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Abstract — A. Frank introduced a concept of generalized polymatroid. We show that a generalized polymatroid is a projection of a base polytope of a submodular system and vice versa. The intersection theorem for generalized polymatroids easily follows from this fact.

1. Introduction

A. Frank [5] introduced a concept of generalized polymatroid which is defined in terms of a pair of a submodular function and a supermodular function satisfying a certain condition. We shall show that a generalized polymatroid is a projection of a base polytope of a polymatroid [1] or a submodular system [7], [8] and vice versa. It easily follows from this fact that the intersection of two integral generalized polymatroids has the integrality property.

2. Definitions

Let E be a finite set and $\mathcal{F}$ be a family of subsets of E. We say a pair of $X, Y \in \mathcal{F}$ is an intersecting (or a crossing) pair if $X \cap Y \neq \emptyset$ (or $X \cap Y \neq \emptyset$, $X \cap (E-Y) \neq \emptyset$, $(E-X) \cap Y \neq \emptyset$ and $(E-X) \cap (E-Y) \neq \emptyset$). $\mathcal{F}$ is called an intersecting (or a crossing) family if for every intersecting (or crossing) pair of $X, Y \in \mathcal{F}$ we have $X \cup Y, X \cap Y \in \mathcal{F}$. A function $f: \mathcal{F} \to R$(the set of reals) is called a submodular function on an intersecting (or a crossing) family $\mathcal{F}$ if for every intersecting (or crossing) pair of $X, Y \in \mathcal{F}$ we have

$$f(X) + f(Y) \geqq f(X \cup Y) + f(X \cap Y). \qquad (2.1)$$

For a distributive sublattice $\mathcal{D}$ of the Boolean lattice 2^E we call $f: \mathcal{D} \to R$ a submodular function on the distributive lattice $\mathcal{D}$ if for every pair of $X, Y \in \mathcal{D}$ the inequality (2.1) holds. A function g is a supermodular function if $-g$ is a submodular function.

Throughout the present paper we assume that for any function f the empty set is in the domain of f and $f(\emptyset) = 0$.

For a submodular function $f: \mathcal{D} \to R$ on a distributive lattice $\mathcal{D} \subseteq 2^E$ with $E \in \mathcal{D}$ we define the dual supermodular function $f^{\#}: \overline{\mathcal{D}} \to R$, of f, on the dual distributive lattice $\overline{\mathcal{D}} = \{E-X \mid X \in \mathcal{D}\}$ of $\mathcal{D}$ by

$$f^{\#}(E-X) = f(E) - f(X) \quad (X \in \mathcal{D}) \tag{2.2}$$

(cf. [6], [12]).

Suppose that F and G are intersecting families formed by subsets of E satisfying

(*) for any $X \in F$ and $Y \in G$ if $X - Y \neq \emptyset$, then $X - Y \in F$ and if $Y - X \neq \emptyset$, then $Y - X \in G$. (2.3)

Also, suppose that $f: F \to R$ and $g: G \to R$ are, respectively, a submodular function on F and a supermodular function on G and that for any $X \in F$ and $Y \in G$ we have

$$f(X) - g(Y) \geq f(X-Y) - g(Y-X). \tag{2.4}$$

Then the polytope Q given by

$$Q = \{x \mid x \in R^E, x(X) \leq f(X)\,(X \in F), x(Y) \geq g(Y)\,(Y \in G)\} \tag{2.5}$$

is called a generalized polymatroid by Frank [5], where for $x \in R^E$ and $X \in F$ or G we define

$$x(X) = \sum_{e \in X} x(e). \tag{2.6}$$

Given a distributive lattice $\mathcal{D} \subseteq 2^E$ and a submodular function $f: \mathcal{D} \to R$ on $\mathcal{D}$, we call the pair $(\mathcal{D}, f)$ a submodular system [7], [8]. When $E \in \mathcal{D}$, the polytope $B(f)$ given by

$$B(f) = \{x \mid x \in R^E, x(X) \leq f(X)\,(X \in \mathcal{D}), x(E) = f(E)\} \tag{2.7}$$

is called the base polytope of the submodular system $(\mathcal{D},f)$.

3. A Simple Characterization of Generalized Polymatroids

Let Q be a generalized polymatroid given by (2.5). Here, without loss of generality we assume that $\mathcal{F}$ and $\mathcal{G}$ are distributive sublattices of 2^E because for any submodular function f' on an intersecting family $\mathcal{F}' \subseteq 2^E$ there exists a submodular function f on a distributive lattice $\mathcal{F} \subseteq 2^E$ such that

$$\{x \mid x \in R^E, x(X') \leq f'(X')(X' \in \mathcal{F}')\}$$
$$= \{x \mid x \in R^E, x(X) \leq f(X)(X \in \mathcal{F})\}, \tag{3.1}$$

where $\mathcal{F}$ and f uniquely exist, $\mathcal{F}$ is formed by those sets $X \subseteq E$ for which there is at least one partition $\{X_1, X_2, \ldots, X_n\}$ of X with $X_i \in \mathcal{F}'$ $(i=1,2,\ldots,n)$ and f is given by

$$f(X) = \min\{\sum_{i=1}^{k} f'(X_i) \mid \{X_1, X_2, \ldots, X_n\}\text{:a partition of } X,$$
$$X_i \in \mathcal{F}'(i=1,2,\ldots,n)\} \tag{3.2}$$

for each $X \in \mathcal{F}$ (see [8] and also [4]).

Moreover, let e be a new element not in E and define

$$\hat{E} = E \dot{\cup} \{e\}, \tag{3.3}$$

$$\overline{\mathcal{G}}^\circ = \{\hat{E} - X \mid X \in \mathcal{G}\}, \tag{3.4}$$

$$\hat{\mathcal{D}} = \mathcal{F} \cup \overline{\mathcal{G}}^\circ. \tag{3.5}$$

Note that $\hat{\mathcal{D}}$ is a distributive lattice with $\hat{E} \in \hat{\mathcal{D}}$ since $\emptyset \in \mathcal{G}$ and since for any $X \in \mathcal{F}$ and $Y \in \overline{\mathcal{G}}^\circ$ we have

$$X \cap Y = X - (E - Y) \in \mathcal{F}, \tag{3.6}$$

$$X \cup Y = \hat{E} - ((E - Y) - X) \in \overline{\mathcal{G}}^\circ \tag{3.7}$$

because of (2.3). Also define a function $\hat{f}\colon \hat{\mathcal{D}} \to R$ by

$$\hat{f}(X) = f(X) \qquad (X \in \mathcal{F}), \tag{3.8}$$

$$\hat{f}(X) = \hat{f}(\hat{E}) - g(E - X) \qquad (X \in \overline{\mathcal{G}}^{\circ}) \tag{3.9}$$

and

$$\hat{f}(\hat{E}) = c, \tag{3.10}$$

where c is an arbitrary but fixed real.

Now, we have

<u>Theorem 1</u>: The following (i) and (ii) are valid.

(i) The function $\hat{f}\colon \hat{\mathcal{D}} \to R$ defined by (3.8) - (3.10) is a submodular function on the distributive lattice $\hat{\mathcal{D}}$.

(ii) The generalized polymatroid Q given by (2.5) is expressed as

$$Q = \{x \mid x \in R^{E}, \exists\, \alpha \in R\colon (x,\alpha) \in B(\hat{f})\}, \tag{3.11}$$

where $B(\hat{f})$ is the base polytope of the submodular system $(\hat{\mathcal{D}},\hat{f})$.

(Proof) (i): It is sufficient to show the inequality

$$\hat{f}(X) + \hat{f}(Y) \geqq \hat{f}(X \cup Y) + \hat{f}(X \cap Y) \tag{3.12}$$

for any $X \in \mathcal{F}$ and $Y \in \overline{\mathcal{G}}^{\circ}$. It follows from (2.4), (3.8) and (3.9) that for $X \in \mathcal{F}$ and $Y \in \overline{\mathcal{G}}^{\circ}$ we have

$$\begin{aligned}\hat{f}(X) + \hat{f}(Y) &= f(X) + \hat{f}(\hat{E}) - g(E - Y)\\ &\geqq f(X - (E - Y)) + \hat{f}(\hat{E}) - g((E - Y) - X)\\ &= f(X \cap Y) + \hat{f}(\hat{E}) - g(E - (X \cup Y))\\ &= \hat{f}(X \cap Y) + \hat{f}(X \cup Y).\end{aligned} \tag{3.13}$$

(ii): We see that $(x,\alpha) \in B(f)$ if and only if

$$x(X) \leqq \hat{f}(X) = f(X) \qquad (X \in \mathcal{F}), \tag{3.14}$$

$$x(X - \{e\}) + \alpha \leqq \hat{f}(X) = \hat{f}(\hat{E}) - g(E - X) \quad (X \in \overline{\mathcal{G}}^{\circ}), \tag{3.15}$$

$$x(E) + \alpha = \hat{f}(\hat{E}). \tag{3.16}$$

Eliminating α in (3.15) by using (3.16), we have

$$x(X - \{e\}) + \hat{f}(\hat{E}) - x(E) \leqq \hat{f}(\hat{E}) - g(E - X) \quad (X \in \overline{G}^{\circ}) \tag{3.17}$$

or

$$x(Y) \geqq g(Y) \qquad (Y \in \mathcal{G}). \tag{3.18}$$

Therefore, we have (3.11). □

From (3.11) we say that the generalized polymatroid Q is a projection, along the e-axis, of $B(\hat{f})$.

Conversely, we have

Theorem 2: Let $\hat{f}: \hat{\mathcal{D}} \to R$ be an arbitrary submodular function on an arbitrary distributive lattice $\hat{\mathcal{D}} \subseteqq 2^{\hat{E}}$ with $\hat{E} \in \hat{\mathcal{D}}$. Then, for any $e \in \hat{E}$ the projection along the e-axis of the base polytope $B(\hat{f})$ of the submodular system $(\hat{\mathcal{D}}, \hat{f})$ is a generalized polymatroid.

(Proof) Let us define distributive sublattices $\mathcal{F}$ and $\mathcal{G}$ of $\hat{\mathcal{D}}$ by

$$\mathcal{F} = \{X \mid e \notin X \in \hat{\mathcal{D}}\}, \tag{3.19}$$

$$\mathcal{G} = \{\hat{E} - X \mid e \in X \in \hat{\mathcal{D}}\}. \tag{3.20}$$

Also define a submodular function $f: \mathcal{F} \to R$ and $g: \mathcal{G} \to R$ by

$$f(X) = \hat{f}(X) \qquad (X \in \mathcal{F}), \tag{3.21}$$

$$g(Y) = \hat{f}^{\#}(Y) \qquad (Y \in \mathcal{G}). \tag{3.22}$$

We easily see that $\mathcal{F}$ and $\mathcal{G}$ given by (3.19) and (3.20) satisfy (2.3), that f and g given by (3.21) and (3.22) satisfy (2.4) and that $x \in R^E$ satisfies the inequalities

$$x(X) \leqq f(X) \qquad (X \in \mathcal{F}), \tag{3.23}$$

$$x(Y) \geqq g(Y) \quad (Y \in \mathcal{G}) \tag{3.24}$$

if and only if for some $\alpha \in R$ we have $(x,\alpha) \in B(\hat{f})$. □

Here, it should be noted that if $\hat{f}$ appearing in Theorem 2 is taken as $\hat{f}$ in Theorem 1, then f and g defined by (3.21) and (3.22), respectively, coincide with f and g which define $\hat{f}$ in Theorem 1 by (3.8)-(3.10).

Remark 1: For the generalized polymatroid Q and the base polytope $B(\hat{f})$ associated with it in Theorems 1 and 2, Q and $B(\hat{f})$ have the same combinatorial structure and there is a one-to-one correspondence between the set of extreme points of Q and the set of extreme points of $B(\hat{f})$. It should be noted that for any extreme point $\hat{x}$ of $B(\hat{f})$ there exists a maximal chain

$$\emptyset = S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_n = \hat{E} \tag{3.25}$$

in the distributive lattice $\hat{\mathcal{D}}$ such that

$$|S_i - S_{i-1}| = 1 \quad (i=1,2,\ldots,n), \tag{3.26}$$

$$\hat{x}(S_i - S_{i-1}) = \hat{f}(S_i) - \hat{f}(S_{i-1}) \quad (i=1,2,\ldots,n) \tag{3.27}$$

(cf. [8]), from which follow formulae for extreme points of Q. Moreover, the intersection theorem for generalized polymatroids in [5] easily follows from [1] and Theorems 1 and 2.

Remark 2: Generalized polymatroids are also considered by R. Hassin [10], though the domains of the relevant submodular functions and supermodular functions are Boolean lattices, and "greedy" and "generous" solutions for polytopes of generalized polymatroids on Boolean lattices are examined in [10, pp. 18-20]. The result is, however, immediately deduced from the greedy algorithm for polymatroids ([1], [2]) and Theorems 1 and 2 of the present paper.

Remark 3: It may be well-known that if $\mathcal{D}$ is a distributive sublattice of the Boolean lattice 2^E and f is an integer-valued submodular function on $\mathcal{D}$, then the system of inequalities in (2.7) is totally dual integral (see [3] and [11] for the definition of total dual integrality). Therefore, from Theorem 2 the system of inequalities in (2.5) is totally dual integral if f and g are integer-valued and defined on distributive lattices. Note that if f' and g' are, respectively, a submodular function on an intersecting family $\mathcal{F}' \subseteq 2^E$ and a supermodular function on an intersecting family $\mathcal{G}' \subseteq 2^E$, then there exist a submodular function f on a distributive lattice $\mathcal{F} \subseteq 2^E$ given by (3.2) and a supermodular function g on a distributive lattice $\mathcal{G} \subseteq 2^E$ given similarly as (3.2) such that

$$\{x \mid x \in R^E, x(X') \leq f'(X')(X' \in \mathcal{F}'), x(Y') \geq g'(Y')(Y' \in \mathcal{G}')\}$$
$$= \{x \mid x \in R^E, x(X) \leq f(X)(X \in \mathcal{F}), x(Y) \geq g(Y)(Y \in \mathcal{G})\}. \quad (3.28)$$

By virtue of (3.1) and (3.2) and similar relations between g' and g we see that the system of inequalities in the left-hand side of (3.28) is totally dual integral if and only if the one in the right-hand side of (3.28) is totally dual integral and thus that the former is totally dual integral.

Remark 4: If F and G are intersecting families of subsets of E, then $\hat{\mathcal{D}}$ given by (3.3)-(3.5) is a crossing family of subsets of $\hat{E}$ and, conversely, if $\hat{\mathcal{D}}$ is a crossing family of subsets of $\hat{E}$, then F and G, respectively, given by (3.19) and (3.20) are intersecting families of subsets of $E = \hat{E} - \{e\}$. Since each submodular function on a crossing family determines a base polytope of a submodular system [8], each pair of a submodular function and a supermodular function on intersecting families determines a generalized polymatroid as in [5].

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