has three discount rates, only a maximal rate has an economic significance as the internal rate of return, and the old puzzle is now resolved.

3. We now deal with multiple projects, apply our method of analysis discussed up until now to a generalized case, and develope a rule for investment selection. When we restrict our attention to a single project, its discount rates other than a maximal rate might seem to have no important economic significance. In contrast with this, when we put multiple projects in a correct order, all discount rates, whether positive or not, of all given projects must be taken into account. By a discount rate of a project we mean a solution of equation (1) such that $r > -1$ is met.

Let $(-S^{(i)}, Q_1^{(i)}, \ldots, Q_n^{(i)})$ be given Project i, $i = 1, \ldots, \omega$, where the life of Project i varies with i, namely, $n = n(i)$. Let us put $m = \max\ (n(1), \ldots, n(\omega))$, and consider $(-S^{(i)}, Q_1^{(i)}, \ldots, Q_m^{(i)})$, where $Q_\tau^{(i)} = 0$ if $\tau > n(i)$. Let $u_0^{(i)}, u_1^{(i)}, \ldots$ be a sequence of activity levels of Project i. Then a feasible sequence of cash flows generated by the ω projects given to us may be written as follows:

$$c_0 \leqq - \Sigma\ S^{(i)}u_0^{(i)},$$

$$c_1 \leqq \Sigma\ Q_1^{(i)}u_0^{(i)} - \Sigma\ S^{(i)}u_1^{(i)},$$

$$\begin{aligned} &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &c_t \leqq \Sigma\, Q_m^{(i)} u_{t-m}^{(i)} + \ldots + \Sigma\, Q_1^{(i)} u_{t-1}^{(i)} - \Sigma\, S^{(i)} u_t^{(i)}, \\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned} \qquad \ldots (9)$$

where the summation goes from 1 to ω. The positive prices $v_0, \ldots, v_{T-1}, v_T$ associated with the efficient frontier of the feasible set of cash flows must satisfy the following equations:

$$\begin{aligned} &- v_t S^{(i)} + v_{t+1} Q_1^{(i)} + \ldots + v_{t+m} Q_m^{(i)} + s_t^{(i)} = 0, \\ &\qquad\qquad t = 0, 1, \ldots, T-m, \\ &- v_{T-m+1} S^{(i)} + \ldots + v_T Q_{m-1}^{(i)} + s_{T-m+1}^{(i)} = 0, \\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &- v_{T-1} S^{(i)} + v_T Q^{(i)} + s_{T-1}^{(i)} = 0, \\ &- v_T S^{(i)} + s_T^{(i)} = - S^{(i)}, \\ &\qquad\qquad i = 1, \ldots, \omega, \end{aligned} \qquad \ldots (10)$$

where $s_0^{(i)}, \ldots, s_T^{(i)}$ are the nonnegative slack variables. As before, suppose that we find the positive prices $v_0, \ldots, v_{T-1}, 1$ satisfying (10) for any T with $s_t^{(i)} > 0$ for $t = 0, \ldots, T-1$, $i = 1, \ldots, \omega-1$, and with $s_t^{(\omega)} = 0$ for $t = 0, \ldots, T-k$, where k is independent of T. Then, efficient is a cash flow satisfying (9) with strict equality, with $u_t^{(i)} = 0$ for all t and for $i \neq \omega$, and with an arbitrary $u_t^{(\omega)} \geqq 0$ for $0 \leqq t \leqq T-k$ and $u_t^{(\omega)} = 0$ for $t > T-k$. Thus if we are about to select a project at $t = 0$, we must select Project ω, since $u_t^{(i)} = 0$ except for $i = \omega$. A question now arises. Are there any simple means or any characteristics of projects through which we can derive the same result?

Let us consider the polynomial associated with Project i,

$$f^{(i)}(z) = S^{(i)} z^n - Q_1^{(i)} z^{n-1} - \ldots - Q_n^{(i)}, \qquad i = 1, \ldots, \omega.$$

r is a discount rate, if and only if $(1+r)$ is a positive root of $f^{(i)}$.

Now let $\lambda_1^{(i)}, \ldots, \lambda_{k_i}^{(i)}$ be all the positive roots of $f^{(i)}$, arranged in descending order. Putting $\nu = \max(k_1, \ldots, k_\omega)$, let us consider

$$\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_\nu^{(i)}), \qquad i = 1, \ldots, \omega,$$

where $\lambda_j^{(i)} = 0$ for $j > k_i$. If $f^{(i)}$ has no positive roots, $\lambda^{(i)}$ is a zero vector. We now assert the following rule:

Selection Rule or Ordering Rule of Projects.

Let Project i, $i = 1, \ldots, \omega$, be given to us. For each i, let us consider a ν-dimensional nonnegative vector $\lambda^{(i)}$ comprising all the positive roots of $f^{(i)}$ arranged in descending order and zeros. Project i is preferred to Project j, if and only if $\lambda^{(i)}$ is lexicographically greater than $\lambda^{(j)}$.

This rule says that we must first choose a project whose internal rate of return is maximal, that if some projects have the same internal rate, we must then choose a project whose second discount rate is maximal among those projects having the same internal rate, etc. and that we need not take into account the negative or complex roots of $f^{(i)}$. The relation between this rule and the positive prices supporting the efficiency frontier of cash flows will be discussed in Theorem 4 in the appendix. We now give an example to illustrate the essential point of our argument.

Example. Consider two projects (-1, 3) and (-1, 4, -1, -6). Project 1 is simple and $1 = \frac{3}{1+r}$ gives $r = 2$ at once. Project 2 has two discount rates, $r = 2$ and 1, since the polynomial $f^{(2)}$ is

$$z^3 - 4z^2 + z + 6 = (z-3)(z-2)(z+1),$$

which has two positive roots. Obviously (3, 2) is lexicographically greater than (3, 0). Let us now consider the efficiency prices which simultaneously satisfy the following (11) and (12):

$$(v_0, \ldots, v_{T-1}, 1)\begin{bmatrix} 1 & & & & & & \\ -4 & 1 & & & & & \\ 1 & -4 & 1 & & & & \\ 6 & 1 & -4 & 1 & & & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & & 6 & 1 & -4 & 1 \end{bmatrix} = (\psi_0, \ldots, \psi_{T-1}, 1), \qquad \ldots (11)$$

and

$$(v_0, \ldots, v_{T-1}, 1)\begin{bmatrix} 1 & & & & & \\ -3 & 1 & & & & \\ & -3 & 1 & & & \\ & & -3 & 1 & & \\ & & & \cdot & \cdot & \cdot \\ & & & & -3 & 1 \end{bmatrix} = (s_0, \ldots, s_{T-1}, 1), \qquad \ldots (12)$$

where ψ_t and s_t are the nonnegative slack variables. (11) and (12) give

$$(s_0, \ldots, s_{T-1}, 1)\begin{bmatrix} 1 & & & & & & \\ -1 & 1 & & & & & \\ -2 & -1 & 1 & & & & \\ & -2 & -1 & 1 & & & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & & & -2 & -1 & 1 \end{bmatrix} = (\psi_0, \ldots, \psi_{T-1}, 1). \qquad \ldots (13)$$

First of all, we will examine whether (11) and (12) are satisfied at the same time for the slack variables such that $s_t = 0$ for at least one t. If $s_k = 0$ holds, we have from (13) that $\psi_k = - s_{k+1} - 2s_{k+2}$. Since the slack variables are nonnegative, we must have $s_{k+1} = s_{k+2} = 0$. But this means that $s_t = 0$ for all $t \geq k$, and $\psi_{T-1} = -1$, a contradiction.

Hence, $s_t > 0$ for all t. On the other hand, (11) and (12) are met simultaneously for positive prices with $\psi_t = 0$ and $s_t > 0$ for all t. In fact, we have a positive solution

$$v_{T-t} = [27(3^t) - 16(2^t) + (-1)^t]/12, \qquad t = 0, 1, \ldots, T,$$

and a positive slack variable for (12)

$$s_{T-t} = [2(2^t) + (-1)^t]/3, \qquad t = 1, 2, \ldots, T.$$

(See, e.g., relation (14) in the appendix for a systematic derivation of the solutions.) Hence, we may undertake Project 2, but not Project 1. This holds whatever large T is taken.

So far, our analysis is based on the assumption that any of the given ω projects is available to the firm at any point of time. It seems that this assumption is plausible for almost all industries. It is always possible, at least potentially, for a firm to undertake the same project repeatedly if the firm wants to do so. As far as the projects on exhaustible resources are concerned, however, the situation may be somewhat different. Consider, for example, a project to dig a specific gold vein, which is available once and for all. Suppose now that Project 1 is available only for $t = 0, 1, \ldots, N$. In this case, a feasible cash flow is obtained from (9), putting $u_t^{(1)} = 0$ identically for all $t > N$. The shadow prices associated with the efficient cash flows are also obtained from (10), deleting the equations associated with Project 1, except for the first N+1 equations, corresponding to $t = 0, 1, \ldots, N$. It is obvious that the same shadow prices, satisfying all the original equations in (10), are valid even in this case. Thus, as long as we focus our attention on the same shadow prices, our conclusion whether Project 1 is worth undertaking or not is unaltered. Suppose, however, that $\lambda^{(1)}$ is lexicographically greatest among $\lambda^{(1)}, \ldots, \lambda^{(\omega)}$, and that e.g., $\lambda^{(2)}$ is lexicographically greater than $\lambda^{(3)}, \ldots, \lambda^{(\omega)}$. If Project 1 is not taken into consideration at all, there exist the shadow prices for which we can conclude that only Project 2 is worth undertaking. A question is if it is possible for these shadow prices to satisfy also

$$- v_t S^{(1)} + v_{t+1} Q_1^{(1)} + \ldots + v_{t+m} Q_m^{(1)} + s_t^{(1)} = 0,$$

$$t = 0, 1, \ldots, N,$$

with some slack variables $s_0^{(1)}, \ldots, s_N^{(1)}$. The answer is affirmative, if the second element of $\lambda^{(1)}$ is greater than the first element of $\lambda^{(2)}$, or if they are equal but $\lambda^{(2)}$ has no other positive element. In such a case, whether we choose Project 1 or Project 2 should be determined by some other factors of the firm.[9]

Example. Consider three projects $(-1, 4, -1, -6)$, $(-1, \lambda)$ and $(-1, 1)$, where $1 < \lambda \leq 2$. Suppose that Projects 2 and 3 are always available, but not Project 1. As before, focusing our attention on $v_{T-t} = [27(3^t) - 16(2^t) + (-1)^t]/12$, we can say that an efficient cash flow is generated, using Project 1 only. But consider also $v_t = \lambda^{T-t}$, $t = 0, 1, \ldots, T$. Since we have $- v_t + 4v_{t+1} - v_{t+2} - 6v_{t+3} \leq 0$ for $t < T-3$, and $- v_t + v_{t+k} < 0$, an efficient cash flow is generated also, using Project 2 only.

In this example, Project 3 implies that any money put in a pocket today can be taken out tomorrow. In Project 2, $\lambda-1$ may be considered as a market interest rate. Notice that the present value of Project 1 for $\lambda < 2$ is less than the supply price, but we do not necessarily reject to undertake Project 1.

University of Tsukuba

Footnotes

† I am grateful to my coleagues, Professors Satoru Fujishige, Mamoru Kaneko and Yoshitsugu Yamamoto for frequent and useful discussions.

1. See, e.g., Karmel [4], or Flemming and Wright [3], p.259.

2. If it is possible produce output y from input x, then it is also possible to produce $\tilde{y}$ from $\tilde{x}$ for any $\tilde{x} \geqq x$ and $\tilde{y} \leqq y$. This is the assumption of free disposal discussed in capital theory. Applying this to the investment projects, we may have the following assumption: If a series of net yields $Q_1, \ldots, Q_n$ are obtained from an initial outlay S, then it is also possible to have $Q_1', \ldots, Q_n'$ from S' provided that $Q_t \geqq Q_t'$ and $S' \geqq S$ hold.

3. Let us consider a project given by Karmel [4]:

$$S = \$4{,}000, \quad Q_1 = \$13{,}200, \quad Q_2 = -\$14{,}510, \quad Q_3 = \$4{,}313.$$

This project has three values for the discount rate, 5, 10, and 15 per cent. It is entirely harmless to assume that Q_3 can be freely discarded, and the project embraces a two-period project:

$$S = \$4{,}000, \quad Q_1 = \$13{,}200, \quad Q_2 = -\$14{,}510.$$

This two-period project does not have a meaningful discount rate, since it gives only two complex roots. It is, however, certainly implausible that Q_2 can be freely disposed of again. It is reasonable to think that Q_1, Q_2 and Q_3 are mutually related, and e.g., one can have, for a while, a positive profit of \$13,200 in period 1 because of a loss of \$14,510 in period 2, and that if one wishes to dispose of the loss in period 2, one has to pay a price for it. Suppose that if we terminate the project in period 1, we can have a one-period project:

$$S = \$4{,}000, \quad Q_1 = \$4{,}560.$$

Q_1 is now only \$4,560 instead of \$13,200. That is, \$8,640 is the 'winding-up cost' (or the winding-up cost minus a second-hand value, or a scrap value, of the project's asset). The discount rate now becomes 14 per cent. Which project should we choose among these three? It is impossible to tell until we clarify the internal rate of return of the original project or until we develope a theory of ordering of projects. The assumption of variable life of investment discussed in the literature requires the truncation of projects without cost. According to this assumption, it is possible to have a one-period project:

$$S = \$4{,}000, \quad Q_1 = \$13{,}200,$$

whose rate of return is 230 per cent. This is the unique and the maximal rate asserted by Soper, Karmel and others. The same assumption is adopted in the works by Arrow, Levhari, Flemming and Wright. It seems to me that it is a clever diversion of argument to introduce the assumption of variable life of investment. Not only is it unnecessary to solve the original problem itself, but also we can hardly accept that the negative net yields are freely discarded. This does not corresponds to the assumption stated in footnote 2, but rather to the one that $\tilde{y}$ can be produced from $\tilde{x}$ even if $\tilde{x} \leq x$ or $\tilde{y} \geq y$.

4. It is true that in a perfect capital market the investment project must have a market value equal to the capitalized value, but this value must also equal to the supply price of the assert in such a perfect capital market. In my opinion, what they really assert is this: when we are given a set of projects, each having a capitalized

value not greater than the supply price, we may undertake any project whose capitalized value equals the supply price. This situation resembles the one in which the constant returns to scale prevails in production; under a perfect competition, the profits of firms are nils and firms may adopt any production processes yielding no profits. Let us also recall that it is meaningless to say that a firm maximizes the profits when the price of a product exceeds its unit cost, but meaningful to say that the firm minimizes the unit cost even in such a situation. Notice that since $(p.x - c(x))/c(x) = p.x/c(x) - 1$ and p is given, where p is the price and $c(x)$ is the cost function, minimizing the average cost $c(x)/x$ is equivalent to maximizing the rate of profits. As far as I know, there is no authorized theory which justifies the present value as a selection criterion in a general situation where the projects need not be marketable, or where a firm has several patent projects, and others have no access to them. May we adopt the internal rate of return as a criterion, then? Recall the examples of three projects in footnote 3, where one has three rates of discount, another has two complex rates, and still other has a unique rate. I say at the risk of being tedious that it is groundless and selfcontradictory to compare these three projects unless we know the true internal rate of return of the first projects, and unless we know how to compare the projects in general.

5. Actually, the reinvestment of net yields need not be carried out to understand the internal rate of return, as we shall see, though we must assume that it is possible to do so potentially. It is a most important to realize that the rate of returns of the initial outlay

depends upon the investment opportunities in the subsequent periods open to the investors. Let us consider a franchise business such as the Kentucky Fried Chicken, the Seven-Eleven, etc., where entrepreneurs continue to build the chain stores. This shows an example that we invest and reinvest the net yields in the same kind of project. As another extreme, let us imagine a tied loan from a bank such that if one borrows S dollars to carry a project into effect, one has to deposit all the net yields in the bank. Let i represent the rate of interest. If one does undertake the project, one has to pay an amount of $(1+i)^n S$ to the bank at the end of period n, but at the same time one's asset in the bank will be

$$A = Q_1(1+i)^{n-1} + Q_2(1+i)^{n-2} + \dots + Q_n.$$

The rate of return may be calculated by solving $A = (1+r)^n S$. If one says that the project is worth undertaking as long as A is not less than $(1+i)^n S$, one adopts the present value as a guide in making the decision. But this is an extreme case of the 100% cruel tied loan.

6. As we shall see later, the assumption of divisibility is not required at all for our purpose. It is solely for expository use. (See, rule (5) in the appendix or the argument in the last part of section 1 in the appendix.) Lifting the assumption of the constant returns to scale, we can construct a nonlinear model of investment, with decreasing returns to scale, say, corresponding to section 11-2-2 in [2]. In this case, it will be possible to derive the Keynesian investment function, relating the level of investment and the rate of interest. This case, however, is not attempted in the present work. It is true that the value of the internal rate of return of a project, in general,

varies depending on the volume of investment, but the concept of the internal rate of return itself has traditionally been discussed independently of the level of investment. I believe the assumption of the constant returns to scale obeys this spirit most faithfully and is appropriate for the present analysis.

7. By a feasible sequence of cash flow, we mean a cash flow over T periods, whether T is finite or infinite, generated from a basic cash flow $(-S, Q_1, \ldots, Q_n)$ and given by (4). The fact that we consider the activity levels $u_t \geq 0$ for $t = 0, 1, 2, \ldots,$ means that it is possible to undertake the project at any time if we want to do so. If the project is available, say, once and for all, then $u_1 = u_2 = \ldots = u_t = \ldots = 0$ identically, or we do not consider u_t except for u_0. This problem is treated in the final section.

8. Let i be a market rate of interest. Then an objective function, maximizing the present value of the sequence of cash flows

$$\Sigma_{t=0}^{\infty} c_t(1+i)^{-t}$$

is not compatible with linear models unless some financial restrictions are introduced from outside or unless i happens to equal the internal rate of return as is the case with the perfect capital market. This objective function will determine the level of investment in a non-linear model, however. The similar situation occurs in production theory when the constant returns to scale prevails, as is well known.

9. If all of the ω projects are unavailable for $t > N$, and have common discount factors $\delta_1, \ldots, \delta_r$, then $v_t = \delta_i^{-t}$, $t = 0, 1, \ldots,$ are the shadow prices for each discount factor. The old puzzle revives. The same is true when $T = \infty$, if the projects are available.

Mathematical Appendix

1. Let $(-S, Q_1, \ldots, Q_n)$ be a given investment project. To simplify, we will divide each component of the project by S, and put $a_i = Q_i/S$ for all i, obtaining $(-1, a_1, \ldots, a_n)$. Throughout this appendix, we will deal with this normalized project, since the argument in our linear model of investment is not affected at all. Let us now consider the polynomial associated with the project

$$f(z) = z^n - a_1 z^{n-1} - \cdot \cdot \cdot \cdot - a_n, \tag{1}$$

and let us introduce the triangular matrix M of order $T+1$, which corresponds to the polynomial $f(z)$ and which is given by

$$\begin{bmatrix} 1 & & & & & & & \\ -a_1 & 1 & & & & & & \\ -a_2 & -a_1 & 1 & & & & & \\ & \cdot & \cdot & \cdot & & & & \\ -a_n & -a_{n-1} & \cdot \cdot & -a_1 & 1 & & & \\ & & \cdot & \cdot & \cdot & \cdot & & \\ & & -a_n & -a_{n-1} & \cdot \cdot & -a_1 & 1 \end{bmatrix}, \tag{2}$$

where any element of the principal diagonal is unity, any element of the next i^{th} lower parallel diagonal is $-a_i$ $(i = 1, \ldots, n)$, and all other elements are zeros. We will name M *the polynomialoid matrix*. The reason for this new term will be understood in section 3. In general, we may consider the triangular matrix of the same type for an arbitrary polynomial. We will write M_ψ or M_ϕ if it is necessary to designate the polynomialoid matrix corresponding with the polynomial $\psi(z)$ or $\phi(z)$, respectively.

The matrix M, if multiplied by -1, is the coefficient matrix of constraints (4.1) in the text. A pair of linear programs discussed in section 2 in the text will be represented as

$$
\begin{aligned}
&& \text{maximize} \quad & e^T u \\
&& \text{subject to} \quad & Mu \leqq -c, \quad u \geqq 0, \\
\text{and} && \text{minimize} \quad & -vc \\
&& \text{subject to} \quad & vM \geqq e^T, \quad v \geqq 0,
\end{aligned} \tag{3}
$$

where e^T and v are row vectors, u and c are column vectors, and in particular, $e^T = (0, \ldots, 0, 1)$. It will be our main interest to study the optimal price vector v associated with the efficient cash flow c, or to study the relation between the positive price vector characterizing the efficiency and the internal rate of returns.

To begin with, consider the set of integers $\{0, 1, \ldots, T\}$, and let the letter J denote a subset of this. It is easy to confirm that

(A) There exists some J for which we have

$$Mx \leqq 0 \quad \text{and} \quad x_i \geqq 0 \quad \text{for} \quad i \in J \quad \text{imply} \quad x = 0. \tag{4}$$

A simplest case for this to be true is the one in which J is the whole set from 0 to T. For, any diagonal element of M being unity, the inequalities $Mx \leqq 0$ and $x \geqq 0$ give $x = 0$ at once. There are of course other cases in which only some coordinates of x are required to be nonnegative while others may be arbitrary, but we ultimately have $x \geqq 0$ and hence $x = 0$. For example, let the zeroth column of M have a negative component and consider $J = \{1, \ldots, T\}$. Let a_k be the first positive net yield, so that $a_1, \ldots, a_{k-1}$ are nonpositive. Since $x_i \geqq 0$ for $i = 1, \ldots, T$, the k^{th} row of $Mx \leqq 0$ or the inequality

$$- a_k x_0 - a_{k-1} x_1 - \cdots - a_1 x_{k-1} + x_k \leqq 0$$

implies $-a_k x_0 \leqq 0$. We therefore have $x_0 \geqq 0$ as well.

We now suppose that (4) holds for a proper subset J, and consider a feasible cash flow $c = -Mu^*$ determined by the rule

$$\begin{aligned} u_i^* &= 0 \quad \text{if } i \in J, \\ u_i^* &\geqq 0 \quad \text{if } i \notin J. \end{aligned} \tag{5}$$

Let us place this particular c in the above linear programs (3). Since $Mu \leqq -c$ means $M(u - u^*) \leqq 0$, and since $u \geqq 0$ means $u_i - u_i^* \geqq 0$ for $i \in J$, it follows from (4) that $u = u^*$. Thus, c is an efficient cash flow, u^* being a unique feasible vector. There exist a price vector v and a nonnegative vector s for which we have

$$vM = s, \quad v > 0 \quad \text{and} \quad \begin{aligned} s_i &> 0 \quad \text{for } i \in J, \\ s_i &= 0 \quad \text{for } i \notin J. \end{aligned} \tag{6}$$

(See, e.g., Nikaido [5], p.128.) Conversely, suppose that (6) holds for some J. Then $c = -Mu^*$ where u^* obeys rule (5), gives an efficient cash flow. For, we have from (5) and (6) that $s(u - u^*) = su \geqq 0$ for any $u \geqq 0$. Since v is strictly positive, we have that $M(u - u^*) \leqq 0$ and $u \geqq 0$ imply $u = u^*$. (If $M(u - u^*) = 0$ does not hold we have that $vM(u - u^*) < 0$, contradicting to $su \geqq 0$.) Thus, c is efficient. In fact, we have just confirmed the following proposition.

(B) There exists a price vector v for which (6) holds for some J, if, and only if, (4) holds for the same J.

Once we have a positive price vector v and a nonnegative u^* satisfying, respectively, (6) and (5) for the same J, and once a cash flow $c = -Mu^*$ is adopted in linear programs (3), then v and u^* are a pair of optimal vectors, giving $e^T u^* = -vc$. (We may normalize v, without loss of generality, so that $v_T = s_T = 1$.) It is important to notice that even if i does not belong to J, u_i^* need not be positive, and even if u_i^* is positive, it may be an integer. We shall derive the internal rate of return from the optimal price vector v of the dual program, and it is in this sense that the reinvestment of the net yields is not always required and that the divisibility of the project in the assumption 1 in the text is solely for expository convenience.

2. It is obvious that if the i^{th} column of M does not have negative elements then this i must belong to J for (4) to be true, or we must have $s_i > 0$ for (6) to be true. For example, T always belongs to J or we may assume that $s_T = 1$ always holds. Exclusive of this last component of s, s_i represents a slack variable, and so $s - e^T$ is a slack vector. We now introduce an important idea concerning the optimal slack variables. We will say that s is 'invariant' or 'finite', if s consists of a row of T-k zeros followed by k+1 non-negative elements that are invariant even if T varies. That is, s is 'invariant' or 'finite' if s takes the form of $(0, \ldots, 0, s_k, ., s_1, 1)$ and if $s_1, \ldots, s_k$ are constant independently of order of s. To put it in a different way, let $\psi(z)$ be a polynomial of, e.g., degree k, having nonnegative coefficients. Let M_ψ stand for the polynomialoid matrix corresponding to ψ. Suppose that we have for any T that $vM = s$, $v > 0$ and $s = e^T M_\psi$, namely, s is the last row of M_ψ. Then s is 'finite', since it is derived from the coefficients of a polynomial; it is to be remarked that the degree of any polynomial is necessarily finite. A question now arises. Is it possible to have (6) for such 'invariant' slack variables? The following two examples will illustrate the cases for or against such 'invariant' or 'finite' s.

Example 1. Consider a two-period project given by

$$(-1, a_1, a_2) = (-1, 3, -2).$$

The equation $z^2 - 3z + 2 = 0$ has two positive roots 1 and 2. If $T = 4$, M is written as

$$M = \begin{bmatrix} 1 & & & & \\ -3 & 1 & & & \\ 2 & -3 & 1 & & \\ & 2 & -3 & 1 & \\ & & 2 & -3 & 1 \end{bmatrix}.$$

Premultiply M by a positive vector $p = (1, 1, 1, 1, 1)$, obtaining $(0, 0, 0, -2, 1)$. For any feasible x, satisfying $Mx \leqq 0$, we have $pMx \leqq 0$ or $2x_3 \geqq x_4$. Now, using $(1, 1, 1, 1, 0)$ in place of p, and repeating the same argument, we have $2x_2 \geqq x_3$. Similarly, using $(1, 1, 1, 0, 0)$ or $(1, 1, 0, 0, 0)$, we have $2x_1 \geqq x_2$ or $2x_0 \geqq x_1$, respectively. Hence, $Mx \leqq 0$ and $x_4 \geqq 0$ imply $x \geqq 0$ or $x = 0$. It is also easy to see that $Mx \leqq 0$ and $x_T \geqq 0$ imply $x = 0$ for any large T. That is, (6) holds for $s = e^T$ for any T. The matrix M_ψ must be an identity matrix I, or $\psi(z)$ must be the polynomial of degree zero, namely, $\psi(z) = 1$.

Example 2. A two-period project considered is

$$(-1, a_1, a_2) = (-1, 2, -2).$$

The polynomial of this project $f(z) = z^2 - 2z + 2$ has two complex roots, 1+i and 1-i. In this case, a cash flow generated by $c = - Mu$ is not efficient, provided that $u_i > 0$ for any consecutive three i's. For example, let $T = 4$ and $u = \{0, 1, 1, 1, 0\}$. But it is easy to see that we have a better cash flow, upon substituting $\{0, 0.5, 0, 0, 0\}$

for u. Suppose next that $u = \{1, 1, 1, 0, \ldots, 0\}$, say. We will note that the following relation holds:

$$\begin{bmatrix} 1 & & & & & & & & \\ -2 & 1 & & & & & & & \\ 2 & -2 & 1 & & & & & & \\ & 2 & -2 & \cdot & & & & & \\ & & 2 & & \cdot & 1 & & & \\ & & & & & -2 & 1 & & \\ & & & & & 2 & -2 & 1 & \\ & & & & & & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is written more compactly as $Mp^* = s^*$. It is easy to see that

$$-M(u - \theta p^*) = c + \theta s^* \geq c \quad \text{and} \quad u - \theta p^* \geq 0 \quad \text{for} \quad .5 > \theta > 0.$$

Hence, c is not efficient. We may have a positive vector v such that $vM = s$ for some nonnegative s. But obviously s is not 'invariant'.

As this example shows, if s is not 'invariant', we are not able to determine the amount of investment at time 0, or the activity level u_0 without considering the subsequent levels of investment, $u_1, u_2, \ldots,$ simultaneously so as to obtain the efficient cash flows. For instance, it is not a wise decision for a firm to make investment in every period and to produce a steady flow of outputs. On the other hand, if s is 'invariant', the first T-k components of s are zeros and according to rule (5) we may freely choose a nonnegative u_t for any t between $0 \leq t < T-k$. We may even remove this condition on t essentially. For, whatever large t one may take, we are allowed to increase T further so that $T-k > t$ always holds. If the activity level of an n-period project is unity in each period, a steady-state is attained from the n period on. Such calm state is efficient. The internal price vector associated with the efficient cash flows is determined by $v = sM^{-1}$ for some s. It is true that some components of s are positive and v depends on them. Suppose, however, that as T goes to an infinity, the price ratio v_0/v_1 (or it is seemingly more general to consider the price ratio v_t/v_{t+1} at any period of time) converges to a unique value λ for any v determined by some 'invariant' s. We may then say that the internal rate of retrun is determined as $\lambda-1$, independently of the time pattern of investment. Traditionally, the internal rate of return is calculated by equation (1) in the text, as soon as a basic cash flow $(-1, a_1, \ldots, a_n)$ is given. In our opinion, this is justified only if one chooses a maximum rate of return among multiple solutions. Indeed, the primary purpose of this appendix is to establish the following theorems.

Theorem 1. There exists a positive price vector v satisfying (6) for some 'invariant' s, if, and only if, the polynomial $f(z)$ has at least one positive root.

Theorem 2. Let λ be a maximal positive root of $f(z)$. There exists a nonnegative vector s for which $v = sM^{-1}$ is positive and v_0/v_1 converges to λ.

Theorem 3. Suppose that $f(z)$ does not have a negative or complex root whose absolute value is λ. Then v_0/v_1 converges to λ for any positive v satisfying (6) with some 'invariant' s.

3. We are now in a position to return to the polynomial $f(z)$ and the triangular matrix M defined at the outset. We wish to clarify the properties of the matrix M and the reason why it is named the polynomialoid matrix. Let $z_1, z_2, \ldots, z_n$ be the roots of $f(z)$, so that $f(z)$ is uniquely factorized as

$$f(z) = (z-z_1)(z-z_2)\ldots(z-z_n). \tag{7}$$

Let us single out $(z-z_i)$, any linear factor of $f(z)$, and consider the triangular matrix of the same type as (2), corresponding with the polynomial $z - z_i$. This matrix of order T+1 will be given by

$$M_i = \begin{bmatrix} 1 & & & & \\ -z_i & 1 & & & \\ & -z_i & 1 & & \\ & \cdot & \cdot & \cdot & \cdot \\ & & & -z_i & 1 \end{bmatrix}.$$

Similarly, let $z^2 + \alpha_1 z + \alpha_2 = (z-z_i)(z-z_j)$ be a quadratic factor of $f(z)$, and let M_{ij} denote the corresponding triangular matrix of the same order:

$$M_{ij} = \begin{bmatrix} 1 & & & & \\ \alpha_1 & 1 & & & \\ \alpha_2 & \alpha_1 & 1 & & \\ & \cdot & \cdot & \cdot & \cdot \\ & & \alpha_2 & \alpha_1 & 1 \end{bmatrix}.$$

Notice that $\alpha_1 = -(z_i+z_j)$ and $\alpha_2 = z_i z_j$. Now by matrix multiplication we have $M_{ij} = M_i M_j$. M_i and M_j are commutable. To repeat a similar argument, let $(z-z_i)(z-z_j)(z-z_k)$ be a cubic factor of $f(z)$, and let M_{ijk} denote the corresponding triangular matrix. The reader can easily verify that $M_{ijk} = M_{ij}M_k = M_i M_j M_k$. In general, we have

$$M = M_1 M_2 . \; . \; . \; M_n . \qquad (8)$$

This is a factor representation of matrix (2). (8) corresponds to (7). A scalar 1 is a trivial factor of $f(z)$, and it is considered a polynomial of degree zero. It is clear that the unit matrix, I, corresponds with the polynomial $a_0 = 1$. The unit matrix is a trivial factor of M also. It is easy to see that the inverse matrix of M_i takes the form of

$$M_i^{-1} = \begin{bmatrix} 1 & & & & \\ z_i & 1 & & & \\ z_i^2 & z_i & 1 & & \\ & \cdot & \cdot & \cdot & \cdot \\ z_i^T & z_i^{T-1} & \cdot\ \cdot & z_i & 1 \end{bmatrix}. \tag{9}$$

Since $M_1, M_2, \ldots, M_n$ commute, so do their inverses. Hence by matrix multiplication we obtain that M^{-1} also takes the form of

$$M^{-1} = \begin{bmatrix} 1 & & & & \\ b_1 & 1 & & & \\ b_2 & b_1 & 1 & & \\ & \cdot & \cdot & \cdot & \cdot \\ b_T & b_{T-1} & \cdot\ \cdot & b_1 & 1 \end{bmatrix}. \tag{10}$$

It is to be noted that since $M_i M_i^{-1} = I$ or $MM^{-1} = I$, we can derive the last row of M_i^{-1} or M^{-1} by considering $1 \div (z-z_i)$ or $1 \div f(z)$. Indeed, let us put, e.g.,

$$1 = f(z)(z^{-n} + b_1 z^{-n-1} + b_2 z^{-n-2} + \ldots \quad).$$

Expanding the right-hand side and comparing the coefficients, we have

$$\begin{array}{cccccc} 1 & b_1 & b_2 & b_3 & \cdots\cdots & b_T \\ & -a_1 & -a_1 b_1 & -a_1 b_2 & \cdots\cdots & -a_1 b_{T-1} \\ & & -a_2 & -a_2 b_1 & \cdots\cdots & -a_2 b_{T-2} \\ & & & & \cdots\cdots & \\ + & & & & -a_n \quad \cdot\ \cdot & -a_n b_{T-n} \\ \hline 1 & 0 & 0 & 0 & \cdots\cdots & 0 \end{array} \tag{11}$$

Therefore, the last row of M^{-1} is obtained from the coefficients of the infinite series, $z^{-n} + b_1 z^{-n-1} + \ldots + b_t z^{-n-t} + \ldots$, by truncating it at $t = T$. The inverse of a polynomial is not a polynomial but an infinite series, while the inverse of M is again the triangular matrix of the same type, corresponding with the truncated polynomial. When we increase T, however, we will see that the last row of M^{-1} is not 'finite' in the sense defined in section 2, since it is obtained from the infinite series. In our analysis, the triangular matrix M plays a vital role rather than the polynomial $f(z)$ itself. But it is interesting to remark that the triangular matrices, which we termed *polynomialoid*, bear a striking resemblance to the polynomials.

4. Having clarified the properties of matrix (2), we now proceed to prove our main theorems. First, it is easy to demonstrate the following three propositions.

(C) (6) holds for $vM = e^T$, if and only if $b_i > 0$ for all i.

In fact, (6) requires v be strictly positive. But $v = e^T M^{-1}$ or $v = (b_T, \ldots, b_1, 1)$. Hence, (C) is obvious.

(D) Let polynomial (1) have positive roots only. Then (6) holds for $vM = e^T$.

Since $M^{-1} = M_1^{-1} \ldots M_n^{-1}$, and since $z_i > 0$ for all i, it is obvious from (9) that $(b_T, \ldots, b_1, 1)$ is strictly positive.

(E) Let $(-1, a_1, \ldots, a_n)$ be an n-period project such that $a_i \geq 0$ for all i, and $a_1 > 0$. Then (6) holds for $vM = e^T$.

It is obvious from (11) that $b_1 = a_1$, which in turn implies that $b_2 > 0$, etc. We ultimately have $b_i > 0$, $i = 1, \ldots, T$.

In general, the polynomial $f(z)$ has negative and complex roots. The following proposition plays an important role for our analysis.

(F) Let $n_1, .., n_q$ be negative numbers, let $c_1, .., c_r$ be complex numbers, and let $\bar{c}_1, .., \bar{c}_r$ be their conjugate numbers. Let us define the polynomial of degree q+2r given by

$$h(z) = (z-n_1)\dots(z-n_q)(z-c_1)(z-\bar{c}_1)\dots(z-c_r)(z-\bar{c}_r).$$

Then there exists a polynomial with non-negative coefficients

$$\psi(z) = z^k + s_1 z^{k-1} + \dots + s_k,$$

such that $\psi(z)$ is divisible by $h(z)$ and the integral quotient $\phi(z) = \psi(z)/h(z)$ has positive coefficients only.

If $h(z)$ happens to have nonnegative coefficients, we will put $\psi(z) = h(z)$ and $\phi(z) = 1$, and (F) is proved at once. This is the case where each complex number c has a nonpositive real part. In general, some of $c_1, .., c_r$ have positive real parts, and we will consider the following polynomials for any given c:

$$\Delta = z^{2t} - (c^t+\bar{c}^t)z^t + (c\bar{c})^t, \qquad t = 1, 2, \dots, \tau,$$

where τ is a minimal positive integer for which c^τ has a nonpositive real part. Namely, $c^t+\bar{c}^t$ is positive for $t < \tau$, but nonpositive for $t = \tau$. τ depends on c. We will write Δ as Δ^* when $t = \tau$. Since c and $\bar{c}$ are roots of Δ, Δ is the product of the two polynomials

$$(z^2 - az + b) \text{ and } Q(z;t) = q_0 z^{2t-2} + q_1 z^{2t-3} + \dots + q_{2t-2},$$

where $a = c+\bar{c}$ and $b = c\bar{c}$. We will show that the coefficients of Q are all positive for $t \leq \tau$. Note that if α is a root of Δ, so is b/α. The same must be true for Q. Thus we have

$$q_{2t-2} = q_0 b^{t-1}, \quad q_{2t-3} = q_1 b^{t-2}, \dots, q_t = q_{t-2} b.$$

It is sufficient to show that $q_0, \dots, q_{t-1}$ are positive. Expanding the product of $(z^2 - az + b)$ and Q, and comparing the coefficients with those of Δ, we obtain

$$q_0 = 1,\ q_1 = a,\ q_2 = aq_1 - b,\ .\ .\ .\ .\ ,\ q_{t-1} = aq_{t-2} - bq_{t-3},$$
$$\text{and}\quad q_t = aq_{t-1} - bq_{t-2} - (c^t+\bar{c}^t).$$

It will be seen at once that $q_0, q_1, \ldots, q_{m-1}$ are the same for all $Q(z;t)$ if $t \geq m$. Moreover, as long as $(c^m+\bar{c}^m)$ is positive, q_m is increasing as t alters from m to $m+1$. It is obvious that $q_0 = 1$ for all t, and when $t = 2$, we have $q_1 = a$ and $q_2 = b$. When t increases further, we have $q_2 = a^2 - b$, and this is greater than b. Repeating the same arguments for $t = 3, \ldots, \tau$, we see that all the coefficients of Q are positive for all $t \leq \tau$.

Let Q^* represent $Q(z;t)$ for $t = \tau$. We may consider the similar polynomials $\Delta_1^*, .., \Delta_r^*$ and $Q_1^*, .., Q_r^*$ for all $c_1, .., c_r$, respectively, and we will put

$$\psi(z) = (z-n_1).\ .\ .(z-n_q).\Delta_1^*\ .\ .\ .\ \Delta_r^*.$$

Since all the coefficients of Δ_i^* are nonnegative, and $-n_1, .., -n_q$ are positive, all the coefficients of ψ are nonnegative. Moreover, we have $\psi(z)/h(z) = Q_1^*\ .\ .\ .\ Q_r^*$, all the coefficients of which are positive. This completes the proof.

The relation $\psi(z) = \phi(z)h(z)$ established above will effectively be utilized, once it is given in matrix form. Let M_ψ, M_ϕ and M_h be the polynomialoid matrices of order T+1, corresponding with three polynomials ψ, ϕ and h, respectively. We then have $M_\phi M_h = M_\psi$. Consider also the following four nonnegative vectors of order T+1:

$$\begin{aligned} s &= (0,\ \ldots,\ 0,\ s_k,\ \ldots,\ s_1,\ 1), \\ p &= (0,\ \ldots.,\ 0,\ p_\ell,\ ..,\ p_1,\ 1), \\ s^* &= \{1,\ s_1,\ \ldots,\ s_k,\ 0,\ \ldots,\ 0\}, \\ p^* &= \{1,\ p_1,\ ..,\ p_\ell,\ 0,\ \ldots.,\ 0\}, \end{aligned} \qquad (12)$$

where $k-\ell = q+2r$, and we will let s or p represent the last row of M_ψ or M_ϕ, respectively. Let s^* or p^* represent the first column of M_ψ or M_ϕ, respectively. Since M_ϕ and M_h commute, we have

$$pM_h = s \qquad \text{and} \qquad M_h p^* = s^*. \tag{13}$$

We are now able to prove Theorem 1.

Proof of Theorem 1. We first wish to show the sufficiency. Suppose that polynomial (1) has at least one positive root. Let us factorize (1) so as to have $f(z) = a(z)h(z)$, where $a(z)$ has positive roots only and $h(z)$ has negative or complex roots only. Then, corresponding with this factorization, we also have $M = M_a M_h$. By (D) any element of M_a^{-1}, not above the diagonal, is positive. On the other hand, by (F) there exist four vectors (12) for which relations (13) hold. We now put $v = pM_a^{-1}$. Obviously $v > 0$, and we have $v = sM_a^{-1}M_h^{-1}$ or $vM = s$. Since T is arbitrary, and $s = (0, \ldots, 0, s_k, \ldots, s_1, 1)$, s is 'invariant'.

We next wish to prove the necessity. Suppose that $f(z)$ does not have any positive roots. Namely, $f(z) = h(z)$ and $M = M_h$. We have $Mp^* = s^*$. Suppose that we have $vM = s$ and $v > 0$ for some 'invariant' s. Since p^* consists of a column of $\ell+1$ nonnegative elements followed by $T-\ell$ zeros, we may assume, when T is large, that $sp^* = 0$ for any 'invariant' s. (Indeed, s even need not be nonnegative.) However, $Mp^* \geq 0$ and $v > 0$ imply $vMp^* = sp^* > 0$, contradicting to $sp^* = 0$. Hence, $v > 0$ cannot hold for any 'invariant' s.

We shall now proceed to prove Theorems 2 and 3. As a preliminary, we shall prove proposition (G).

(G) Let $M_1, M_2, \ldots, M_r$ be polynomialoid matrices corresponding with linear polynomials $z-z_1, z-z_2, \ldots, z-z_r$, respectively, where z_i's are mutually distinct. Let k_i be a nonnegative integer, representing the degree of multiplicity of M_i. We then have

$$(M_1^{k_1} \ldots M_r^{k_r})^{-1} = f_1(M_1^{-1}) + \ldots + f_r(M_r^{-1}), \tag{14}$$

where f_i is a polynomial, of degree k_i, in a matrix M_i^{-1} with scalar coefficients, and without a constant term.

To verify (G), we will first note that

$$z_1M_2 - z_2M_1 = (z_1-z_2)I,$$

or

$$(M_1M_2)^{-1} = c_1M_1^{-1} + c_2M_2^{-1},$$

where $c_1 = z_1/(z_1-z_2)$ and $c_2 = -z_2/(z_1-z_2)$. This proves (14) for the case where $k_1 = k_2 = 1$ and $k_3 = \ldots = k_r = 0$. Let us increase any k_i by 1. We have, e.g., for $k_1 = 2$ that

$$\begin{aligned}(M_1^2M_2)^{-1} &= M_1^{-1}(c_1M_1^{-1} + c_2M_2^{-1}), \\ &= c_1M_1^{-2} + c_2(c_1M_1^{-1} + c_2M_2^{-1}), \\ &= (c_1M_1^{-2} + c_1c_2M_1^{-1}) + c_2^2M_2^{-1}.\end{aligned}$$

Repeating the similar argument, we will see that (14) is true for any given $k_1, \ldots, k_r$.

Let $(\rho_T, \ldots, \rho_1, 1)$ represent the last row of the matrix of the left-hand side of (14). How does ρ_T vary as T increases? Let us write $x = (x_T, \ldots, x_1, 1)$ and $y = (y_T, \ldots, y_1, 1)$ for the last rows of M_i^{-k} and $f_i(M_i^{-1})$, respectively. It is obvious from (9) that for $k = 1$, we have $x = (z_i^T, \ldots, z_i, 1)$. We also have for $k = 2$ that $x = ((T+1)z_i^T, \ldots, 2z_i, 1)$. It may be verified from routine calculations that we have in general

$$x_t = z_i^t(t+k-1)\ldots(t+1)/(k-1)!, \qquad \text{for } k = 2, 3, \ldots, k_i.$$

Since f_i is of degree k_i, we have $y_t = z_i^t . g_i(t)$, where $g_i(t)$ is a polynomial of degree k_i-1 in t. Thus it follows from (14) that

$$\rho_t = \sum_{i=1}^{r} g_i(t) z_i^t. \tag{15}$$

Proof of Theorem 2. Let $z_1, z_2, \ldots, z_r$ be all the positive roots of (1), and let $a(z)$ be written as

$$a(z) = (z-z_1)^{k_1} \ldots (z-z_r)^{k_r}.$$

By Theorem 1, we have $vM_a = p$ and $v > 0$ for any T. Suppose that $z_1 = \lambda$ and $z_i < \lambda$ for $i = 2, \ldots, r$. We will show that $v_0/v_1 \to \lambda$. Let $(\rho_T, \ldots, \rho_1, 1)$ represent the last row of M_a^{-1}. It follows from (15) that we have

$$\rho_t = \sum_{i=1}^{r} g_i(t) z_i^t$$

or

$$\rho_t/\lambda^t = g_1(t) + \sum_{i=2}^{r} g_i(t)(z_i/\lambda)^t.$$

Hence ρ_t/λ^t converges to $g_1(t)$. We see that $\rho_{t+1}/\lambda\rho_t$ converges to $g_1(t+1)/g_1(t)$, which, by L'Hospital's rule, converges to unity. Thus $\rho_{t+1}/\rho_t \to \lambda$. Since $vM_a = p$ and p is 'invariant', v_0/v_1 converges to λ as well.

Proof of Theorem 3. Suppose that we have $vM = s$ and $v > 0$ for any T and for some 'invariant' s. This means that there is a polynomial $\sigma(z)$ with nonnegative coefficients for which we have $s = e^T M_\sigma$. Since the factorization of the polynomial or of the polynomialoid matrix is unique, M_i is a common factor of M and M_σ and can be eliminated from both sides of $vM = e^T M_\sigma$, if and only if z_i is a common root of $f(z)$ and $\sigma(z)$. It is easy to see that $\sigma(z)$ with nonnegative coefficients does not have any positive roots, so that any factor of M_a cannot be eliminated. It is also easy to see that $e^T M_\sigma M_i^{-1}$ is again 'invariant' (though some of its nonzero components may be negative), if and only if $\sigma(z)$ is divisible by $z-z_i$. Suppose that a highest common factor of M and M_σ is eliminated from both sides of $vM = e^T M_\sigma$, and we obtain

$vM_\alpha = e^T M_\xi$. Since $f(z)$ and $\sigma(z)$ have real coefficients, so do $\alpha(z)$ and $\xi(z)$. This is so, because if z_i is a complex root common to both $f(z)$ and $\sigma(z)$, so is its conjugate number. If the absolute value of any negative or complex root of $\alpha(z)$ is less than λ, we see that since $e^T M_\xi$ is 'invariant', v_0/v_1 converges to λ as in the proof of Theorem 2. Hence, it is sufficient to show that $\alpha(z)$ has no root whose absolute value is greater than λ.

Let $z_1, \ldots, z_k$ be pairwise distinct roots of $\alpha(z)$, and let z_i be r_i-multiple. Suppose that $|z_i| = m$ for $i \leqq \nu$, and $|z_i| < m$ for $i > \nu$, including $z_i = \lambda < m$ for some i. It follows from (14) that

$$M_\alpha^{-1} = f_1(M_1^{-1}) + \ldots + f_k(M_k^{-1}),$$

where f_i is of degree r_i. Let $e^T M_\alpha^{-1} = (\alpha_T, \ldots, \alpha_1, 1)$. (15) gives

$$\alpha_t = \textstyle\sum_{i=1}^k g_i(t) z_i^t,$$

or

$$\alpha_t/m^t = \textstyle\sum_{i=1}^{\nu} g_i(t)(z_i/m)^t + \sum_{i=\nu+1}^k g_i(t)(z_i/m)^t,$$

where g_i is of degree r_i-1. Since $|z_i| < m$ for $i > \nu$, and $|z_i| = m$ for $i \leqq \nu$, the second term of the right-hand side of the last equation goes to zero, while the first term does not, as is well-known.

We now put $v' = e^T M_\xi(f_1(M_1^{-1}) + \ldots + f_\nu(M_\nu^{-1}))$. Since $e^T M_\xi M_\alpha^{-1} = v$, and since $e^T M_\xi$ is 'invariant', v_0/m^T converges to v'_0/m^T as T goes to an infinity. Define a polynomial

$$H(z) = (z-z_1)^{r_1} \cdot \cdot \cdot (z-z_\nu)^{r_\nu}.$$

By proposition (F), there exist $\Psi(z)$ and $\Phi(z)$ with nonnegative coefficients such that $\Psi(z) = H(z)\Phi(z)$. We also have $M_\Psi = M_H M_\Phi$. Now consider

$$v' M_\Psi = e^T M_\xi(f_1(M_1^{-1}) + \ldots + f_\nu(M_\nu^{-1}))M_\Psi. \tag{16}$$

$\Psi(z)$ is a polynomial of degree, e.g., K, with nonnegative coefficients, so that the first component of $v' M_\Psi$ is positive, provided that the first

K components of v' are positive. On the other hand, $f_i(M_i^{-1})$ is a polynomial, of degree r_i, in a matrix M_i^{-1} with scalar coefficients. Since M_i^r is a factor of M_ψ for $r = 1, 2, \ldots, r_i$, the right-hand side of (16) is an 'invariant' row vector. Its first component is zero. Hence, v' has a negative component among its first K elements. This means that $v > 0$ cannot hold. Whenever we have $vM_\alpha = e^T M_\xi$ and $v > 0$ for any T, $\alpha(z)$ does not have any root whose absolute value is greater than λ. This completes the proof.

5. Suppose that we are given ω investment projects,

$$(-1, a_1^{(i)}, \ldots, a_n^{(i)}), \qquad i = 1, \ldots, \omega,$$

where the life of Project i varies with i, viz., $n = n(i)$. Let $M^{(i)}$ be the polynomialoid matrix corresponding with the polynomial

$$f^{(i)}(z) = z^n - a_1^{(i)} z^{n-1} - \ldots - a_n^{(i)}, \qquad i = 1, \ldots, \omega,$$

and consider the following pair of linear programs:

$$\begin{aligned} &\text{maximize} && \Sigma_{i=1}^{\omega} e^T u^{(i)} \\ &\text{subject to} && \Sigma_{i=1}^{\omega} M^{(i)} u^{(i)} \leqq -c, \quad u^{(i)} \geqq 0, \\ \text{and} \quad &\text{minimize} && -vc \\ &\text{subject to} && vM^{(i)} \geqq e^T, \quad i = 1, \ldots, \omega, \quad v \geqq 0. \end{aligned} \tag{17}$$

Let $\lambda_1^{(i)}, \ldots, \lambda_{k_i}^{(i)}$ be all the positive roots of $f^{(i)}$ arranged in descending order. Define $\nu = \max(k_1, \ldots, k_\omega)$ and consider a vector $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_\nu^{(i)})$, $i = 1, \ldots, \omega$, putting $\lambda_j^{(i)} = 0$ for $j > k_i$.

We may now prove

Theorem 4. There exists a positive vector v satisfying

$vM^{(1)} = e^T M_\sigma$, and $vM^{(i)} = s^{(i)} > 0$, $i = 2, \ldots, \omega$, for any T,

where σ is a polynomial with nonnegative coefficients, if and only if $\lambda^{(1)}$ is lexicographically greater than $\lambda^{(i)}$, $i = 2, \ldots, \omega$.

Proof. We will first show the necessity. Suppose that we have

$$vM^{(1)} = e^T M_\sigma \quad \text{and} \quad vM^{(i)} = s^{(i)} > 0 \quad \text{for} \quad i = 2, \ldots, \omega,$$

but some $\lambda^{(i)}$ is lexicographically as great as or greater than $\lambda^{(1)}$. Suppose that we are given $\lambda^{(1)}$ and $\lambda^{(i)}$ for this particular i as

$$\lambda^{(1)} = (x_1, .., x_r, 0, .., 0) \quad \text{and} \quad \lambda^{(i)} = (z_1, .., z_k, 0, .., 0),$$

and let M_i or N_j be the polynomialoid matrix corresponding with $z - x_i$ or $z - z_j$, respectively. Factorize $M^{(1)}$ and $M^{(i)}$ as

$$M^{(1)} = M_1 \ldots M_r H_1 \quad \text{and} \quad M^{(i)} = N_1 \ldots N_k H_i,$$

where e.g., H_i corresponds with $h_i(z)$, the factor of $f^{(i)}$ which has no positive roots. We may consider the relation $\psi_1 = \phi_1 h_1$ or $\psi_i = \phi_i h_i$ assured by proposition (F), and the corresponding matrix relation $M_{\psi_1} = M_{\phi_1} H_1$ or $M_{\psi_i} = M_{\phi_i} H_i$. Since $s^{(i)} M^{(1)} = e^T M_\sigma M^{(i)}$ must hold, we will multiply the both sides by $M_{\phi_1} M_{\phi_i}$, obtaining

$$s^{(i)} M_1 \ldots M_r M_{\psi_1} M_{\phi_i} = e^T M_\sigma N_1 \ldots N_k M_{\psi_i} M_{\phi_1}. \tag{18}$$

First, suppose that $\lambda^{(i)} = \lambda^{(1)}$. Then $M_1 \ldots M_r = N_1 \ldots N_k$ holds, and (18) reduces to

$$s^{(i)} M_{\psi_1} M_{\phi_i} = e^T M_\sigma M_{\psi_i} M_{\phi_1}.$$

Obviously the right-hand side is 'invariant', and it is impossible to have $s^{(i)} > 0$. Hence $\lambda^{(i)} = \lambda^{(1)}$ cannot hold. Second, suppose that $x_1 < z_1$. (18) is equivalent to

$$s^{(i)} (M_1^{-1} M_1) \ldots (M_1^{-1} M_r)(N_2 \ldots N_k)^{-1} M_c = e^T M_d M_1^{-r} N_1, \tag{19}$$

where $M_c = M_{\psi_1} M_{\phi_i}$ and $M_d = M_\sigma M_{\psi_i} M_{\phi_1}$. Since x_i and z_j are positive, we have $M_i^{-1} \geqq I$ and $N_j^{-1} \geqq I$. Also $x_1 \geqq x_i$ means $M_i \geqq M_1$ or $M_1^{-1} M_i \geqq I$. Since ψ_1 and ϕ_i have nonnegative coefficients, $M_c \geqq I$ also holds. We also have $M_d \geqq I$. We see that the left-hand side of (19) is strictly positive according to our hypothesis $s^{(i)} > 0$. As to the right-hand side, we will write $(\alpha_T, \ldots, \alpha_1, 1)$, $(\beta_T, \ldots, \beta_1, 1)$ and $(\gamma_T, \ldots, \gamma_1, 1)$ for $e^T M_1^{-r}$, $e^T M_1^{-r} N_1$ and $e^T M_d M_1^{-r} N_1$, respectively. As was showed before, α_t/α_{t-1} converges to x_1. But this means, since $\beta_t = \alpha_t - z_1\alpha_{t-1}$ and $x_1 < z_1$, that $\beta_t < 0$ holds for all t greater than some t^*. Since $e^T M_d$ is 'invariant' and nonnegative, we have $\gamma_t < 0$ for any sufficiently large t as well. This is a contradiction. If $x_1 = z_1$, then $M_1 = N_1$ and we will eliminate M_1 and N_1 from the both sides of (18), and will repeat the same arguments to show $x_2 < z_2$ does not hold, etc. Hence, $\lambda^{(1)}$ must be lexicographically greater than $\lambda^{(i)}$ for $i = 2, \ldots, \omega$.

We will next show the sufficiency. As a preliminary, we will demonstrate the following proposition.

(H) Let $x_1, \ldots, x_r$ and $z_1, \ldots, z_k$ be two groups of positive numbers, each being arranged in descending order. Let M_i and N_j be the polynomialoid matrices for $z - x_i$ and $z - z_j$, respectively. Then there exists a polynomial $d(z)$ with positive coefficients for which we have for any T

$$sM_1 \ldots M_r = e^T M_d N_1 \ldots N_k \quad \text{and} \quad s > 0, \tag{20}$$

provided that $x_1 > z_1$ holds.

We will consider $wM_1 = e^T M_d N_1^k$ and $w > 0$ instead of (20). For these and $s = w(M_2 \ldots M_r)^{-1}(N_1^{-1}N_1)\ldots(N_1^{-1}N_k)$ will give (20) at once. We will also assume without loss of generality that $x_1 = 1$. For, if $x_1 = m \neq 1$, then we introduce a diagonal matrix D given by

$$D = \begin{bmatrix} 1 & & & & \\ & m & & & \\ & & m^2 & & \\ & & & \ddots & \\ & & & & m^T \end{bmatrix},$$

and consider $p(D^{-1}M_1D) = e^T(D^{-1}M_dD)(D^{-1}N_1D)^k$ and $p > 0$, which are equivalent to $wM_1 = e^TM_dN_1^k$ and $w > 0$, since $p = wD/m^T$. We will see that $D^{-1}M_1D$ or $D^{-1}N_1D$ corresponds to $z - 1$ or $z - z_1/m$, respectively.

Write $(\delta_T, \ldots, \delta_1, 1)$, $(d_T, \ldots, d_1, 1)$ and $(\varepsilon_T, \ldots, \varepsilon_1, 1)$ for $e^TM_1^{-k}$, $e^TM_1^{-k+1}$ and $e^TM_1^{-k}N_1^k$, respectively. First we note that $1 > z_1$ means $e^TM_1^{-1}N_1 > 0$, so that $e^TM_1^{-k}N_1^k > 0$ also. We now consider M_d such that $e^TM_d = (0, \ldots, 0, d_\tau, \ldots, d_1, 1)$ for some fixed τ. Then we obtain $e^TM_dM_1^{-1} = (\delta, \ldots, \delta, \delta_\tau, \ldots, \delta_1, 1)$, where $\delta = \delta_\tau$. Let $(1, \eta_1, \ldots, \eta_k)$ represent the coefficients of $(z - z_1)^k$ when expanded, which corresponds with N_1^k. Obviously, $1 + \eta_1 + \ldots + \eta_k = (1 - z_1)^k = a > 0$. We therefore have

$$e^TM_dM_1^{-1}N_1^k = (\delta a, \ldots, \delta a, \theta_{\tau+k}, \ldots, \theta_{\tau+1}, \varepsilon_\tau, \ldots, \varepsilon_1, 1).$$

Thus, to verify (H), we will show that there is some τ for which we have that $\theta_{\tau+1}, \ldots, \theta_{\tau+k}$ are all positive. This is seen to be true as soon as we notice that

$$\begin{aligned}\theta_{\tau+i} &= \delta + \eta_1\delta + \ldots + \eta_i\delta + \eta_{i+1}\delta_{\tau-1} + \ldots + \eta_k\delta_{\tau-k+i}\\ &= \delta(1 + \eta_1 + \ldots + \eta_i + \eta_{i+1}\delta_{\tau-1}/\delta + \ldots + \eta_k\delta_{\tau-k+i}/\delta).\end{aligned}$$

Since δ_t/δ_{t-1} converges to unity, and since $\delta = \delta_\tau$, we have that $\theta_{\tau+i}/\delta$ converges to $a > 0$ as τ increases.

We will now return to prove Theorem 4. Let $f^{(i)}$ be factorized as $f^{(i)} = a_i(z)h_i(z)$, where $a_i(z)$ has positive roots only while $h_i(z)$ has no positive roots. Correspondingly, $M^{(i)}$ is factorized as $M^{(i)} = A_iH_i$. Suppose that $\lambda^{(1)}$ is lexicographically greater than $\lambda^{(i)}$. We will then consider a polynomial $d_i(z)$ with positive coefficients for which we have from (H) that $p^{(i)}A_1 = e^TM_{d_i}A_i$ for $i = 2, \ldots, \omega$. As asserted by (H), we have $p^{(i)} > 0$. Let us define $d = d_2 \ldots d_\omega$. d is also a polynomial with positive coefficients. Consider also the relation $\psi_i = \phi_i h_i$ asserted by proposition (F). ϕ_i has positive coefficients, and ψ_i has nonnegative coefficients. Define two polynomials with non-negative coefficients, $\phi = \phi_1\phi_2\ldots\phi_\omega$ and $\sigma_i = h_i\phi_1\phi_2\ldots\phi_\omega d_2\ldots d_\omega$. We will show that the following vectors v and $s^{(i)}$ given by

$$vM^{(1)} = e^T M_{\sigma_1} \quad \text{and} \quad s^{(i)} = vM^{(i)}$$

satisfy the conditions that $v > 0$ and $s^{(i)} > 0$ for $i = 2, \ldots, \omega$. It is clear that $v > 0$. For, $vM^{(1)} = e^T M_{\sigma_1}$ reduces to $vA_1 = e^T M_d M_\phi$. Since $p^{(i)}A_1 = e^T M_{d_i} A_i$ and since d_i is a factor of d, we have $q^{(i)}A_1 = e^T M_d A_i$, and $q^{(i)} > 0$, where $q^{(i)} = p^{(i)} M_d M_{d_i}^{-1}$. On the other hand, we have $s^{(i)} = vA_i H_i = e^T M_d M_\phi A_1^{-1} A_i H_i = e^T M_d A_1^{-1} A_i M_{\sigma_i} = q^{(i)} M_{\sigma_i} > 0$.

We see from Theorem 4 that if $\lambda^{(1)}$ is lexicographically greater than any other $\lambda^{(i)}$, then efficient is any cash flow generated by $u^{(i)} = 0$ for $i \neq 1$, and $u_t^{(1)} \geqq 0$ for $0 \leqq t \leqq T-\ell$ for some fixed ℓ. This is valid whatever large T is taken. It is clear that if we want to choose a project, independently of the time pattern of investment, we must choose Project 1. What will happen if T is infinite? When matrix (2) has an infinite order, it is no longer a polynomialoid matrix but is equivalent to polynomial (1) itself. For, when T is finite, only the first j number of coefficients of (1) appear in the j^{th} column, from the right, of (2). The inverse matrix of (2) has the similar characteristics. But if T is infinite, no such truncation of coefficients is made; e.g., the coefficients of an infinite series appear in every column of the inverse of (2). We simply have to return to the old arguments dealing directly with polynomial (1) so far. The polynomialoid matrix distinguishes our arguments from the old ones.

References

[1] Arrow, K. J. and D. Levhari, 'Uniqueness of the Internal Rate of Return with Variable Life of Investment', Economic Journal, September 1969, pp. 560-6.

[2] Dorfman, Samuelson and Solow, Linear Programming and Economic Analysis, McGraw-Hill Book Company, Inc., New York, 1958.

[3] Flemming, J. S. and J. F. Wright, 'Uniqueness of the Internal Rate of Return: A Generalization', Economic Journal, June 1971, pp. 256-63.

[4] Karmel, P. H., 'The Marginal Efficiency of Capital', with Appendix by B. C. Rennie, Economic Record, vol. XXXV, #72 (1959), pp. 423-34.

[5] Nikaido, H., Convex Structures and Economic Theory, Academic Press, New York and London, 1968.

[6] Norstrøm, C. J., 'Uniqueness of the Internal Rate of Return with Variable Life of Investment: A Comment', Economic Journal, December 1970, pp. 983-4.

[7] Pitchford, J. D. and A. J. Hagger, 'A Note on the Marginal Efficiency of Capital', Economic Journal, September 1958, pp. 597-600.

[8] Samuelson, P. A., 'Some Aspects of the Pure Theory of Capital', Quarterly Journal of Economics (1937), pp. 469-96.

[9] Samuelson, P. A., 'An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money', Journal of Political Economy, December 1958, pp. 467-82.

[10] Soper, C. S., 'The Marginal Efficiency of Capital: A Further Note', Economic Journal, March 1959, pp. 174-7.