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A UNIFIED APPROACH TO SEVERAL RESTART FIXED POINT ALGORITHMS FOR THEIR IMPLEMENTATION AND A NEW VARIABLE DIMENSION ALGORITHM

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Abstract: We present a unified description of a class of restart fixed point algorithms including Merrill's method and several variable dimension algorithms for their implementation on computers. Based on this description we show that some pivot-saving techniques originally developed for the homotopy methods can be applied to the class. We also propose a new variable dimension algorithm having 3^n-1 rays along which we can move toward a solution. Some numerical comparions of the simplicial restart algorithms, Merrill's method, the 2n-method, the octahedral method and the new one, support that the latter two methods are more efficient than the others.

Key words: fixed point algorithms, systems of nonlinear equations, variable dimension algorithms, piecewise smooth mappings, piecewise linear mappings, complementary pivoting

1. Introduction

In these several years a class of new computational methods has been developed for approximating solutions to the systems of nonlinear equations

$$f(x) = 0, \qquad x \in R^n, \tag{1.1}$$

where R^n is an n-dimensional Euclidean space and f is a continuous mapping from R^n into itself (10,11,21) (see also (5,8,9,12,13,19,25,27,28)). The methods in this class are called variable dimension algorithms (will be abbreviated by vd algorithms in this paper) after their common feature that they generate a sequence of simplicies with varying dimensions. Till now we have several different interpretations for the vd algorithms. Todd (21) has shown that the vd algorithms can be viewed as methods for tracing a solution path of piecewise linear mapping defined on a subdivision of a certain subset of $R^n \times (0, 1)$. Freund (3) has given a different interpretation by introducing the V-complexes and H-complexes. We have also presented interpretations for the vd algorithms in (7), where the new notion of a primal-dual pair of subdivided manifolds (abbreviated by PDM) plays a fundamental and important role. Thus one vd algorithm now receives several different interpretations. In fact, we observed in (7) that the 2n-method (Van der Laan and Talman (13)) is described in two different ways, the one with the PDM associated with the complementarity problem and the other with the PDM consisting of a pair of conical subdivisions of R^n (see Section 5.2 and 5.3 of (7),

respectively). Also it has been described in another way with the use of a PDM in Kojima (6). Besides these three we have of course the original one in (13). Such differences in description may not be essential when we study the theoretical aspects of a given vd algorithm such as sufficient conditions for global convergence. However, they causes us inconvenience when we implement the vd algorithm on a computer because different descriptions often yield different families of systems of equations which might greatly affect the computational efficiency of the implementation. Hence we have to answer several important questions to choose a suitable family: Which family has less variables, equalities and inequalities than the others? Which family requires less pivoting operations and function evaluations than the others? How can we apply the pivot-saving technique developed by Todd (22,23,24,26) to the families when the system of equations to be solved has a special structure such as partial linearity, separability and sparsity?

One of the purposes of this paper is to propose a unified description for a class of restart algorithms including Merrill's method (Merrill (15)), the (n+1)-method (Van der Laan and Talman (13) and Todd (21)), the 2n-method (Van der Laan and Talman (13)) and the 2^n-method (or the octahedral method, Wright (28)). Namely we shall show that each method is obtained as a combination of the one-parameter family of systems of equations into which the system (1.1) is embedded and a suitable PDM (The vd algorithm using "one-function-at-a-time" homotopy by

Saigal (18) and the 2-ray-method by Yamamoto (29) are not obtained in this manner. Modifications are needed). Such description will show that the pivot-saving technique developed by Todd (22,23,24,26) can be efficiently applied to all these restart algorithms.

The vd algorithms were originally developed in the framework of simplicial approximation methods. Recently, Kojima (6) has proposed a class of piecewise smooth vd algorithms, which exhibit the "variable dimension structure" of the vd algorithms in simpler and clearer manner than the simplicial vd algorithms do. Because of this reason we shall first assume that the mapping f in (1.1) is continuously differentiable on R^n and introduce the piecewise smooth vd algorithms, and then build the simplicial vd algorithms on the foundation of them. The major emphasis is placed on the simplicial ones in this paper.

The second purpose of this paper is to propose a new vd algorithm. To explain its noteworthy feature we shall outline the piecewise smooth version of a restart algorithm. We take the 2n-method as an example.

We first embed the system (1.1) into the system

$$y + tf(x) = 0, \qquad (x, y, t) \in L \times R_+, \qquad (1.2)$$

where R_+ is the set of nonnegative numbers and L is an n-dimensional piecewise linear manifold in R^{2n} that is composed of a couple of subdivided manifolds $\mathcal{P}$ and $\mathcal{D}$ in R^n according to a certain rule d. $\mathcal{P}$ consists of all the orthants of R^n and $\mathcal{D}$ consists of a unit cube $\{ y \in R^n : -1 \leqq y_j \leqq 1$ for

$j=1,2,\ldots,n \}$ in R^n. Figure 1.1 shows $\mathcal{P}$ and $\mathcal{D}$ in the two dimensional case. The triplet ($\mathcal{P}$, $\mathcal{D}$; d) is a special case of the PDM given in (7). We assume that f is continuously differentiable. Then the left side of the system (1.2) is smooth on each piece of $L \times R_+$, i.e., piecewise smooth on $L \times R_+$. Hence the solution set of (1.2) consists of several 1-dimensional piecewise smooth manifolds. The piecewise linear manifold L is so designed that $(0, 0) \in L$. Therefore $(x^0, y^0, t^0) = (0, 0, 0) \in L \times R_+$ is a trivial solution of the system (1.2). Starting from this point, we trace a piecewise smooth path S consisting of solutions (x, y, t) of (1.2). Under a certain assumption the path S "converges" to $(\bar{x}, \bar{y}, +\infty)$. In this case, if $(x, y, t) \in S$ and t is sufficiently large,

$$f(x) = - y / t \fallingdotseq 0;$$

x is an approximate solution of (1.1). The path S moves through the (n+1)-dimensional piecewise linear manifold $L \times R_+$ all the way from (x^0, y^0, t^0). But we can observe the variable dimension structure of the path S in the projections S_x and S_y of S on the two distinct Euclidean spaces of the variable vector x and of the variable vector y. Darkened lines in Figure 1.1 show S_x and S_y for some solution path S. We see that S_x leads to a solution of (1.1) along several faces of $\mathcal{P}$ with varying dimensions. Since y serves as a slack variable vector, the complexity of computation of S deeply depends on the dimension of faces along which the variable vector

x moves. At earlier several stages of the computaion, x moves along lower dimensional faces of P. In fact x moves toward a solution along a 1-dimensional face or an unbounded ray of P at the initial stage. This contributes to saving a certain amount of works. Therefore it is reasonable to expect that the vd algorithm with larger number of unbounded rays in P locates a solution of (1.1) more rapidly. We shall propose a vd algorithm on a conical subdivision having 3^n-1 rays extending in all directions. We will refer it as the (3^n-1)-method (The numbers $n+1$, $2n$ and 2^n attached to the three methods referred above are the numbers of rays of conical subdivisions used in the methods, respectively). Note that the number 3^n-1 grows more rapidly than the numbers $n+1$, $2n$ and 2^n as the dimension n increses; for example $(3^{10}-1)/2^{10} \fallingdotseq 58$.

The third and final purpose of this paper is to report computaional results on the class of simplicial restart algorithms (except the $(n+1)$-method) referred above. They support that the 2^n-method and the (3^n-1)-method work more efficiently than the others. But we need further numerical tests to judge which of the 2^n- and (3^n-1)-methods is superior.

In Section 2, we introduce several terminologies and notations which will be necessary in our succeeding discussions, and we state the basic theorem for piecewise smooth (homotopy) continuation methods.

In Section 3, we give specifications of PDM which will be used in the piecewise smooth and simplicial vd algorithms.

In Section 4, we give a new PDM on which the (3^n-1)-method works.

The piecewise smooth vd algorithms are explained in Section 5.

Section 6 and 7 are devoted to simplicial vd algorithms. We also show how to apply the pivot-saving technique developed by Todd [22,23,24,26] to the simplicial vd algorithms when the system (1.1) has a special structure.

Numerical results are reported in Section 8.

2. Preliminaries

We call a convex polyhedral set C in R^k a cell or an m-cell, where m is the dimension of the affine subspace spanned by C. Let B be a cell such that $B \subset C$. If

$$\lambda x + (1-\lambda) y \in B$$

$$x, y \in C, \quad \lambda \in (0, 1)$$

always imply that $x, y \in B$, then we say that B is a face of C and we write $B < C$.

Let $\mathcal{M}$ be a finite or countable collection of m-cells in R^k. Let

$$\tilde{\mathcal{M}} = \{ B : B < C \text{ for some } C \in \mathcal{M} \},$$

$$|\mathcal{M}| = \bigcup \{ C : C \in \mathcal{M} \}.$$

$\mathcal{M}$ is a subdivided m-manifold if it satisfies the following three conditions:

(i) For each pair $B, C \in \mathcal{M}$, either $B \cap C = \emptyset$ or $B \cap C$ is a common face of B and C.

(ii) Each $(m-1)$-cell of $\tilde{\mathcal{M}}$ lies in at most two m-cells of $\mathcal{M}$.

(iii) $\mathcal{M}$ is locally finite; each point $x \in |\mathcal{M}|$ has a neighborhood that intersects with finitely many m-cells of $\mathcal{M}$.

Define the boundary $\partial\mathcal{M}$ of a subdivided m-manifold $\mathcal{M}$ by

$$\partial\mathcal{M} = \{ B : B \text{ is an } (m-1)\text{-cell of } \tilde{\mathcal{M}} \text{ that lies in exactly one m-cell of } \mathcal{M} \}.$$

Let M be a subset of R^k. If $M = |\mathcal{M}|$ for some subdivided m-manifold $\mathcal{M}$, then M is an m-dimensional piecewise linear manifold, or simly, an m-manifold and $\mathcal{M}$ is a subdivision of M. Let $\mathcal{M}'$ be a subdivision of M. If each cell of $\mathcal{M}'$ lies in some cell of $\mathcal{M}$, $\mathcal{M}'$ is a refinement of $\mathcal{M}$. A subdivision $\mathcal{M}$ of M (or a refinement $\mathcal{M}'$ of $\mathcal{M}$) is simplicial if every cell of $\mathcal{M}$ (or $\mathcal{M}'$) is a simplex.

By a unit interval, we mean $(0, 1)$, $[0, 1)$ or $[0, 1] \subset R^1$, and by the unit circle, the set $\{ x \in R^2 : x^t x = 1 \}$. It is well-known that a 1-manifold M is homeomorphic to either a unit interval or the unit circle. We call M a path in the former case, and a loop in the latter case. A path M is open, semi-closed, or closed if it is homeomorphic to $(0, 1)$, $[0, 1)$ or $[0, 1]$, respectively. A point x of a path M is an endpoint of M if x corresponds to a boundary point of the unit interval to which M is homeomorphic. Hence a path M has no, one and two endpoints if it is open, semi-closed and closed, respectively.

Let $\mathcal{M}$ be a subdivided m-manifold in R^k. A continuous mapping h from $|\mathcal{M}|$ into R^n is piecewise continuously differentiable (abbreviated by PC^1) on $\mathcal{M}$ if the restriction of h to each cell C of $\mathcal{M}$ can be extended to a continuously differentiable mapping defined on an open neighborhood of C. To clarify the underlying subdivided m-manifold $\mathcal{M}$ on which a PC^1 mapping h is defined, we shall write $h : |\mathcal{M}| \to R^n$. We denote the Jacobian matrix of a PC^1 mapping $h : |\mathcal{M}| \to R^n$ at $x \in C \in \mathcal{M}$ by $Dh(x;C)$. Specifically, a PC^1 mapping $F : |\mathcal{M}| \to R^n$ is piecewise linear (abbreviated by PL) if it is affine on each cell C of $\mathcal{M}$, i.e.,

$$F(\lambda x + (1-\lambda) y) = \lambda F(x) + (1-\lambda) F(y)$$

for every $x, y \in C$ and any $\lambda \in (0,1)$. $c \in R^n$ is a <u>regular value</u> of a PC^1 mapping $h : |\mathcal{M}| \to R^n$ if $x \in B < C \in \mathcal{M}$ and $h(x) = c$ always imply

$$\dim \{ Dh(x;C)\, y : y \in B \} = n.$$

By using Sard's theorem, we can easily show that almost every $c \in R^n$ (in Lubesgue measure) is a regular value of a PC^1 mapping $h : |\mathcal{M}| \to R^n$. In our succeeding discussion, we will be concerned with only the case where $m = \dim \mathcal{M} = n+1$. In this case, if $c \in R^n$ is a regular value of h, then the set $h^{-1}(c) = \{ x \in |\mathcal{M}| : h(x) = c \}$ has no intersection with any cell of $\mathcal{M}$ which has dimension less than n. Furthermore, we can establish the following result by using the implicit function theorem. The proof is omitted here. See Allexander [1] for more general discussion.

<u>Theorem 2.1.</u> Let $\mathcal{M}$ be a subdivided $(n+1)$-manifold in R^k, and $h : |\mathcal{M}| \to R^n$ be a PC^1 mapping on $\mathcal{M}$. Suppose that $c \in R^n$ is a regular value of the PC^1 mapping h. Then $h^{-1}(c)$ is a disjoint union of paths and loops satisfying the following properties:

(i) If $C \in \mathcal{M}$ and $h^{-1}(c) \cap C \neq \emptyset$ then $h^{-1}(c) \cap C$ is a disjoint union of smooth 1-manifolds. If, in addition, $h : |\mathcal{M}| \to R^n$ is PL, then $h^{-1}(c) \cap C$ is a 1-cell, i.e., it is either a line segment, a half line or a line.

(ii) Each loop has no intersection with $|\partial\mathcal{M}|$.

(iii) $x \in h^{-1}(c)$ is an endpoint of a path if and only if $x \in |\partial\mathcal{M}|$.

(iv) If $|\mathcal{M}|$ is closed, every open or semi-closed path is unbounded.

3. Specification of PDM

Suppose that the triplet ($\mathcal{P}$, $\mathcal{D}$; d) satisfies the following conditions:

(i) $\mathcal{P}$ is a subdivision of R^n.

(ii) $\mathcal{D}$ is a subdivision of a polyhedral subset D of R^n such that each cell of $\mathcal{D}$ is bounded.

(iii) d is an operator from $\bar{\mathcal{P}} \cup \bar{\mathcal{D}}$ into itself such that $X^d \in \bar{\mathcal{D}}$ for every $X \in \bar{\mathcal{P}}$ and $Y^d \in \bar{\mathcal{P}}$ for every $Y \in \bar{\mathcal{D}}$.

(iv) If $Z \in \bar{\mathcal{P}} \cup \bar{\mathcal{D}}$ then $(Z^d)^d = Z$ and $\dim Z + \dim Z^d = n$.

(v) If $X_1, X_2 \in \bar{\mathcal{P}}$ and $X_1 < X_2$, then $X_2^d < X_1^d$.

(v)' If $Y_1, Y_2 \in \bar{\mathcal{D}}$ and $Y_1 < Y_2$, then $Y_2^d < Y_1^d$.

The triplet ($\mathcal{P}$, $\mathcal{D}$; d) above is a special case of <u>primal-dual pair of subdivided manifolds</u> (abbreviated by PDM) which was introduced by the authors (7). See Section 3 of (7) for the general definition of PDM and its fundamental properties. We call $\mathcal{P}$ (resp. $\mathcal{D}$) the <u>primal</u> (resp. <u>dual</u>) <u>subdivided manifold</u>, d the <u>dual operator</u> and Z^d the <u>dual</u> of Z for each $Z \in \bar{\mathcal{P}} \cup \bar{\mathcal{D}}$. Note that the condition (iv) implies that the dual operator d is one-to-one and onto, and that its inverse is d itself. Define

$$< \mathcal{P} , \mathcal{D} ; d > = \{ X \times X^d : X \in \bar{\mathcal{P}} \},$$

or equivalently

$$< \mathcal{P} , \mathcal{D} ; d > = \{ Y^d \times Y : Y \in \bar{\mathcal{D}} \}.$$

Let $\mathcal{L} = < \mathcal{P} , \mathcal{D} ; d >$.

<u>Theorem 3.1.</u> $\mathcal{L}$ is a subdivided n-manifold with no boundary, i.e., $\partial \mathcal{L} = \emptyset$, and $|\mathcal{L}|$ is closed.

<u>Proof.</u> The first assertion follows directly from Theorem 3.1, 3.2 and 3.3 in Kojima and Yamamoto (7), and the second from the closedness of the sets $|\mathcal{P}| = R^n$ and $|\mathcal{D}| = D$ and the locally finite property of the collection $\mathcal{P}$ and $\mathcal{D}$. Q.E.D.

We further impose the following two conditions on the PDM ($\mathcal{P}$, $\mathcal{D}$; d):

(vi) $\{0\} \in \bar{\mathcal{P}}$, and $0 \in \operatorname{int} Y_0$, where $Y_0 = \{0\}^d \in \mathcal{D}$.

(vii) There exists a positive number α such that

$$x^t y \geqq \alpha \|x\| \quad \text{for every} \quad (x,y) \in |\mathcal{L}|,$$

where $\|x\| = (x^t x)^{1/2}$. From the condition (vi), we see that (0, c) lies in $|\mathcal{L}|$ whenever $\|c\|$ is sufficiently small. This point (0, c) will be used as a starting point of the vd algorithms. The condition (vii) will play an important role when we derive a sufficient condition for the global convergence of the vd algorithms. The following lemma will be used in Section 6.

<u>Lemma 3.2.</u> (Theorem 4.1 in Kojima and Yamamoto (7))

Let $\mathcal{P}^*$ be a refinement of $\mathcal{P}$, and

$$\mathcal{L}^* = \{ \sigma \times X^d : \sigma \subset X \in \bar{\mathcal{P}} , \sigma \in \bar{\mathcal{P}}^*, \dim \sigma = \dim X \}.$$

Then $\mathcal{L}^*$ is a refinement of $\mathcal{L}$.

Now we show four examples of PDM's satisfying the conditions (i) - (vii).

<u>Example 3.1.</u> The first PDM was constructed in Section 5.4 of Kojima and Yamamoto (7) for an efficient implementation of Merrill's method with the use of the Union Jack triangulation of $R^n \times (0, 1)$ (Todd (20)), and was utilized in a generalization of

the triangulaion J_3 of $R^n \times (0, 1)$ (Todd (20)) in Section 6 of (7). This PDM is also closely related to a subdivision of $R^n \times (0, 1)$ given by Todd (22) (see also Todd (24,26)) for saving many pivots in Merrill's method. In the case of $n=2$, the primal and dual subdivided manifolds both look like a "checkerboard". See Figure 3.1.

Let δ be an arbitrarily fixed positive number and let Q be the set of all n-dimensional vectors $q = (q_1, q_2, \ldots, q_n)$ such that each q_i is an integral multiple of δ. For every $q \in Q$, let

$$I_e(q) = \{ i : q_i \text{ is an even multiple of } \delta \},$$

$$I_o(q) = \{ i : q_i \text{ is an odd multiple of } \delta \},$$

$$X(q) = \{ x \in R^n : \begin{array}{ll} x_i = q_i & \text{for } i \in I_e(q) \\ q_j - \delta \leqq x_j \leqq q_j + \delta & \text{for } j \in I_o(q) \}, \end{array} \quad (3.1)$$

$$Y(q) = \{ y \in R^n : \begin{array}{ll} q_i - \delta \leqq y_i \leqq q_i + \delta & \text{for } i \in I_e(q) \\ y_j = q_j & \text{for } j \in I_o(q) \}. \end{array} \quad (3.2)$$

It is readily seen that

$$\mathcal{P} = \{ X(q) : q \in Q \text{ and } I_e(q) = \emptyset \},$$

$$\mathcal{D} = \{ Y(q) : q \in Q \text{ and } I_o(q) = \emptyset \}$$

are subdivisions of R^n and that

$$\bar{\mathcal{P}} = \{ X(q) : q \in Q \},$$

$$\bar{\mathcal{D}} = \{ Y(q) : q \in Q \}.$$

Thus we have a PDM satisfying the conditions (i) - (vii) by defining the dual operator d as

$$X(q)^d = Y(q), \quad Y(q)^d = X(q) \quad \text{for every } q \in Q.$$

Note that the cell Y_0 in the condition (vi) is $Y(0)$.

Example 3.2. Figure 3.2 illustrates the two dimensional case of this example. In a general n-dimensional case, the primal subdivided manifold $\mathcal{P}$ has (n+1)-rays. This PDM was implicitly used in the vd algorithm by Todd (21), which we call the (n+1)-method in this paper.

Let e^i be the i-th unit vector of R^n and let $e = \sum_{i=1}^{n} e^i$. Define the n+1 vectors $p^0, p^1, \ldots, p^n$ as

$$p^0 = -e,$$

$$p^i = e^i \quad \text{for } 1 \leqq i \leqq n.$$

For every nonempty proper subset I of $N^* = \{0, 1, \ldots, n\}$, let

$$X(I) = \{ x \in R^n : x = \sum_{i \in I} \lambda_i p^i \text{ for some } \lambda_i \geqq 0, (i \in I) \},$$

and specifically $X(\emptyset) = \{0\}$. Then

$$\mathcal{P} = \{ X(I) : I \subset N^* \text{ and } |I| = n \}$$

is a subdivision of R^n and

$$\bar{\mathcal{P}} = \{ X(I) : I \subsetneqq N^* \},$$

where $|I|$ means the number of elements in I.

To make the dual subdivided manifold $\mathcal{D}$, let

$$Y_0 = \{ y \in R^n : (p^i)^t y \leqq 1 \text{ for } i \in N^* \}. \quad (3.3)$$

Then Y_0 is a polyhedral set containing the origin in its interior. For every nonempty proper subset I of N^*, let

$$Y(I) = Y_0 \cap \{ y \in R^n : (p^i)^t y = 1 \text{ for } i \in I \}, \quad (3.4)$$

and specifically $Y(\emptyset) = Y_0$. Let

$$\mathcal{D} = \{ Y_0 \},$$

then we see

$$\bar{\mathcal{D}} = \{ Y(I) : I \subsetneqq N^* \}.$$

Thus by defining the dual operator d as

$$X(I)^d = Y(I),\ Y(I)^d = X(I) \quad \text{for every} \quad I \subsetneq N^*$$

we have a PDM satisfying all the conditions (i) - (vii).

Example 3.3. Figure 3.3 illustrates the two dimensional case of this example. Generally, the primal subdivided manifold $\mathcal{P}$ consists of all the orthants of R^n, and the dual $\mathcal{D}$ consists of a single unit cube. This PDM was implicitly used in the vd algorithm given by Van der Laan and Talman (13). Here we call it the 2n-method because $\mathcal{P}$ has 2n rays.

Let $N = \{1,2,\ldots,n\}$, $N^- = \{-i : i \in N\}$, and $N^{**} = N \cup N^-$. Define 2n vectors as

$$p^i = e^i \qquad \text{for } i \in N,$$
$$p^j = -e^{-j} \qquad \text{for } j \in N^-.$$

We say that a subset I of N^{**} is consistent if $\{i, -i\} \not\subset I$ for any $i \in N$. For each consistent subset I of N^{**} let

$$X(I) = \{x \in R^n : x = \sum_{i \in I} \lambda_i p^i \text{ for some } \lambda_i \geqq 0\ (i \in I)\},$$
$$\mathcal{P} = \{X(I) : I \subset N^{**} \text{ is consistent and } |I| = n\},$$

where we assume that $X(\emptyset) = \{0\}$. Then $\mathcal{P}$ is a subdivision of R^n consisting of all the orthants and

$$\bar{\mathcal{P}} = \{X(I) : I \subset N^{**} \text{ is consistent}\}.$$

Let

$$Y_0 = \{y \in R^n : (p^i)^t y \leqq 1 \quad \text{for} \quad i \in N^{**}\}, \qquad (3.5)$$
$$= \{y \in R^n : -1 \leqq y_i \leqq 1 \text{ for } i \in N\},$$

and for each consistent $I \subset N^{**}$

$$Y(I) = Y_0 \cap \{y \in R^n : (p^i)^t y = 1 \quad \text{for} \quad i \in I\}. \qquad (3.6)$$

Then $\mathcal{D} = \{Y_0\}$ is a subdivided manifold and

$$\bar{\mathcal{D}} = \{Y(I) : I \subset N^{**} \text{ is consistent}\}.$$

Defining the dual operator d by

$$X(I)^d = Y(I), \quad Y(I)^d = X(I) \quad \text{for every consistent} \quad I \subset N^{**},$$

we obtain a PDM satisfying all the conditions (i) - (vii).

Example 3.4. The last example is the PDM which was implicitly used in the octahedral method by Wright (28), which we here call the 2^n-method. Figure 3.4 illustrates the two dimensional case. Generally, the primal subdivided manifold $\mathcal{P}$ has 2^n rays. An n-dimensional vector $s = (s_1, s_2, \ldots, s_n)$ is said to be a sign vector if $s_i \in \{-1, 0, 1\}$ for every $i \in N$. For each sign vector s, let

$$I(s) = \{i : s_i \neq 0\},$$
$$J(s) = \{i : s_i = 0\}.$$

Let

$$\Sigma = \{s \in R^n : s \text{ is a sign vector such that } I(s) \neq \emptyset\},$$

i.e., Σ is the set of all nonzero sign vectors, and let

$$\Pi = \{p \in \Sigma : J(p) = \emptyset\}.$$

For two vectors $s, t \in \Sigma$, we say that s conforms to t and write $s \preccurlyeq t$ if $s_i \neq 0$ implies that $s_i = t_i$. For each $s \in \Sigma$, let

$$P(s) = \{p \in \Pi : s \preccurlyeq p\}.$$

Now let

$$X(s) = \{x \in R^n : x = \sum_{p \in P(s)} \lambda_p p \text{ for some } \lambda_p \geqq 0\ (p \in P(s))\},$$

and specifically $X(0) = \{0\}$. Then

$$\mathcal{P} = \{X(s) : s \in \Sigma, |I(s)| = 1\}$$
$$= \{X(e^i) : i \in N\} \cup \{X(-e^i) : i \in N\}$$

is a subdivision of R^n and

$$\bar{P} = \{ X(s) : s \in \Sigma \cup \{0\} \}.$$

Let

$$Y_0 = \{ y \in R^n : p^t y \leqq 1 \quad \text{for} \quad p \in \Pi \} . \qquad (3.7)$$

Then Y_0 is a polyhedral set containing the origin in its interior. Note that Y_0 is an octahedron when the dimension n is three. For each vector $s \in \Sigma$, let

$$Y(s) = Y_0 \cap \{ y \in R^n : p^t y = 1 \quad \text{for} \quad p \in P(s) \} \qquad (3.8)$$

and specifically $Y(0) = Y_0$. Let

$$\mathcal{D} = \{ Y_0 \} ,$$

then we see

$$\bar{\mathcal{D}} = \{ Y(s) : s \in \Sigma \cup \{0\} \}.$$

Thus the dual operator d such that

$$X(s)^d = Y(s), \quad Y(s)^d = X(s) \quad \text{for every} \quad s \in \Sigma \cup \{0\}$$

completes a PDM ($\mathcal{P}$, $\mathcal{D}$; d). We can easily see that this PDM also satisfies all the conditions (i) - (vii).

4. A New PDM

In this section we explain a new PDM on which we will construct a new vd algorithm, (3^n-1)-method. Let Σ be the set of all nonzero sign vectors. For $s, t \in \Sigma$ such that $s \preccurlyeq t$ we define an interval $[s, t]$ as

$$[s, t] = \{ p \in \Sigma : s \preccurlyeq p \preccurlyeq t \} ,$$

where $s \preccurlyeq p$ means that s conforms to p. For $s, t \in \Sigma$ such that $s \preccurlyeq t$ let

$$X(s, t) = \{ x \in R^n : x = \sum_{p \in [s,t]} \lambda_p p \text{ for some } \lambda_p \geqq 0 \ (p \in [s,t]) \ \} \qquad (4.1)$$

Then $X(s, t)$ forms a convex polyhedral cone. Figure 4.1 shows the intersections of some $X(s, t)$ with a segment of the octahedron having the origin as a center. It is readily seen that

$$\dim X(s, t) = |I(t) \setminus I(s)|+1, \qquad (4.2)$$

where $I(s)$ denotes the support $\{i \in N : s_i \neq 0\}$ of a sign vector s. We have the following lemma as for the facial structure of $X(s, t)$.

<u>Lemma 4.1.</u> Let $s, t \in \Sigma$ such that $s \preccurlyeq t$ and let $X \subset R^n$. Then $X < X[s, t]$ if and only if X is either $\{0\}$ or $X[s', t']$ for some $s', t' \in \Sigma$ such that $s' \preccurlyeq t'$ and $[s', t'] \subset [s, t]$.

<u>Proof.</u> We first show the "if" part. Since $X = \{0\}$ is trivially a face of $X[s, t]$, we have only to consider the case where $X = X[s',t']$. Let $u^1,\dots, u^h, u^{h+1},\dots, u^k, u^{k+1},\dots, u^m$ be sign vectors in the interval $[s, t]$. We here assume without

loss of generality that

$$\begin{aligned} &u^i \notin (s',t) && \text{for } 1 \leqq i \leqq h, \\ &u^i \in (s', t') && \text{for } h+1 \leqq i \leqq k, \\ &u^i \notin (s, t') && \text{for } k+1 \leqq i \leqq m. \end{aligned}$$

Let $x \in X(s', t')$ and $x^1, x^2 \in X(s, t)$, then

$$x = \sum_{i=h+1}^{k} \alpha_i u^i ,$$

$$x^j = \sum_{i=1}^{m} \alpha_i^j u^i \qquad \text{for } j = 1, 2,$$

for some nonnegative coefficients α_i, α_i^j. Suppose $x = \lambda x^1 + (1-\lambda)x^2$ for some $\lambda \in (0, 1)$. Since $I(u^i) \setminus I(t') \neq \emptyset$ for $k+1 \leqq i \leqq m$, we can easily see that $\alpha_i^1 = \alpha_i^2 = 0$ for any $k+1 \leqq i \leqq m$. Therefore, if $s = s'$, we have proved the "if" part. Hence we consider the case where $s \neq s'$. Then $I(s') \setminus I(u^i) \neq \emptyset$ for $1 \leqq i \leqq h$. Since

$$|x_j| = \sum_{i=h+1}^{k} \alpha_i \, |u_j^i| = \sum_{i=h+1}^{k} \alpha_i = \text{constant}$$

for any $j \in I(s')$, we have $\alpha_i^1 = \alpha_i^2 = 0$ for $1 \leqq i \leqq h$. This implies that $x^1, x^2 \in X(s',t')$ and completes the proof of the "if" part.

Next we show the "only if" part. Let $V = \{ v^i : i = 1, 2, .., h' \}$ be the set of all nonzero sign vectors in X. Since X is a face of the polyhedral cone $X(s, t)$,

$$X = \{ x \in R^n : x = \sum_{i=1}^{h'} \alpha_i v^i \text{ for some } \alpha_i \geqq 0 \ (i=1,2,\ldots,h') \}.$$

Let s' be a maximal element in V with respect to the conformal order $\preccurlyeq$. Then we see that s' is unique. Suppose on the contrary there were two distinct maximal elements, say q^1 and q^2, in V. Let q^3 be a sign vector such that

$$q_i^3 = \begin{cases} q_i^1 & \text{if } q_i^1 \neq 0 , \\ q_i^2 & \text{otherwise.} \end{cases}$$

Then it is clear that $q^1, q^2 \preccurlyeq q^3$, $q^3 \neq q^1, q^2$ and $q^3 \in (s, t) \subset X(s, t)$. Let $q^4 = q^1 + q^2 - q^3$, then $q^4 \in (s, t) \subset X(s, t)$ and $(1/2)q^3 + (1/2)q^4 = (1/2)q^1 + (1/2)q^2 \in X$. Since X is a face of $X(s, t)$, this implies that $q^3 \in X$, and hence $q^3 \in V$. This contradicts the assumption that q^1 and q^2 were maximal elements in V. We can also see that the unique minimal element, say t', exists in V. Now we show that $X = X(s', t')$. By the choice of s' and t', it is clear that $X \subset X(s', t')$. To see that $X(s', t') \subset X$, it is sufficient to show that $(s', t') \subset X$. Suppose the contrary, i.e., there exists a nonzero sign vector $r \in (s', t') \setminus X$. Let $r' = s' + t' - r$, then we see that $r' \in (s, t)$ and $(1/2)r + (1/2)r' = (1/2)s' + (1/2)t' \in X$. This is contrary to the assumption that X is a face of $X(s, t)$.

Q.E.D.

<u>Lemma 4.2.</u> Let s, t, s' and t' be nonzero sign vectors such that $s \preccurlyeq t$ and $s' \preccurlyeq t'$. Let $X = X(s, t) \cap X(s', t')$. Then $X = \{0\}$ or $X = X(p, q)$ for some $p, q \in \Sigma$ such that $p \preccurlyeq q$ and $(p, q] \subset (s, t) \cap (s', t')$.

<u>Proof.</u> Supoose first that $[s, t] \cap [s', t'] = \emptyset$. Let x be an arbitrary point in X. If x were a nonzero vector, we could define a nonzero sign vector r as follows:

$$r_i = \begin{cases} +1 & \text{if } x_i > 0, \\ -1 & \text{if } x_i < 0, \\ 0 & \text{if } x_i = 0. \end{cases}$$

It is readily seen that $r \in [s, t] \cap [s', t']$, which is a contradiction. Hence we have seen that $X = \{0\}$ if $[s, t] \cap [s', t'] = \emptyset$. Next suppose that $[s, t] \cap [s', t'] \neq \emptyset$. Let p and q be sign vectors such that

$$p_i = \begin{cases} s_i & \text{if } s_i = s_i', \\ s_i & \text{if } s_i \neq 0 \text{ and } s_i' = 0, \\ s_i' & \text{if } s_i = 0 \text{ and } s_i' \neq 0, \end{cases}$$

$$q_i = \begin{cases} t_i & \text{if } t_i = t_i', \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows from the assumption $[s, t] \cap [s', t'] \neq \emptyset$ that p is well defined, $q \neq 0$ and $p \preccurlyeq q$. We also see from the construction of p and q that $[p, q] \subset [s, t] \cap [s', t']$. Hence it is straightforward to see that $X[p, q] \subset X$. Let x be an arbitrary point of X, then

$$x_i \begin{cases} = 0 & \text{if } t_i = 0 \text{ or } t_i' = 0, \\ \geqq 0 & \text{if } s_i = 1 \text{ or } s_i' = 1, \\ \leqq 0 & \text{if } s_i = -1 \text{ or } s_i' = -1. \end{cases}$$

Therefore x is clearly a point of $X[p, q]$. Q.E.D.

<u>Lemma 4.3.</u> Let s and t be nonzero sign vectors such that $s \preccurlyeq t$. If $X[s, t]$ is an $(n-1)$-cell, then there exist exactly two n-cells $X[s^1, t^1]$ and $X[s^2, t^2]$ such that

$$X[s, t] < X[s^1, t^1], \quad X[s^2, t^2].$$

<u>Proof.</u> By (4.2), we have $|I(t) \setminus I(s)| = n-2$, i.e., one of the following two cases occurs:

(i) $|I(t)| = n$ and $|I(s)| = 2$,

(ii) $|I(t)| = n-1$ and $|I(s)| = 1$.

If (i) occurs, then there exist exactly two nonzero sign vectors s^1 and s^2 such that $s^1, s^2 \preccurlyeq s$, $|I(s^1)| = |I(s^2)| = 1$. Hence by Lemma 4.1, we see that $X[s^1, t]$ and $X[s^2, t]$ are the desired n-cells. If (ii) occurs, then there exist exactly two nonzero sign vectors t^1, t^2 such that $t \preccurlyeq t^1, t^2$ and $|I(t^1)| = |I(t^2)| = n$. Hence by Lemma 4.1, $X[s, t^1]$ and $X[s, t^2]$ are the desired n-cells. Q.E.D.

Let

$$\mathcal{P} = \{ X[s, t] : s, t \in \Sigma \text{ such that } s \preccurlyeq t \quad |I(t)| = n \text{ and } |I(s)| = 1 \} \tag{4.3}$$

By Lemma 4.1, 4.2 and 4.3 we see that $\mathcal{P}$ is a subdivided n-manifold and that

$$\bar{\mathcal{P}} = \{0\} \cup \{ X[s, t] : s, t \in \Sigma \text{ such that } s \preccurlyeq t \}. \tag{4.4}$$

To make the dual subdivided manifold $\mathcal{D}$ of $\mathcal{P}$, let γ be an arbitrarily chosen number such that $0 < \gamma < 1/n$ and $\beta = 1 - (n-1)\gamma$. Let

$$Y_0 = \{ y \in R^n : p^t y \leqq \beta + (|I(p)|-1)\gamma \text{ for } p \in \Sigma \}, \tag{4.5}$$

and for $s, t \in \Sigma$ such that $s \preccurlyeq t$ let

$$Y(s, t) = Y_0 \cap \{ y \in R^n : p^t y = \beta + (|I(p)|-1)\gamma \text{ for } p \in [s,t] \} \tag{4.6}$$

Then we see that Y_0 is a compact polyhedral set having the origin in its interior and that $Y(s, t)$ is a nonempty $(n - |I(t)\setminus I(s)| - 1)$-dimensional face of Y_0. In fact it is not difficult to see that

$$Y(s, t] = \{ y \in R^n : \begin{array}{ll} \sum_{i \in N} t_i y_i = \beta + (|I(t)|-1)\gamma, & \\ \gamma \leqq t_i y_i \leqq \beta & \text{for } i \in I(s), \\ t_i y_i = \gamma & \text{for } i \in I(t)\setminus I(s), \\ |y_i| \leqq \gamma & \text{for } i \not\in I(t) \}. \end{array} \quad (4.7)$$

In three dimensional case Y_0 is a polyhedron with $3^3 - 1 = 26$ facets. Figure 4.2 shows several $Y(s, t)$ in three dimensional case. Let

$$\mathcal{D} = \{Y_0\} , \qquad (4.8)$$

then $\mathcal{D}$ is a subdivided n-manifold and

$$\bar{\mathcal{D}} = \{Y_0\} \cup \{ Y(s, t] : s, t \in \Sigma \text{ such that } s \preccurlyeq t \}. (4.9)$$

The next lemma is immediate from the definition of $Y(s, t)$.

Lemma 4.4. Let $s, t \in \Sigma$ such that $s \preccurlyeq t$ and let $Y \subset R^n$. Then Y is a face of $Y(s, t]$ if and only if $Y = Y(s', t')$ for some s' and $t' \in \Sigma$ such that $s' \preccurlyeq t'$ and $[s, t] \subset [s', t']$.

Now to complete the PDM we have only to define the dual operator d as follows:

$$\begin{array}{ll} \{0\}^d = Y_0, & Y_0^d = \{0\}, \\ X(s, t]^d = Y(s, t), & Y(s, t]^d = X(s, t). \end{array} \qquad (4.10)$$

It is readily seen that $(\mathcal{P} , \mathcal{D} ; d)$ defined in (4.3), (4.8) and (4.10) satisfies the conditions (i) - (vi). The next lemma shows that it also satisfies the last condition (vii).

Lemma 4.5. Let $\mathcal{L} = < \mathcal{P} , \mathcal{D} ; d >$. Then for every $(x, y) \in |\mathcal{L}|$

$$x^t y \geqq \gamma \|x\| . \qquad (4.11)$$

Proof. If x is a zero vector, (4.11) holds trivially. Hence suppose that $x \in X(s, t]$ and $y \in Y(s, t]$ for some $s, t \in \Sigma$ such that $s \preccurlyeq t$. Then

$$\begin{aligned} \sum_{i \in N} x_i y_i &= \sum_{i \in I(t)} x_i y_i \quad (\text{since } x_i = 0 \text{ for any } i \not\in I(t)) \\ &= \sum_{i \in I(t)} (t_i x_i).(t_i y_i) \\ &\geqq \gamma \sum_{i \in I(t)} t_i x_i \quad (\text{since } t_i y_i \geqq \gamma \text{ for any } i \in I(t) \text{ by (4.7)}) \\ &\geqq \gamma \|x\| . \end{aligned}$$

Q.E.D.

Therefore the PDM introduced here satisfies all the conditions (i) - (vii) in Section 3. Note that the number of subdivided 1-manifolds of $\bar{\mathcal{P}}$, or the number of facets of Y_0, is equal to the number of the sign vectors in Σ, which amounts to $3^n - 1$.

5. Piecewise Smooth Vd Algorithm

Let $(\mathcal{P} , \mathcal{D} ; d)$ be a PDM which satisfies the conditions (i) - (vii) given in Section 3, and let

$$\mathcal{L} = < \mathcal{P} , \mathcal{D} ; d > = \{ X \times X^d : X \in \bar{\mathcal{P}} \}.$$

Define the subdivided (n+1)-manifold

$$\mathcal{M} = \{ Z \times R_+ : Z \in \mathcal{L} \} .$$

By Theorem 3.1, we see

$$\partial\mathcal{M} = \{ Z \times \{0\} : Z \in \mathcal{L} \} \qquad (5.1)$$

$$= \{ X \times X^d \times \{0\} : X \in \bar{\mathcal{P}} \},$$

and that $|\mathcal{M}|$ is closed. Let w^0 be an arbitrarily chosen point of R^n. When a rough approximation of a solution of the system (1.1) is known, take it as w^0. Throughout this section we assume that the mapping f appearing in the left hand side of the system (1.1) is continuously differentiable on R^n. For simplicity of notation, letting

$$g(x) = f(w^0+x) \qquad \text{for every} \quad x \in R^n, \qquad (5.2)$$

we shall deal with the system of equations

$$g(x) = 0, \qquad x \in R^n \qquad (5.3)$$

instead of the system (1.1).

Now we define the PC^1 maping $h : |\mathcal{M}| \to R^n$ by

$$h(x, y, t) = y + tg(x) \quad \text{for every } (x, y, t) \in |\mathcal{M}| . \qquad (5.4)$$

Let c be an interior point of the cell Y_0 of $\mathcal{D}$. By the condition (vi), any point c with sufficiently small norm $\|c\|$ lies in the interior of Y_0. In the introduction we have taken $c = 0$. We shall consider the system of equations

$$h(x, y, t) = c, \qquad (x, y, t) \in |\mathcal{M}|. \qquad (5.5)$$

It can be readily verified that the point

$$z^0 = (x^0, y^0, t^0) = (0, c, 0)$$

lies on $h^{-1}(c) \cap |\partial\mathcal{M}|$ (see the condition (vi) in Section 3). This point will serve as an initial point for the piecewise smooth vd algorithms as well as the simplicial vd algorithms which will be given in the next section.

Lemma 5.1. $h^{-1}(c) \cap |\partial\mathcal{M}|$ consists of the single point $z^0 = (x^0, y^0, t^0)$.

Proof. Let $z = (x, y, t)$ be an arbitrary point in $h^{-1}(c) \cap |\partial\mathcal{M}|$. Then we immediately have $t = 0$ from (5.1). This implies that $y = c$. Since $c \in \text{int } Y_0$ by the condition (vi), $x \in Y_0^d = \{0\}$. This completes the proof. Q.E.D.

In the remainder of this section, we assume:

Condition 5.2. c is a regular value of the PC^1 mapping $h : |\mathcal{M}| \to R^n$.

Then the solution set $h^{-1}(c)$ of the system (5.5) forms a disjoint union of piecewise smooth paths and loops which satisfy the properties (i) - (iv) of Theorem 2.1. We focus our attention to the connected component of $h^{-1}(c)$ which contains the initial point z^0. We denote it by S.

Lemma 5.3. Assume the conditions (i) - (vii) in Section 3 and Condition 5.2. Then S is a semi-closed and unbounded path. If, in addition, there exists a bounded subset U of R^n such that

$$x \in U \quad \text{for all } (x, y, t) \in S, \qquad (5.6)$$

then there exists a bounded subset W of $R^n \times D$ such that

$$S \subset W \times R_+. \quad (5.7)$$

Proof. Since $z^0 \in |\partial\mathcal{M}|$, S is either a semi-closed path or a closed path. If the latter case occurred, the path S would have two distinct endpoints on $|\partial\mathcal{M}|$. This contradicts Lemma 5.1. Thus S is a semi-closed path. Furthermore, since $|\mathcal{M}|$ is closed, we see by (iv) of Theorem 2.1 that S is unbounded. Thus we have shown the first assertion.

Assume that (5.6) holds for some bounded subset U of R^n. Let

$$\mathcal{Q} = \{ X \in \bar{\mathcal{P}} : X \cap U \neq \emptyset \},$$
$$\mathcal{E} = \{ X^d \in \bar{\mathcal{D}} : X \in \mathcal{Q} \} .$$

Then it is clear that

$$S \subset U \times |\mathcal{E}| \times R_+.$$

Since $\mathcal{P}$ is locally finite (see the definition of subdivided manifolds in Section 2), $\mathcal{Q}$ is finite, and so is $\mathcal{E}$. By the condition (ii) imposed on the PDM in Section 3, we have that $|\mathcal{E}|$ is bounded. Therefore $W = U \times |\mathcal{E}|$ is the desired subset of $R^n \times D$. Q.E.D.

If (5.7) holds for some bounded subset W of $R^n \times D$ as in Lemma 5.3, each $(\bar{x}, \bar{y}, \bar{t}) \in S$ with sufficiently large $\bar{t}$ satisfies

$$g(\bar{x}) = - \bar{y} / \bar{t} \fallingdotseq 0;$$

$\bar{x}$ is an approximate solution of the system (5.3). For tracing the path S numerically, we can employ various predictor-corrector procedures developed in the homotopy continuation methods (Li and Yorke [14], Allgower and Georg [2], Georg [4], etc.). See Kojima [6] for more detail.

Remark 5.4. When we trace the solution path S of the system (5.5), we may encounter numerical unstability as the value of t increases. It might be better to employ the system of equations

$$(1-t)y + tg(x) = 0, \quad (x, y, t) \in |\mathcal{L}| \times [0, 1).$$

In this case we obtain a solution to the system (5.3) when the variable t attains 1.

In what follows we explain the variable dimension structure of the solution path S. The system of equations (5.5) is decomposed into the family of systems of equations:

$$y + tg(x) = c, \quad (x, y, t) \in X \times X^d \times R_+ \quad (X \in \bar{\mathcal{P}}). \quad (5.8)$$

Suppose S moves from an (n+1)-cell $X_1 \times X_1^d \times R_+$ into another (n+1)-cell $X_2 \times X_2^d \times R_+$ of $\mathcal{M}$ penetrating their common n-face. As shown in Section 3 of Kojima and Yamamoto [7] (see also Kojima [6]), either

$$X_1 < X_2 \quad \text{and} \quad \dim X_1 = \dim X_2 - 1$$

or

$$X_2 < X_1 \quad \text{and} \quad \dim X_2 = \dim X_1 - 1$$

occurs, i.e., the dimension of cells of $\bar{\mathcal{P}}$ varies by unity as the path S moves into a new (n+1)-cell of $\mathcal{M}$. To make the explanation more concrete let us focus our attention to the new PDM introduced in Section 4. Let $z^0 = (x^0, y^0, t^0) = (0, c, 0) \in |\partial\mathcal{M}|$, where $\| c \|$ is so small that $c \in \text{int } Y_0$. Then z^0 satisfies

$$y + tg(x) = c, \quad (5.9)$$

$$(x, y, t) \in \{0\} \times Y_0 \times R_+.$$

The variable vector y plays as a slack variable, so that we eliminate y as well as the variable vector x which is fixed to zero. Then we obtain

$$c - tg(0) \in Y_0, \qquad t \in R_+.$$

We may assume that $g(0) \neq 0$. By the ratio test we easily find

$$t^1 = \sup \{ t : c - tg(0) \in Y_0, t \in R_+ \} .$$

Let $y^1 = c - t^1 g(0)$. Then y^1 is in some (n-1)-face , say $Y(p, p)$, of Y_0. Hence, by Condition 5.2, we see that $z^1 = (0, y^1, t^1)$ lies in the relative interior of the n-cell $\{0\} \times Y(p, p) \times R_+$. This n-cell is a common face of $\{0\} \times Y_0 \times R_+$ and $X(p, p) \times Y(p, p) \times R_+$. Thus we move into the (n+1)-cell $X(p, p) \times Y(p, p) \times R_+$ along the path S. In this case the dimension of cells of $\bar{\mathcal{P}}$ increases and that of cells of $\bar{\mathcal{D}}$ decreses by one. In general, let $z^k = (x^k, y^k, t^k)$ be an intersecting point of S with a common n-face of $X(s, t) \times Y(s, t) \times R_+$ and $X(s', t') \times Y(s', t') \times R_+$. Suppose that

$$X(s', t') < X(s, t) \quad \text{and} \quad \dim X(s', t') = \dim X(s, t) - 1.$$

Then by (4.2) and Lemma 4.1, either

$$|I(t) \setminus I(t')| = 1 \quad \text{or} \quad |I(s') \setminus I(s)| = 1$$

occurs. In this way as we trace the path S, one of the sign vectors s and t increases or decreases its nonzero components by one. Figure 5.1 illustrates the path S in a two dimensional case.

The following theorem provides a sufficient condition for

the global convergence of the smooth vd algorithms.

Theorem 5.5. (Theorem 5.1 in Kojima [6])

In addition to the conditions (i) - (vii) imposed on the PDM in Section 3, suppose that for some $\mu > 0$ and for every $x \in R^n$ with $\| x \| \geqq \mu$ there exists an $\hat{x} \in R^n$ such that

$$\| \hat{x} \| \leqq \mu \quad \text{and} \quad (x - \hat{x})^t g(x) > 0$$

(a weaker version of Merrill's condition, see [15]). Then there exists a bounded set U such that (5.6) holds.

Remark 5.6. Let $\mathcal{P}^*$ be a refinement of the primal subdivided manifold $\mathcal{P}$. Suppose that the mapping $g : R^n \rightarrow R^n$ is PC^1 on $\mathcal{P}^*$. Let

$$\mathcal{P}^* | X = \{ \sigma \in \bar{\mathcal{P}}^* : \sigma \subset X, \dim \sigma = \dim X \}$$

for every $X \in \bar{\mathcal{P}}$. All the results obtained in this section remain valid if we replace $\mathcal{L}$ and $\mathcal{M}$ by

$$\mathcal{L}^* = \{ \sigma \times X^d : \sigma \in \mathcal{P}^* | X, X \in \bar{\mathcal{P}} \} \tag{5.10}$$

and

$$\mathcal{M}^* = \{ Z \times R_+ : Z \in \mathcal{L}^* \} , \tag{5.11}$$

respectively. Note that $\mathcal{L}^*$ and $\mathcal{M}^*$ form refinements of $\mathcal{L}$ and $\mathcal{M}$, respectively (Lemma 3.2). This observation will lead us to the simplicial vd algorithms.

6. Simplicial Vd Algorithms

Throughout this section, we shall assume that $g : R^n \to R^n$ is continuous but not necessarily continuously differentiable. We employ the same symbols and notations as in the previous section. Let T be a simplicial refinement of the primal subdivided manifold $\mathcal{P}$. We assume that T has a finite mesh size δ;

$$\delta = \sup_{\sigma \in T} \sup \{ \|u-v\| : u, v \in \sigma \} < +\infty \tag{6.1}$$

For example, we can employ the triangulations J_1 and K' of R^n (see Todd [20]) in the case of the PDM's given in the four examples in Section 3 and the new PDM in Section 4. That is, they are legitimate triangulations for Merrill's method and the simplicial (n+1)-, 2n-, 2^n-, and (3^n-1)-methods. For every cell X of $\bar{\mathcal{P}}$, let

$$T \mid X = \{ \sigma \in \bar{T} : \sigma \subset X, \dim \sigma = \dim X \} .$$

Then $T \mid X$ forms a simplicial subdivision of X. Define

$$\mathcal{N} = \{ \sigma \times X^d \times R_+ : \sigma \in T \mid X, X \in \bar{\mathcal{P}} \} .$$

By Lemma 3.2 (see also Remark 5.6) we see that $\mathcal{N}$ is a refinement of $\mathcal{M}$, and that

$$|\partial \mathcal{N}| = |\partial \mathcal{M}| = \bigcup \{ X \times X^d \times \{0\} : X \in \bar{\mathcal{P}} \}.$$

We shall approximate the PC^1 mapping $h : |\mathcal{M}| \to R^n$ by another PC^1 mapping $H : |\mathcal{N}| \to R^n$ as follows. Let $G : |T| \to R^n$ be a PL approximation of g with respect to the triangulation T of R^n, i.e., for every $x \in \sigma \in T$, define

$$G(x) = \sum_{i=0}^{n} \lambda_i g(v^i) \tag{6.2}$$

where

$$x = \sum_{i=0}^{n} \lambda_i v^i, \qquad \sum_{i=0}^{n} \lambda_i = 1 \tag{6.3}$$

$$\lambda_i \geqq 0 \quad \text{for } i = 0,1,\dots,n$$

and $v^0, v^1, \dots, v^n$ are the vertices of the simplex σ. Then we can locate an approximate solution of the system (5.3) by solving the system of PL equations

$$G(x) = 0, \qquad x \in |T| = R^n. \tag{6.4}$$

The simplicial vd algorithm described below can be regarded as a piecewise smooth vd algorithm to the system (6.4). In a major cycle of the application of the simplicial vd algorithm to the system (5.3), we calculate an exact solution x^* of the system (6.4); hence it is an approximation of a solution of the system (5.3). If we need an approximate solution with a higher accuracy, we will restart a major cycle after replacing w^0 by $w^0 + x^*$ in the definition (5.2) of the mapping g and the triangulation T by a finer triangulation T'. When we restart a major cycle, we may utilize Saigal's acceleration technique (Saigal [17]).

We now replace the mapping $g : R^n \to R^n$ by its PL approximation $G : |T| \to R^n$ in the definition (5.4) of the mapping $h : |\mathcal{M}| \to R^n$ to obtain the PC^1 mapping $H : |\mathcal{N}| \to R^n$;

$$H(x, y, t) = y + tG(x) \quad \text{for every } (x, y, t) \in |\mathcal{N}| , \tag{6.5}$$

and consider the system of equations

$$H(x, y, t) = c, \qquad (x, y, t) \in |\mathcal{N}| . \tag{6.6}$$

Note that H is always PC^1 even if g is not continuous. In the remainder of this paper, we shall assume:

<u>Condition 6.1.</u> $c \in \text{int } Y_0$ is a regular value of the PC^1 mapping $H : |\mathcal{N}| \to R^n$.

Then we can apply the similar arguments to the system (6.6) as we have done to the system (5.5) in the previous section (see also Remark 5.6). Specifically, we obtain the two lemmas below:

<u>Lemma 6.2.</u> $H^{-1}(c) \cap |\partial\mathcal{N}|$ consists of the single point $z^0 = (x^0, y^0, t^0) = (0, c, 0)$.

Let $\hat{S}$ denote the connected component of $H^{-1}(c)$ which contains the point z^0.

<u>Lemma 6.3.</u> Assume the conditions (i) - (vii) in Section 3 and Condition 6.1. Then $\hat{S}$ is a semi-closed and unbounded path. If, in addition, there exists a bounded subset U of R^n such that

$$x \in U \quad \text{for all} \quad (x, y, t) \in \hat{S} \tag{6.7}$$

then there exists a bounded subset W of $R^n \times D$ such that

$$\hat{S} \subset W \times R_+. \tag{6.8}$$

The simplicial vd algorithm traces the path $\hat{S}$. If (6.8) holds for some bounded subset W of $R^n \times D$, then each $(\bar{x}, \bar{y}, \bar{t}) \in \hat{S}$ with a sufficiently large $\bar{t}$ satisfies

$$G(\bar{x}) = - \bar{y} / \bar{t} \fallingdotseq 0.$$

Thus $\bar{x}$ is an approximate solution to the system (6.4). Furthermore, the direction of an unbounded ray of $\hat{S}$ provides us with an exact solution of the system (6.4). This will be shown in

Theorem 6.4.

The mapping $H : |\mathcal{N}| \to R^n$ in (6.5) is not PL generally. But note that for each fixed t, $H(., ., t)$ is PL with respect to the variable vector (x, y) and that for each fixed x, $H(., x, .)$ is linear with respect to the variable vector (y, t). We will exploit these features of the mapping H in Section 7 to trace the path S by applying complementary pivoting to the family of linear systems induced from the system (6.6).

If a continuous mapping $\phi : R^n \to R^m$ has a special structure such as (partial) linearity, separability and sparsity, the PL approximation Φ of ϕ on the triangulation K' (or J_1) is PL on a coarser subdivision than the triangulaion; it is affine on the union of some simplices of K' (or J_1). Todd [22,23,24,26] applied this result to a homotopy associated with Merrill's method, and proposed some techniques for increasing computaional efficiency of the method. In our succeeding discussions, we shall deal with the case where the mapping $g: R^n \to R^n$ has such a special structure. So it will be convenient to introduce a coarser subdivision on which the mapping G is PL. Let $\mathcal{P}^*$ be a refiniment of $\mathcal{P}$ such that each cell $\sigma \in \mathcal{P}^*$ is the intersection of a piece of linearity of the PL mapping $G : |T| \to R^n$ with some $X \in \mathcal{P}$. Define the subdivided manifolds $\mathcal{L}^*$ and $\mathcal{N}^*$ by (5.10) and (5.11), respectively. Then we see that the mapping H is PC^1 on the subdivided manifold $\mathcal{N}^*$ which is a coarser refinement of $\mathcal{N}$ than $\mathcal{N}$. For simplicity of discussion we assume:

Condition 6.4. Each $\sigma \in P^*$ is bounded.

Theorem 6.5. Suppose that $Z = \sigma \times X^d \times R_+ \in \mathcal{H}^*$ and that $Z \cap H^{-1}(c) \neq \emptyset$. Then

(i) $Z \cap H^{-1}(c)$ consists of a single smooth path which is either closed (if it is bounded) or semi-closed (if it is unbounded),

and

(ii) if $Z \cap H^{-1}(c)$ is a semi-closed path, then it can be written as

$$\{ ((\bar{x}+\theta x^*)/(\bar{t}+\theta),\ \bar{y},\ \bar{t}+\theta) : \theta \geqq 0 \} \tag{6.9}$$

for some $\bar{x} \in R^n$, $x^* \in G^{-1}(0)$, $\bar{y} \in R^n$ and $\bar{t} \in R_+$. Here we assume $0/0 = 0$.

Proof. First we deal with the case that $X = \{0\}$. Then, $Z \cap H^{-1}(c)$ turns out to be the set of solutions (x, y, t) to the linear system

$$y + tG(x) = c, \qquad y \in Y_0 \text{ and } x = 0. \tag{6.10}$$

Hence we have

$$Z \cap H^{-1}(c) = \{(0, c-tg(0), t) : 0 \leqq t \leqq t^1 \}, \tag{6.11}$$

where

$$t^1 = \sup \{ t : c - tg(0) \in Y_0,\ t \in R_+ \}. \tag{6.12}$$

Hence $Z \cap H^{-1}(c)$ is a line segment if $G(0) = g(0) \neq 0$ (because Y_0 is a bounded polyhedral set), and it is a half line if $G(0) = g(0) = 0$. Thus (i) as well as (ii) follows if we take $\bar{x} = x^* = 0$, $\bar{y} = c$ and $\bar{t} = 0$.

Now suppose that $X \neq \{0\}$. Then, by Lemma 6.2, $t > 0$ for any $(x, y, t) \in Z \cap H^{-1}(c)$. Since G is affine on σ, there exists an $n \times n$ matrix A and an n-dimensional vector a such that

$$G(x) = Ax + a \qquad \text{for every } x \in \sigma. \tag{6.13}$$

Hence, if we restrict the variable vector (x, y, t) to the cell $Z = \sigma \times X^d \times R_+$, the system (6.6) can be written as

$$\begin{aligned} & y + A(tx) + ta = c, \\ & x \in \sigma \in P^*|X, \quad y \in X^d, \quad t \in R_+; \end{aligned} \tag{6.14}$$

$Z \cap H^{-1}(c)$ is the set of points (x, y, t) satisfying these relations.

On the one hand, since σ and X^d are cells in R^n, we have

$$\begin{aligned} \sigma &= \{ x \in R^n : B^1x + C^1w^1 + b^1 = 0,\ w^1 \geqq 0 \}, \\ X^d &= \{ y \in R^n : B^2y + C^2w^2 + b^2 = 0,\ w^2 \geqq 0 \}, \end{aligned} \tag{6.15}$$

where B^j, C^j and b^j are matrices and a vector with appropriate sizes and C^j is of column full rank for $j = 1, 2$. Thus if $(x, y, t) \in R^{2n+1}$ satisfies (6.14), then $(u, v, s) = (tx, y, t)$ satisfies the linear system

$$\begin{aligned} & v + Au + sa = c, \\ & B^1u + C^1w^1 + sb^1 = 0, \quad w^1 \geqq 0, \quad s \geqq 0 \\ & B^2v + C^2w^2 + b^2 = 0, \quad w^2 \geqq 0, \end{aligned} \tag{6.16}$$

for some $w^1 \geqq 0$ and $w^2 \geqq 0$. Conversely, if (u, v, s) with $s > 0$ satisfies (6.16) for some $w^1 \geqq 0$ and $w^2 \geqq 0$ then $(x, y, t) = (u/s, v, s)$ satisfies (6.14).

Let L denote the set of all solutions (u, v, s) of (6.16). Then L forms a polyhedral set with a dimension at least one because L contains at least two distinct points

(tx, y, t) and $(t'x', y', t')$ with $t > 0$ and $t' > 0$, where (x, y, t) and (x', y', t') are distinct points of $Z \cap H^{-1}(c)$. If L were more than one dimensional, the set $Z \cap H^{-1}(c)$ could not be the disjoint union of paths and loops. This is a contradiction. Thus we have seen that L is a line segment, a half line or a line. We also see

$$Z \cap H^{-1}(c) \subset \{ (u/s, v, s) ; (u, v, s) \in L, s > 0 \}. \quad (6.17)$$

Note that L contains a point (u, v, s) with $s > 0$. So if L were a line, then s should be a positive constant for all $(u, v, s) \in L$. This contradicts the boundedness of $\sigma \times X^d$.

Assume that s is zero at some $(u, v, s) \in L$. Then from the second equality of (6.16) we have $u = 0$ and $w^1 = 0$ because the cell σ represented as in (6.15) is bounded (Condition 6.4). It follows from the first equality of (6.16)' that $v = c$, which implies $X^d = Y_0$ or $X = \{0\}$ (the condition (vi) in Section 3). This contradicts the assumption that $X \neq \{0\}$. Thus we have shown

$$s > 0 \quad \text{for all} \quad (u, v, s) \in L$$

and

$$Z \cap H^{-1}(c) = \{(u/s, v, s) : (u, v, s) \in L \} . \quad (6.18)$$

This ensures that $Z \cap H^{-1}(c)$ is connected and is homeomorphic to neither the open interval nor the unit circle. By Theorem 2.1, we obtain (i).

To see (ii), assume that L is a half line. Then it can be

written as

$$L = \{ (\bar{u}+\theta\ \Delta u,\ \bar{v}+\theta\ \Delta v,\ \bar{s}+ \Delta s) : \theta \geqq 0 \}$$

for some $(\bar{u}, \bar{v}, \bar{s}) \in R^{2n+1}$ with $\bar{s} > 0$ and some nonzero $(\Delta u, \Delta v, \Delta s) \in R^{2n+1}$. It follows from $\bar{v}+\theta\ \Delta v \in X^d$ for all $\theta \geqq 0$ and the condition (ii) in Section 3 that $\Delta v = 0$. We can also see that $\Delta s > 0$. In fact, if Δs were zero, we would have

$$(\Delta u, \Delta w) \neq 0, \quad B^1\ \Delta u + C^1\ \Delta w = 0 \quad \text{and} \quad \Delta w \geqq 0,$$

which is contrary to the boundedness of σ again (Condition 6.4). Therefore we may assume without loss of generality that $\Delta s = 1$; replace $(\Delta u, \Delta v, \Delta s)$ by $(\Delta u/ \Delta s, , \Delta v/ \Delta s, 1)$ if necessary. Finally we observe that $(\Delta u, \Delta v, \Delta s)$ satisfies

$$\Delta v + A\ \Delta u + \Delta s a = 0.$$

Since $\Delta s = 1$ and $\Delta v = 0$, this equality implies

$$G(\Delta u) = 0. \quad (6.19)$$

Let $\bar{x} = \bar{u}$, $x^* = \Delta u$, $\bar{y} = \bar{v}$ and $\bar{t} = \bar{s}$. Then the desired result follows directly from (6.18) and (6.19). Q.E.D.

Remark 6.6. Condition 6.4 is not restrictive. If it is not satisfied, then we may replace $\mathcal{P}^*$ by its appropriate refinement which consists of bounded cells. Specifically, the theorem above remains valid if we replace $\mathcal{P}^*$ and $\mathcal{M}^*$ by T and $\mathcal{N}$, respectively.

Remark 6.7. In the four examples of Section 3, each dual cell $X^d \in \bar{\mathcal{D}}$ is represented by a simple system of linear equalities and inequalities (see (3.2), (3.4), (3.6) and (3.8)). As for the new PDM introduced in Section 4, we have seen in (4.5) and

(4.6) that

$$X(s,\ t)^d = \{\ y\ :\ p^t y \leqq \beta + (|I(p)| - 1)\gamma \quad \text{for } p \in \Sigma \setminus \{s,t\},$$
$$q^t y = \beta + (|I(q)| - 1)\gamma \quad \text{for } q \in \{s,t\}\ \}.$$

Furthermore, we have also seen that the above system is simplified as in (4.7) which consists of only one linear equality and several bounded variable constraints. These simplicities will contribute to keeping the basis matrix of the system (6.16) small.

Corollary 6.8. Suppose that the condition (i) - (vii) in Section 3, Condition 6.1 and Condition 6.4 hold. If there exists a bounded subset U of R^n for which (6.7) holds, then $\hat{S}$ consists of a finite number of smooth closed paths and a single semi-closed path of the form (6.9).

Proof. By Lemma 6.3, (6.8) holds for some bounded subset W of $R^n \times D$. Hence the path $\hat{S}$ intersects with finitely many cells of $\mathcal{M}^*$. Thus the desired result follows from Theorem 6.5.

Q.E.D.

The result above ensures that we can compute a solution of the PL system (6.4) by tracing the path $\hat{S}$ if the assumption (6.7) holds for some bounded subset U of R^n. The following theorem gives a sufficient condition on the mapping g for (6.7) to hold.

Theorem 6.9. In addition to the conditions (i) - (vii) in Section 3, suppose that for every positive δ, some positive μ (which may be dependent on δ) and every $x \in R^n$ with $\|x\| \geqq \mu$ there exists an $\hat{x} \in R^n$ such that $\|\hat{x}\| \leqq \mu$ and

$(x - \hat{x})^t g(v) > 0$ whenever $\|v - x\| < \delta$. Then there exists a bounded subset U of R^n for which (6.7) holds.

Proof. By the assumption (6.1) of the triangulation T, it has a finite mesh size δ. We shall establish that for every $x \in R^n$ with $\| x \| \geqq \mu$ there exists an $\hat{x} \in R^n$ such that $\|\hat{x}\| \leqq \mu$ and that $(x - \hat{x})^t G(x) > 0$. Then the desired result follows from Theorem 5.5 (see Remark 5.6).

Suppose that $x \in R^n$ and $\|x\| \geqq \mu$. Then, by the definition of the PL mapping $G : |T| \to R^n$, we can find a simplex σ and numbers λ_i ($i = 0,1,\ldots,n$) for which (6.2) and (6.3) hold. By the assumption, there exists an $\hat{x} \in R^n$ such that

$$\| \hat{x} \| \leqq \mu ,$$
$$(x - \hat{x})^t g(v^i) > 0 \quad \text{for } i = 0,1,\ldots,n.$$

The last inequalities and (6.2) imply $(x - \hat{x})^t G(x) > 0$. Q.E.D.

7. Numericl Methods for Tracing the Path $\hat{S}$

In this section we shall briefly show two methods of computing the path $\hat{S}$ numerically.

7.1. The Global Method

The essential idea of the first method is based on the paper (22) by Todd in which he has proposed a "global" technique for fixed point algorithms to trace piecewise linear paths efficiently when the system of equations to be solved has special structures such as (partial) linearity, separability and sparsity. This method works effectively for systems of equations having such structures.

As we have stated just before Condition 6.4, if the mapping $g : R^n \rightarrow R^n$ enjoys such structures and if we employ K' or J_1, the refinement $\mathcal{M}^*$ of $\mathcal{M}$ is coarser than $\mathcal{N}$; each piece of $\mathcal{M}^*$ is the union of several cells of $\mathcal{N}$. The proof of Theorem 6.5 shows a possibility that we can traverse a piece of $\mathcal{M}^*$ in one iteration.

Suppose that $\sigma \in P^*|X$, $X \in \bar{P}$, $Z = \sigma \times X^d \times R_+ \in \mathcal{M}^*$ and that $Z \cap \hat{S} \neq \emptyset$. If $X = \{0\}$, we have (6.11) and (6.12). Thus the computation of $Z \cap \hat{S}$ can be carried out very easily by the ordinary ratio test.

Now suppose that $X \neq \{0\}$. Assuming that an endpoint $(\bar{x}, \bar{y}, \bar{t})$ of the path $Z \cap \hat{S}$ is already known, we shall show how to generate the path $Z \cap \hat{S}$. If we write the restriction of the mapping G to the piece σ as in (6.13), $Z \cap \hat{S} = Z \cap H^{-1}(c)$ coincides with the set of solutions to the system (6.14). Furthermore, representing the cells σ and X^d by using linear equalities and inequalities as in (6.15), we have the equality (6.18), where L denotes the set of solutions to the linear system (6.16). Hence the set $Z \cap \hat{S}$ can be calculated by solving the linear system (6.16). As we have observed, L forms either a half line or a line segment. Note that the endpoint $(\bar{x}, \bar{y}, \bar{t})$ of $Z \cap \hat{S}$ corresponds to the endpoint $(\bar{u}, \bar{v}, \bar{s}) = (\bar{t}\bar{x}, \bar{y}, \bar{t})$ of L. Hence if we maintain basis inverse matix of the linear system (6.16) at the point $(\bar{u}, \bar{v}, \bar{s})$, we can easily compute the direction $(\Delta u, \Delta v, \Delta s)$ of L and apply the ratio test to obtain

$$\hat{\theta} = \sup \{ \theta : (\bar{u},\bar{v},\bar{s}) + \theta(\Delta u, \Delta v, \Delta s) \in \sigma \times X^d \times R_+ \} .$$

Thus we have the other endpoint $(\hat{u}, \hat{v}, \hat{s}) = (\bar{u}, \bar{v}, \bar{s}) + \hat{\theta}(\Delta u, \Delta v, \Delta s)$ if $\hat{\theta} < +\infty$ or we find that L is unbounded if $\hat{\theta} = +\infty$. In the latter case we see that $v = 0$ and that $\Delta u / \Delta s$ is a solution to the system (6.4) (i.e., an approximate solution of the system (5.3)); hence we stop the computation of the path $\hat{S}$ and then restart the method after updating the initial point and the mesh size of the triangulation if we need a finer approximate solution of (5.3).

In the former case the point $(\hat{x}, \hat{y}, \hat{t}) = (\hat{u}/\hat{s}, \hat{v}, \hat{s})$ is the other endpoint of the $Z \cap \hat{S}$; $Z \cap \hat{S}$ is a smooth path jointing the two endpoints $(\bar{x}, \bar{y}, \bar{t})$ and $(\hat{x}, \hat{y}, \hat{t})$. In this case the path $\hat{S}$ intersects with an n-dimensional face V of $Z = \sigma \times X^d \times R_+$. Hence we need to generate a new (n+1)-cell of $\mathcal{M}^*$ which the path $\hat{S}$ moves in. There are three possibilities:

(i) $V = \tau \times X^d \times R_+$, $\tau < \sigma$, $\dim \tau = \dim \sigma - 1$ and τ does not lie on the relative boundary of X.

(ii) $V = \tau \times X^d \times R_+$, $\tau < \sigma$, $\dim \tau = \dim \sigma - 1$ and τ lies on the relative boundary of X.

(iii) $V = \sigma \times Y' \times R_+$, $Y' < X^d$, $\dim Y' = \dim X^d - 1$.

If (i) occurs then let $\hat{Z} = \hat{\sigma} \times X^d \times R_+$, where $\hat{\sigma}$ is a cell of $P^* \mid X$ which shares a common face τ with the cell σ. If (ii) occurs, then let $\hat{Z} = \tau \times \hat{X}^d \times R_+$, where $\hat{X}$ is a facets of X containing τ, i.e., $\tau \subset \hat{X} < X$ and $\dim \hat{X} = \dim X - 1$. If (iii) occurs, then let $\hat{Z} = \hat{\sigma} \times Y' \times R_+$, where $\hat{\sigma}$ is a cell of $P^* \mid (Y')^d$ containing σ as its face. In all the cases, replacing $(\bar{x}, \bar{y}, \bar{t})$ by $(\hat{x}, \hat{y}, \hat{t})$ and Z by $\hat{Z}$, we repeat the same procedure. We may apply some other techniques developed by Todd (26). The details are omitted here.

7.2. The Local Method

For general systems of equations without special structures, the global method described above is not efficient; we have to represent the cells σ and X^d by linear equalities and inequalities as in (6.15) and have to update the matrix A and the vector a associated with the restriction of the PL mapping G to the piece $\sigma \in P^*$. The amount of computaion required by these works is worthy of being comsumed only when we can traverse a very large piece of $\mathcal{M}^*$ each of which consists of several cells of $\mathcal{N}$. The local method gives us an efficient way of traversing a piece of $\mathcal{N}$.

Suppose that $\sigma \in \mathcal{T} \mid X$, $X \in \bar{P}$, $Z = \sigma \times X^d \times R_+ \in \mathcal{N}$ and

that $Z \cap \hat{S} \neq \emptyset$. Assuming that an endpoint $(\bar{x}, \bar{y}, \bar{t})$ of $Z \cap \hat{S}$ has already been computed, we shall show how to generate $Z \cap \hat{S}$. Consider the system of equatins

$$\begin{aligned} &y + tG(x) = c \\ &x \in \sigma, \quad y \in X^d \quad \text{and} \quad t \in R_+. \end{aligned} \tag{7.1}$$

$Z \cap \hat{S}$ coincides with the set of all solutions to this system. Since σ is a simplex, every $x \in \sigma$ is represented as

$$\begin{aligned} &x = \sum_{i=0}^{k} \lambda_i v^i, \\ &\sum_{i=0}^{k} \lambda_i = 1; \qquad \lambda_i \geqq 0 \quad \text{for} \quad i = 0,1,\ldots,k, \end{aligned} \tag{7.2}$$

where k is the dimension of the simplex σ and $v^0, v^1, \ldots, v^k$ are the vertices of σ. By the construction, we have

$$G(x) = \sum_{i=0}^{k} \lambda_i g(v^i) \tag{7.3}$$

for every $x \in \sigma$ satisfying (7.2). Substituting (7.3) into (7.1), we have

$$\begin{aligned} &y + t \sum_{i=0}^{k} \lambda_i g(v^i) = c, \\ &\sum_{i=0}^{k} \lambda_i = 1; \quad \lambda_i \geqq 0 \quad \text{for } i = 0,1,\ldots,k, \\ &y \in X^d. \end{aligned} \tag{7.4}$$

Furthermore letting

$$\mu_i = t\ \lambda_i \quad \text{for } i = 0,1,\ldots,k,$$

and representing the cell X^d as in (6.15), we have the system of linear equations

$$y + \sum_{i=0}^{k} \mu_i g(v^i) = c,$$

$$B^2 y + C^2 w + b^2 = 0, \qquad (7.5)$$

$$\mu = (\mu_0, \mu_1, \ldots, \mu_k) \geqq 0, \quad w \geqq 0,$$

where B^2, C^2 and b^2 are matrices and a vector with appropriate sizes, say $m \times n$, $m \times n'$ and m, and $w \in R^{n'}$ is a slack variable vector. Note that we assume that C^2 is of column full rank. The system (7.1) is equivalent to the linear system (7.5) in the sense that (x, y, t) is a solution of the former if and only if there exists a solution (μ, y, w) of the latter which satisfies

$$t = \sum_{i=0}^{k} \mu_i,$$

$$x = \begin{cases} 0 & \text{if } t = 0 \\ \sum_{i=0}^{k} \mu_i v^i / t & \text{if } t > 0 \end{cases} \qquad (7.6)$$

Thus the computation of $Z \cap \hat{S}$ has been reduced to solving the linear system (7.5), which is simpler than the system (6.16) used in the global method. Note that (7.5) does not involve the constraint corresponding to $x \in \sigma$.

By the regurality assumption, the set of solutions (μ, y, w) of the linear system (7.5), which will be denoted by L', forms either a line segment or a half line. If it is a half line, we obtain its direction vector $(\Delta\mu, \Delta y, \Delta w) \neq 0$ satisfying the homogeneous linear system:

$$\Delta y + \sum_{i=0}^{k} \Delta\mu_i g_i(v^i) = 0,$$

$$B^2\ \Delta y + C^2\ \Delta w = 0 \qquad (7.7)$$

$$\Delta\mu \geqq 0, \quad \Delta w \geqq 0.$$

Since X^d is bounded (see the condition (ii) in Section 3), we have that $\Delta y = 0$. Since C^2 is of culumn full rank, it follows that $\Delta w = 0$ and

$$\sum_{i=0}^{k} \Delta\mu_i g(v^i) = 0$$

$$\Delta\mu \geqq 0, \quad \sum_{i=0}^{k} \Delta\mu_i > 0.$$

Letting

$$x^* = \left(\sum_{i=0}^{k} \Delta\mu_i v^i\right) / \left(\sum_{i=0}^{k} \Delta\mu_i\right)$$

and recalling the definition of the PL mapping $G : |T| \to R^n$, we thus obtain $G(x^*) = 0$; hence we stop the computation or restart the method in the same way as in the local method.

Suppose now that the set L' of solutions (μ, y, w) to the system (7.5) is a line segment, and let $(\hat{\mu}, \hat{y}, \hat{w})$ be the endpoint of L' which is different from $(\bar{\mu}, \bar{y}, \bar{w})$. The point $(\hat{\mu}, \hat{y}, \hat{w})$ is computed by performing the usual pivoting operation to the system (7.5). Let $(\hat{x}, \hat{y}, \hat{t})$ denote the point induced from $(\hat{\mu}, \hat{y}, \hat{w})$ by the transformation (7.6). Then $(\hat{x}, \hat{y}, \hat{t})$ turns out to be an endpoint of $Z \cap \hat{S}$; $Z \cap \hat{S}$ is a smooth path jointing the two points $(\bar{x}, \bar{y}, \bar{t})$ and $(\hat{x}, \hat{y}, \hat{t})$. The point $(\hat{x}, \hat{y}, \hat{t})$ lies on an n-dimensional face V of the cell Z. Then we find a new cell $\hat{Z}$ into which the path S moves. Therefore in the same way as in the global method, replacing Z and $(\bar{x}, \bar{y}, \bar{t})$ by $\hat{Z}$ and $(\hat{x}, \hat{y}, \hat{t})$, we will continue the same procedure.

When the mapping g has special structures, we can incorporate the "local" pivot-saving technique developed by Todd [23] into the method described above. Here we shall refer to Section 5.4 of Allgower and Georg [2] where a simple explanation of the local pivot-saving technique was given, and briefly show how we utilize it in our local method for saving some pivoting operations consumed while we move around inside each cell $X \times X^d \times R_+$ of $\mathcal{M}$. We shall write the linear system (7.5) as

$$B \begin{pmatrix} \mu \\ y \\ w \end{pmatrix} = b,$$

$$\mu = (\mu_0, \ldots, \mu_k) \geqq 0, \quad w = (w_1, \ldots, w_{n'}) \geqq 0. \tag{7.8}$$

Here B is an $(n+m) \times (1+k+n+n')$ matrix and b is an $(m+n)$-dimensional vector. The matrix B plays the role of the matrix L in (1) of Section 5.4 of [2]. Since the linear system (7.8) is nondegenerate and has a 1-dimensional solution set, the number $n+m$ of the rows of B is less than the number $1+k+n+n'$ of its columns by one; hence $m = k + n'$. Suppose that $\mu_1, \ldots, \mu_k, y_1, \ldots, y_n, w_1, \ldots, w_{n'}$ are basic variables and μ_0 is a nonbasic variable at a basic solution. We assume that we have already computed an $(n+m+1) \times (n+m)$ matrix C and a vector $d \in R^{n+m+1}$ such that $BC = I$, $Bd = 0$, the first row of C is zero and the first component of d is -1, where I denotes the $(n+m) \times (n+m)$ identity matrix. Note that the last $n+m$ columns of B (resp. C) form a basis matrix (resp. its inverse). Then we compute a new basic solution $(\hat{\mu}, \hat{y}, \hat{w})$, if it exists, as follows:

$$(\hat{\mu}, \hat{y}, \hat{w}) = (\bar{\mu}, \bar{y}, \bar{w}) - \theta^* d,$$

$$\theta^* = \sup \{ \theta : \theta \geqq 0, \ \mu - \theta \Delta\mu \geqq 0, \ \bar{w} - \theta \Delta w \geqq 0 \} .$$

Here $d = (\Delta\mu, \Delta y, \Delta w) \in R^{1+k+n+n'}$. Furthermore, we know that the path $\hat{S}$ intersects with a face V of Z at a point $(\hat{x}, \hat{y}, \hat{t})$ induced from $(\hat{\mu}, \hat{y}, \hat{w})$ by the transformation (7.6). Assume that the case (i) of subsection 7.1 occurs and that G is affine on $\sigma \cup \hat{\sigma}$, where $\hat{\sigma} \in T|X$ is a k-dimensional simplex which shares a common face τ with σ. In this case we can utilize the same technique given in Section 5.4 of [2] to update the matrix C and the vector d. The details are omitted here.

8. Computational Results

To compare the efficiency of the new vd algorithm, (3^n-1)-method with those of other fixed point algorithms, we have made programs of four methods in FORTRAN IV using double precision arithmetic:

(i) Merrill's method (Example 3.1),
(ii) the 2n-method (Example 3.3),
(iii) the 2^n-method (Example 3.4), and
(iv) the (3^n-1)-method (Section 4).

Merrill's method was originally proposed as a method for tracing a solution path of a PL mapping from $R^n \times (0, 1]$ into R^n. We have, however, shown in [7] that it can be viewed as a vd algorithm on the PDM consisting of the two checkerboard subdivisions (see Example 3.1 in Section 3) of R^n, and have proposed an efficient implementation, which requies only a triangulation of R^n and saves a large amount of pivoting operations in the artificial level. The program tested here is based on this efficient implementation. The (3^n-1)-method has a parameter γ which the behavior of the method will depend on. If γ becomes close to zero, the solution path generated in the method becomes almost the same as in the 2^n-method, while if γ becomes close to $1/n$, the solution path becomes almost the same as in the 2n-method. Hence we have tested the (3^n-1)-method for three variants of γ: $0.2/(n+1)$, $0.5/(n+1)$ and $0.8/(n+1)$.

In all the methods we have used K' as a triangulation T of R^n because it is legitimate for all the primal subdivided manifolds in the methods (i) - (iv). We have employed the system (7.5) (the local method) for all the methods but we have not used the pivot-saving technique explained in Subsection 7. . We have built in all the methods Saigal's acceleration technique [17], which improves the efficiency of the algorithm at final several stages.

The computation is carried out as follows. For an $\varepsilon_1 > 0$, let x^0 be an n-dimensional vector such that

$$x_i^0 = -\varepsilon_1 (n+1-i)/(n+1) \quad \text{for } i=1,2,\dots,n.$$

For an arbitrarily chosen point $w^0 \in R^n$ let

$$g^1(x) = f(x + w^0 + x^0) \quad \text{for every } x \in R^n,$$

and let $G^1(x) : |\varepsilon_1 K'| \to R^n$ be a PL approximation of $g^1(x)$ with respect to $\varepsilon_1 K' = \{ \varepsilon_1 \sigma : \sigma \in K' \}$. In the first cycle we solve the system of PL equations

$$G^1(x) = 0, \quad x \in |\varepsilon_1 K'| = R^n$$

starting from the origin. Then we obtain an approximate solution w^1 of (1.1) and an approximate Jacobian inverse W^1 of the mapping f at w^1. Let

$$\varepsilon_2 = \min \{ \varepsilon_1 / 2, \quad 4n \| W^1 f(w^1) \| \},$$

$$x_i^1 = -\varepsilon_2 (n+1-i)/(n+1) \quad \text{for } i=1,2,\dots,n,$$

$$g^2(x) = W^1 f(x + w^1 + x^1) \quad \text{for every } x \in R^n$$

and $G^2(x) : |\varepsilon_2 K'| \to R^n$ be a PL approximation of $g^2(x)$ with respect to $\varepsilon_2 K'$. In the second cycle we solve the system of PL equations

$$G^2(x) = 0, \quad x \in |\varepsilon_2 K'| = R^n.$$

In general, at the k-th cycle

$$\varepsilon_k = \min \{ \varepsilon_{k-1}/2, \ 4n\| W^{k-1}f(w^{k-1}) \| \},$$
$$x_i^{k-1} = - \varepsilon_k(n+1-i)/(n+1) \quad \text{for } i+1,2,\dots,n,$$
$$g^k(x) = W^{k-1}f(x + w^{k-1} + x^{k-1}) \quad \text{for every } x \in R^n,$$

$G^k(x) : |\varepsilon_k K'| \to R^n$ is the PL approximation of $g^k(x)$ with respect to $\varepsilon_k K'$ and we solve the system of PL equations

$$G^k(x) = 0, \quad x \in |\varepsilon_k K'| = R^n$$

starting from the origin.

Saigal's acceleration technique improves the efficiency of the methods at final several stages; however, we have found after some preliminary experiments that the technique is not always stable, especially when the aproximate Jacobian inverse W^{k-1} of f is badly conditioned. Hence we have not used the technique when $|\det W^{k-1}| < 10^{-4}$ or $|\det W^{k-1}| > 10^4$, i.e. we have replaced W^{k-1} by the $n \times n$ identity matrix and have set $\varepsilon_k = \varepsilon_{k-1}/2$.

Table 8.1 - 8.3 present the results of solving the three test problems in Allgower and Georg (2) for several dimensions:

(P1) $$f_i(x) = x_i - (\sum_{j=1}^{n} x_j^3 + i) / 2n,$$

(P2) $$f_i(x) = x_i - \exp (\cos (i \sum_{j=1}^{n} x_j)),$$

(P3) $$f_i(x) = \begin{cases} \prod_{j=1}^{n} x_j - 1 & \text{if } i = 1, \\ \sum_{j=1}^{n} x_j + x_i - (n + 1) & \text{otherwise.} \end{cases}$$

In all runs initial grid size ε_1 is 0.5 and $w^0 = 0$. We have stopped the computaions, with several exceptions indicated

by special symbols, when we have obtained an approximate solution x satisfying $\| f(x) \| \leqq 10^{-8}$. In these results there is no distinct difference between the (3^n-1)-method and the 2^n-method, but they appear faster than the other two methods. Especially, the advantage of the former two methods over the others bocomes more remarkable in Table 8.2, which shows the results of solving the highly nonlinear problem (P2). The mapping in (P3) is Brown's almost linear mapping which is known to have a badly conditioned Jacobian. It is noticable that Merrill's method is as fast or faster than the (3^n-1)-method and the 2^n-method for this problem.

Figure 8.1 shows one of the typical examples of solution histories of the four methods for the problem (P2). All the methods are accelerated by Saigal's acceleration technique. Hence they behave in almost the same way at final several stages. At initial stages, however, the (3^n-1)-method and the 2^n-method locate a roughly approximate solution by smaller number of pivoting operations than the other two methods. We attribute this advantage to the number of rays or subdivided 1-manifolds of the primal subdivided manifolds used in these methods.

To make clear the above advantage of the (3^n-1)- and 2^n-methods, we have solved the problem (P2) starting from a hundred initial points w^0 chosen at random such that all their components lie in the interval $[0, 3]$. Figure 8.2 and 8.3 show the averages and standard deviations of the numbers of pivoting operaions required for finding the first approximate solutions.

Here the parameter γ of the (3^n-1)-method has been fixed to $0.5/(n+1)$ and $\varepsilon_1 = 0.5$ in all runs. In these figures we report the statistics of only the number of pivoting operations because the numbers of function evaluations have been nealy equal to them in all runs. From these figures we can confirm that the (3^n-1)-method and the 2^n-method are more efficient than the other two methods in finding the first approximate solution.

We have generated a hundred systems of linear equations of the form

$$x - b = 0, \quad x \in R^n$$

for the dimensions n = 2,4,6,8,10,15 and have solved them, where the vector b is randomly chosen such that each component lies in (-10, 10). In Figure 8.4 and 8.5 we show the averages and standard deviations of the number of pivoting operations required by the methods for finding the first approximate solutions. We have taken $\gamma = 0.5/(n+1)$ in the (3^n-1)-method and $w^0 = 0$, $\varepsilon_1 = 1$ in all runs. There is no difference among the three vd algorithms for these problems. We have, however, obtained Figure 8.6 and 8.7 as the results of solving the systems of linear equations of the form

$$A^tA\ (\ x - b\ ) = 0, \quad x \in R^n.$$

For the dimensions n=2,4,6,8,10,15,20 a hundred problems have been generated by randomly choosing each component of the square matrix A and the vector b from the interval (-10, 10).

Figure 8.6 and 8.7 show the averages and standard deviations of the number of pivoting operations required by each method for finding the first approximate solutions. Here $\gamma = 0.5/(n+1)$, $w^0 = 0$ and $\varepsilon_1 = 1$. For these problems there exists a remarkable difference between the 2n-method and the more efficient two methods, (3^n-1)- and 2^n-methods.

The computational results reported above are only for the restricted test problems. But they give adequate reasons to conclude that the (3^n-1)-method and the 2^n-method are more efficient than Merrill's method and the 2n-method.

Finally we have to point out that there might be compatibility of the method and the triangulation used in it. In the tests reported above we have used K' as a triangulation of R^n. But in the preliminary tests we have made a program of Merrill's method with J_1 and have found it much less efficient for Problem (P1) than Merrill's method with K'. This fact poses a question which triangulation is suitable to which method. More research and more computational tests are needed to answer this question.

After we had finished typing this manuscript, we received the paper: G. van der Laan and A. J. J. Talman, "Simplicial algorithms for finding stationary points, a unifying description", Free University, Amsterdam, Jan. 1982.
In the paper they also point out that the 2n-method and the 2^n-method are both extreme cases of the (3^n-1)-method.

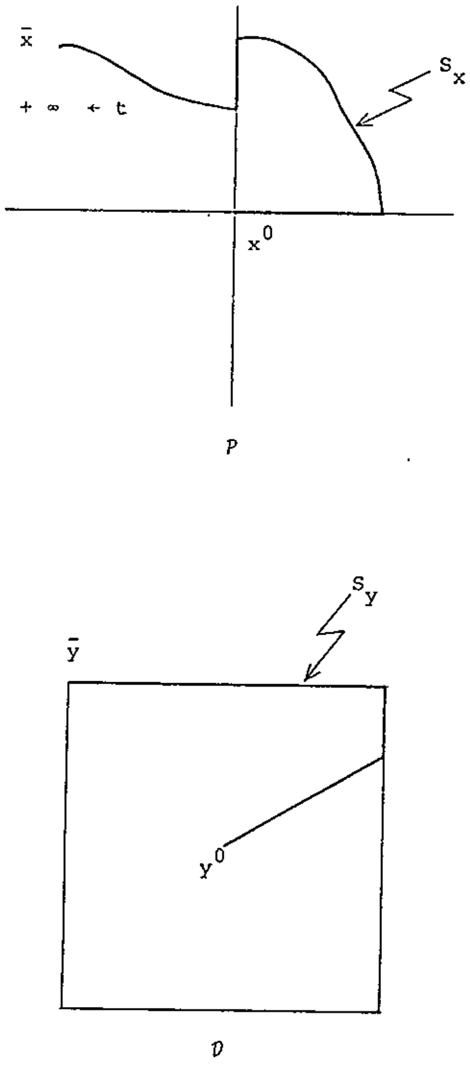


Figure 1.1

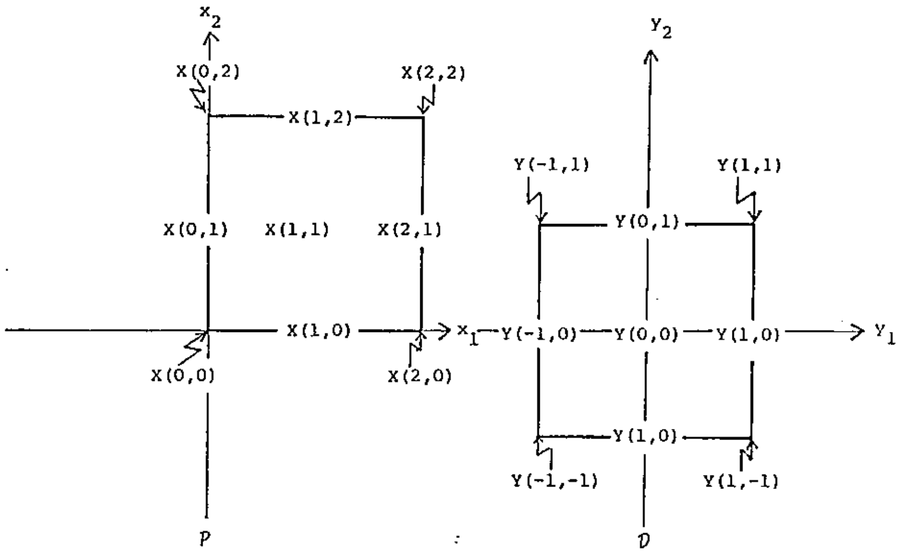


Figure 3.1

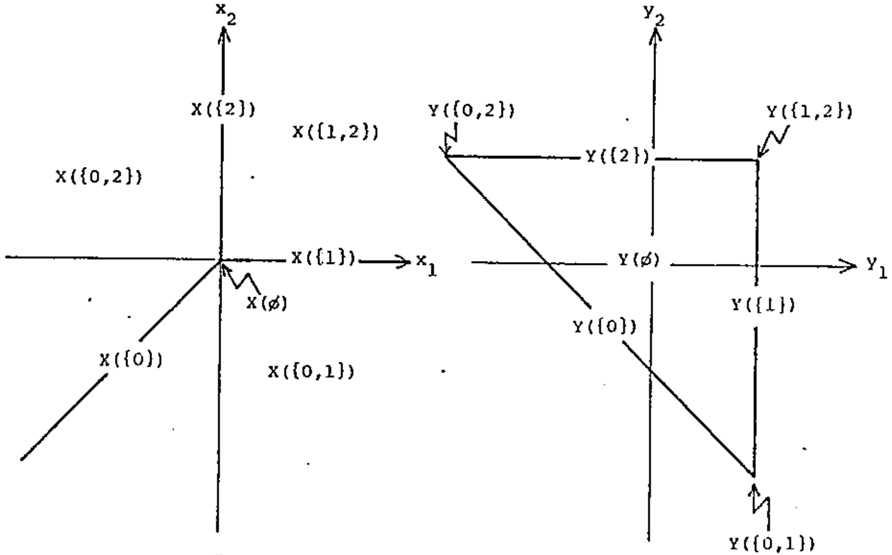


Figure 3.2

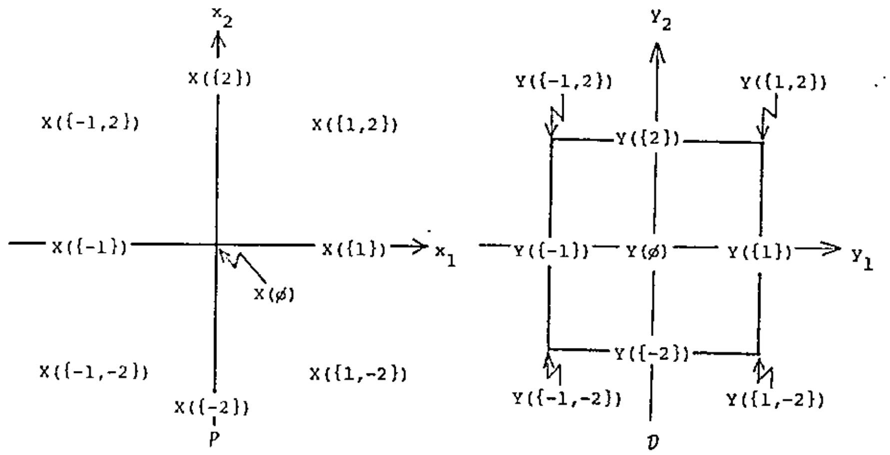


Figure 3.3

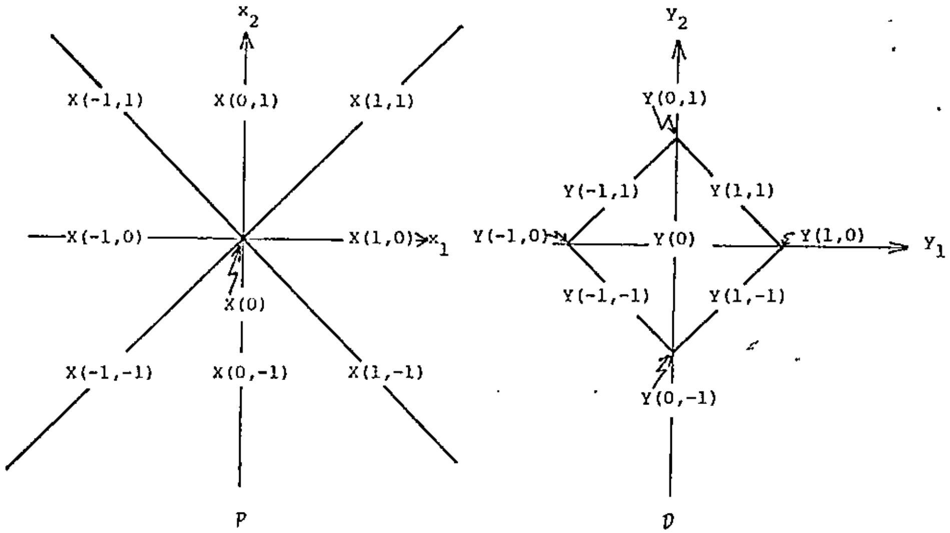


Figure 3.4

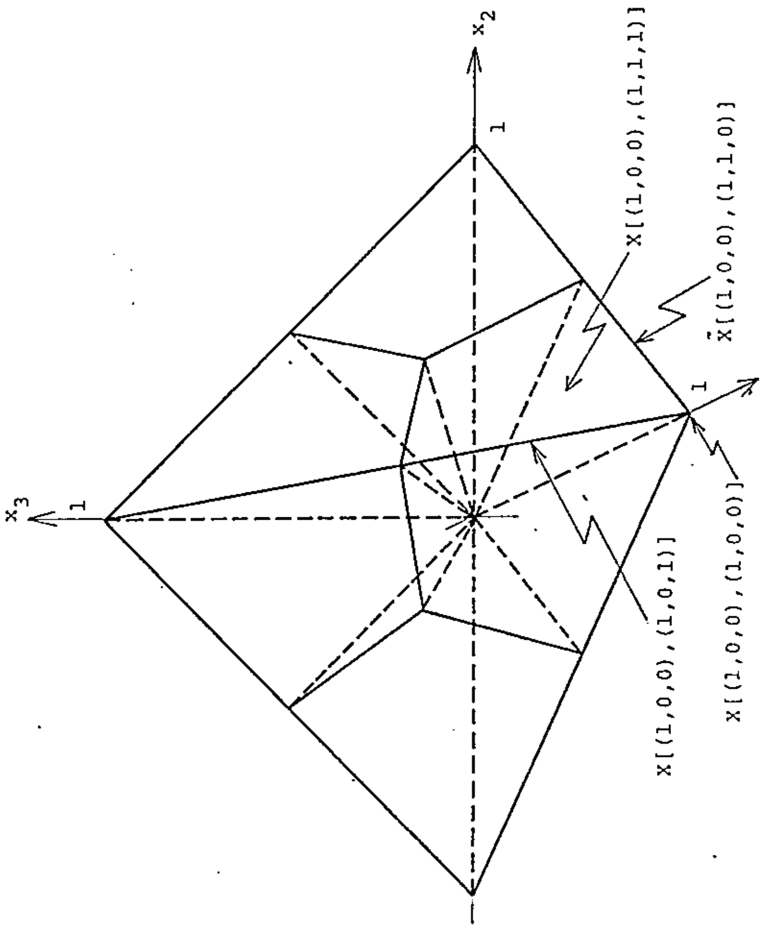
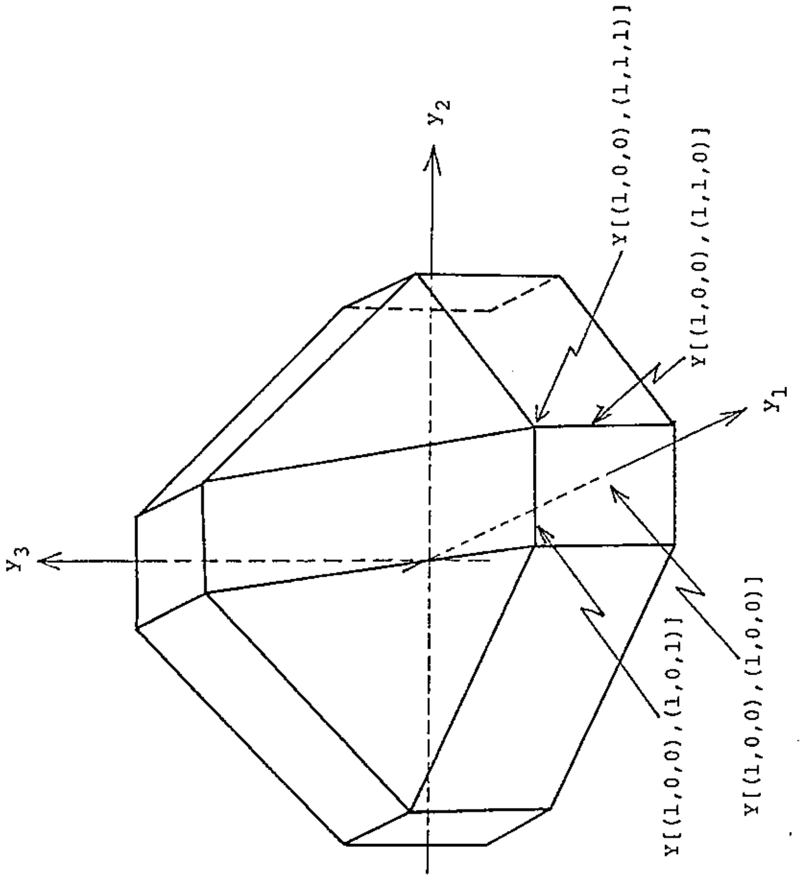


Figure 4.1 Intersections of some X[s, t] with a segment of the octahedron



· Figure 4.2

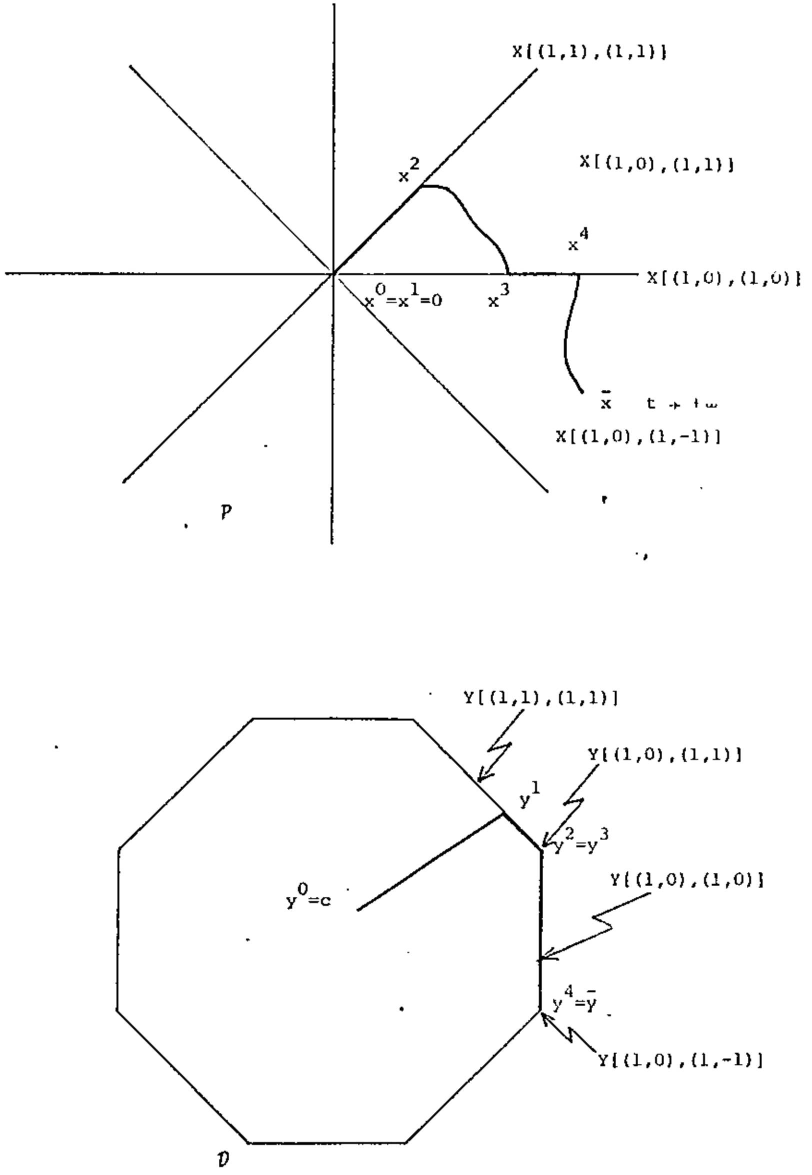


Figure 5.1

Table 8.1 Comparison of the four methods for Problem (P1)

n		Merrill	2n	2^n	3^n-1		
					0.2/(n+1)	0.5/(n+1)	0.8/(n+1)
10	# pivot	98	98	81	81	81	84
	# function	106	106	97	97	97	100
20	# pivot	209	209	182	183	181	186
	# function	218	218	200	200	199	204
30	# pivot	340	340	302	303	301	307
	# function	350	350	322	322	321	327
40	# pivot	451	451	404	404	402	407
	# function	461	461	422	423	422	427
50	# pivot	612	612	554	555	553	555
	# function	623	623	576	576	575	577

Table 8.2 Comparison of the four methods for Problem (P2)

n		Merrill	2n	2^n	3^n-1		
					0.2/(n+1)	0.5/(n+1)	0.8/(n+1)
1	# pivot	10	9	6	6	6	6
	# function	12	12	12	12	12	12
2	# pivot	26	25	18	19	19	21
	# function	32	32	32	32	32	34
3	# pivot	91	79	48	53	64	82
	# function	93	89	68	71	81	98
4	# pivot	354	229	132	131	200	205
	# function	325	240	158	153	210	213
5	# pivot	544	561	256	254	357	345
	# function	554	571	268	249	347	342
6	# pivot	906	1246	327	331	350	530
	# function	807	1240	346	345	367	521
7	# pivot	2363	*4	647	731	*3	1023
	# function	2078		649	706		961
8	# pivot	6519	*2	1828	3778	25242	2134
	# function	5801		1800	3419	25077	1946

*k We stopped the computation because the total number of pivot operations had exceeded 50,000 at the k-th major cycle.

Table 8.3 Comparison of the four methods for Problem (P3)

n		Merrill	2n	2^n	3^n-1		
					0.2/(n+1)	0.5/(n+1)	0.8/(n+1)
10	# pivot	382	*3	649	674	342	*4
	# function	358		670	681	359	
20	# pivot	1021	*3	978	1004	975	913
	# function	976		1005	1006	985	925
30	# pivot	2032	*4	2098	2078	1992	*6
	# function	1943		2128	2079	1995	
40	# pivot	3571	$	3650	3624	3460	4319
	# function	3431		3680	3605	3453	4279
50	# pivot	4997	$	6334	6165	5508	#9
	# function	4827		6353	6112	5489	

*k We stopped the computation because the total number of pivot operations had exceeded 50,000 at the k-th major cycle.

#k We stopped the computation because the variable vector x in the solution path had moved out of the cube $\{ x \in R^n : -3 \leqq x_j \leqq 3 \text{ for } j \in N \}$ at the k-th major cycle.

$ not tested.

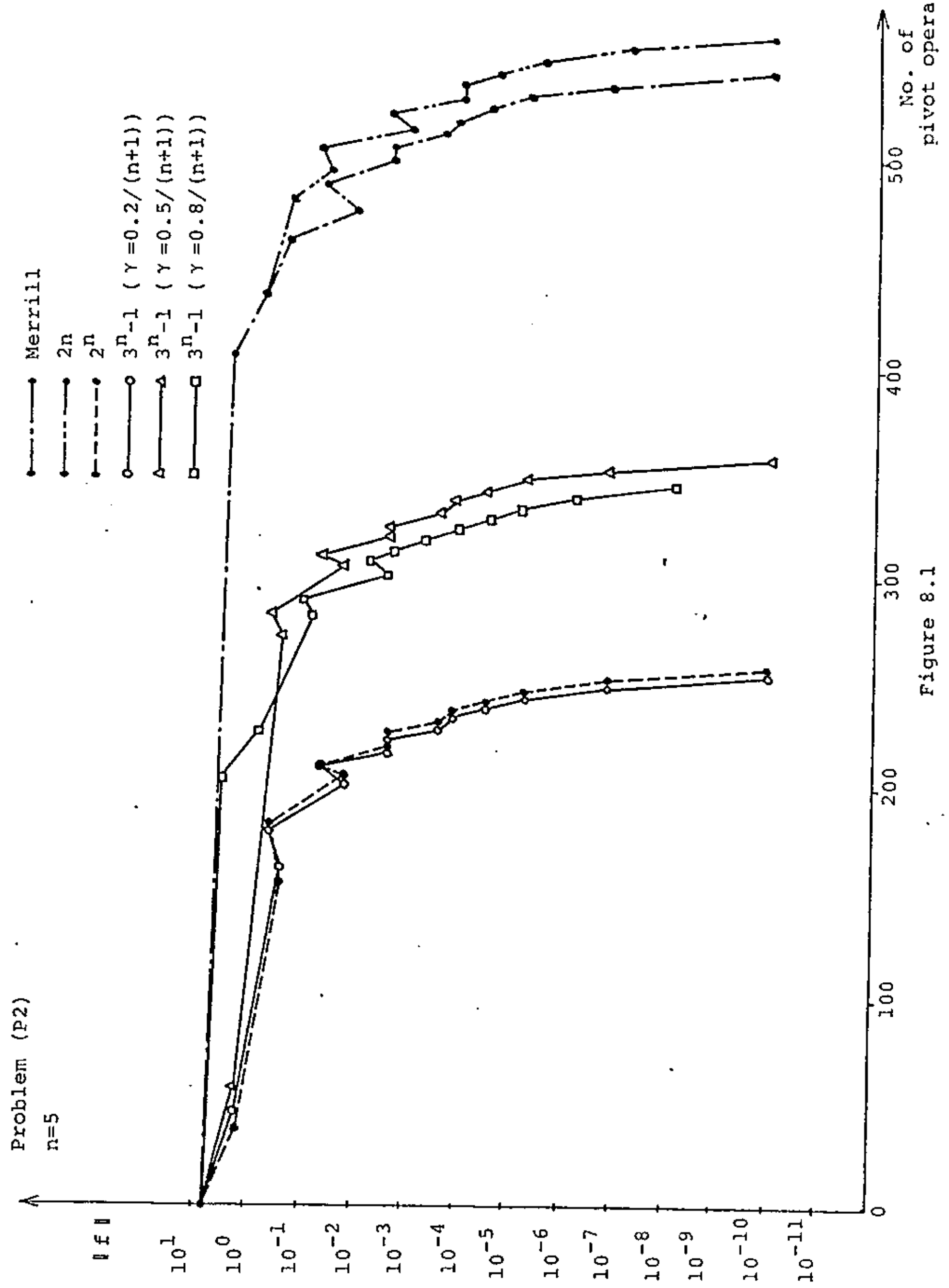


Figure 8.1

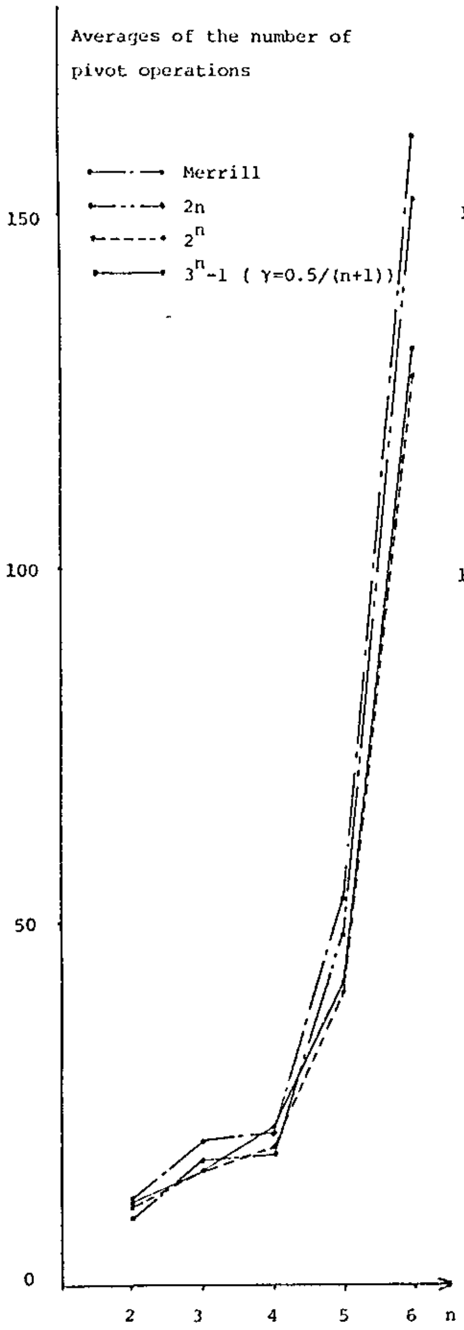


Figure 8.2

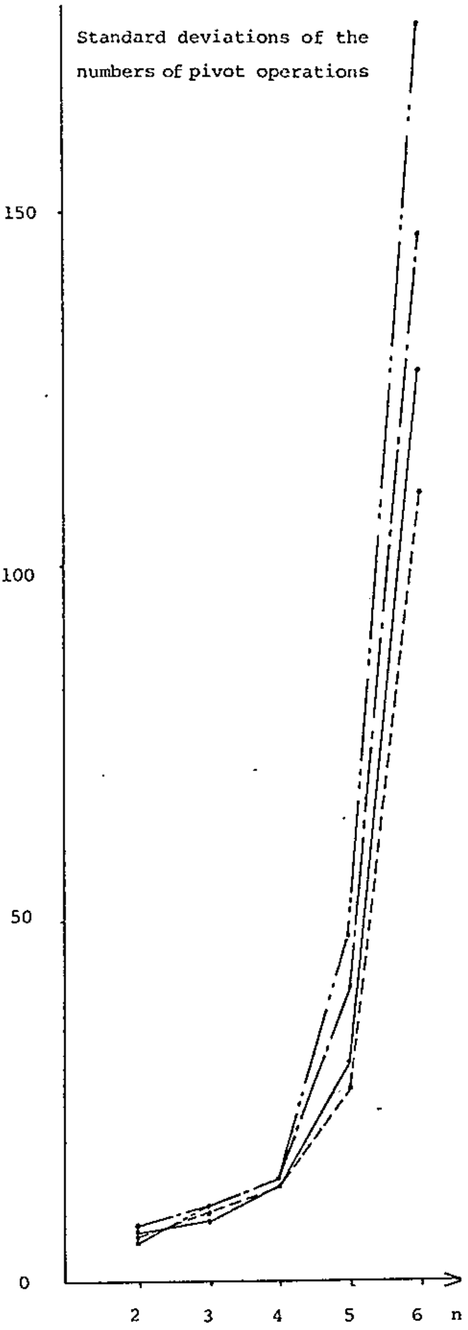


Figure 8.3

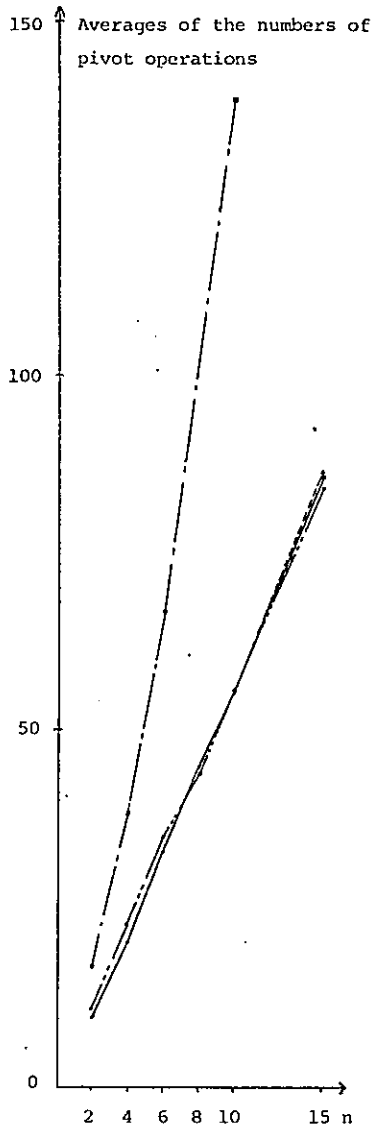


Figure 8.4

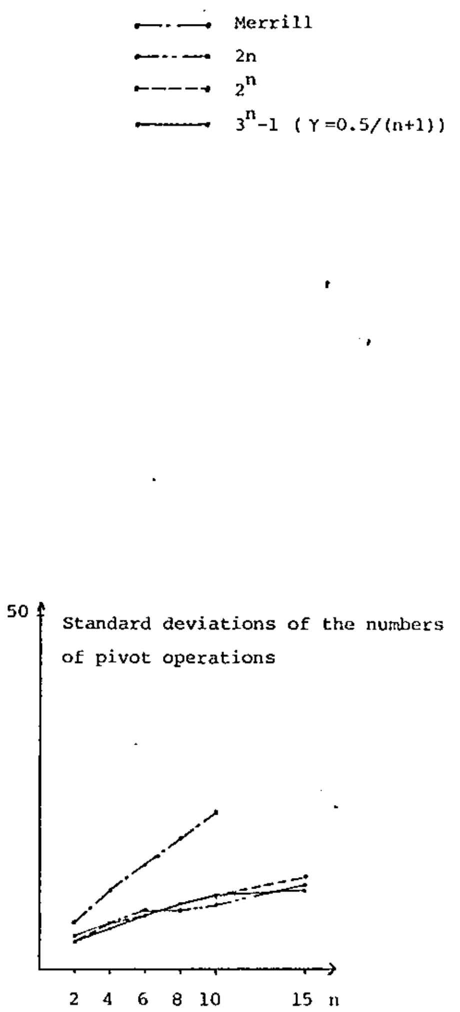


Figure 8.5

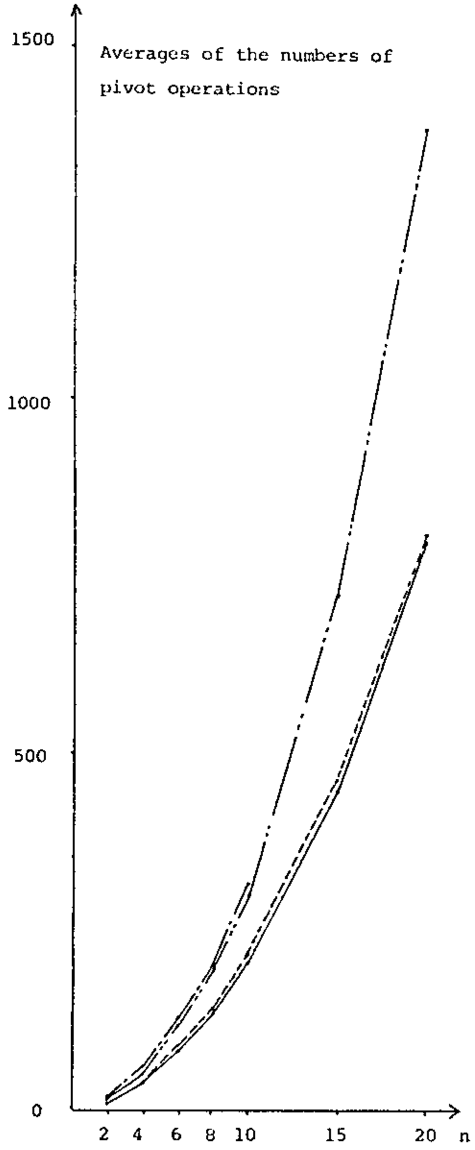


Figure 8.6

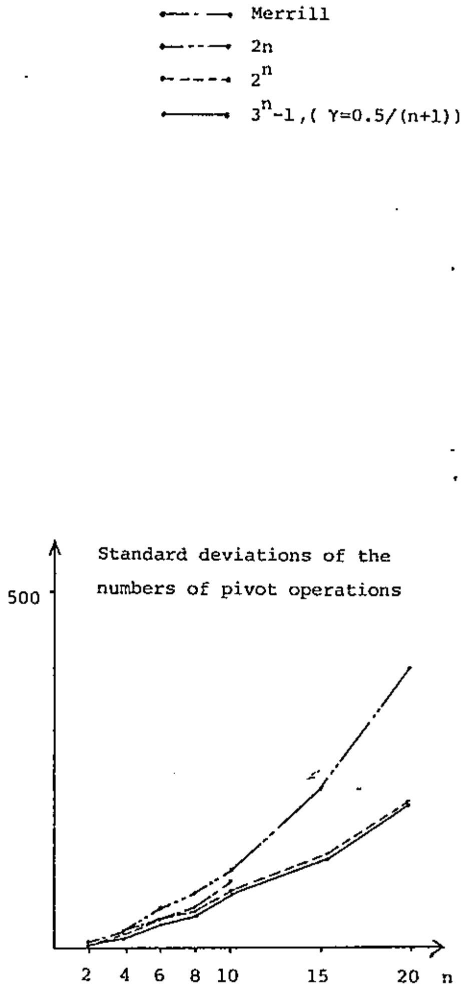


Figure 8.7

References

(1) J. C. Allexander, "The topological theory of an embedding method", in: H. Wacker, ed., Continuation Methods (Academic Press, New York, 1978) pp.36-67.

(2) E. Allgower and K. Georg, "Simplicial and continuation methods for approximaing fixed points and solutions to systems of equations", SIAM Review 22 (1980) 28-85.

(3) R. Freund, "Variable-dimension complexes with applications", Ph. D. Dissertation, Stanford University, June 1980.

(4) K. Georg, "On tracing an implicitly defined curve by quasi-Newton steps and calculating bifurcation by local perturbations", University of Bonn, Bonn, Jan. 1980.

(5) M. Kojima, "A note on 'A new algorithm for computing fixed points' by van der Laan and Talman", in: W. Forster, ed., Numerical Solution of Highly Nonlinear Problems, Fixed Point Algorithms and Complementarity Problems (North-Holland, New York, 1980) pp.37-42.

(6) M. Kojima, "An introduction to variable dimension algorithms for solving systems of equations", in: E. L. Allgower, K. Glashoff and H. -O. Peitgen, ed., Numerical Solution of Nonlinear Equations, Lecture Note in Mathematics 878 (Springer, Berlin, 1980) pp.199-237.

(7) M. Kojima and Y. Yamamoto, "Variable dimension algorithms: Basic theory, interpretations and extensions of some existing methods", to appear in Mathematical Programming.

[8] G. van der Laan, "Simplicial fixed point algorithms", Ph. D. Dissertation, Free University, Amsterdam, 1980.

[9] G. van der Laan and A. J. J. Talman, "On the computation of fixed points in the product space of the unit simplices and an application to non cooperative n-person games", Free University, Amsterdam, Oct. 1978.

[10] G. van der Laan and A. J. J. Talman, "A restart algorithm for computing fixed point without extra dimension", Mathematical Programming 17 (1979) 74-84.

[11] G. van der Laan and A. J. J. Talman, "A restart algorithm without an artificial level for computing fixed points on unbounded regions", in: H. -O. Peitgen and H. -O. Walther, ed., Functional Differential Equations and Approximation of Fixed Points, Lecture Notes in Mathematics 730 (Springer, Berlin, 1979) pp.247-256.

[12] G. van der Laan and A. J. J. Talman, "Convergence and properties of recent variable dimension algorithms", in: W. Forster, ed., Numerical Solution of Highly Nonlinear Problems, Fixed Point Algorithms and Complementarity Problems (North-Holland, New York, 1980) pp.3-36.

[13] G. van der Laan and A. J. J. Talman, "A class of simplicial restart fixed point algorithms without an extra dimension", Mathematical Programming 20 (1981) 33-48.

[14] T. Y. Li and J. A. Yorke, "A simple reliable numerical algorithm for following homotopy paths", in: S. M. Robinson, ed., Analysis and Computation of Fixed Points (Academic Press, New York, 1980).

[15] O. H. Merrill, "Applications and extensions of an algorithm that computes fixed points of a certain upper semi continuous point to set mapping", Ph. D. Dissertation, University of Michigan, 1972.

[16] R. M. Reiser, "A modified integer labelling for complementarity algorithms", Mathematical Programming 6 (1981) 129-139.

[17] R. Saigal, "On the convergence rate of algorithms for solving equations that are based on methods of complementarity pivoting", Mathematics of Operations Research 2 (1977) 108-124.

[18] R. Saigal, "A homotopy for solving large, sparse and structured fixed point problems", Northwestern University, Evanston, Illinois, Jan. 1981.

[19] A. J. J. Talman, "Variable dimension fixed point algorithms and triangulations", Ph. D. Dissertaion, Free University, Amsterdam, 1980.

[20] M. J. Todd, The Computation of Fixed Points and Applications, Lecture Notes in Economics and Mathematical Systems (Springer, New York, 1976).

[21] M. J. Todd, "Fixed-point algorithms that allow restarting without an extra dimension", Technical Report No.379, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York, Sept. 1978.

[22] M. J. Todd, "Traversing large pieces of linearity in algorithms that solve equations by following piecewise-linear paths", Mathematics of Operations Research 5 (1980) 242-257.