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Yoshitsugu YAMAMOTO

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University of Tsukuba

Sakura, Ibaraki 305, Japan

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Yoshitsugu Yamamoto

Institute of Socio-Economic Planning, University of Tsukuba

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Abstract   We present a model based on retraction for several fixed point algorithms. The model embraces the interpretation of variable dimension algorithms in terms of stationary point problem by van der Laan and Talman and fully explains the 2-ray method.

Recently van der Laan and Talman [5] have presented a unifying description of several restart fixed point algorithms, the $(n+1)$-ray, $2n$-ray, 2^n-ray and (3^n-1)-ray mehtods and Merrill's method, for approximating a solution of systems of n equations in n variables:

$$f(x) = 0, \qquad x \in R^n, \tag{1}$$

where $f: R^n \to R^n$ is continuous. But they have failed in fully explaining the 2-ray method propsed by Saigal [6] and Yamamoto [8,9]. In this note we will present a model based on 'retraction' such that each of the methods is considered as a realization of it. We will describe the model in the framework of homotopy continuation methods to dispense with concepts like piecewise linear approximation and complementarity pivoting, which would obscure the idea of the model. For this purpose we will assume that $f: R^n \to R^n$ is continuously differentiable (abbreviated by C^1).

Let $v \in R^n$ be a best known approximate solution of (1) and consider a family $\{ D_t: t \geq 0 \}$ of compact convex subsets of R^n satisfying the following three conditions:

(i) $D_0 = \{ v \}$,

(ii) $D_t \subseteq D_s$ if $0 \leq t \leq s$,

(iii) for any bounded subset E of R^n, there is a $t \geq 0$ such that $E \subseteq D_t$.

We say that for each $t \geq 0$, a continuous mapping r_t: $R^n \to R^n$ is a <u>retraction</u> onto D_t if

(iv) $r_t(R^n) \subseteq D_t$,

(v) $r_t(x) = x$ for any $x \in D_t$.

For the same reason as we have assumed that f is C^1 we assume that $r_t(x)$ is PC^1 in $(x, t) \in R^{n+1}$, i.e. $r_t(x)$ has an C^1 extension on each piece of some subdivision of R^{n+1}. See Alexander [1] for detailed discussion.

Let

$$h(x, t) = x - r_t(x - f(x)) \quad \text{for each } (x, t) \in R^n \times R^1_+,$$

where R^1_+ is the set of nonnegative real values. We consider the system of eqations:

$$h(x, t) = 0, \qquad (x, t) \in R^n \times R^1_+ . \qquad (2)$$

<u>Proposition 1.</u> The system (2) has a solution for each $t \geq 0$.

proof. For each $t \geq 0$, $r_t(x - f(x))$ is a continuous mapping from R^n into a compact subset D_t of R^n. Therefore by Schauder's fixed point theorem (see, for example, Smart [7]) it has a fixed point in D_t. □

<u>Proposition 2.</u> Let $(x, t) \in R^n \times R^1_+$ be a solution of (2). Then $x - f(x) \in D_t$ if and only if x is a solution of (1).

proof. The assertion immediately follows from the condition (v) of the retraction $r_t(x)$. □

<u>Proposition 3.</u> When $t = 0$, the system (2) has a unique solution $(x^0, t^0) = (v, 0)$.

proof. Since $D_0 = \{ v \}$, the assertion is trivial. □

Assume that zero is a regular value of h: $R^{n+1} \to R^n$. See Alexander [1] and Kojima [4] for the definition. Then we can see that

$$S = \{ (x, t) : (x, t) \text{ is a solution of } (2) \}$$

is a disjoint union of one-dimensional manifolds and each connected component of S homeomorphic to a one-dimensional sphere does not intersect $R^n \times \{ 0 \}$. Now let S^0 be a connected component of S having the trivial solution $(x^0, t^0) = (v, 0)$. Then S^0 is a path and is homeomorphic to a half-open interval $[0, 1)$.

Proposition 4. S^0 intersects $R^n \times \{ s \}$ for any $s \geq 0$.

proof. For each $0 \leq t \leq s$, let $R_t(x)$ be a restriction of $r_t(x - f(x))$ to D_s. Then $R_t(x)$ is a continuous mapping from a compact convex set D_s into itself. Applying Browder's theorem [2] to $\{ R_t(x) : 0 \leq t \leq s \}$ we see that there is a connected subset S^1 of $D_s \times [0, s]$ such that

$$x - R_t(x) = 0 \quad \text{for each } (x, t) \in S^1,$$

$$S^1 \cap (D_s \times \{ 0 \}) \neq \phi, \tag{3}$$

$$S^1 \cap (D_s \times \{ s \}) \neq \phi. \tag{4}$$

By (3) S^1 shares (x^0, t^0) with S^0, and hence S^1 is a subset of S^0. Consequently (4) implies the desired result.□

Proposition 5. If S^0 is contained in $C \times [0, +\infty)$ for some bounded subset C of R^n, then there is an $s \geq 0$ and an $x^1 \in R^n$ such that

$$S^0 \cap (R^n \times [s, +\infty)) = \{ x^1 \} \times [s, +\infty).$$

Furthermore x^1 is a solution of (1).

proof. Let

$$E = \{ x - f(x) : x \in C \},$$

then E is also bounded. By the condition (iii) of $\{ D_t : t \geq 0 \}$ there is an $s \geq 0$ such that $E \subseteq D_s$. Therefore $h(x, t) = f(x)$ for each $(x, t) \in C \times [s, +\infty)$. Since S^0 intersects $R^n \times \{ s \}$ by Proposition 4, choose an arbitrary point, say (x^1, s), from $S^0 \cap (R^n \times \{ s \})$. Then

$$0 = h(x^1, s) = f(x^1) = h(x^1, t) \quad \text{for any } t \geq s.$$

Therefore

$$\{ x^1 \} \times [s, +\infty) \subseteq S^0 \cap (R^n \times [s, +\infty)).$$

Suppose

$$S^0 \cap (R^n \times [s, +\infty)) \setminus \{ x^1 \} \times [s, + \infty) \neq \phi$$

and choose an arbitrary point (x^2, t) from the difference. Then by the same argument as above we see that

$$\{ x^1, x^2 \} \times [t, +\infty) \subseteq S^0 \cap (R^n \times [t, +\infty))$$

This is contrary to the fact that S^0 is homeomorphic to a half-open interval. Hence we have the desired result. □

Therefore tracing the path S^0 we can locate a solution of (1) if we come up to a point $(x, t) \in S^0$ such that $x - f(x) \in D_t$. Suppose that r_t has the additional property that

(vi) $r_t(R^n \setminus D_t) \subseteq \mathrm{bd}.D_t$,

where $\mathrm{bd}.D_t$ is the set of boundary points of D_t. Then x is a solution of (1) if and only if there is a $t \geq 0$ such that $(x, t) \in S$ and $x \in \mathrm{int}.D_t$, where $\mathrm{int}.D_t$ is the set of interior points of D_t. Therefore we can stop following the path S^0 when we find a point $(x, t) \in S^0$ with $x \in \mathrm{int}.D_t$.

Now let us confine ourselves to the case where

$$D_t = t\,D = \{\ tz : z \in D\ \} \tag{5}$$

for some compact convex set $D \subseteq R^n$ having zero in its interior. For each point $x \in D_t$, let

$$Y_t(\ x\ ) = \{\ y \in R^n : r_t(\ y\ ) = x\ \}. \tag{6}$$

Then $(\ x,\ t\ )$ satisfies (2) if and only if there is a $y \in R^n$ such that

$$y + f(\ x\ ) = 0 \quad \text{and} \quad y \in Y_t(\ x\ ) - x. \tag{7}$$

We here assume that r_t satisfies the condition (vi) and further that

(vii) $Y_t(\ x\ ) - x = Y_1(\ z\ ) - z$ for each $x \in D_t$ with $t > 0$, where z is a point of D such that $x = t\,z$, i.e. $z = x\,/\,t$.

Note that the condition (vi) implies that $Y_t(\ x\ ) - x = \{\ 0\ \}$ for any point $x \in \text{int}.D_t$. If $f(\ x\ ) \neq 0$, $x \in \text{bd}.D_t$, and hence $z = x\,/\,t$ is an intersecting point of the half line $\{\ \lambda x : \lambda \geq 0\ \}$ and bd.D. Therefore we can eliminate the variable t from (7) and obtain the following proposition.

<u>Proposition 6.</u> Suppose that D_t is defined by (5) and the conditions (vi) and (vii) are satisfied. Suppose further that $f(\ x\ ) \neq 0$. Then x satisfies (2) with some $t > 0$ if and only if there is a $y \in R^n$ satisfying

$$y + f(\ x\ ) = 0 \quad \text{and} \quad y \in Y(\ q(x)\ ), \tag{8}$$

where $q(x)$ is the unique point in $\{\ \lambda x : \lambda \geq 0\ \} \cap \text{bd}.D$ and $Y(\ z\ ) = Y_1(\ z\ ) - z$.

To present a unifying description of several variable dimension algorithms, van der Laan and Talman [5] have chosen a small compact polyhedral set B having zero in its interior, put $B_t = (\ 1 + t\ )\,B$ and considered the following family of stationary point problems:

$$\text{find } x(t) \in B_t \text{ such that } \langle\ x(t) - x,\ f(\ x(t)\ )\ \rangle \geq 0 \quad \text{for all } x \in B_t. \tag{9$_t$}$$

Let us define a projection p_t onto B_t by
$p_t(y) \in B_t$ and $\| p_t(y) - y \|_2 = \min \{ \| x - y \|_2 : x \in B_t \}$.
Then p_t is a retraction onto B_t satisfying the condition (vi), and x is a stationay point of $(9)_t$ if and only if x solves

$$x - p_t(x + f(x)) = 0 \qquad (10)$$

(see Eaves [3] for a relation of stationary points and fixed points). It is readily seen that $Y_t(x)$ in (6) with r_t replaced by p_t satisfies the condition (vii). Therefore we can apply Proposition 6 to stationay point problems, and hence to the (n+1)-ray, 2n-ray, 2^n-ray and (3^n-1)-ray methods with aid of the results in [5] (Note that the sign of f in (10) is not significant here).

In the remainder we will show that appropriate set D and retraction r_t derive the 2-ray method from the system (2). Let

$$D = \{ x \in R^n : \| x \|_1 \leq 1 \}$$

and let $r_t(y)$ be defined to be a point $x \in D_t$ which lexicographically minimizes the vector

$$(| y_1 - x_1 |, | y_2 - x_2 |, \dots, | y_n - x_n |).$$

Then $r_t : R^n \to D_t$ is a retraction onto D_t satisfying the condition (vi). For each point $x \in \mathrm{bd}.D_t$, let

$$j(x) = \min \{ k: \Sigma_{i=1}^{k} |x_i| = t \}.$$

Then we see that

$$Y_t(x) = \{ x \} \quad \text{if } x \in \mathrm{int}.D_t,$$
$$Y_t(x) = \{ y : y_i = x_i \quad \text{for } 1 \leq i \leq j(x) - 1,$$
$$\mathrm{sgn}(x_{j(x)})\, y_{j(x)} \geq |x_{j(x)}| \} \quad \text{if } x \in \mathrm{bd}.D_t.$$

Consequently $Y_t(x) - x$ is independent of t, i.e. the condition (vii) is satisfied and

$$Y(q(x)) = \{ 0 \} \quad \text{if } x \in \mathrm{int}.D_t,$$
$$Y(q(x)) = \{ y : y_i = 0 \quad \text{for } 1 \leq i \leq j(x) - 1,$$
$$x_{j(x)}\, y_{j(x)} \geq 0 \} \quad \text{if } x \in \mathrm{bd}.D_t.$$

Hence the system (8) is rewritten as

$$f_i(x) = 0 \qquad \text{for} \quad 1 \leq i \leq j(x)-1,$$
$$y_{j(x)} + f_{j(x)}(x) = 0,$$
$$x_{j(x)} y_{j(x)} \geq 0.$$

These relations show that the system (2) with D and r_t defined above corresponds to the 2-ray method.

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