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A FAIR COST ALLOCATION SCHEME OF A PUBLIC PROJECT

by

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ABSTRACT

In this paper we propose a fair cost allocation scheme for a large scale public project. This scheme, which we shall call "Proton", is derived uniquely from the axioms of efficiency, null-property, and linearity can be obtained by means of game theory. The Proton is the only cost allocation scheme which has the null-property essential to give users incentives to join the project as compared with other fair solutions such as Shapley value, Nucleolus or Rawlsian solution. The Proton is a realistic fair cost allocation scheme applicable in several kinds of decision making processes associated with large-scale public projects.

1. INTRODUCTION

What is the optimal way of producing services characterized by economies of scale, such as cooperative public projects or large-scale common facilities? One of the most important problems in achieving optimality is to assign costs to the participants of the project or facility. A transportation system, a water resource development, a cooperative waterbasin sewage system, etc., are typical large-scale public projects which exhibit the property of scale economies. It is important to determine the appropriate cost assignment scheme for these kinds of large-scale public projects. Otherwise, charges that are too high will act as disincentives for joining the cooperative project, while charges that are too law will mean that the total cost of the project will not be covered. Cost assignments must be considered "fair" so that the potential participants will agree to join in the cooperative project.

There is much debate concerning the proper approach to optimal pricing of public projects. Ruggles (1949-1950) summarizes some of the arguments against the use of marginal cost pricing. We cannot judge the welfare issues of pricing rules solely on efficiency principles such as marginal cost pricing. Since almost all public projects exhibit decreasing marginal costs, marginal cost pricing will not cover the total costs of producing such services. For projects in which deficits would be covered by lump sum government subsidies, the use of marginal cost pricing implies some redistribution of income from nonusers to users of those services. In such cases, making a decision about the fairness of marginal cost pricing as opposed to other pricing rules requires some concept of fairness regarding who should and should not pay for the use of a common facility.

Baumal and Bradford (1970) have suggested departures from marginal cost pricing. Their price rules require uniform market prices for the use of a public facility, and the covering of the cost of the facility. This requirement of uniform prices represents an additional constraint on the problem of optimal pricing, and it is one which is seldom realistic. Approaches using cost-benefit analysis primarily ignore the cost assignment problem, concentrating in stead on efficiency, and therefore cannot supply an appropriate solution to the cost assignment problem of a public project.

In order to include equity issues of distribution, the optimal pricing problem of a public project can be viewed by game-theoretic approaches. Developing a cooperative large-scale public project or a large-scale common facility can be considered as a cooperative game. From the viewpoint of the theory of games, some such work has already been carried out.

Loehman and Whinston (1971) developed a cost assignment scheme which makes the assumption of uniform prices. Certain other reasonable properties in the form of axioms are also required. In Loehman and Whinston (1974), the same charge formula was derived axiomatically using an axiom stating that users with the same incremental costs should be charged the same amount; that is, the demands, but not the ordering, of users is important in computing the charge. The axiom system serves as a constitution to which users of a common facility must agree before deciding to cooperate in building such a facility. Obviously, if a user would suffer a loss by participating in this project, he would not join the project. This cost allocation scheme proposed by Loehman and Whinston is an application of the Shapley value (Shapley (1953)) of cooperative n-person game theory to the cost allocation problem of a public project.

Suzuki and Nakayama (1976) applied the Nuleolus of a cooperative game to the cost assignment problem of a cooperative water resource development. They insisted that the Nucleolus "is an expression of the difference principle, or Maximin prinicple, of social justice suggested by Rawls (1971) in terms of game theory in characteristic function form." However, this interpretation is not correct. In general the Nucleolus is not identical with the Rawlsian solution. It is true that the Rawlsian solution in the core is identical with the Nucleolus.

Yasuda and Watanabe (1979)(1980) examined several equitable cost allocation schemes for a cooperative waterbasin sewage system using the concept of the Shapley value and the Nucleolus and compared them with the actual cost allocation scheme.

As we pointed out, both the Shapley value and the Nucleolus gave a very curious cost allocation scheme. In the cost allocation problem of the cooperative sewage system, the Shapley value and the Nucleolus gave negative cost assignments to some players. The Shapley value cost assignment formula is obtained by weighting each possible incremental cost by the probability of that order occurring. This is identical to the expected incremental cost due to a user when all orders of arrival are equally likely. In such a case, a powerful player can lower his cost through cooperation, but a powerless player can lower his cost only to a lesser degree through cooperation. Therefore, the Shapley value cost allocation scheme is not fair in the above sense in the cost assignment problem of a public project. The Nucleolus cost assignment scheme is also not fair for a similar reason. Therefore, it is not always valid to apply the solutions of cooperative games to the cost allocation problems of actual public projects.

In this paper we propose a new concept of fairness that is called "Proton", and we apply this fair solution to the cost allocation problem of a public project. This fair solution is derived from the axioms which have the desirable properties of fairness. The null-property is the essential feature of the Proton, and it means that players with positive demand will always be assigned a positive assignment of total savings. Shapley value and Nucleolus as fair solutions are dependent on coalitional power. This is especially true of Shapley value, which is actually considered an equilibrium determined by coalitional power. However, a fair allocation must be independent of coalitional power. The Rawlsian solution is one of the fair allocations which is independent from coalitional power, but it does not have the null-property that is essential to provide incentives for participating in a public project. For these two reasons, namely the null-property and the independence property of coalitional power, the Proton will be the only fair cost allocation scheme of a public project.

2. A COST ALLOCATION GAME

Suppose that there are n behavioral units who intend to develop a common large-scale public project or facility such as a cooperative sewage system, rubbish treatment facility, water resource development, transportation system, etc., which has the property of increasing returns to scale. Assume that the demand of each behavioral unit for the services from the common project or facility is given a priori, the most important problem is then to divide the total cost needed for the common project among all behavioral units.

This cost allocation problem can be analyzed by a game-theoretic approach. The agreement to develop a common project and to allocate the total cost among all units is then viewed as a cooperative n-person game among the users (e.g., municipal governments) of the common project.

Behavioral units such as the users of a common facility are called players in the theory of games. Each player $i \epsilon N$ participating in the common public project has a demand z_i for the service, where N is a set of players, i.e., $N = \{1, 2, ---i, ---n\}$.

$C(z)$ is the cost function for producing the service, and this is identical for all players.

$z_N = \sum_{i \epsilon N} z_i$ is defined as the total sum of the demands.

A cost allocation game is defined so as to obtain a vector $B = (B_1, B_2, ---, B_n)$, where each component B_i is a mapping from the subset of games in V to the real number R such that

$$B_i ; v \epsilon V \rightarrow R \quad (2.1)$$

$$\sum_{i \epsilon N} B_i(v) = C(z_N) \quad (2.2)$$

A game in characteristic function form is defined by a set function v from subsets (coalitions) S of the set of total players N into the real number R, that is,

$$v: S \subseteq N \rightarrow R \tag{2.3}$$

With addition and multiplication by positive scalars defined for games, the set of all games on N is a subset of a vector space V of all mappings from subsets of N to the real numbers. (The set of all games on N is not a subspace of V since only multiplication by positive scalars is allowed.)

In general the real-valued set function $v \varepsilon V$ is required to satisfy the property of superadditivity such that

$$v(S \cup T) \geq v(S) + v(T) \text{ for any } S, T \subseteq N, S \cap T = \phi. \tag{2.4}$$

The cost allocation problem of a public project involves two types of decision problems. One is the selection of a game v from a vector space V, and the other is the selection of a vector $B = (B_1, B_2, ---, B_n)$ where B_i is a cost assignment to the player i.

It is possible to define various kinds of characteristic function for the game. We shall use the characteristic function game defined as follows:

$$v^t(S) = \sum_{i \varepsilon N} C(z_i) - C(z_S) \text{ for every } S \subseteq N \tag{2.5}$$

where $z_S = \sum_{i \varepsilon S} z_i$

The characteristic function of this game V^t indicates the savings or financial surplus to the coalition S. This definition satisfies property (2.4) of a super additive game if the cost function is assumed to satisfy the following property of increasing returns to scale:

$$C(z_S) + C(z_T) \geq C(z_{S \cup T}) \text{ for any } S, T \subseteq N, S \cap T = \phi. \tag{2.6}$$

The characteristic function of all players indicates the total savings which are obtained by the cooperation of all players participating a common public project. In this way the cost allocation game becomes the total savings allocation game.

Another definition of the characteristic function is given as follows;

$$v^d(S) = -C(z_S) \quad \text{for every } S \subseteq N. \tag{2.7}$$

where $z_S = \sum_{i \in S} z_i$

The characteristic function v^d, of this game is equivalent to the utility function defined as the disutility of the cost for a coalition.

Usually the characteristic function indicates the coalitional power. The characteristic function $v^t(S)$ denotes the ability of the coalition to decrease the cost, so that players can lower the cost by forming a coalition. However, the meaning of the characteristic function v^d is not clear. Therefore, we selected the game v^t as the cost allocation game of the public project.[1]

Hereafter we will without remark omit the suffix t of the game v^t, since, the cost allocation game is equivalent of the savings allocation game.

The solution to this game is the allocation of the total savings among players. Savings for each player are equal to the cost of building his own facility minus the charge to him for use of the common facility, as follows:

$$x_i(v) = C(z_i) - B_i(v) \quad \text{for every } i \in N \tag{2.8}$$

The allocation of costs to a player is found from

$$B_i(v) = C(z_i) - x_i(v) \quad \text{for every } i \in N \tag{2.9}$$

The sum of cost shares defined in this way equals total cost. That is,

$$\sum_{i\epsilon N} B_i(v) = \sum_{i\epsilon N} C(z_i) - \sum_{i\epsilon N} x_i(v) = \sum_{i\epsilon N} C(z_i) - v(N)$$

$$= \sum_{i\epsilon N} C(z_i) - \left(\sum_{i\epsilon N} C(z_i) - C(z_N)\right) = C(z_N) \qquad (2.10)$$

$x_i(v)$ is the Paret optimal allocation of the total savings v(N) among players. Thus,

$$\sum_{i\epsilon N} x_i(v) = v(N) \qquad (2.11)$$

where $v(N) = \sum_{i\epsilon N} C(z_i) - C(z_N)$

The cost allocation scheme is then found once $x_i(v)$ is determined. This game has the property of individual rationality since a vector $x = (x_1, x_2, ---x_n)$ satisfies the following inequality (2.12):

$$x_i \geqq v(\{i\}) \; (= C(z_i) - C(z_i)) = 0 \quad \text{for every } i\epsilon N \qquad (2.12)$$

A vector x which satisfies both Pareto optimality (2.11) and individual rationality (2.12) is called an imputation. Since individual rationality holds in this game, the cost assignment to each player cannot be greater than the cost of building his own facility. That is,

$$B_i(v) \leqq C(z_i) \quad \text{for every } i \,\epsilon\, N \qquad (2.13)$$

We assume that the cost function has the property of increasing returns to scale in the following sense.

Assumption 2.1

The cost function $C(z_i)$ satisfies the following conditions:

If z = 0, then C(z) = 0. That is, C(0) = 0. (2.14)

C(z) is non-decreasing with respect to z. That is,

if $z^1 \leq z^2$, then $C(z^1) \leq C(z^2)$ (2.15)

$$C(z_{S\cup j}) - C(z_{S\cup j-i}) \leqq C(z_S) - C(z_{S-i}) \tag{2.16}$$

for any $S \subsetneq N - \{i,j\}$ and every $i,\ j \in N,\ i \neq j$.

From this definition of the cost function there will exist incentives for new members. Inequality (2.16) coincides with the following inequality (2.17).

$$C(z_{S\cup T}) + C(z_{S\cap T}) \leqq C(z_S) + C(z_T) \text{ for any } S,\ T \subseteq N. \tag{2.17}$$

Inequality (2.6) is derived from the inequality (2.17) if we set $S \cap T = \phi$. Thus, this game has the property of the super-additivity (2.4). Moreover, as we will now show, this game becomes the convex game which is stronger than the super-additive game.

Conditions (2.16) and (2.17) are identical with the following condition (2.18).

$$C(z_{T\cup i}) - C(z_T) \leqq C(z_{S\cup i}) - C(z_S) \tag{2.18}$$

for any $S \subsetneq T \subsetneq N - \{i\}$ and every $i \in N$

<u>Theorem 2.1</u>

The game (v:x) is a convex game such that

$$v(S \cup i) - v(S) \leqq v(T \cup i) - v(T) \tag{2.19}$$

for every $i \in N$ and any $S \subsetneq T \subsetneq N - \{i\}$

Proof) It is easy to find that

$$v(T \cup i) - v(T) - \left[v(S \cup i) - V(S)\right]$$
$$= C(z_{S\cup i}) - C(z_S) - \left[C(z_{T\cup i}) - C(z_T)\right] \geqq 0$$

Q.E.D.

Note that condition (2.19) of the convex game coincides with both of the following conditions (2.20) and (2.21).[2]

$$V(S \cup j) - v(S \cup j - i) \geqq v(S) - v(S - i)$$

for any $S \subsetneq N - \{i,j\}$ and every $i,\ j \in N,\ i \neq j$. (2.20)

$$v(S \cup T) + v(S \cap T) \geqq v(S) + v(T) \tag{2.21}$$

for any $S, T \subseteq N$

Condition (2.21) is the original condition of the convex game defined by Shapley (1971).

Lemma 2.1

The cost function C(z) is an average cost decreasing function. That is,

$$\frac{C(z_S)}{z_S} \geqq \frac{C(z_T)}{z_T} \geqq \cdots \geqq \frac{C(z_N)}{z_N}$$

for any $S \subseteq T \subseteq N$

Proof)

If we set $S = T - \{i\}$ in the condition (2.18), then we obtain

$$C(z_T) - C(z_{T-i}) \geqq C(z_{T \cup i}) - C(z_T)$$

Thus, we get,

$$\tfrac{1}{2}\ \{C(z_{T \cup i}) + C(z_{T-i})\} \leqq C(z_T) \quad \text{for any } T \subseteq N - \{i\}$$

From the generalization of this condition, we obtain easily,

$$C(Zv) \geqq tC(z_T) + (1-t)\ C(z_S) \tag{2.23}$$

for any $S \subseteq U \subseteq T \subseteq N$

where $t = \dfrac{z_U - z_S}{z_T - z_S}$, $\quad 0 \leqq t \leqq 1$

In the condition (2.23), set $S = \phi$, $T = T$, $= S$, to obtain

$$C(z_S) \geqq \frac{z_S - z_\phi}{z_T - z_\phi}\ C(z_T) + \frac{z_T - z_S}{z_T - z_S}\ C(z\phi)$$

as $z\phi = 0$, $C(C_{z\phi}) = 0$. Thus, we obtain

$$C(z_S) \geqq \frac{z_S}{z_T}\ C(z_T)$$

$$\therefore \quad \frac{C(z_S)}{z_S} \geq \frac{C(z_T)}{z_T} \quad \text{for any } S \subseteq T \subseteq N \qquad \text{Q.E.D.}$$

From this lemma we obtain the following property of the cost allocation game.

Corrolary 2.1

The game (v:x) is an average coalitional value increasing game. That is,

$$\frac{v(S)}{z_S} \leq \frac{v(T)}{z_T} \leq \cdots \leq \frac{v(N)}{z_N} \quad \text{for any } S \subseteq T \subseteq N \qquad (2.24)$$

3. A Fair Cost Allocation Scheme

A fair cost allocation scheme, which we propose in this paper, is the unique solution which satisfies the proper axioms of efficiency, null-property and linearity. Since the cost assignment B_i for a player i is defined as $B_i = C(z_i) - x_i(v)$, the fair cost allocation is obtained from the fair savings assignment. Thus, we have to obtain the fair assignment of total savings. The fair savings assignment scheme, as denoted by $x^P = (x_1^P, x_2^P, ---x_n^P)$, is derived from the following three axioms.

Axiom 1. (Efficiency)

$$\sum_{i \varepsilon N} x_i^P (v) = v(N) \tag{3.1}$$

Axiom 2. (Null-property)

$$x_i(v_z) = 0 \text{ if and only if } z_i = 0 \text{ for every } i \varepsilon N \tag{3.2}$$

Axiom 3. (Linearity)

For any two games v_{z^1} and v_{z^2}

$$x_i^P (v_{\lambda z^1 + (1-\lambda) z^2}) = \lambda x_i^P (v_{z^1}) + (1 - \lambda) x_i(v_{z^2}) \tag{3.3}$$

for every $i \varepsilon N$

Axiom 1 means that charges for participants of the common public project must cover total costs. This corresponds to the Pareto efficient condtion. Axiom 2 implies that if the fixed demand of a player is zero, then the charge for the player must be zero, and vice versa. Axiom 3 implies that if the game is divided into two games according to the fixed demands, then the charge for a player in the original game is the expected value of the divided games.[3] In axiom 3, the parameter λ can be regarded as the probability distribution of the occurrence of the divided games.

Using these axioms, the following theorem of the existence and the uniqueness of the fair cost allocation scheme of the public project will be proven.

Theorem 3.1

A unique vector function $x^p = (x_1^p(v), x_2^p(v), ---, x_n^p(v))$ exists and satisfies axioms 1-3 for the cost allocation problem. It is given by the formula,

$$x_i^p(v_z) = \frac{z_i}{z_N} v(N) = \frac{z_i}{z_N} \left(\sum_{i \in N} C(z_i) - C(z_N) \right) \text{ for every } i \in N$$

Proof)

We can assume $v(N) = 1$ without loss of generality (or set a new game $u = \frac{1}{v(N)} v$ with scalar multiplication of the game v).

For a fixed demand z_i of a player i, we define w_i as follows;

$$w_i = \frac{z_i}{z_N} \text{ for every } i \in N$$

where $z_N = \sum_{i \in N} z_i$

We easily obtain,

$$\sum_{i \in N} w_i = 1$$

Thus, w_i is the relative weight of the demand of player i.

Let us define the set of weight vectors as follows;

$$W_N = \{(w_1, w_2, ---, w_n): \sum_{i \in N} w_i = 1, w_i \geq 0 \text{ for every } i \in N\}$$

W_N is the surface of the n dimensional simplex which has non-negative components.

Define the vector e_j as follows;

$$e_j = (\underset{1}{0}, ---0, \underset{j}{1}, 0, ---\underset{n}{0}) \text{ for any } j\varepsilon N \quad (3.5)$$

Then, using axiom 2, we obtain

$$x_i\ (v_{ej}) = 0 \quad \text{for any } i\varepsilon N,\ i\neq j. \quad (3.6)$$

Using axiom 1, we get

$$\sum_{i\varepsilon N} x_i(v_{ej}) = 1 \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$x_j(v_{ej}) = 1 \text{ for any } j\ \varepsilon\ N \quad (3.8)$$

(3.6) and (3.8) implies that

$$x(v_{ej}) = ej \text{ for any } j\ \varepsilon\ N$$

If $w\,\varepsilon\, W_N$, then w is expressed as the convex combination of vectors $e_i(i = 1, 2,---n)$.

$$w = \sum_{i\varepsilon N} w_i e_i$$

Using axiom 3, we get

$$x(v_w) = x(\sum_{i\varepsilon N} w_i e_i) = \sum_{i\varepsilon N} w_i x(e_i)$$

$$= \sum_{i\varepsilon N} w_i e_i = w$$

Q.E.D.

Note that the cost assignment is defined as follows;

$$B_i^P = C(z_i) - x_i^P\ (v) \quad (3.4)$$

$$= c(z_i) - \frac{z_i}{z_N}\ v(N) = C(z_i) - \frac{z_i}{z_N} \left(\sum_{i\varepsilon N} C(z_i) - C(z_N) \right)$$

for every $i\ \varepsilon\ N$

This unique fair cost allocation scheme is called hereafter the "Proton".

The Proton satisfies the desirable properties of fairness.

Corrolary 3.2

The Proton of the game satisfies the property of symetry.

(3.10) (Symmetry)

$z_i = z_j$ if and only if $x_i(v_z) = x_u(v_z)$

for every i, jεN, i$\neq$j.

Using the property of individual rationality $x_i(v_z) \geqq 0$ and axiom 2, we obtain

$$z_i > 0 \text{ if and only if } x_i^p(v_z) > 0 \text{ for every } i\varepsilon N \tag{3.11}$$

Condition (3.11) implies that any player can obtain a positive saving assignment if this demand of the player is positive, and vice versa. This gives all behavioral units the strong incentive to join the common project. Note that other fair solutions such as the Shapley value, the Nucleolus, and the Rawlsian do not satisfy the null property.

The Proton as a fair allocation of the savings is the total savings weighted by the relative ratio of the demand of a player to the total demands.

4. The Stable Solution (Core) of the Problem

The concept of fairness in the cost assignment problem of a public project must contain the concept of stability. The concept of stability which we chose is that of the core of the game theory.

An imputation can be blocked by a coalition of players if every member of the coalition could be made better off from the given imputation by trading only among members of the coalition. An imputation belongs to the core of the game if it can be blocked by no coalition. The core has the property of coalitional stability.

Any fair imputation should be contained in the core for the reason of coalitional stability.

Let us give a formal definition of the core.

Def. 4.1

An imputation $x = (x_1, x_2, \text{---} x_n)$ is in the core if

$$\sum_{i \in S} x_i(v) \geqq v(S) \quad \text{for every } S \subseteq N \tag{4.1}$$

Condition (4.1) is the generalization of individual rationality to a coalition, and can be called coalitional rationality.

Theorem 4.1

The cost allocation game (v:x) has non-empty core. The Proton always exists and is in the core.

Proof)

Let us put,

$$x_i = x_i^p = \frac{z_i}{z_N} v(N) \quad \text{for every } i \in N$$

Using the average coalitional value increasing property, we obtain

$$\sum_{i \in S} x_i = \sum_{i \in S} x_i^p = \sum_{i \in S} \frac{z_i}{z_N} v(N) = \frac{z_S}{z_N} v(N) \geqq v(S) \text{ for every } S \subseteq N$$

Q.E.D.

All of the convex games have non-empty core in general as shown by Shapley (1971). However, the above proof is much simple for the cost allocation game than the complicated and the difficult proof by Shapley. The Proton has the property of stability in the sense of the core.

5. Other Fair Solutions

There are many kinds of fair solutions to the cost allocation problem. We consider the several concepts of fairness and compare them with the Proton. First of all, we define the "process equitable solution" of the game by applying the concept of the Shapley value solution of the n-person game.

Definition 5.1

A vector $x^s = (x_1^s, x_2^s, ---, x_n^s)$, given by the following formula (5.1) is called the "process equitable solution" (Shapley value) of the game v_z.

$$x_i^s(v) = \sum_{T \subseteq N} \gamma\ n\ (t) \quad \{v(T) - v(T-i)\} \text{ for every } i \varepsilon N \qquad (5.1)$$

where,

$$\gamma_n(t) = \frac{(t-1)!(n-t)!}{n!} \qquad (5.2)$$

$t = |T|$ is the number of members in coalition $T \subseteq N$. Shapley value is a unique solution satisfying the axioms 5.1 --- 5.3 in general.

Axiom 5.1 (Efficiency)

$$\sum_{i \varepsilon N} x_i^s\ (v) = v(N) \qquad (5.3)$$

Axiom 5.2 (Symmetry)

For each ¶ in ¶ (N)

$$x_{\P i}^s\ \{\P v\} = x_i^s\ \{v\} \text{ for every } i \varepsilon N \qquad (5.4)$$

where ¶(N) denotes the set of permutations of N.

Axiom 5.3 (Law of aggregation; Linearity)

For any two games v and w

$$x^s\ \{v+w\} = x^s\ \{v\} + x^S\ \{w\} \qquad (5.5)$$

The first axiom is the same as axiom 1 of the Proton. The second axiom has the same meaning as corrolary 3.2 of the axioms of the Proton. The third axiom has a meaning similar to that of axiom 3 of the axioms of the Proton. It states that when two independent games are combined, their Shapley values must be added player by player, as explained in Shapley (1953).

The Shapley value of the player for the game equals the expected value of imputations which the player can expect.

$v(T) - v(T - \{i\})$ represents the marginal payoff which player i obtained by conspiring with a coalition $T \subseteq N$. The parameter $\gamma_n(t)$ is defined so that all orders in which each player conspires with coalition T happen with equal probability in all coalitions $T \subseteq N$.

The fair cost allocation scheme proposed by Loehman and Whinston (1974) is equivalent to this Shapley value.

The coalition power defined by the characteristic function plays an important role in the definition of the Shapley value. Shapley value is an equilibrium solution determined by coalitional power. Moreover, in general, Shapley value does not have the null property. Without the null property it is possible that a player will not obtain positive assignment of the savings, so that this player will not have the incentive to participate in the cooperative project. For these two reasons, Shapley value is not appropriate for a fair cost allocation scheme of a public project. Note that the Nash bargaining solution is identical to the Shapley value in an n-person cooperative game with sidepayment.

Another fair solution is the "outcome equitable solution," which is an application of the Nucleolus of an n-person cooperative game.

Definition 5.2

An "outcome equitable solution" (Nucleolus) of the savings assignment game (v:x) is defined as a unique imputation;

$x^1 = (x_1^1, x_2^1, ---, x_n^1)$

such that

$$\underset{x}{\text{Min}} \; \underset{S}{\text{Max}} \; e(x, S) \tag{5.6}$$

where $e(x, S) = v(S) - \sum_{i \in S} x_i$

This outcome equitable solution (Nucleolus) is a minimax saving allocation in that it minimizes large disparities between the savings allocated to members of any coalition and the savings to that coalition of acting alone. We can obtain the Nucleolus as a vector of imputations which minimizes the maximum dissatisfaction, e(x,s) by linear programming

$$\text{Min } y \tag{5.7}$$

subject to

$$\left.\begin{array}{l} v(S) - \sum_{i \in S} x_i \leqq y \quad (S \neq N) \\ v(N) - \sum_{i \in N} x_i = 0 \\ \qquad x_i \geqq 0, \text{ for every } i \in N \end{array}\right\} \tag{5.8}$$

Theorem 5.2

The outcome equitable solution (Nucleolus) of the game (v:x) is in the core.

Note that the Nucleolus is always in the core if the core of a game is non-empty, as shown by Kohlberg (1971).

The Nucleolus is an equilibrium solution that is determined by coalitional powers. Because of the definition Nucleolus gives an extreme imputation.

The Rawlsian solution of the game (v:x) is found by applying the concept of fairness proposed by Rawls (1971).

Definition 5.3

The Rawlsian fair solution of the game (v:x) is defined as an imputation

$$x^r = (x_1^r, x_2^r, ---, x_n^r)$$

such that

$$\max_x \min_{i \in N} (x_1(v), x_2(v), ---, x_n(v)) \tag{5.9}$$

Suzuki and Nakayama(1976) insisted that "This (Nucleolus) is an expression of the difference principle, or Maximim principle, of social justice suggested by Rawls (1971) in terms of game theory in characteristic function form". However, this interpretation is not correct, and in general the Nucleolus is not equal to the Rawls. It is correct that the Rawls in the core is identical with the Nucleolus. However, the Rawlsian solution is seldome in the core, and therefore it does not have the coalitional stability in the sense of the core.

Moreover, the Rawlsian solution does not have the null property. In fact, it is possible that a player with no demand will obtain the maximum payoff.

As a result, the Rawlsian solution of the game cannot be a fair scheme of the allocation problem.

In the real world of the cost allocation problem of the cooperative sewage system in Japan, the demand proportional cost assignment scheme is generally used. This demand proportional cost assignment scheme is defined formally as follows.

Definition 5.4

The solution of the demand proportional cost assignment of the total cost is defined uniquely as an imputation

$$x^d = (x_1^d, x_2^d, ---, x_n^d)$$

such that

$$B_i^d (v) = \frac{z_i}{z_N} C(z_N) \quad \text{for every } i \in N$$

This demand proportional solution has desirable properties of fairness as follows. Using the non-decreasing property of the cost function, we obtain:

$x_i^p(v) \geqq 0$. In other words, savings assignment to each player is non-negative. Moreover, if $z_i = 0$ for some $i \in N$, then $x_i^d(v) = 0$. However, $x_i^d(v) = 0$ for some $i \in N$ does not necessarily imply $z_i = 0$. The demand proportional solution has only one necessary condition of the null property.

The demand proportional solution of the game v^d has the property of coalitional stability in the sense of the core.

Theorem 5.3

The demand proportional solution $x^d = (x_1^d, x_2^d, ---, x_n^d)$ is in the core.

Proof)

Using the average cost decreasing property of Lemma 2.1, we obtain

$$\begin{aligned} \sum_{i \in S} x_i^d &= \sum_{i \in S} \left(C(z_i) - \frac{z_i}{z_N} C(z_N)\right) \\ &= \sum_{i \in S} C(z_i) - \frac{z_S}{z_N} C(z_N) \geqq \sum_{i \in S} C(z_i) - C(z_S) = v(S) \end{aligned}$$

for any $S \subseteq N$ Q.E.D.

Table 1 shows the result of a comparison of the alternative fair solutions according to the axioms and desirable properties of fairness.

If fairness requires organizational stability, then the Rawlsian solution is not appropriate for the fair solution.

It is clear that the null property is the most essential for fairness to be achieved in allocation problems such as this cost allocation problem of a large-scale public project.

Last of all, we would like to clarify the relationship between coalitional power and fair solutions. In both the Shapley value and the Nucleolus, coalitional power plays an important role. However in both the Rawlsian and the Proton, the solutions are not affected by coalitional power. The concept of fairness needs independence from coalitional power.

As a result, we propose that the Proton is the best fair cost allocation scheme of a public project.

Table 1. A Comparison among fair solutions

Solutions \ Properties	Core	Efficiency	Null-Property	Linearity	Symmetry
1. Proton	0	0	0	0	0
2. Shapley Value	0	0	X	0	0
3. Nucleolus	0	0	X	X	X
4. Rawlsian	X	0	X	X	X
5. Demand Proportional	0	0	Δ	0	0

where,

0: satisfies

X: does not satisfy

Δ: does not necessarily satisfy

6. Concluding Remarks

In this paper we propose a fair cost allocation scheme for a public project using a game theoretic approach. This scheme, called "Proton" is a unique fair cost allocation scheme satisfying the three axioms of efficiency, null property, and linearity which seem to be necessary conditions for fairness.

In the following paper, we will show the validity of the Proton by applying it to actual public projects such as a cooperative waterbasin sewage system.

In our cost allocation problems, the fixed demand is assumed to be given a priori. However this assumption is not generally true, so that we should determine the optimal level of the demand for players using a more general model of cost allocation problems.

We also assumed that the number of participants for a public project was given a priori. However, one of the most important issues is to determine the optimal number of members in a large-scale public project which exhibits scale economies. This will be the optimal size of public projects of public facilities.

The Proton in the weighted majority game will have desirable properties of fairness comparing with other fair solutions such as Shapley value as shown in the following paper.

Finally, we must investigate the definition of the Proton in the general n-person cooperative game.

Footnotes

[1]If the demand function of player i is given by $p_i = f_i(z_i)$ and $\bar{z}_i$ is the given fixed demand of player i, then the benefit for a coalition S in general will be given as the following formula.

$$v^g(S) = \sum_{i \in S} \int_0^{\bar{z}_i} (f_i(z))\, dz + (\sum_{i \in S} C(\bar{z}_i) - C(\bar{z}_S))$$

The first term is the consumer surplus and the second term is the savings due to a coalition. The characteristic function game $v^t(S)$ is the game which considers only the second term in the general form of the characteristic function $v^g(S)$.

[2]This proof is found in Shapley (1971)

[3]Axiom 3 will become a subject of discussion similar to axiom 3 of the Shapley value.

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REFERENCES

Baumol, E.J., Bradford, D.F., 1970 "Optimal Departures for Marginal Cost Pricing" American Economic Review, 60, 265-283

Hillman, A.L., 1976 "Discussion on a Generalized Cost Allocation Scheme" in Theory and Measurement of Economic Externalities, Ed. SA Lin (Academic Press, New York) pp. 103-106

Ichiishi, T., 1981 "Super-modularity: Applications to Connex Games and to The Greedy Algoriths for LP" Journal of Economic Theory 25, 283-286

Kohlberg, E., 1971 "On the Nucleolus of a Characteristic Function Game" SIAM Journal of Applied Mathematics, 20, 62-66

Loehman, E.T., Whinston, A. 1971 "A New Theory of Pricing and Decision-making in Public Investments" Bell Journal of Economis and Management 2, 606-628

Loehman, E.T., Whinston, A., 1974 "An Axiomatic Approach to Cost Allocation for Public Investment" Public Finance Quarterly 2, 236-251.

Loehman, E.T., Whinston, A., 1976 "A Generalized Cost Allocation Scheme" in Theory and Measurement of Economic Externalities Ed. SA Lin (Academic Press, New York) pp. 87-101 .

Myerson, R.B., 1980 "Conference Structure and Fair Allocation Rules" International Journal of Game Theory 9, 169-182

Ratich, S.J., 1979 Efficient, Equitable and Stable Solutions to N-Person Bargaining Game Ph.D. thesis,——————, Johns Hopkins University, Baltimore, Maryland, USA

Rawls, J., 1971 A Theory of Justice (Oxford University Press, Oxford)

Ruggles, N., 1949-1950 "Recent Developments in the Theory of Marginal Cost Pricing" Review of Economic Studies 1, 29-46

Shapley, L.S., 1953 "A Value for N-person Games" in Contributions to the Theory of Games, Vol. 28 of Annals of Mathematical Studies Ed. H.W. Kuhn and A.W. Tucker (Princeton University Press, Princeton, New Jersey) pp. 307-317

Shapley, L.S., 1971 "Cores of Convex Game" International Journal of Game Theory 1, 11-26

Schmeidler, D., 1961 "The Nucleolus of a Characteristic Function Game" SIAM Journal of Applied Mathematics 17, 1163-1170

Sorenson, J., Tschishart, J., Whinston, A., 1978 "A Theory of Pricing Under Decreasing Costs" American Economic Review 68, 614-629

Suzuki, M., Nakazama, M., 1976 "The Cost Assignment of the Cooperative Water Resource Development: A Game Theoretic Approach" Management Science 22, 1081-1086

Yasuda, Y., 1972 "A Concept of Fairness in the Socio-Economic Systems" Journal of Mita Economic Society 65, (in Japanese)

Yasuda, Y., Watanabe, K., 1979 "An Equitable Cost Allocation of a Cooperative Sewage System as Regional Public Goods" Discussion Paper Series, No. 109 Institute of Socio-Economic Planning, Sakura, Ibaraki, JAPAN

Yasuda, Y., Watanalu, K., 1980 "An Equitable Cost Allocation of Cooperative Sewage System" Annals of Regional Science, Japan 10 103-118 (in Japanese)

Young, H.P., Okada N., Hashimata, T., 1981 "Sharing Costs Fairly - A Practical Introduction to Cooperative Game Theory" Executive Report 5, International Institute for Applied Systems Analysis, Laxenburg, Austria