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THE 2-RAY METHOD: A NEW VARIABLE DIMENSION FIXED POINT ALGORITHM WITH INTEGER LABELLING

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Abstract: In this paper we propose a new variable dimension fixed point algorithm with integer labelling. We show that the algorithm provides an approximate fixed point of a continuous mapping f from a compact convex subset D of R^n into R^n if f maps the set of boundary points of D into D. We define a k-complete simplex and its orientation for $k = 1, 2, \ldots, n$ to characterize the sequence of simplices generated by the algorithm. Two techniques are given which possibly generate several approximate fixed points.

Key Words: Rothe's fixed point theorem, variable dimension algorithms orientation

1. INTRODUCTION

In these several years many new simplicial restart algorithms have been developed for approximating a fixed point of a continuous mapping f from an n-dimensional Euclidean space R^n into itself. They are called variable dimension algorithms after their common feature that they generate a sequence of simplices with varying dimensions. They start from a single point, 0-dimensional simplex, and leave it along one of the rays extending in several directions. They are classified according to the number of rays possibly followed by the method: the (n+1)-ray and 2n-ray methods by Van der Laan and Talman [8,9,10,11], the 2^n-ray method by Wright [15], the (3^n-1)-ray method by Kojima and Yamamoto [4,5], and the 2-ray method by Saigal [13] and Yamamoto [16]. The first two methods have both vector labelling and integer labelling versions while the others now have only vector labelling versions. The major purpose of this paper is to develop an integer labelling version of the 2-ray method. We also argue the modifications of the method to approximate several fixed points.

In Section 2 we introduce a labelling function and define a k-complete simplex which provides an approximate solution of the k-dimensional subproblem

$$f_i(x) = x_i \quad \text{for } i = 1, 2, \ldots, k.$$

In Section 3 we define the adjacency relation of simplices and give an existence theorem of an n-complete simplex and a fixed point. In Section 4 we propose the 2-ray method with integer labelling. In Section 5 an illustrative example is given. In Section 6 we show how to apply the method to approximating a fixed point of a continuous mapping from a compact convex subset of R^n into R^n. We also give a constructive proof of Rothe's fixed point theorem. In Section 7 we investigate the orientation of complete simplices. In Section 8 we propose two techniques of the method for approximating several fixed points.

We explain below some terminologies and notations used in this paper.

$I(k) = \{ 1, 2, \ldots, k \}$ for a nonnegative integer k, where $I(0) = \emptyset$,

$\| x \|_\infty = \max \{ |x_i| : i \in I(n) \}$ for $x \in R^n$,

$\| x \| = (x^t x)^{1/2}$ for $x \in R^n$,

$C^n = \{ x \in R^n : \| x \|_\infty \leq 1 \}$,

$\{ \pm 1 \} = \{ -1, +1 \}$,

$C^n(k, \alpha) = \{ x \in C^n : \alpha x_k \geq 0, x_j = 0$ for $j \in I(n) \setminus I(k) \}$

for $k \in I(n) \cup \{0\}$ and $\alpha \in \{\pm 1\}$,

$C^n(k) = C^n(k, -1) \cup C^n(k, +1)$,

$B^n(k, \alpha) = \{ x \in C^n : \alpha x_k = 1 \}$,

$H^n(k, \alpha) = \{ x \in R^n : \alpha x_k \leq 1 \}$,

bd X : the boundary set of $X \subset R^n$,

Let T be a locally finite triangulation of C^n. For T let

$$\bar{T} = \{ \tau : \tau < \sigma \text{ for some } \sigma \in T \},$$

where $\tau < \sigma$ means that τ is a face of σ. For a convex subset $X \subset C^n$ let

$$T|X = \{ \tau \in \bar{T} : \tau \subset X, \dim \tau = \dim X \}.$$

For notational convenience we shorten $T|C^n(k, \alpha)$ to $T(k, \alpha)$ and $T(k, -1) \cup T(k, +1)$ to $T(k)$. We assume in this paper that $T(k, \alpha)$ forms a subdivision of $C^n(k, \alpha)$ for $k \in I(n) \cup \{0\}$ and $\alpha \in \{\pm 1\}$.

2. THE LABELLING FUNCTION AND k-COMPLETE SIMPLICES

Let f be a continuous mapping from C^n into R^n. In this paper we assume the following condition.

Condition 2.1.

$$f(B^n(k, \alpha)) \subset H^n(k, \alpha) \quad \text{for each } k \in I(n) \text{ and each } \alpha \in \{\pm 1\}.$$

Note that if f maps bd C^n into C^n, the condition is satisfied.

For each point $x \in C^n$, let

$$J(x) = \{ j \in I(n) : x_j \geq f_j(x) \text{ and } x_j \neq -1 \} \cup \{n+1\} \tag{2.1}$$

and define the labelling function $L : C^n \to I(n+1)$ as

$$L(x) = \min \{ j : j \in J(x) \}. \tag{2.2}$$

For each simplex τ of $\bar{T}$ let

$$L(\tau) = \bigcup \{ L(u) : u \text{ is a vertex of } \tau \}.$$

We say that τ is a k-complete simplex if

$$I(k) \cup \{m\} \subset L(\tau) \quad \text{for some } m \in I(n+1) \setminus I(k). \tag{2.3}$$

Note that $\dim \tau \geq k$ if τ is a k-complete simplex, and any simplex is a 0-complete simplex. We simply say that τ is a complete simplex when $\dim \tau = k$ and τ is a k-complete simplex. Then we have the following lemma which states that a k-complete simplex has an approximate fixed point of a k-dimensional subproblem of the original fixed point problem. The readers may refer to Kryński [6] for the approximation accuracy of the labelling function (2.2) with a very slight modification. In what follows let δ be the mesh size of the triangulation T of C^n, i.e.

$$\delta = \max \{ \max \{ \| x - y \| : x, y \in \sigma \} : \sigma \in T \} \tag{2.4}$$

and let

$$\varepsilon = \max \{ \| f(x) - f(y) \|_\infty : x, y \in C^n, \| x - y \| \leq \delta \}. \tag{2.5}$$

Lemma 2.2. Suppose that f satisfies Condition 2.1. If $\tau \in \bar{T}$ is a k-complete simplex for some $k \in I(n)$, then

$$\max \{ \mid x_i - f_i(x) \mid : i \in I(k) \} \leq \delta + \varepsilon$$

for any point $x \in \tau$. Especially, if $\sigma \in T$ is an (n-) complete simplex, then

$$\| x - f(x) \|_\infty \leq \delta + \varepsilon$$

for any point $x \in \sigma$.

Proof. Let u^i be a vertex of τ with $L(u^i) = i$ for $i \in I(k)$ and let v be a vertex of τ with $L(v) = m$ for some $m \in I(n+1) \setminus I(k)$. For the vertex v define

$$J_1 = \{ j \in I(k) : v_j < f_j(v) \},$$
$$J_2 = \{ j \in I(k) : v_j \geq f_j(v) \text{ and } v_j = -1 \}.$$

Let x be an arbitrary point of τ. Then for $j \in J_1$,

$$x_j - f_j(x) \geq u_j^j - f_j(u^j) - (\delta + \varepsilon) \geq -(\delta + \varepsilon)$$

and

$$x_j - f_j(x) \leq v_j - f_j(v) + (\delta + \varepsilon) \leq \delta + \varepsilon.$$

Hence

$$\mid x_j - f_j(x) \mid \leq \delta + \varepsilon \quad \text{for any } j \in J_1. \qquad (2.6)$$

For $j \in J_2$, since $f_j(v) \geq -1 = v_j$ from Condition 2.1, we have

$$x_j - f_j(x) \leq v_j - f_j(v) + \delta + \varepsilon \leq \delta + \varepsilon,$$

and

$$x_j - f_j(x) \geq -1 - f_j(u^j) - \varepsilon \geq -1 - u_j^j - \varepsilon$$
$$\geq -1 - v_j - \delta - \varepsilon = -(\delta + \varepsilon).$$

Therefore we obtain that

$$\mid x_j - f_j(x) \mid \leq \delta + \varepsilon \quad \text{for any } j \in J_2. \qquad (2.7)$$

Since $I(k) = J_1 \cup J_2$, the lemma follows from (2.6) and (2.7). Q.E.D.

Since $f : C^n \to R^n$ is a continuous mapping on a compact set C^n, f is uniformly continuous on C^n. Therefore we can make ε arbitrarily small by choosing a triangulation with a sufficiently small mesh size δ.

3. EXISTENCE OF AN n-COMPLETE SIMPLEX AND A FIXED POINT

In this section we show that T always has an n-complete simplex if $f : C^n \to R^n$ satisfies Condition 2.1. For this purpose we define an adjacency relation among simplices of $\bar{T}$, and show that the 0-dimensional simplex $\{0\}$ is linked to an n-complete simplex by a sequence of adjacent simplices. Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a given sequence of -1's and +1's.

Definition 3.1. Let σ be a simplex of $T(k)$ and let τ be its facet, i.e. $\tau < \sigma$ and $\dim \tau = \dim \sigma - 1$. τ is a complete facet of σ if τ is a complete simplex and

$$\begin{aligned} k &\notin L(\tau) \quad \text{if} \quad \sigma \in T(k, \gamma_k), \\ k &\in L(\tau) \quad \text{if} \quad \sigma \in T(k, -\gamma_k). \end{aligned} \tag{3.1}$$

We write $\tau \lessdot \sigma$ when τ is a complete facet of σ.

Lemma 3.2. Let σ be a simplex of $T(k)$. If σ is a complete simplex, then it has a unique complete facet.

Proof. Let $\{ u^i : i \in I(k+1) \}$ be the vertex set of σ. We assume without loss of generality that $L(u^i) = i$ for $i \in I(k)$ and $L(u^{k+1}) = m$ for some $m \in I(n+1) \setminus I(k)$. Let τ^i be a facet of σ which does not have the vertex u^i. Then for $i \in I(k-1)$, τ^i is neither a complete simplex, hence nor a complete facet of σ. If $\sigma \in T(k, \gamma_k)$, τ^k is the unique complete facet of σ and if $\sigma \in T(k, -\gamma_k)$, τ^{k+1} is the unique complete facet of σ. Q.E.D.

Lemma 3.3. Let σ be a simplex of $T(k)$. If σ is a (k-1)-complete but not k-complete simplex, then σ has either exactly two or no complete facets.

Proof. Let $u^1, u^2, \ldots, u^k$ and v be the vertices of σ. We assume without loss of generality that $L(u^i) = i$ for $i \in I(k-1)$. To prove the lemma we have only to show the following two assertions:

(a) If $L(u^k) = k$ and $L(v) \le k$, then σ has no and exactly two complete facets when $\sigma \in T(k, \gamma_k)$ and $\sigma \in T(k, -\gamma_k)$, respectively.

(b) If $L(u^k) \ge k+1$ and $L(v) \ne k$, then σ has exactly two and no complete simplices when $\sigma \in T(k, \gamma_k)$ and $\sigma \in T(k, -\gamma_k)$, respectively.

Let τ be any complete facet of σ. Note that $\dim \tau = k-1$ and τ has k vertices of $\{ u^1, u^2, \ldots, u^k, v \}$. We first show the assertion (a). When $\sigma \in T(k, \gamma_k)$, $k \notin L(\tau)$ by (3.1) in Definition 3.1. Therefore $L(\tau) \subset I(k-1)$, which is contrary to the fact that $\tau \lhd \sigma$. When $\sigma \in T(k, -\gamma_k)$, let u^i be the vertex with $L(u^i) = L(v)$. Then both $\{ u^1, u^2, \ldots, u^k \}$ and $\{ u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^k, v \}$ forms complete facets of σ. Since it is clear that σ has no other complete facets, the assertion (a) follows.

The assertion (b) can be seen by the similar arguments and the proof is omitted.

Q.E.D.

<u>Lemma 3.4.</u> Let τ be a complete simplex of $T(k-1)$. Then there exists a unique simplex $\sigma \in T(k)$ such that $\tau \lhd \sigma$.

<u>Proof.</u> Since τ is a complete simplex, $L(\tau) = I(k-1) \cup \{m\}$ for some $m \in I(n+1) \setminus I(k-1)$. By the construction of $T(k)$, there exist exactly two simplices $\sigma_+ \in T(k, \gamma_k)$ and $\sigma_- \in T(k, -\gamma_k)$ which have τ as a facet. From Definition 3.1, σ_+ is the desired simplex if $m \ne k$, and σ_- is the desired simplex if $m = k$.

Q.E.D.

<u>Definition 3.5.</u> Let τ and τ' be two distinct simplices of $\bar{T}$. τ and τ' are <u>adjacent</u> when one of the following three cases occurs:

(i) τ and τ' are complete facets of the same simplex $\sigma \in T(k)$,

(ii) τ is a complete simplex of $T(k)$ and $\tau' \lhd \tau$,

(iii) τ' is a complete simplex of $T(k)$ and $\tau \lhd \tau'$.

We write $\tau \sim \tau'$ if τ and τ' are adjacent.

Lemma 3.6. Let τ be a simplex of $\bar{T}$. If τ is a complete facet of no simplices of $\bar{T}$, then τ has no adjacent simplices.

Proof. We have only to show that the case (ii) in Definition 3.5 will never occur. If $\tau \notin T(k)$ for any $k \in I(n)$, the lemma is immediate (see Definition 3.1). If $\tau \in T(k)$, then τ is not a complete simplex. Since otherwise by Lemma 3.4 there would exist a simplex $\sigma \in T(k+1)$ such that $\tau \triangleleft \sigma$. This is contrary to the assumption. Hence the case (ii) will not occur. Q.E.D.

Lemma 3.7. Let τ be a complete facet of some simplex $\sigma \in T(k)$. If $\tau \neq \{0\}$ and $\tau \not\subset \text{bd } C^n$, then τ has exactly two adjacent simplices of $\bar{T}$.

Proof. Since τ is a complete simplex, σ is a (k-1)-complete simplex. If σ is a complete simplex, then $\tau \sim \sigma$ because the case (iii) in Definition 3.5 occurs. By Lemma 3.2, σ has no other complete facets. If σ is not a complete simplex, by Lemma 3.3, σ has exactly two complete facets τ and τ'. Hence $\tau \sim \tau'$ because the case (i) in Definition 3.5 occurs. Thus we have shown that τ has an adjacent simplex, which is either σ or another complete facet τ' of σ.

To show that τ has another adjacent simplex, we consider the following two cases:

(a) $\tau \notin T(k-1)$,

(b) $\tau \in T(k-1)$.

Since $\tau \not\subset \text{bd } C^n$, there is a unique simplex $\sigma' \in T(k)$ such that $\tau < \sigma'$ and $\sigma' \neq \sigma$. In the case (a), $\sigma \in T(k, \alpha)$ if and only if $\sigma' \in T(k, \alpha)$. Hence by applying the same argument to σ' as above, we see that τ has another adjacent simplex. Note that τ has no complete facets because $\tau \notin T(k-1)$. Hence we have seen that τ has exactly two adjacent simplices in the case (a).

In the case (b), we see that τ has a unique complete facet η, by Lemma 3.2.

Hence $\tau \sim \eta$. By Lemma 3.4, we also see that σ is the unique simplex having τ as a complete facet. Thus we have seen that τ has exactly two adjacent simplices in the case (b).

Q.E.D.

Lemma 3.2, 3.4, 3.6 and 3.7 are summarized in the following theorem.

Theorem 3.8. Let τ be a simplex of $\bar{T}$ such that $\tau \not\subset \text{bd } C^n$. If τ is neither a 0-dimensional simplex $\{0\}$ nor an n-complete simplex, it has either no or exactly two adjacent simplices. Furthermore, the 0-dimensional simplex $\{0\}$ and each n-complete simplex have only one adjacent simplex.

From the proceeding discussions we have seen that we reach either an n-complete simplex of T or a simplex of $\bar{T}$ lying in bd C^n by generating a sequence of adjacent simplices which starts from the 0-dimensional simplex $\{0\}$. It follows from Lemma 3.6 that each simplex of the sequence is a complete facet of some simplex of $\bar{T}$. Therefore if any simplex of $T(k)|\text{bd } C^n = \{ \tau \in \bar{T} : \tau < \sigma \text{ for some } \sigma \in T(k),\ \tau \subset \text{bd } C^n,\ \dim \tau = k-1 \}$ is not a complete facet of any simplex $\sigma \in T(k)$, the sequence leads to an n-complete simplex, a simplex having an approximate fixed point (see Lemma 2.2).

Definition 3.9. The labelling function $L : C^n \to I(n+1)$ is proper if there is no (k-1)-dimensional simplex τ of $T(k)|\text{bd } C^n$ such that $\tau \lhd \sigma$ for some $\sigma \in T(k)$, i.e.

(i) for each simplex $\tau \in T(k, \gamma_k)|\text{bd } C^n = \{ \tau \in \bar{T} : \tau < \sigma \text{ for some } \sigma \in T(k, \gamma_k),\ \tau \subset \text{bd } C^n,\ \dim \tau = k-1 \}$, $L(\tau) \neq I(k-1) \cup \{m\}$ for any $m \in I(n+1) \cup I(k)$, and

(ii) for each simplex $\tau \in T(k, -\gamma_k)|\text{bd } C^n$, $L(\tau) \neq I(k)$.

Lemma 3.10. Suppose that f satisfies Condition 2.1. If $\gamma_k = 1$ for all $k \in I(n)$, then the labelling function L defined by (2.1) and (2.2) is proper.

Proof. Let τ be a simplex of $T(k, \alpha)|\text{bd } C^n$ for some $k \in I(n)$ and for some $\alpha \in \{\pm 1\}$ and let $u^1, u^2, \ldots, u^{k-1}, v$ be the vertices of τ. We assume without loss of generality that $L(u^i) = i$ for $i \in I(k-1)$. We show that v does not have such a label as to let τ be a complete facet of some $\sigma \in T(k, \alpha)$. Since $\tau \in T(k, \alpha)|\text{bd } C^n$, $\tau \subset B^n(h, \beta)$ for some $h \in I(k)$ and some $\beta \in \{\pm 1\}$. If $h \in I(k-1)$, then $\beta = 1$ since otherwise u^h could not have the label h (see the definition (2.1) and (2.2) of L). Since f satisfies Condition 2.1, we have $L(v) \leq h \leq k-1$, and consequently $L(\tau) \subset I(k-1)$. Hence in this case τ is not a complete simplex. If $h = k$, then $\beta = \alpha$ because $\tau \in T(k, \alpha)|B^n(k, \beta)$. When $\alpha = 1$, $v_k = 1$. Since $f(B^n(k, 1)) \subset H^n(k, 1)$, we have $L(v) \leq k$, i.e. $L(\tau) \subset I(k)$. Hence the condition (i) of Definition 3.9 holds. When $\alpha = -1$, $v_k = -1$. Therefore by the definition (2.1) and (2.2) of L, $L(v) \neq k$ and $L(\tau) \neq I(k)$. Hence the condition (ii) of Definition 3.9 holds. Q.E.D.

Theorem 3.11. If f satisfies Condition 2.1, then $\mathcal{T}$ has at least one n-complete simplex.

Proof. By Lemma 3.10, the labelling function L is proper if $\gamma_k = 1$ for all $k \in I(n)$. Therefore the sequence of adjacent simplices starting from $\{0\}$ always leads to an n-complete simplex. Q.E.D.

Corollary 3.12. If f maps bd C^n into C^n, then $\mathcal{T}$ has at least one n-complete simplex.

Proof. $f(B^n(k, \alpha)) \subset f(\text{bd } C^n) \subset C^n \subset H^n(k, \alpha)$ for any $k \in I(n)$ and any $\alpha \in \{\pm 1\}$. Q.E.D.

If f satisfies Condition 2.1, we can generate a sequence of arbitrarily small n-complete simplices by taking triangulations of C^n finer and finer. Since C^n is compact, the simplices generated have at least one accumulation point in C^n, which is a fixed point of f. Hence we obtain the following theorem.

Theorem 3.13. Let f be a continuous mapping from C^n into R^n satisfying Condition 2.1. Then f has a fixed point. Especially, if f maps bd C^n into C^n, then f has a fixed point.

4. THE 2-RAY METHOD WITH INTEGER LABELLING

As we have seen in the proof of Theorem 3.11, we have only to generate a sequence of adjacent simplices starting from the 0-dimensional simplex $\{0\}$ in order to obtain an n-complete simplex. There is no guarantee that the sequence leads to an n-complete simplex if some γ_k is -1, i.e. it possibly bumps against the boundary of C^n. But we describe the method without assuming that $\gamma_k = 1$ for all $k \in I(n)$.

The 2-ray method with integer labelling

Step 0 (initialization) : $p := 0$, $\tau_p := \emptyset$, $v^+ := 0$.

Step 1 (increasing the dimension) : If $\tau_p \cup \{v^+\}$ is a complete simplex, then $\tau_{p+1} := \tau_p \cup \{v^+\}$. Otherwise go to Step 4.

Step 2 (termination) : If $\dim \tau_{p+1} = n$, stop. Otherwise go to Step 3.

Step 3 : $p := p+1$, $k := \dim \tau_p + 1$,

$$\alpha := \begin{cases} \gamma_k & \text{if } k \notin L(\tau_p), \\ -\gamma_k & \text{otherwise.} \end{cases}$$

Find a vertex v^+ of T such that $\tau_p \cup \{v^+\} \in T(k, \alpha)$. Go to Step 1.

Step 4 (replacement) : Find a vertex v^- of τ_p such that

$$L(v^-) = \begin{cases} L(v^+) & \text{if } L(v^+) \in L(\tau_p), \\ \max \{ L(u) : u \text{ is a vertex of } \tau_p \} & \text{otherwise.} \end{cases}$$

$\tau_{p+1} := \tau_p \cup \{v^+\} \setminus \{v^-\}$.

Step 5 : $p := p+1$. If $\tau_p \subset \text{bd } C^n$, then stop.

Step 6 : $k := \dim \tau_p$. If $\tau_p \in T(k, \beta)$ for some $\beta \in \{\pm 1\}$, go to Step 7. Otherwise go to Step 8.

Step 7 (decreasing the dimension) : Find a vertex v^- of τ_p such that

$$L(v^-) = \begin{cases} k & \text{if } \beta = \gamma_k, \\ \max \{ L(u) : u \text{ is a vertex of } \tau_p \} & \text{otherwise.} \end{cases}$$

$\tau_{p+1} := \tau_p \setminus \{v^-\}$, $\alpha := \beta$, and go to Step 5.

Step 8 : $k := \dim \tau_p + 1$. Find a vertex v^+ of T such that $\tau_p \cup \{v^+\} \in T(k, \alpha)$ and $\tau_p \cup \{v^+\} \neq \tau_p \cup \{v^-\}$. Go to Step 1.

Now we shall show that the sequence $\{ \tau_p : p = 1, 2, \ldots \}$ generated by the method is a sequence of adjacent simplices. For this purpose we first show that each simplex τ_p is a complete simplex in the following lemma. In its proof we also see that the simplex τ_p in Step 7 always has a vertex with label k.

Lemma 4.1. The simplex τ_p generated by the method is a complete simplex for $p = 1, 2, \ldots$.

Proof. First note that $\tau_1 = \{0\}$ is a complete simplex. If τ_{p+1} is generated in Step 1, it is trivially a complete simplex. Assuming that τ_p is a complete simplex, we show that τ_{p+1} is also a complete simplex. First consider the simplex τ_{p+1} generated in Step 4. If $L(v^+) \in L(\tau_p)$, then $L(\tau_{p+1}) = L(\tau_p)$. Since $\dim \tau_{p+1} = \dim \tau_p$, we see that τ_{p+1} is also a complete simplex by the induction hypothesis. If $L(v^+) \notin L(\tau_p)$, then $L(v^+) \notin I(k)$ and $L(v^-) \notin I(k)$, where $k = \dim \tau_p$. Therefore $L(\tau_{p+1}) = I(k) \cup L(v^+)$, i.e. τ_{p+1} is a complete simplex. Next consider the simplex τ_{p+1} generated in Step 7. Note that $k \in L(\tau_p)$ since τ_p is a complete simplex, where $k = \dim \tau_p$. Therefore $L(\tau_{p+1}) = L(\tau_p) \setminus L(v^-) = I(k-1) \cup \{m\}$ for some $m \in I(n+1) \setminus I(k-1)$ when $\beta = \gamma_{k-1}$. Since $\dim \tau_{p+1} = k-1$, τ_{p+1} is proved to be a complete simplex. When $\beta \neq \gamma_{k-1}$, $L(\tau_{p+1}) = I(k)$; τ_{p+1} is a complete simplex. Q.E.D.

Lemma 4.2. $\tau_p \sim \tau_{p+1}$ for $p = 1, 2, \ldots$.

Proof. Since we have seen in Lemma 4.1 that each τ_p is a complete simplex, we have only to show the following three assertions:

(a) When τ_{p+1} is generated in Step 1, $\tau_p \triangleleft \tau_{p+1}$.

(b) When τ_{p+1} is generated in Step 7, $\tau_{p+1} \triangleleft \tau_p$.

(c) When τ_{p+1} is generated in Step 4, $\tau_p \triangleleft \sigma$ and $\tau_{p+1} \triangleleft \sigma$ for some σ.

(i) Consider the case where v^+ is found in Step 3 and τ_{p+1} is generated in Step 1. Since α is defined according to the rule in Step 3 and $\tau_{p+1} = \tau_p \cup \{v^+\} \in T(k, \alpha)$, we easily see that $\tau_p \lhd \tau_{p+1}$.

(ii) Consider the simplex τ_{p+1} generated in Step 7. Since τ_p is a complete simplex, $L(\tau_p) = I(k) \cup \{m\}$ for some $m \in I(n+1) \setminus I(k)$. By the choice rule of v^-, we have

$$L(\tau_{p+1}) = \begin{cases} I(k-1) \cup \{m\} & \text{if } \beta = \gamma_k, \\ I(k) & \text{otherwise.} \end{cases}$$

Therefore $k \notin L(\tau_{p+1})$ when $\tau_p \in T(k, \gamma_k)$ and $k \in L(\tau_{p+1})$ when $\tau_p \in T(k, -\gamma_k)$. Hence $\tau_{p+1} \lhd \tau_p$.

(iii) Consider the case where the vertex v^+ is found in Step 3 and τ_{p+1} is generated in Step 4. Let $\sigma = \tau_p \cup \{v^+\}$. Then we see that $\tau_p \lhd \sigma$ in the same way as in (i). If $L(v^+) \in L(\tau_p)$, then $L(\tau_{p+1}) = L(\tau_p)$, which implies that $\tau_{p+1} \lhd \sigma$. If $L(v^+) \notin L(\tau_p)$, we will see that $k+1 \notin L(\tau_p)$, where $k = \dim \tau_p$. Suppose the contrary, then we see that $L(\sigma) = I(k+1) \cup L(v^+)$ because τ_p is a complete simplex. This contradicts the fact that σ is not a complete simplex. Therefore we have that $L(\tau_p) = I(k) \cup \{m\}$ for some $m \in I(n+1) \setminus I(k+1)$. We also have that $L(v^+) \neq k+1$, since otherwise σ would be a complete simplex again. Hence $k+1 \notin L(\tau_{p+1})$ and $\tau_{p+1} \lhd \sigma$.

(iv) Consider the case where v^+ is found in Step 8 and τ_{p+1} is generated in Step 4. Let $\sigma = \tau_p \cup \{v^+\}$. We shall show that $\tau_p \lhd \sigma$. Then the same argument as in (iii) yields the desired result that $\tau_{p+1} \lhd \sigma$. Let τ_q be the last simplex generated in either Step 1 or Step 7 before τ_{p+1}. Then there is a sequence $\{ \tau_i : i = q, q+1, \dots, p \}$ of $(k-1)$-dimensional simplices and $\{ \sigma_i : i = q, q+1, \dots, p-1 \}$ of k-dimensional simplices of $T(k, \alpha)$ such that $\tau_i \lhd \sigma_i$ and $\tau_{i+1} \lhd \sigma_i$ for $i = q, q+1, \dots, p-1$. If τ_q was generated in Step 1, we see that $\tau_q \triangleleft \sigma_q$ in the same way as in (iii). If τ_q was generated in Step 7, $\tau_q \triangleleft \tau_{q-1}$ as shown in (ii). Since τ_{q-1}, $\sigma_q \in T(k, \alpha)$ by the choice of v^+ in Step 8, we also obtain $\tau_q \triangleleft \sigma_q$. Then by the same argument as in (iii) we have $\tau_{q+1} \triangleleft \sigma_q$. Since σ_{q+1} is also a simplex of $T(k, \alpha)$, it is clear that $\tau_{q+1} \triangleleft \sigma_{q+1}$. Therefore we have $\tau_{q+2} \triangleleft \sigma_{q+1}$ again by the same argument as in (iii). Repeating this argument we at last have $\tau_p \triangleleft \sigma_{p-1}$. Since v^+ was found in Step 8, σ is also a simplex of $T(k, \alpha)$. Hence we see that $\tau_p \triangleleft \sigma$.

(v) Consider the case where v^+ is found in Step 8 and τ_{p+1} is generated in Step 1. In this case we can see that $\tau_p \triangleleft \tau_{p+1}$ in the similar way as in (iv).

Q.E.D.

Theorem 4.3. Suppose that $f : C^n \to R^n$ satisfies Condition 2.1. If $\gamma_k = 1$ for all $k \in I(n)$, the method always generates an n-complete simplex.

Proof. The theorem directly follows from the proof of Theorem 3.11 and Lemma 4.2.

Q.E.D.

Remark 4.4. When the method generates simplices in $C^n(k)$, the function value $f_j(.)$ does not affects the behavior of the method for $j \in I(n) \setminus I(k)$. Moreover all that we need in Step 3 is whether $L(\tau_p)$ has the label k or not. Therefore we can save the function evaluations by the following modifications: For each $k \in I(n)$ let

$$J_k(x) = \{ j \in I(k) : x_j \geq f_j(x), \quad x_j \neq -1 \} \cup \{n+1\},$$

$$L_k(x) = \min \{ j : j \in J_k(x) \}.$$

As long as the method generates (k-1)-dimensional simplices in $C^n(k)$, we employ L_k instead of L as the labelling function. Let τ_p be a complete simplex of $T(k)$ found by the method with L_k. Then τ_p always has a vertex v with the label $n+1$. Before choosing α in Step 3, we evaluate the (k+1)-st component of $f(v)$ and relabel the vertex v accoring to the labelling function L_{k+1}. The other parts remain unchanged.

5. EXAMPLE

Figure 5.1 shows the sets $\{ x \in C^2 : L(x) = k \}$ for $k = 1, 2, 3$ and the sequences of simplices generated by the method. We have not pictured the triangulation for the sake of clarity. If $\Gamma = \{1, 1\}$, the sequence of simplices generated follows ——>——. Since the labelling function in this figure is proper, the sequence finally reaches an n-complete simplex. The sequences for $\Gamma = \{1, -1\}$, $\{-1, 1\}$ and $\{-1 -1\}$ follow - - >- - , —·>·—, and —··>··—, respectively. Note that the sequence for $\{-1, 1\}$ collides against bd C^2 and that the sequence for $\{-1, -1\}$ reaches the same n-complete simplex as the sequence for $\{1, i\}$ does.

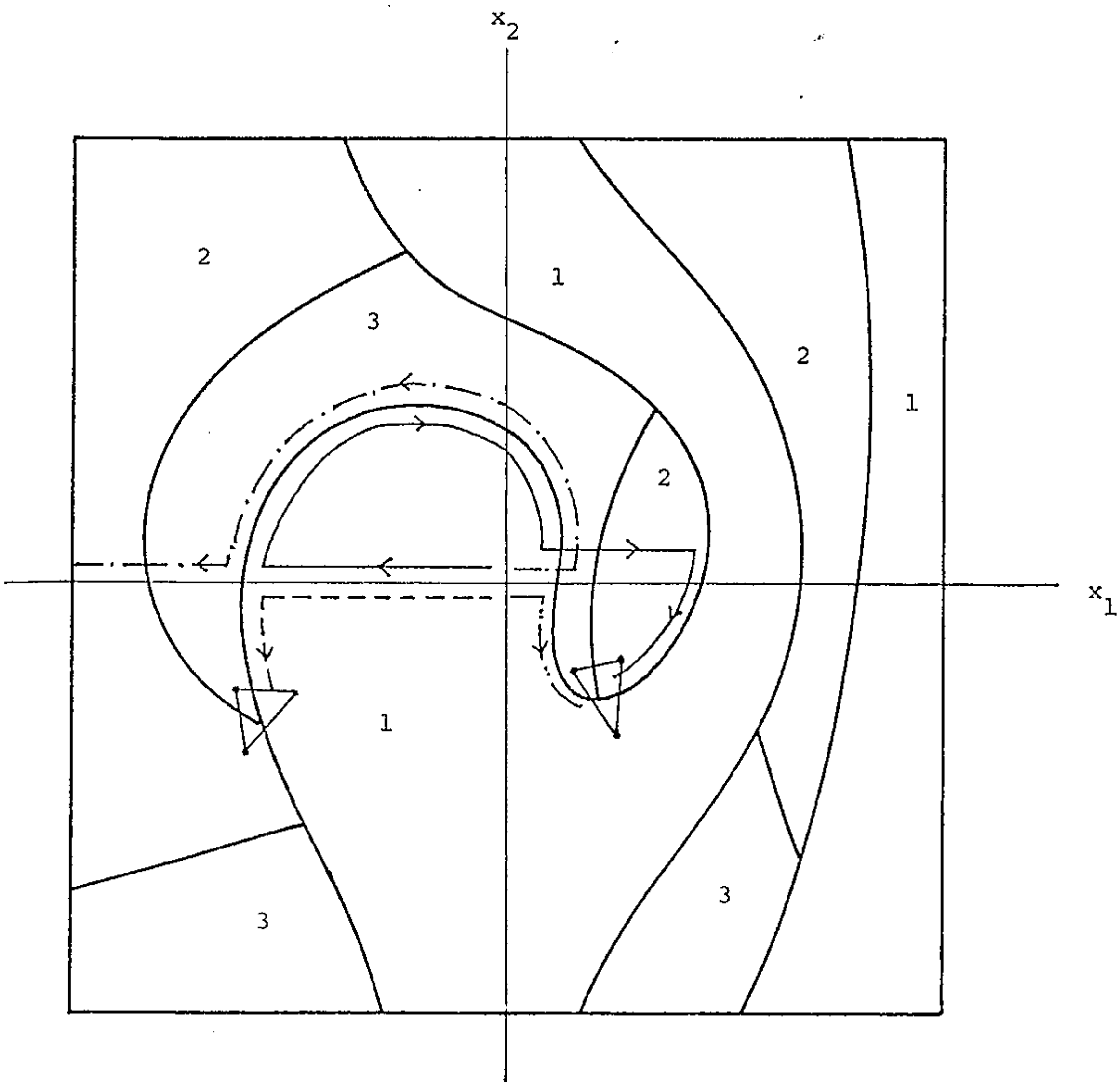

Figure 5.1

6. CONTINUOUS MAPPINGS ON A COMPACT CONVEX SET AND ROTHE'S FIXED POINT THEOREM

Let D be a compact convex subset of R^n and let $g : D \to R^n$ be a continuous mapping. We here assume without loss of generality that $\dim D = n$ and $D \subset C^n$. In this section we show how to apply the 2-ray method with integer labelling to approximating a fixed point of g if exist. The basic idea is to make a continuous extension $f : C^n \to R^n$ of g such that $f(x) = g(x)$ for $x \in D$ and an approximate fixed point of f provides an approximate fixed point of g. There may be various ways of making f; we, however, confine ourselves to the following one which utilizes a projection mapping P onto D from the view point of approximation accuracy.

For each point $x \in C^n$ let $P(x)$ be the closest point in D to x, i.e.

$$\| x - P(x) \| = \min \{ \| x - y \| : y \in D \}. \tag{6.1}$$

Since D is a compact convex subset of C^n, $P(x)$ is well defined for each $x \in C^n$ and $P : C^n \to D$ is continuous. Let $f : C^n \to R^n$ be

$$f(x) = g(P(x)) \quad \text{for} \quad x \in C^n. \tag{6.2}$$

Note that f is also continuous.

Lemma 6.1. The projection mapping P is nonexpansive, i.e.

$$\| P(x) - P(y) \| \le \| x - y \| \quad \text{for} \quad x, y \in C^n.$$

Proof. Let x and y be arbitrary points of C^n. If $P(x) = P(y)$, the lemma is trivial. We assume that $P(x) \neq P(y)$. By the definition (6.1) of P, for any $\lambda \in [0, 1]$,

$$\begin{aligned} \| x - P(x) \|^2 &\le \| x - (\lambda P(x) + (1-\lambda)P(y)) \|^2 \\ &= \| x - P(x) + (1-\lambda)(P(x) - P(y)) \|^2 \\ &= \| x - P(x) \|^2 + 2(1-\lambda)(x - P(x))^t (P(x) - P(y)) \\ &\quad + (1-\lambda)^2 \| P(x) - P(y) \|^2. \end{aligned}$$

Therefore

$$(x - P(x))^t (P(y) - P(x)) \le (1/2)(1 - \lambda) \| P(x) - P(y) \|^2$$

holds for any $\lambda \in [0, 1)$. Hence we obtain

$$(x - P(x))^t (P(y) - P(x)) \le 0. \tag{6.3}$$

In the same way we also obtain

$$(y - P(y))^t (P(x) - P(y)) \le 0. \tag{6.4}$$

By (6.3) and (6.4)

$$\begin{aligned} \| P(x) - P(y) \|^2 &= (x - y + P(x) - x - P(y) + y)^t (P(x) - P(y)) \\ &= (x - y)^t (P(x) - P(y)) \\ &\quad + (P(x) - x)^t (P(x) - P(y)) \\ &\quad - (P(y) - y)^t (P(x) - P(y)) \\ &\le (x - y)^t (P(x) - P(y)) \\ &\le \| x - y \| \, \| P(x) - P(y) \|. \end{aligned} \tag{6.5}$$

Since $\| P(x) - P(y) \| \neq 0$, the desired result follows from (6.5). Q.E.D.

<u>Lemma 6.2.</u> Let

$$\varepsilon = \max \{ \| f(x) - f(y) \|_\infty : x, y \in C^n, \| x - y \| \le \delta \}$$

$$\varepsilon' = \max \{ \| g(x) - g(y) \|_\infty : x, y \in D, \| x - y \| \le \delta \}.$$

Then $\varepsilon = \varepsilon'$.

<u>Proof.</u> By Lemma 6.1 we have

$$\begin{aligned} \varepsilon &= \max \{ \| f(x) - f(y) \|_\infty : x, y \in C^n, \| x - y \| \le \delta \} \\ &= \max \{ \| g(P(x)) - g(P(y)) \|_\infty : x, y \in C^n, \| x - y \| \le \delta \} \\ &\le \max \{ \| g(P(x)) - g(P(x)) \|_\infty : x, y \in C^n, \| P(x) - P(y) \| \le \delta \} \\ &= \max \{ \| g(x) - g(y) \|_\infty : x, y \in D, \| x - y \| \le \delta \} = \varepsilon'. \end{aligned}$$

Since it is clear that $\varepsilon' \le \varepsilon$, the lemma follows. Q.E.D.

Let σ be an n-complete simplex with respect to the labelling function induced by f. Then

$$\| P(x) - g(P(x)) \|_\infty = \| P(x) - f(x) \|_\infty$$
$$\leq \| P(x) - x \|_\infty + \| x - f(x) \|_\infty$$
$$\leq \| P(x) - x \|_\infty + \delta + \varepsilon, \qquad (6.6)$$

for any point $x \in \sigma$. Therefore if $\sigma \cap D \neq \emptyset$, we have

$$\| x - g(x) \|_\infty \leq \delta + \varepsilon \quad \text{for any} \quad x \in \sigma \cap D;$$

an approximate fixed point of g. However, an n-complete simplex does not always have an intersection with D. The following lemma gives the upper bound of $\| P(x) - x \|_\infty$ in (6.6).

Lemma 6.3. Suppose that g maps bd D into D. If σ is an n-complete simplex with respect to the labelling function induced by f, then

$$\sigma \cap U(D, \delta + \varepsilon) \neq \emptyset,$$

where $U(D, \mu) = \{ x \in R^n : \| x - P(x) \|_\infty \leq \mu \}$.

Proof. Let $u^1, u^2, \ldots, u^n$ and v be the vertices of σ which are numbered as $L(u^i) = i$ for $i \in I(n)$ and $L(v) = n+1$. Since the lemma is trivial if $\sigma \cap D \neq \emptyset$, we assume that $\sigma \cap D = \emptyset$. Since $u_i^i \geq f_i(u^i)$ for $i \in I(n)$, we see that

$$v_i - f_i(v) = v_i - u_i^i + u_i^i - f_i(u^i) + f_i(u^i) - f_i(v)$$
$$\geq - | v_i - u_i^i | + (u_i^i - f_i(u^i)) - | f_i(u^i) - f_i(v) |$$
$$\geq - (\delta + \varepsilon) \qquad (6.7)$$

holds for $i \in I(n)$. Since $f(C^n \setminus D) \subset g(\text{bd } D) \subset D \subset C^n$, we also see

$$v_i - f_i(v) \leq 0 \quad \text{for} \quad i \in I(n). \qquad (6.8)$$

Hence by (6.7) and (6.8)

$$\| v - f(v) \|_\infty \leq \delta + \varepsilon.$$

Since $f(v) \in f(\sigma) \subset f(C^n \setminus D) \subset D$, we see that $v \in U(D, \delta + \varepsilon)$. Q.E.D.

Thus we can find a point x of an n-complete simplex σ such that its projection $y = P(x)$ satisfies

$$\| y - g(y) \|_\infty \leq 2 (\delta + \varepsilon).$$

By the same argument as we have made before Theorem 3.13, we have the following Euclidean space version of Rothe's fixed point theorem [12] (see also Smart [14]).

Theorem 6.4. Let D be a compact convex subset of R^n and $g : D \to R^n$ be a continuous mapping which maps bd D into D. Then g has a fixed point.

7. THE ORIENTATION OF SIMPLICES

In this section we define the orientation of a complete simplex and characterize the orientations of n-complete simplices obtained by the 2-ray method.

Definition 7.1. Let $\tau \in T(k)$ be a complete simplex and let $u^1, u^2, \ldots, u^{k+1}$ be its vertices, where they are numbered so that $L(u^i) = i$ for $i \in I(k)$ and $L(u^{k+1}) = m$ for some $m \in I(n+1) \setminus I(k)$. Let

$$W(\tau) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ u^1_{(k)} & u^2_{(k)} & \cdots & u^{k+1}_{(k)} \end{bmatrix},$$

where $u^i_{(k)}$ is the k-dimensional vector consisting of the fist k components of u^i for $i \in I(k+1)$. Then the orientation of τ, denoted by $or(\tau)$, is

$$or(\tau) = (-1)^k \text{ sign det } W(\tau). \tag{7.1}$$

Definition 7.2. Let σ be a simplex of $T(k, \alpha)$ for some $\alpha \in \{\pm 1\}$ and τ be its complete facet. Let $u^1, u^2, \ldots, u^{k+1}$ be the vertices of σ, where they are numbered so that $L(u^i) = i$ for $i \in I(k-1)$, $L(u^k) = m$ for some $m \in I(n+1) \setminus I(k-1)$ and $u^{k+1} \notin \tau$. Let

$$W(\tau;\, \sigma) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ u^1_{(k)} & u^2_{(k)} & \cdots & u^k_{(k)} & u^{k+1}_{(k)} \end{bmatrix}.$$

Then the orientation of τ with respect to σ, denoted by $or(\tau;\, \sigma)$, is

$$or(\tau;\, \sigma) = (-1)^{k-1}\, \alpha \text{ sign det } W(\tau;\, \sigma). \tag{7.2}$$

Lemma 7.3. Let $\sigma \in T(k)$ be a complete simplex. If $\tau \lhd \sigma$, then

$$or(\sigma) = \gamma_k\, or(\tau;\, \sigma).$$

Proof. If $\sigma \in T(k, -\gamma_k)$, then $L(\tau) = I(k)$ and $L(\sigma) = I(k) \cup \{m\}$ for some $m \in I(n+1) \setminus I(k)$. Therefore $W(\sigma) = W(\tau;\, \sigma)$. This yields the desired result. If $\sigma \in T(k, \gamma_k)$, $L(\tau) = I(k-1) \cup \{m\}$ and $L(\sigma) = I(k) \cup \{m\}$ for some $m \in I(n+1) \setminus I(k)$. Therefore sign det $W(\sigma) = -$ sign det $W(\tau;\, \sigma)$.

Hence

$$\gamma_k \operatorname{or}(\tau;\ \sigma) = \gamma_k\ (-1)^{k-1}\ \gamma_k \text{ sign det } W(\tau;\ \sigma)$$
$$= (-1)^k \text{ sign det } W(\sigma) = \operatorname{or}(\sigma). \qquad \text{Q.E.D.}$$

Lemma 7.4. Let $\sigma \in T(k)$ and $\tau \in T(k-1)$. If $\tau \triangleleft \sigma$, then

$$\operatorname{or}(\tau) = \operatorname{or}(\tau;\ \sigma).$$

Proof. Let $\{ u^i : i \in I(k+1) \}$ and $\{ u^i : i \in I(k) \}$ be the sets of vertices of σ and τ, respectively. Here we assume without loss of generality that $L(u^i) = i$ for $i \in I(k-1)$ and $L(u^k) = m$ for some $m \in I(n+1) \setminus I(k-1)$. Since $\sigma \in T(k,\ \alpha)$ for some $\alpha \in \{\pm 1\}$ and $\tau \in T(k-1)$, $u_k^i = 0$ for $i \in I(k)$ and $\alpha\ u_k^{k+1} > 0$. Therefore

$$\operatorname{or}(\tau) = (-1)^{k-1} \text{ sign det } W(\tau)$$
$$= (-1)^{k-1}\ \alpha \text{ sign det} \left[\begin{array}{ccc|c} 1 & \cdots & 1 & \\ u^1_{(k-1)} & \cdots & u^k_{(k-1)} & w \\ \hline 0 & \cdots & 0 & \alpha \end{array}\right]$$
$$= (-1)^{k-1}\ \alpha \text{ sign det } W(\tau;\ \sigma) = \operatorname{or}(\tau;\ \sigma),$$

where w is an arbitrary k-dimensional vector. Q.E.D.

Lemma 7.5. Let τ_1 and τ_2 be two distinct complete facets of $\sigma \in T(k)$. Then

$$\operatorname{or}(\tau_1;\ \sigma) = -\operatorname{or}(\tau_2;\ \sigma).$$

Proof. Let v^i be the vertex of $\sigma \setminus \tau_i$ for $i = 1, 2$. If $L(v^1) = L(v^2)$, $W(\tau_1;\ \sigma)$ coincides with $W(\tau_2;\ \sigma)$ except that the $L(v^1)$-th column and the (k+1)-st column are exchanged. If $L(v^1) \neq L(v^2)$, then $k \leq L(v^i)$ for $i = 1, 2$. Since otherwise either τ_1 or τ_2 would not be a complete facet of σ. Therefore $W(\tau_1;\ \sigma)$ coincides with $W(\tau_2;\ \sigma)$ except that the last two columns are exchanged. Hence in either case we have sign det $W(\tau_1;\ \sigma) = -$ sign det $W(\tau_2;\ \sigma)$.

Q.E.D.

Lemma 7.6. If σ_1 and $\sigma_2 \in T(k)$ share a common complete facet τ, then

$$or(\tau;\ \sigma_1) = -\ or(\tau;\ \sigma_2).$$

Proof. First note that $\sigma_1,\ \sigma_2 \in T(k,\ \alpha)$ for some $\alpha \in \{\pm 1\}$. Therefore we have only to see that sign det $W(\tau;\ \sigma_1) = -$ sign det $W(\tau;\ \sigma_2)$. Let v^i be the vertex of $\sigma_i \setminus \tau$ for i = 1, 2. Let $aff(\tau)$ be the affine subspace spanned by τ and $tng^*(\tau)$ be the orthogonal complement of the tangential space $aff(\tau) - \tau$ of τ. Since v^1 and v^2 lie on the opposite sides of $aff(\tau)$, we can find a nonzero vector $s \in C^n(k) \cap tng^*(\tau)$, $x^1,\ x^2 \in aff(\tau)$, and scalars $\lambda_1 > 0$ and $\lambda_2 < 0$ such that

$$v^i = x^i + \lambda_i s \quad \text{for} \quad i = 1,\ 2.$$

Hence by suitably defining a $(k+1) \times k$ matrix W we see that

$$\begin{aligned}
\text{sign det } W(\tau;\ \sigma_1) &= \text{sign det} \left[\ W \ \middle|\ \begin{matrix} 1 \\ v^1_{(k)} \end{matrix} \right] \\
&= \text{sign det} \left[\ W \ \middle|\ \begin{matrix} 1 \\ x^1_{(k)} + \lambda_1 s_{(k)} \end{matrix} \right] \\
&= \text{sign det} \left[\ W \ \middle|\ \begin{matrix} 0 \\ \lambda_1 s_{(k)} \end{matrix} \right] \\
&= -\ \text{sign det} \left[\ W \ \middle|\ \begin{matrix} 0 \\ \lambda_2 s_{(k)} \end{matrix} \right] \\
&= -\ \text{sign det} \left[\ W \ \middle|\ \begin{matrix} 1 \\ x^2_{(k)} + \lambda_2 s_{(k)} \end{matrix} \right] \\
&= -\ \text{sign det } W(\tau;\ \sigma_2).
\end{aligned}$$

Q.E.D.

Corollary 7.7. Let $\eta \in T(k-1)$ and $\eta' \in T(k)$ be complete simplices. If there is a sequence of adjacent simplices $\{\ \tau_q : q = 1,\ 2, \ldots,\ t\ \}$ such that

$\tau_1 = \eta$, $\tau_t = \eta'$ and $\dim \tau_q = k-1$ for $q = 2, 3, \ldots, t-1$, then

$$or(\eta') = \gamma_k or(\eta).$$

Proof. Since $\tau_q \sim \tau_{q+1}$ for $q = 1, 2, \ldots, t-1$, there is a sequence of simplices $\tau_1, \sigma_1, \tau_2, \ldots, \sigma_{t-2}, \tau_{t-1}, \tau_t$ such that $\tau_q \lhd \sigma_q$, $\tau_{q+1} \lhd \sigma_q$ for $q = 1, 2, \ldots, t-2$ and $\tau_{t-1} \lhd \tau_t$. Then

$$\begin{aligned}
or(\eta') &= \gamma_k or(\tau_{t-1}; \tau_t) && \text{(by Lemma 7.3)}\\
&= -\gamma_k or(\tau_{t-1}; \sigma_{t-2}) && \text{(by Lemma 7.6)}\\
&= \gamma_k or(\tau_{t-2}; \sigma_{t-2}) && \text{(by Lemma 7.5)}\\
&\ \ \vdots\\
&= \gamma_k or(\tau_1; \sigma_1)\\
&= \gamma_k or(\eta). && \text{(by Lemma 7.4)}
\end{aligned}$$

Q.E.D.

Corollary 7.8. Let η and η' be distinct complete simplices of $T(k)$. Suppose that there is a sequence of adjacent simplices $\{ \tau_q : q = 1, 2, \ldots, t \}$ such that $\tau_1 = \eta$, $\tau_t = \eta'$ and $\tau_q \not\in T(k)$ for $q \neq 1, t$. If all τ_q's except τ_1 and τ_t are of the same dimension, then

$$or(\eta') = - or(\eta).$$

Proof. First note that $\dim \tau_q = k$ or $k-1$ for all $q = 2, 3, \ldots, t-1$. If all τ_q's are k-dimensional simplices, there is a sequence of simplices $\tau_1, \sigma_1, \ldots, \sigma_{t-1}, \tau_t$ such that $\tau_q \lhd \sigma_q$ and $\tau_{q+1} \lhd \sigma_q$ for $q = 1, 2, \ldots, t-1$. Then

$$\begin{aligned}
or(\eta') &= or(\tau_t; \sigma_{t-1}) && \text{(by Lemma 7.4)}\\
&= - or(\tau_{t-1}; \sigma_{t-1}) && \text{(by Lemma 7.5)}\\
&= or(\tau_{t-1}; \sigma_{t-2}) && \text{(by Lemma 7.6)}\\
&\ \ \vdots\\
&= or(\tau_2; \sigma_1)\\
&= - or(\tau_1; \sigma_1)\\
&= - or(\eta). && \text{(by Lemma 7.4)}
\end{aligned}$$

If $\dim \tau_q = k-1$ for all $q = 2, 3, \ldots, t-1$, there is a sequence of simplices $\tau_1, \tau_2, \sigma_2, \ldots, \sigma_{t-2}, \tau_{t-1}, \tau_t$ such that $\tau_q \triangleleft \sigma_q$, $\tau_{q+1} \triangleleft \sigma_q$ for $q = 2, 3, \ldots, t-2$ and $\tau_2 \triangleleft \tau_1$ and $\tau_{t-1} \triangleleft \tau_t$. Then

$$
\begin{aligned}
or(\eta') &= \gamma_k or(\tau_{t-1}; \tau_t) && \text{(by Lemma 7.3)}\\
&= -\gamma_k or(\tau_{t-1}; \sigma_{t-2}) && \text{(by Lemma 7.6)}\\
&= \gamma_k or(\tau_{t-2}; \sigma_{t-2}) && \text{(by Lemma 7.5)}\\
&\ \vdots\\
&= \gamma_k or(\tau_2; \sigma_2)\\
&= -\gamma_k or(\tau_2; \tau_1)\\
&= -or(\eta). && \text{(by Lemma 7.3)}
\end{aligned}
$$

Q.E.D.

Corollary 7.9. Let η and η' be distinct complete simplices of $T(k)$. Suppose that there is a sequence of adjacent simplices $\{ \tau_q : q = 1, 2, \ldots, t \}$ such that $\tau_1 = \eta$, $\tau_t = \eta'$ and $\tau_q \not\in T(k)$ for $q \neq 1, t$. Then

$$or(\eta') = - or(\eta).$$

Proof. Let $\{ \eta_r : r = 1, 2, \ldots, s \}$ be a subsequence of $\{ \tau_q : q = 1, 2, \ldots, t \}$ such that each η_r is a simplex of some $T(k)$. Then $\dim \eta_{r+1} = \dim \eta_r$ or $\dim \eta_r \pm 1$. Let

$$R_+ = \{ r : \dim \eta_r = \dim \eta_{r-1} + 1 \},$$

$$R_- = \{ r : \dim \eta_r = \dim \eta_{r-1} - 1 \},$$

$$R_0 = \{ r : \dim \eta_r = \dim \eta_{r-1} \}.$$

Then clearly $|R_+| = |R_-|$, where $|R|$ means the number of elements in R. Since $\tau_q \not\in T(k)$ for $q \neq 1, t$, we see that the $|R_0|$ is odd. By Corollary 7.7

$$
\begin{aligned}
or(\eta_r) &= \gamma_k or(\eta_{r-1}) \quad \text{for } r \in R_+,\\
or(\eta_{r+1}) &= \gamma_k or(\eta_r) \quad \text{for } r \in R_-,
\end{aligned}
\tag{7.1}
$$

where $k = \dim \eta_r$. By Corollary 7.8

$$or(\eta_r) = - or(\eta_{r-1}) \quad \text{for } r \in R_0.$$

Since each γ_k appears even times in (7.1), we have

$$or(\eta') = (-1)^{|R_0|} or(\eta) = - or(\eta). \qquad Q.E.D.$$

Theorem 7.10. Let $k \in I(n)$ and let $\{ \eta_r : r = 1, 2, \ldots \}$ be the subsequence of the sequence generated by the method such that each η_r is a simplex of $T(k)$. Then

$$or(\eta_r) = (-1)^{r-1} \pi_k,$$

where $\pi_k = \Pi\{ \gamma_j : j = 1, 2, \ldots, k \}$.

Proof. We show the assertion by the induction over k. If $k = 1$,

$$W(\eta_1) = \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}$$

for some a and $b \in R^1$. It is easy to see that $a > b$ if $\gamma_1 = 1$ and $a < b$ if $\gamma_1 = -1$. Therefore $or(\eta_1) = (-1)^1 \text{sign det } W(\eta_1) = (-1)^0 \pi_1$. By Corollary 7.9, we have $or(\eta_r) = (-1)^{r-1} \pi_1$.

Assuming that the theorem is true for all $j \le k-1$, consider the sequence $\{ \eta_r : r = 1, 2, \ldots \}$ such that $\eta_r \in T(k)$. Since η_1 is generated in Step 1 of the method, we see by Corollary 7.7 that $or(\eta_1) = \gamma_k or(\eta)$, where η is the simplex of $T(k-1)$ which was generated just prior to η_1. It is readily seen that η occupies the odd position in the sequence of simplices of $T(k-1)$. Therefore by the induction hypothesis $or(\eta) = \pi_{k-1}$. Hence we have

$$or(\eta_1) = \pi_k. \qquad (7.2)$$

The theorem follows from (7.2) and Corollary 7.9. Q.E.D.

From Theorem 7.10 we obtain that the n-complete simplex generated by the method has the orientation π_n. Hence different sequences $\Gamma = \{ \gamma_1, \gamma_2, \ldots, \gamma_n \}$ may provide different n-complete simplices.

It is known that any fixed point algorithm with integer labelling can be regarded as one with a suitable vector labelling function. In the sequel we show that the 2-ray method with integer labelling function $L : C^n \to I(n+1)$ generates the same sequence of simplices as the 2-ray method with a certain vector labelling function does. For $k \in I(n)$, let $h^k = (h_1^k, h_2^k, \ldots, h_n^k)$ be a vector of R^n such that

$$h_j^k = \begin{cases} -1 & \text{if } j < k, \\ +1 & \text{if } j = k, \\ 0 & \text{if } j > k, \end{cases}$$

and let $E = \{ h^k : k \in I(n+1) \}$. Let $h : C^n \to E$ be a function defined by

$$h(x) = h^{L(x)} \qquad \text{for } x \in C^n.$$

Let $H : C^n \to R^n$ be a piecewise linear extension of h with respect to T. Then we have the following lemma.

Lemma 7.11. $\tau \in \bar{T}$ is a complete simplex if and only if τ has a point x such that $H_{(k)}(x) = 0$, where $k = \dim \tau$.

Proof. We first show the "only if" part. Let $u^1, u^2, \ldots, u^{k+1}$ be the vertices of τ and we assume without loss of generality that $L(u^i) = i$ for $i \in I(k)$ and $L(u^{k+1}) = m$ for some $m \in I(n+1) \setminus I(k)$. Let $\lambda_i = 2^{-i}$ for $i \in I(k)$ and $\lambda_{k+1} = 2^{-k}$. Then it is readily seen that

$$\Sigma \{ \lambda_i : i \in I(k+1) \} = 1,$$

$$\Sigma \{ \lambda_i h_{(k)}(u^i) : i \in I(k+1) \} = 0.$$

Therefore $x = \Sigma \{ \lambda_i u^i : i \in I(k+1)\}$ is the desired point.

Next we show the "if" part. Let u^{k+1} be the vertex of τ which has the maximum label, say m, of all the vertices of τ. Then we see that $m \geq k+1$. Suppose the contrary, then $h_m(u^i) \geq 0$ for $i \in I(k+1)$ and $h_m(u^{k+1}) = 1$. This is contrary to the assumption that $H_{(k)}(x) = 0$ for some $x \in \tau$. Since $h_{(k)}(u^{k+1}) = (-1, -1, \ldots, -1)$, we can readily see that τ has a vertex u^i with $L(u^i) = i$ for each $i \in I(k)$. Hence τ is a complete simplex. Q.E.D.

By the above lemma and a careful comparison between the 2-ray method with the vector labelling function h (see [16]) and the 2-ray method with integer labelling function L proposed in this paper, we can see that the two methods generate the same sequence of simplices when $\gamma_k = 1$ for all $k \in I(n)$. The proof will be omitted here.

Let τ be a k-dimensional simplex of $\bar{T}$. Then there are $A \in R^{k \times k}$ and $b \in R^k$ such that $H_{(k)}(x) = A\, x_{(k)} + b$ for any $x \in \tau$. The next lemma shows the relation between the orientation of τ and the determinant of the matrix A.

Lemma 7.12. Let τ be a complete simplex of $T(k)$. Then

$$or(\tau) = \text{sign det } A.$$

Proof. Let $u^1, u^2, \ldots, u^{k+1}$ be the vertices of τ such that $L(u^i) = i$ for $i \in I(k)$ and $L(u^{k+1}) = m$ for some $m \in I(n+1) \setminus I(k)$. Let

$$W = \begin{bmatrix} u^1_{(k)} - u^{k+1}_{(k)} & \cdots & u^k_{(k)} - u^{k+1}_{(k)} \end{bmatrix},$$

$$Q = \begin{bmatrix} h^1_{(k)} - h^m_{(k)} & \cdots & h^k_{(k)} - h^m_{(k)} \end{bmatrix}.$$

Then it is readily seen that

$$A\, W = Q.$$

Consequently

$$\text{sign det } A = \text{sign det } W \cdot \text{sign det } Q.$$

Let

$$\bar{W} = \begin{bmatrix} 1 & \cdots & 1 \\ u^1_{(k)} & \cdots & u^{k+1}_{(k)} \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} 1 & \cdots & 1 \\ h^1_{(k)} & \cdots & h^m_{(k)} \end{bmatrix},$$

then

$$\det W = (-1)^{k+2} \det \bar{W},$$

$$\det Q = (-1)^{k+2} \det \tilde{Q} = (-1)^{k+2} (-1)^k 2^k.$$

Hence

$$\text{sign} \det A = (-1)^k \text{ sign} \det \bar{W} = or(\tau).$$ Q.E.D.

8. THE APPROXIMATION OF SEVERAL FIXED POINTS

As we have seen in Section 4, the method with different sequences Γ provide different sequences of adjacent simplices. Furthermore Theorem 7.10 has shown that the orientation of an n-complete simplex generated by the method is $\pi_n = \Pi \{ \gamma_k : k = 1, 2, \ldots, n \}$. Therefore there is a fair possibility that we reach several n-complete simplices by the method with different sequences Γ. In this section we propose two techniques to increase the possibility of finding several approximate fixed points.

For a given $f : C^n \to R^n$ and a sequence $\Gamma = \{ \gamma_1, \gamma_2, \ldots, \gamma_n \}$ let

$$S(\Gamma) = \{ S : S \text{ is a sequence of adjacent simplices with respect to the labels induced by } f \text{ under } \Gamma \}.$$

Then $S(\Gamma)$ can be illustrated as in Figure 8.1, where each horizontal line corresponds to $T(k)$ for $k \in I(n) \cup \{0\}$ and each solid curve shows a sequence $S \in S(\Gamma)$. The symbols $+$ and $-$ show the orientations of simplices. In this figure we have taken $\gamma_k = 1$ for all $k \in I(n)$ (see Corollary 7.7, 7.8 and 7.9).

The first technique is based on the extension of $f : C^n \to R^n$ to a mapping $\bar{f} : C^{n+1} \to R^{n+1}$. For each point $\bar{x} = (x_1, x_2, \ldots, x_{n+1}) \in R^{n+1}$ we denote its projection $(x_1, x_2, \ldots, x_n)$ onto R^n by x. Let $\bar{f} : C^{n+1} \to R^{n+1}$ be an arbitrary continuous mapping satisfying

$$\bar{f}_{(n)}(\bar{x}) = f(x) \quad \text{for } \bar{x} \in C^{n+1} \text{ such that } x_{n+1} = 0. \tag{8.1}$$

For instance we may take $\bar{f}$ such that

$$\bar{f}_i(\bar{x}) = \begin{cases} \phi(x_{n+1}) \, f_i(x) & \text{for } i \in I(n), \\ \psi(\bar{x}) & \text{for } i = n + 1, \end{cases}$$

where $\phi : [-1, +1] \to R^1$ and $\psi : C^{n+1} \to R^1$ are continuous functionals such that $\phi(0) = 1$. Let $\bar{\Gamma}$ be a sequence $\{ \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1} \}$ for some $\gamma_{n+1} \in \{\pm 1\}$ and let

$$\bar{S}(\bar{\Gamma}) = \{ \bar{S} : \bar{S} \text{ is a sequence of adjacent simplices with respect to labels induced by } \bar{f} \text{ under } \bar{\Gamma} \}.$$

By the construction (8.1) of $\bar{f}$, it is readily seen that $S(\Gamma)$ coincides with $\bar{S}(\bar{\Gamma})$ below the horizontal line corresponding to $T(n)$. Therefore $\bar{S}(\bar{\Gamma})$ possibly have some sequences which bend back to $T(n)$ as shown by dashed curves in Figure 8.1. If the sequence of $\bar{S}(\bar{\Gamma})$ starting from $\{0\}$ bends back to $T(n)$, we can find several n-complete simplices along it. To make the sequence of $\bar{S}(\bar{\Gamma})$ more likely to bend back to $T(n)$, it seems preferable to make the extension $\bar{f}$ so that it has no fixed points. It is very important that we can always find an n-complete simplex if f satisfies Condition 2.1 and $\gamma_k = 1$ for all $k \in I(n)$ whether $\bar{f}$ has a fixed point or not (see Theorem 4.3). But note that such an extension involves a risk that the sequence will collide against bd C^{n+1} before finding the second n-complete simplex. This technique shares some features with the approach by Garcia and Gould [2] and Jeppson [3] (see also Allgower and Georg [1]).

The second technique exploits the fact that the structure of $S(\Gamma)$ largely depends upon γ_k's. That is, replacing γ_k by $-\gamma_k$ greatly changes the structure of sequences of adjacent simplices. In Figure 8.2 we show $S(\Gamma)$ by solid curves and $S(\Gamma')$ by dashed curves, where $\Gamma = \{ 1, 1, \ldots, 1 \}$ and $\Gamma' = \{ 1, 1, \ldots, 1, -1, 1, \ldots, 1 \}$. They coincide except the portions between two horizontal lines corresponding to $T(k-1)$ and $T(k)$. It is very important that two simplices of $T(k-1)$ and $T(k)$ have the same orientation if they are linked by a sequence of $S(\Gamma)$, while they have opposite orientations if they are linked by a sequence of $S(\Gamma')$.

Now suppose we have an n-complete simplex σ_1 by generating a sequence of $S(\Gamma)$ which starts from $\{0\}$. Trace back the sequence of $S(\Gamma')$ shown by a dashed curve from σ_1, we may reach another n-complete simplex, say σ_2. In fact, we can see that it will never return to the starting 0-dimensional simplex {0}. Suppose the contrary. Then by Theorem 7.10, $\gamma_1\gamma_2\cdots\gamma_k\cdots\gamma_n = \text{or}(\sigma_1) = \gamma_1\gamma_2\cdots(-\gamma_k)\cdots\gamma_n$, which is a contradiction. Therefore it follows from Theorem 3.8 that this dashed curve leads to another n-complete simplex σ_2 unless it collides against bd C^n. We again trace back the sequence of $S(\Gamma)$, a solid curve, from σ_2. Then we readily see that this sequence of $S(\Gamma)$ will return to neither {0} nor σ_1. Hence we reach another n-complete simplex, say σ_3, if the sequence does not collide against bd C^n. In this way we possibly find several n-complete simplices by tracing sequences of $S(\Gamma)$ and $S(\Gamma')$ alternately. Note that $\text{or}(\sigma_i) = -\text{or}(\sigma_{i+1})$ for $i = 1, 2, \ldots$. If we employ $\{\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, -\gamma_n\}$ as Γ', this technique has a close relation with the extended reflection algorithm by Van der Laan [7].

Figure 8.3 shows an example. If we choose $\Gamma = \{1, 1\}$ and $\Gamma' = \{1, -1\}$, and trace sequences of $S(\Gamma)$ and $S(\Gamma')$ alternately, we successively generate n-complete simplices σ_1, σ_2 and σ_3. If we choose $\Gamma' = \{-1, 1\}$ then we have σ'_1, σ'_2 and σ'_3.

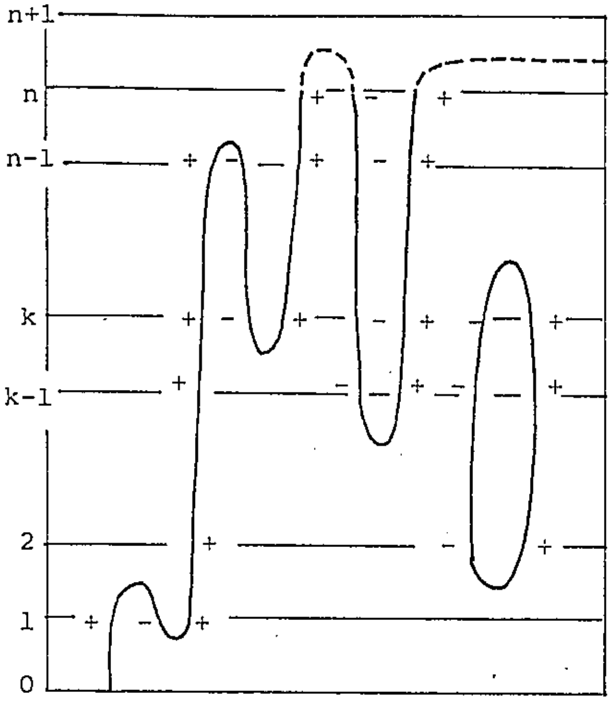

Figure 8.1.

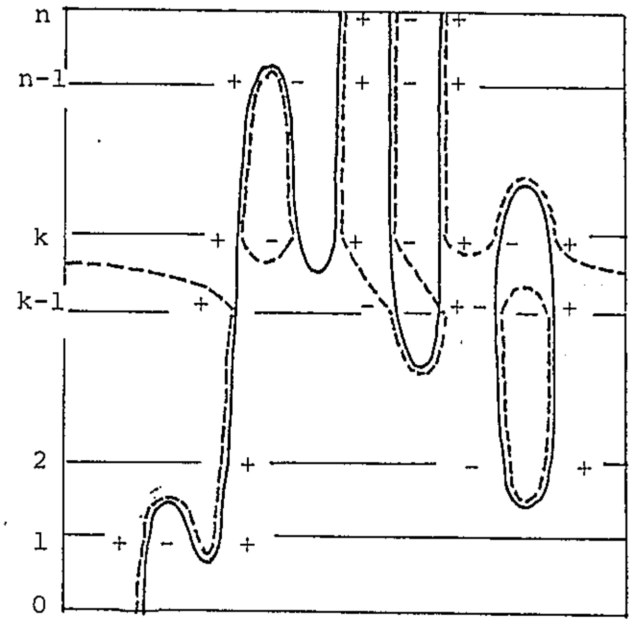

Figure 8.2.

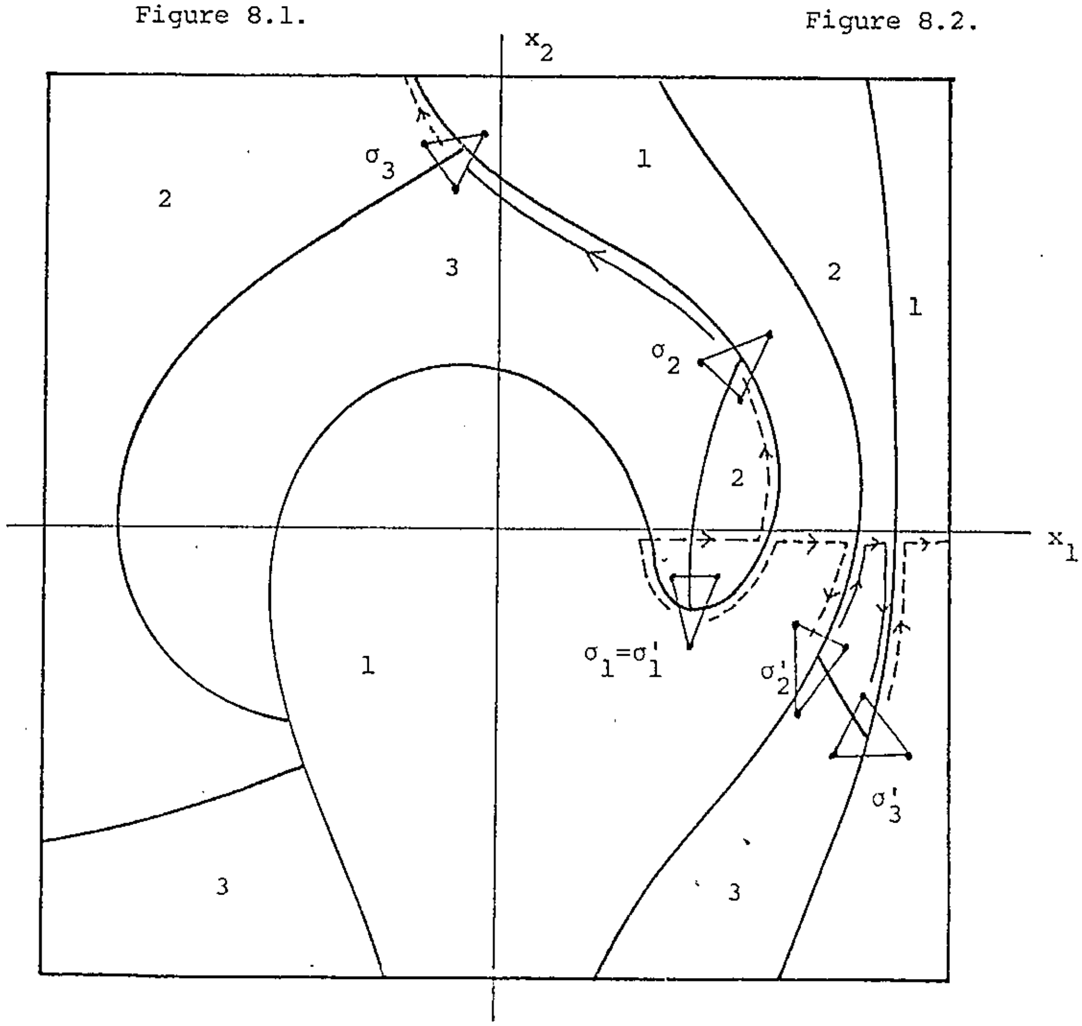

Figure 8.3

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