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A Model of Urban and Rural Growth in a Small Economy

by

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Abstract

This paper analyzes urban and rural growth in a small economy using a two-sector growth model. The urban and rural production are characterized by increasing and constant returns to scale, respectively. The temporary production decisions concern with optimal labor inputs, and profit incomes are reinvested in the two sectors according to relative profitability. Using phase diagrams, we observe a transition of the economy from unlimited to limited labor supply, a tendency for urban concentration to continue once started, and the importance of initial profitability for economic development. The associated trade patterns are also analyzed incorporating demand conditions.

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I. Introduction

Despite recent talks on flight to suburbs and urban decay in the United States and elsewhere, concentration of population and economic activities into a small number of urban areas is far from decelerating but appears to be proceeding without much moderation. The recent trend is characterized by rapid growth of suburban cities and slower growth of the central city, but the "greater" city area continues to expand absorbing the suburban cities and their population. Indeed, the proportion of total population living in urban areas has been continuously increasing in practically all the countries in the world. As Table 1 shows for the past two decades, the ratio of urban to total population increased in every category of countries by type, by income group, and by region, with the highest population concentration observed in the industrialized nations of Western Europe, North America, and the Pacific Basin. Even in developing countries, the process of urbanization is well under way, with an average middle-income country already having 40% of total population concentrated in urban areas.

The continuing urban concentration of population and economic activities is a source of many socio-economic problems that are typical of urban

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* This study is an outgrowth of my earlier work (1974, 1975). I wish to thank Professor Yoshihiko Otani for useful discussions on the subject. However, the author alone is responsible for the remaining errors in the paper.

Table 1: Urban Population as Percent of Total Population

	1960	1970	MRE[1]
Countries by Type			
Developing Countries	21.2%	26.4%	29.3%
Capital-Surplus Oil Exporting Countries	29.8	47.0	58.0
Industrialized Countries	65.1	71.8	74.6
Centrally Planned Economies	46.8	54.3	58.3
Developing Countries by Income Group[2]			
Low Income	10.6	13.7	15.8
Middle Income	30.1	36.8	40.0
Developing Countries by Region			
Africa south of Sahara	9.5	15.9	17.0
Middle East and North Africa	33.5	43.0	47.3
East Asia and Pacific	23.5	27.5	30.3
South Asia	8.4	11.3	13.2
Latin America and Caribbean	48.0	56.9	59.3
Southern Europe	43.3	51.9	56.3

Source: World Tables 1980, World Bank, pp. 437-441.

Notes: [1] Most recent estimates obtained from the closest year to 1977.

[2] Cut-off point between low and middle income is $300 in 1977 U.S. dollars.

areas. Air and noise pollution, traffic congestion, and housing problems are well-known examples. These problems share a common feature that limited (or difficult-to-increase) capacity is required to cope with continuously increasing demand for its services. Unless the capacity is increased as fast as the increase in demand or else the rate of increase in demand is controlled, the system is bound to run into problems. Yet, the capacity of atmosphere to absorb pollutants and the tolerance level of human ears are almost impossible to increase, and the provision of additional road and railway capacities takes time. Thus, controlling for urban concentration is an important element in dealing with many of the problems typical of large urban areas.

Associated with urban concentration is the relative decay of rural areas. In extreme cases, as observed in a number of Japanese rural communities, excessive outflow of productive labor force and population has resulted in serious problems of maintaining living standard and productive functions of those communities [Niida (1971)]. The problem of rural depopulation is a necessary consequence of urban concentration, but it is often overshadowed by the problems taking place in the urban areas and not much joint analysis of urban and rural growth has been done in the past.

Thus, in this paper, we shall explore into the mechanism of urban concentration and rural growth using a two-sector model of economic growth. In doing so, we assume, in contrast to the neoclassical two-sector growth models, that increasing returns to scale prevail in the urban production, while constant returns to scale is assumed for the rural production. An empirical support for the former assumption is provided by recent findings by Segal (1976) and Sveikauskas (1975) that Hicks-neutral production efficiency increases with city size, which provides a conceptual basis for the notion of "agglomeration economies" of large cities.1/

Based on these assumptions on returns to scale, we examine the growth of urban and rural sectors in a small economy with an initial condition of unlimited supply of labor. Capital accumulation is dictated by relative profitability of the two sectors according to the rule proposed by Inada (1971). The model describes growth possibilities of the two sectors under variety of conditions. We observe the transition of the economy from unlimited to limited supply of labor in the sense of Lewis (1954). We find that there is a strong tendency for the economy to concentrate into one of the two sectors with eventual decay of the other sector. That is, urban concentration, once started, is likely to continue at the sacrifice of the rural sector. In turn, the development possiblity of the economy from unlimited to limited labor supply hinges crucially on the initial profitability of the rural sector, which renders support for the importance of rural sector at an early stage of economic development, as often asserted in the development literature.2/ The model is also able to describe the changing patterns of trade that accompany the economy's growth, indicating a tendency for specialization in the products of the sector that becomes dominant.

The present model differs from the neoclassical version of the dual economy models, as represented by Jorgenson (1961, 1967), Dixit (1970), Zarembka (1970), Marino (1975) and Amano (1980), in that non-constant returns to scale is introduced in the urban production. It also differs from Inada's (1971) model of monocultural development as it explicitly incorporates the case of limited labor supply, enabling us to analyze the economy's transition from the unlimited to limited labor supply within a single model. Finally, the saddle-point property of the stationary state in the present model makes a marked contrast with the stable stationary point in the neoclassical growth models [Uzawa (1961, 1963)], and shows that the similar property obtained by Inada (1971) and Kaiyama (1973) for the case of unlimited labor supply also holds even if the labor supply becomes limited.

The plan of the paper is as follows: Section II below describes the model and Section III discusses the temporary equilibrium and comparative statics. The growth paths of the economy are examined in Section IV along with the conditions for development. In Section V, the model is extended to incorporate international trade, and in Section VI, there is a brief summary and conclusion.

II. The Model

We consider an economy with two sectors, designated as urban and rural sectors, respectively. The urban sector produces an aggregated manufactured good, X, which can be either consumed or used as investment goods. The rural sector produces an aggregated agricultural good, Y, which can only be consumed. Both production processes require capital and labor as inputs.

We take the manufactured good as numeraire and denote by w and p, respectively, the wage rate and the price of good Y in terms of the numeraire commodity. The economy is assumed small relative to the world economy, so that p is assumed constant. Labor is assumed immobile internationally but perfectly mobile domestically, so that the real wage rate, w, is determined by the conditions of the domestic labor market.[3/] Producers in each sector are assumed to be price takers in both the product and the factor market. We also assume that capital in this economy is unshiftable between sectors once installed, so that capital stock in each sector is fixed at each moment of time. Hence, the short-run production decision is concerned only with an optimal choice of labor input that maximizes short-run profit.

Let K_x and N_x denote capital and labor input in the urban sector. The technology in the urban sector is assumed to be represented by an aggregate production function

$$(1) \qquad X = F_x(K_x, N_x) \quad , \qquad K_x \geqq 0,\ N_x \geqq 0,$$

which is homogeneous of degree greater than one and satisfies:

$$(2) \quad \begin{cases} \partial F_x/\partial K_x > 0 \quad , & \partial F_x/\partial N_x > 0, \\ \partial^2 F_x/\partial N_x^{\,2} < 0 \quad , & \partial^2 F_x/\partial N_x \partial K_x > 0 \quad , \text{ and } \quad F_x(K_x, 0) = 0. \end{cases}$$

Similarly, let K_y and N_y be capital and labor input in the rural sector. The technology in the rural sector is assumed to be represented by an aggregate production function

$$(3) \qquad Y = F_y(K_y, N_y) \quad , \qquad K_y \geqq 0,\ N_y \geqq 0,$$

which is homogeneous of degree one and satisfies similar conditions as (2).

The homogeneity assumptions on F_x and F_y enable us to rewrite (1) and (3) into a more convenient form. Let $n_x = N_x/K_x$ and $n_y = N_y/K_y$. Then,

$$(4) \qquad F_x(K_x, N_x) = K_x^{\,r} f_x(n_x),$$

and

$$(5) \qquad F_y(K_y, N_y) = K_y f_y(n_y),$$

where $f_x(n_x) = F_x(1, n_x)$, $f_y(n_y) = F_y(1, n_y)$, and r is the degree of homogeneity of F_x, with $r > 1$ by assumption. The assumptions on the derivatives of F_x and F_y imply that f_x and f_y satisfy the following conditions:

$$f_i'(n_i) > 0, \quad f_i''(n_i) < 0, \quad \text{and} \quad \lim_{n_i \to 0} f_i(n_i) = 0, \quad i = x, y.$$

As additional conditions on f_x and f_y, we postulate the following Inada derivative conditions:[4]

$$\lim_{n_i \to 0} f_i'(n_i) = \infty \quad \text{and} \quad \lim_{n_i \to \infty} f_i'(n_i) = 0, \quad i = x, y.$$

We assume that the labor supply in this economy is given by

$$(6) \qquad \begin{cases} N^s = \nu(w) & \text{for } w \geqq w_o > 0 \\ w = w_o & \text{for } 0 \leqq N^s \leqq N_o, \end{cases}$$

where $N_o = \nu(w_o)$ and $\nu'(w) \geqq 0$ for $w > w_o$. Here, the commodity price ratio, p, is assumed subsumed in the functional form of ν, and w_o is the institutionally determined minimum subsistence wage rate, measured in terms of the numeraire commodity.

It is assumed that no saving is made out of wage income and no consumption out of profit. The profit income in each sector is reinvested in the two sectors according to the following rule proposed by Inada (1971): Let R_x and R_y be the average profit rate in the urban and rural sector, respectively. That is,

$$(7) \qquad R_x = K_x^{-1} [\, F_x(K_x, N_x) - wN_x], \qquad K_x > 0; \text{ and}$$

$$(8) \qquad R_y = K_y^{-1} [pF_y(K_y, N_y) - wN_y], \qquad K_y > 0.$$

Let ξ_i $(i = x, y)$ be the fraction of urban sector's profit invested in sector i, and ζ_i $(i = x, y)$ the fraction of rural sector's profit invested in sector i. These fractions are assumed to be continuous functions of the difference in average profit rates, $R_x - R_y$, have finite right-hand and left-hand derivatives at zero,[5] and satisfy:

$$\xi_x + \xi_y = 1, \quad \xi_x \geqq 0, \quad \xi_y \geqq 0,$$
$$\xi_x = 1 \text{ if and only if } R_x - R_y \geqq 0,$$
$$\xi_x \text{ is nondecreasing in } R_x - R_y,$$
$$\zeta_x + \zeta_y = 1, \quad \zeta_x \geqq 0, \quad \zeta_y \geqq 0,$$
$$\zeta_x = 0 \text{ if and only if } R_x - R_y \leqq 0, \text{ and}$$
$$\zeta_x \text{ is nondecreasing in } R_x - R_y.$$

Let μ be the rate of depreciation of capital, assumed the same in both sectors.[6] Then, capital accumulation in the two sectors is dictated by the following accumulation functions:

$$\dot{K}_x = R_x K_x \xi_x (R_x - R_y) + R_y K_y \zeta_x (R_x - R_y) - \mu K_x, \tag{9}$$

and

$$\dot{K}_y = R_x K_x \xi_y (R_x - R_y) + R_y K_y \zeta_y (R_x - R_y) - \mu K_y. \tag{10}$$

III. Temporary Equilibrium and Profit Rate Functions

In the short-run, the producers in each sector choose the level of employment so as to maximize profit. Since capital stock is fixed in the short-run, this is equivalent to maximizing average profit rate $[K_x^{r-1} f_x(n_x) - wn_x$ for the urban and $pf_y(n_y) - wn_y$ for the rural sector] with respect to labor-capital ratio. The optimality requires that[7]

$$f_x'(n_x) = w/K_x^{r-1},$$

and

$$pf_y'(n_y) = w,$$

which can be inverted to entail demand functions for labor per unit of capital as

$$n_x = g_x(z), \tag{11}$$

and

(12) $n_y = g_y(w)$,

where $z = w/K_x^{r-1}$, $g_x = (f'_x)^{-1}$, $g'_x = (f''_x)^{-1} < 0$, $g_y = (pf'_y)^{-1}$, and $g'_y = (pf''_y)^{-1} < 0$.

In the labor market, if the total demand for labor at w_o falls short of N_o, there is surplus labor, and otherwise labor is fully employed. The condition that full employment of labor is achieved at w_o is given by

(13) $g_x(z_o)K_x + g_y(w_o)K_y = N_o$,

where $z_o = w_o/K_x^{r-1}$. It can be easily shown that the locus of (K_x, K_y) satisfying (13) is a downward sloping curve in the (K_x, K_y)-plane with finite intercepts, $\bar{K}_x$ and $\bar{K}_y$, with both axes (Figure 1). We shall call this locus as the "boundary locus." To the southwest of this locus, more labor can be employed without an increase in wage rate. Following Lewis (1954), we shall refer to this as the case of unlimited supply of labor.[8/] To the northeast of this locus, more labor can not be employed without an increase in wage rate. We refer to this as the case of limited supply of labor.

When labor supply is limited, the wage rate is determined so as to equate demand and supply. Hence, the following equilibrium condition must hold:

(14) $g_x(z)K_x + g_y(w)K_y = \nu(w)$.

Note that

$$\Delta \equiv g'_x K_x^{2-r} + g'_y K_y - \nu' < 0.$$

Hence, (14) can be solved for the equilibrium wage rate as

(15) $w = h(K_x, K_y)$,

where

(16) $\partial h/\partial K_x = -(1/\Delta)[g_x - (r-1)g'_x z] > 0$,

and

(17) $\partial h/\partial K_y = -(1/\Delta)g_y > 0.$

With (15) substituted into each sector's production and labor demand functions, the equilibrium levels of output and labor input in each sector are determined as functions of K_x and K_y. Hence, the profit rate in each sector becomes a function of K_x and K_y alone. Specifically, under limited supply of labor, we have

(18) $R_x(K_x, K_y) = K_x^{r-1} f_x(g_x(z)) - w\, g_x(z),$

and

(19) $R_y(K_x, K_y) = pf_y(g_y(w)) - w\, g_y(w),$

where $w = h(K_x, K_y)$. Using the chain rule and the envelope theorem,[9] we can show that

(20) $\partial R_x/\partial K_x = (r-1)K_x^{r-2} f_x - g_x \frac{\partial h}{\partial K_x} \gtreqless 0,$

(21) $\partial R_x/\partial K_y = -g_x \frac{\partial h}{\partial K_y} < 0,$

(22) $\partial R_y/\partial K_x = -g_y \frac{\partial h}{\partial K_x} < 0,$

and

(23) $\partial R_y/\partial K_y = -g_y \frac{\partial h}{\partial K_y} < 0.$

Note that capital accumulation in one sector always reduces the profit rate in the other sector when labor supply is limited. The reason is that the effect of increased capital stock in one sector is transmitted to the other sector in the form of an increased wage rate, resulting in a decrease in its profit rate. Note also that, while the direction of change in the urban profit rate is indeterminate as its capital stock increases, the profit rate of the rural sector always declines with K_y under the limited supply of labor.

The indeterminacy of the sign of $\partial R_x/\partial K_x$ arises from two offsetting factors. The direct effect of an increase in K_x on R_x is positive, but the indirect effect through an increase in wage rate is negative. The direct effect signifies increase in output due to increasing returns to scale, while the indirect effect is the increase in wage outlay caused by an increase in wage rate. The actual direction of change depends on whether the former outweighs, breaks even, or falls short of the latter.

As (16) and (17) show, the equilibrium wage rate, $w = h(K_x, K_y)$, increases monotonically with K_x and K_y. Indeed, $h(K_x, K_y) \to \infty$ as $K_x \to \infty$ and/or $K_y \to \infty$. That is, the equilibrium wage rate ranges from w_o to ∞ under the limited supply of labor. This implies that $R_x(K_x, K_y) \to 0$ as $K_y \to \infty$ for each fixed K_x, and $R_y(K_x, K_y) \to 0$ as $K_x \to \infty$ and/or $K_y \to \infty$ under the limited supply of labor.

When the labor supply is unlimited, the wage rate is fixed at w_o. Hence, the amount of labor employed in each sector is equal to the amount of labor demanded at w_o. Then, the two sectors' profit rates become

(24) $$R_x(K_x, K_y) = K_x^{r-1} f_x(g_x(z_o)) - w_o g_x(z_o),$$

and

(25) $$R_y(K_x, K_y) = pf_y(g_y(w_o)) - w_o g_y(w_o),$$

where $z_o = w_o/K_x^{r-1}$. We readily see that

(26) $$R_y(K_x, K_y) = \bar{R}_y \quad \text{(constant)},$$

(27) $$\partial R_x/\partial K_y = 0,$$

and by the envelope theorem,

(28) $$\partial R_x/\partial K_x = (r-1)K_x^{r-2} f_x(g_x(z_o)) > 0.$$

Thus, under the unlimited supply of labor, the profit rate of neither sector is influenced by the other sector. The absence of repercussions from the labor market enables the urban sector to enjoy full benefit of increasing returns to scale, while the rural sector maintains a constant profit rate. We have $R_x(K_x, K_y) \to 0$ as $K_x \to 0$, and, if the case of limited labor supply is disregarded, $R_x(K_x, K_y) \to \infty$ as $K_x \to \infty$.

IV. Growth of the Economy

A. Profitability Conditions

The dynamic property of the model hinges on the profitability of the two sectors. As a means to classify various cases, we introduce three "profitability conditions." First, let

$$S = \{(K_x, K_y) : g_x(z_o)K_x + g_y(w_o)K_y \leqq \nu(w_o)\}$$

and

$$M = \{(K_x, K_y) : g_x(z_o)K_x + g_y(w_o)K_y \geqq \nu(w_o) \text{ and } R_y(K_x, K_y) \leqq \mu\}.$$

The set S is the unlimited labor supply region in the (K_x, K_y)-plane, while M is the subset of the limited labor supply region where $R_y(K_x, K_y) \leqq \mu$. The three profitability conditions we postulate are:

(A-1) $\bar{R}_y > \mu$;

(A-2) There exists a $(K_x, K_y) \varepsilon S$ such that $R_x(K_x, K_y) \geqq \bar{R}_y$,

and

(A-3) There exists a $(K_x, K_y) \varepsilon M$ such that $R_x(K_x, K_y) \geqq R_y(K_x, K_y)$.

(A-1) states that the rural sector is profitable above depreciation under the unlimited supply of labor. (A-2) states that the urban sector can be at least as profitable as the rural sector when labor supply is unlimited. (A-3) is a similar condition as (A-2) for the limited labor supply case. It states that the urban sector can be at least as profitable as the rural sector as long as the rural sector is not profitable above depreciation. These conditions are seemingly harmless but they play an important role in determining the development possibility of this two-sector economy.

B. Properties of the Profit Rate Functions

For the subsequent discussion, it is convenient to define:

$$L = \{(K_x, K_y): g_x(z_o)K_x + g_y(w_o)K_y \geqq \nu(w_o)\}$$

and

$$L_o = \{(K_x, K_y): g_x(z_o)K_x + g_y(w_o)K_y > \nu(w_o)\},$$

which are the limited labor supply region inclusive and exclusive of the boundary locus, respectively. Equations (14)-(17) imply that, for each $w > w_o$, the locus of (K_x, K_y) satisfying $h(K_x, K_y) = w$ is a dowanward-sloping curve in L_o with finite intercepts with both axes. We shall call this curve as an "iso-wage rate curve." The iso-wage rate curves for $w > w_o$ constitute a one-parameter family of curves in L_o which shifts to the north-east as w increases. Note that the boundary locus is a special iso-wage rate curve with $w = w_o$. Along any iso-wage rate curve, (19) implies that the rural profit rate is constant, declining monotonically from $\bar{R}_y$ to 0 as w increases from w_o to ∞. Hence, for each iso-wage rate curve, one can associate a unique iso-profit rate curve of the rural sector, $R_y(K_x, K_y) = \beta$, with an appropriate $\beta \varepsilon (0, \bar{R}_y)$. In turn, one can show that the urban profit rate is strictly monotone increasing in K_x along any iso-wage rate curve, approaching 0 as $K_x \to 0$ and a finite value as $K_y \to 0$.

The following lemmas can be easily proved:

Lemma 1: Assume (A-1) and (A-2).

(a) The $R_y(K_x, K_y) = \mu$ curve is a downward-sloping curve in L_o having finite intercepts, K_x^{μ} and K_y^{μ}, with both axes.

(b) The locus of (K_x, K_y) satisfying $R_x(K_x, K_y) = \mu$ is a curve represented by $K_x = K_x^*$ in S, where $0 < K_x^* < \bar{K}_x$, and by a curve whose slope has the same sign as $\partial R_x/\partial K_x$ in L. This locus does not locate to the left of $K_x = K_x^*$.

(c) The locus of (K_x, K_y) satisfying $R_x = R_y$ is a curve represented by $K_x = K_x^{**}$ in S, where $K_x^* < K_x^{**} \leqq \bar{K}_x$, and by a curve which locates in the following subset in L:

$$\{(K_x, K_y)\varepsilon L: [R_x(K_x, K_y) - \mu][R_y(K_x, K_y) - \mu] \geqq 0\}.$$

The loci of points satisfying $R_x = \mu$ and $R_x = R_y$ are referred to as the "$R_x = \mu$ curve" and the "$R_x = R_y$ curve", respectively.

Lemma 2: Under the limited supply of labor,

(a) The $R_x = \mu$ and $R_x = R_y$ curves, if they exist, can have at most one intersection with any iso-wage rate curve.

(b) Assume (A-1) and (A-2). Let $\hat{w}$ be the unique wage rate associated with the $R_y = \mu$ curve. Then,

(i) for all $w > \hat{w}$, if the $h(K_x, K_y) = w$ curve has an intersection (K_x^1, K_y^1) with the $R_x = \mu$ curve, then it has a unique intersection (K_x^2, K_y^2) with the $R_x = R_y$ curve, where $K_x^2 > K_x^1$; and

(ii) for all w satisfying $\hat{w} > w > w_o$, if the $h(K_x, K_y) = w$ curve has an intersection (K_x^2, K_y^2) with the $R_x = R_y$ curve, then, it has a unique intersection (K_x^1, K_y^1) with the $R_x = \mu$ curve, where $K_x^1 > K_x^2$.

The shapes of the $R_x = \mu$, $R_y = \mu$ and $R_x = R_y$ curves are illustrated in Figure 2. The figure assumes that $\partial R_x/\partial K_x > 0$ in L_o, entailing an upward-sloping $R_x = \mu$ curve by Lemma 1(b). In this case, the $R_x = \mu$ and $R_y = \mu$ curves have a unique intersection in L_o which the $R_x = R_y$ curve passes through. However, such an intersection may not exist. The following theorem, proved in appendix, provides a sufficient condition for the existence of such an intersection:

Theorem 1: Under (A-1), (A-2), and (A-3), there exists a unique intersection, $(\hat{K}_x, \hat{K}_y)$, of the $R_x = \mu$ and $R_y = \mu$ curves in L_o, where $K_x{}^* < \hat{K}_x < K_x{}^{\mu}$.

C. Growth Patterns under Profitability Conditions

The growth of the present economy is governed by the capital accumulation functions (9) and (10). The continuity of the right-hand side functions of (9) and (10) guarantees the existence of a solution to this system, but the uniqueness does not necessarily follow. However, since the uniqueness is ensured under a fairly general condition,[10] we shall proceed by assuming that the solution is unique. Then, based on the results obtained so far, the growth patterns of the economy can be analyzed using a phase diagram.

When the profitability conditions, (A-1), (A-2), and (A-3), are satisfied, the stationary state of the economy is achieved at the intersection $(\hat{K}_x, \hat{K}_y)$ of the $R_x = \mu$ and $R_y = \mu$ curves in L_o, whose existence is guaranteed by Theorem 1. Indeed, from (9) and (10), $\dot{K}_x = \dot{K}_y = 0$ at this point. By examining the signs of $\dot{K}_x$ and $\dot{K}_y$ in each section of the (K_x, K_y)-plane separated by the $R_x = \mu$, $R_y = \mu$, and $R_x = R_y$ curves and the boundary locus, we find that the locus of points satisfying $\dot{K}_y = 0$ is a downward-sloping curve in L_o which intersects with the K_y-axis at $K_y{}^{\mu}$ and the K_x-axis at a point between $K_x{}^{\mu}$ and $\bar{K}_x$. To the left of $(\hat{K}_x, \hat{K}_y)$, this locus locates above the $R_y = \mu$ curve, and to the right, in the area surrounded by the $R_y = \mu$ and $R_x = R_y$ curves and the boundary locus, as shown in Figure 3.[11] Similarly, the locus of points satisfying $\dot{K}_x = 0$ is a vertical line, $K_x = \tilde{K}_x$ where $K_x{}^* < \tilde{K}_x < K_x{}^{**}$, under the unlimited supply of labor, and stretches into the limited labor supply region passing through $(\hat{K}_x, \hat{K}_y)$. The locus is bounded by the $R_x = \mu$

and $R_x = R_y$ curves except at $(\hat{K}_x, \hat{K}_y)$, as shown also in Figure 3.

The directions of changes in K_x and K_y in each section of the (K_x, K_y)-plane separated by the $\dot{K}_x = 0$ and $\dot{K}_y = 0$ loci are shown in Figure 3 by arrows. We have the following result, which is proved in appendix:

Theorem 2: If (A-1), (A-2), and (A-3) are satisfied, the stationary point $(\hat{K}_x, \hat{K}_y)$ is a saddle point.

The growth trajectories of the economy starting from various initial points are shown in Figure 4. An important implication of Theorem 2 is that the economy has a strong tendency to concentrate in one of the two sectors at the cost of the other sector. For example, suppose that the economy is initially at (K_x^o, K_y^o) in the unlimited labor supply region S. There is a path originating in S and approaching the stationary state, which we call the "critical path" (Figure 4). If the economy is initially at a point in S to the right of this path, the urban sector grows regardless of the initial size of the rural sector. The latter sector may grow initially but will eventually decay as the urban sector grows larger. In turn, if the economy is at a point in S to the left of the critical path, the rural sector grows, and the urban sector decays sooner or later. As time goes, the rural sector approaches a finite size, K_y^{μ}, while the urban sector becomes very small. If (K_x^o, K_y^o) happens to be on the critical path, the economy approaches the stationary state where each sector achieves a finite size. However, even a small disturbance will cuase the economy to move toward predominance of one of the two sectors.

Thus, urban concentration, once started, has a tendency to continue until the rural sector becomes very small. The capital accumulation in the urban sector carries with it the concentration of workers in the urban area and away from the rural sector, as we can see from (11), (12), and (16). That is, the regional employment pattern in this model is endogenously determined by the relative capital accumulation in the two sectors, which contrasts with the exogenous mechanism of rural-urban labor migration as studied by Todaro (1965), Harris and Todaro (1970), and Mas-Colell and Razin (1973), among others. The present model induces urban concentration of labor force through job opportunities created by accumulated capital in the urban area. Since, as observed above, the urban concentration of capital tends to continue once started, the present model suggests that any regional labor redistribution policy must involve appropriate policy measures to control for proper regional distribution of private investments.

It is interesting to note that the existence of increasing returns to scale in the urban technology does not by itself ensure urban growth. The amount of initial urban capital stock, K_x^o, holds a key importance. Indeed, since the critical path is bounded by $K_x = \hat{K}_x$ and $K_x = \tilde{K}_x$ in S, $K_x^o \geqq \max \{\hat{K}_x, \tilde{K}_x\}$ is sufficient to induce urban concentration with rural decay,[12] while $K_x^o \leqq \min \{\hat{K}_x, \tilde{K}_x\}$ results in the opposite growth pattern. In these cases, the initial size of the rural sector, K_y^o, has no effect on the eventual direction of urban and rural growth. However, if the initial urban size is such that $\min \{\hat{K}_x, \tilde{K}_x\} < K_x^o < \max \{\hat{K}_x, \tilde{K}_x\}$, the initial size of the rural sector may have a critical importance in determining where the economy will lead to. Note also that, since the critical path locates to the left of the $R_x = R_y$ curve, urban concentration can result even if the urban sector is not as profitable as the rural sector initially, although it must be profitable above depreciation.

Although increasing returns to scale does not necessarily induce urban growth, it plays an important role in determining the minimum size of the urban capital stock required to ensure urban growth. In fact, under the unlimited supply of labor, (24) and (11) imply that R_x is larger for each K_x the larger the degree of increasing returns to scale, r. Hence, K_x^* and K_x^{**} are smaller. Moreover, one can show that $\hat{K}_x$ is also smaller. Hence the $R_x = \mu$, $R_x = R_y$, and $K_x = \hat{K}_x$ curves all move to the left as r increases. Hence, the initial capital stock K_x^o needed to induce urban concentration becomes smaller, the larger the degree of increasing returns to scale, resulting in a higher possibility of urban concentration.

Finally, note that, when profitability conditions are satisfied, any path starting from an initial point in the unlimited labor supply region eventually crosses the boundary locus and moves into the limited labor supply region. That is, as capital accumulation in the economy proceeds, the surplus labor that existed initially gets absorbed by job opportunities created by increased capital, and full employment of labor is achieved. The point at which the growth path crosses the boundary locus is the turning point of the economy from unlimited to limited supply of labor in the sense similar to that of Lewis (1958).[13/] Indeed, the boundary locus itself is the set of all potential turning points. Thus, a successful development in the present model is always accompanied by a turning point where marginal productivity of labor ceases to be less than the wage rate. Note that, in Figure 4, the growth of the economy from unlimited to limited labor supply is always assured, but such is not necessarily the case if some of the profitability conditions are violated. We will now turn to these cases.

D. Conditions for Development

Variety of cases arise if some of the profitability conditions are not met. Major cases are briefly described below:

(i) If (A-1) holds, the $R_y = \mu$ curve exists in L_o. If (A-3) also holds, it is possible that there is a unique intersection of the $R_x = \mu$ and $R_y = \mu$ curves in L_o, even if (A-2) does not hold. In such a case, Theorem 2 is still valid and the properties of the phase diagram become similar to those obtained for the case discussed in Section IV.C. However, the urban sector is dominated by the rural sector in terms of profitability in S, so that the possibility of urban growth becomes very small for the paths starting in S, while rural expansion to a finite size is likely to take place.

(ii) If (A-1) and (A-2) hold but not (A-3), the $R_x = \mu$ and $R_y = \mu$ curves do not intersect. The $R_x = \mu$ curve exists and extends into the limited labor supply region, L_o, but it does not reach the $R_y = \mu$ curve. A similar case arises if (A-2) alone is violated and if the $R_x = \mu$ and $R_y = \mu$ curves do not intersect. In these cases, almost all paths eventually move toward rural expansion with urban decay. That is, a sustained urban growth is impossible and the economy will become predominantly agrarian.

(iii) If (A-1) does not hold and if $\bar{R}_y < \mu$, then regardless of (A-2) and (A-3), the rural sector will eventually become very small. The urban sector may or may not grow depending on its initial profitability relative to that of the rural sector.

(iv) If $\bar{R}_y = \mu$, then the locus of (K_x, K_y) satisfying $R_y(K_x, K_y) = \mu$ coincides with the unlimited labor supply region, S. If (A-2) holds, the $R_x = \mu$ and $R_x = R_y$ curves coincide in S. The $\dot{K}_x = 0$ and $\dot{K}_y = 0$ curves in L meet at the intersection of these curves with the boundary locus and become identical with them in S. Most of the properties of the phase diagram in this case are similar to those discussed in Section IV.C, but this case presents a possibility of stagnation under unlimited labor supply if the economy is initially at a point on the $R_x = \mu$ curve.

Thus, when some of the profitability conditions are violated, the development possibility of the economy is significantly affected. When the rural sector is at least profitable enough to cover depreciation but if the urban sector cannot achieve a comparable profitability with either unlimited or limited labor supply, the urban sector may not grow, or even if it does, a large initial capital stock is needed to induce urban growth. In such a case, a large injection of capital through borrowing from abroad or foreign aid may help, but other measures can also be used. As (24) shows, the urban profit rate can be increased through increased labor productivity, higher returns to scale, or an increase in capital stock. The labor productivity can be increased through technological improvements or an increase in capital stock. The returns to scale may be increased by technological change but also by increased agglomeration economies, for example, through improvements in transport and communication facilities. All these measures would help increase the urban profit rate and enhance a better chance for urban growth.

If $\bar{R}_y < \mu$, the rural sector has no prospect for sustained growth and the urban sector will also contract unless its initial capital stock is very large. In this case, the policy exclusively designed toward urban growth is not advisable since, if it is not successful, both sectors would contract.

A more preferable approach is to first increase rural profitability through tariff policies to improve its terms of trade and/or improvements in technology, and then, or at the same time, aim at increasing urban profitability. A profitable rural sector generates investment funds that can be partly channeled to investments in the urban sector, which would facilitate better prospects for the growth of the urban sector and of the whole economy.

V. Patterns of Trade

A. Demand Conditions

So far, the demand side of the economy has not been considered, but by doing so, we can analyze the trade implications of the present model. To maintain consistency with the labor supply function (6), we assume that the consumers in the economy behave as if they maximize an aggregate utility function

$$U(C_x, C_y, N^S) = \phi[u(C_x, C_y) + v(N^S)],$$

subject to the budget constraint, where C_x and C_y are consumption of the two goods, and N^S is the amount of labor supply. The function u is assumed to be strictly quasi-concave and homogeneous of degree one with positive marginal utilities, ϕ is a monotone increasing transformation, and v measures disutility of labor with $v'(N^S) < 0$ and $v''(N^S) < 0$.

Since no savings are made out of wage income and no consumption out of profit income, the consumers' budget constraint is given by

$$w N^S = C_x + p C_y .$$

The optimality then requires that

$$(28) \qquad u_y/u_x = p \qquad \text{and} \qquad v'(N^S)/u_x = -w,$$

where u_x and u_y are partial derivatives of u with respect to C_x and C_y. Because of homogeneity, we can write

$$u_y(c,\ 1)/u_x(c,\ 1) = p\ ,$$

where $c = c_x/c_y$, which can be solved for c as

$$c = \sigma(p),$$

with $\sigma'(p) > 0$. Hence, from (28),

$$v'(N^S) = -w\ u_x(\sigma(p),\ 1),$$

which we solve for N^S to obtain

$$N^S = \nu(w),$$

where $\nu'(w) > 0$ and p is assumed subsumed in the functional form of ν.

Now, let the total demand for labor at w be $N^D(w)$. Then, the actual amount of labor employed is determined as

$$\text{(29)} \qquad N = \begin{cases} N^D(w) & \text{if } w = w_o \\ \nu(w) & \text{if } w > w_o\ , \end{cases}$$

and the consumers' income becomes

$$I = w\ N\ .$$

The labor supply function (29) corresponds to the one postulated earlier.

Once the consumers' income is determined, the demand functions for two goods are obtained as

$$\text{(30)} \qquad C_x = \alpha_x(p)I$$

and

$$\text{(31)} \qquad C_y = \alpha_y(p)I,$$

where $\alpha_x(p) = \sigma(p)/(\sigma(p)+p)$ and $\alpha_y(p) = 1/(\sigma(p)+p)$.

Let E_x and E_y be the excess supply of X and Y, respectively. The balance of payments of this economy is given by

$$B = E_x + pE_y.$$

Now, the excess supplies are defined as $E_x = X - C_x - (R_xK_x + R_yK_y)$ and

$E_y = Y - C_y$. Let N_x and N_y be employment in the urban and rural sector, respectively. Then, since all wage income is consumed and all profit income is invested,

$$R_xK_x + R_yK_y = X + pY - w(N_x + N_y) = X + pY - (C_x + pC_y)$$
$$= (X - C_x) + p(Y - C_y).$$

That is, $B = E_x + pE_y = 0$, which shows that the balance of payments is always in equilibrium in the present economy.

B. Trade Patterns

When labor supply is unlimited, the consumers' income is given by

$$I = w_o(N_x + N_y) \ ,$$

where N_x and N_y are demand for labor in the urban and rural sectors at w_o. The excess supply of good Y becomes

$$(32) \qquad E_y(K_x, K_y) = K_y[f_y(g_y(w_o)) - \alpha_y w_o g_y(w_o)] - \alpha_y w_o g_x(z_o)K_x,$$

where $z_o = w_o/K_x^{r-1}$. Clearly, for each fixed K_x,

$$\lim_{K_y \to 0} E_y(K_x, K_y) < 0.$$

We can also show that, for each fixed K_y,[14]

$$\lim_{K_x \to 0} E_y(K_x, K_y) = K_y[f_y(g_y(w_o)) - \alpha_y w_o g_y(w_o)] > 0 \ .$$

The locus of (K_x, K_y) satisfying $E_y(K_x, K_y) = 0$ will be called the "$E_y = 0$ curve." From (32), we see that, along the $E_y = 0$ curve,

$$dK_y/dK_x\Big|_{E_y=0} > 0 \ .$$

The shape of this curve in S is shown in Figure 5. To the left of this curve, $E_y > 0$, so that the economy exports good Y and imports good X, while to the right of this curve, $E_y < 0$, so that the reverse trade pattern emerges.

When labor supply is limited, the excess supply function of Y becomes

$$E_y(K_x, K_y) = \tilde{E}_y(w, K_y) = K_y f_y(g_y(w)) - \alpha_y w\nu(w),$$

where $w = h(K_x, K_y)$. Hence, along the $h(K_x, K_y) = w$ curve,

$$\lim_{K_y \to 0} E_y < 0 .$$

We can solve $\tilde{E}_y(w, K_y) = 0$ for K_y to obtain

$$(33) \qquad K_y = [\alpha_y w \nu(w)]/f_y(g_y(w)) ,$$

which implies that

$$(34) \qquad dK_y/dw\Big|_{\tilde{E}_y=0} > 0.$$

From (14), the K_y-intercept of the $h(K_x, K_y) = w$ curve is given by

$$K_y^{\beta} = \nu(w)/g_y(w),$$

so that, for each $w \in (w_o, \infty)$ and along the $\tilde{E}_y = 0$ curve,

$$(35) \qquad K_y - K_y^{\beta} = \nu(w)[\alpha_y w g_y - f_y]/f_y g_y < \nu(w)[\tfrac{w}{p} g_y - f_y]/f_y g_y < 0 ,$$

since $f_y - \frac{w}{p} g_y > 0$. (33) - (35) imply that, for each $w \in (w_o, \infty)$, there is a unique $(K_x, K_y) > 0$ on the $h(K_x, K_y) = w$ curve satisfying $E_y(K_x, K_y) = 0$.

Differentiating $E_y(K_x, K_y)$ partially, we have $\partial E_y/\partial K_x < 0$. However, the sign of $\partial E_y/\partial K_y$ is indeterminate. Consequently, the locus of $E_y = 0$ in the limited labor supply region may be upward sloping, vertical, or downward sloping. However, this locus is a well-defined curve in L which crosses every iso-wage rate curve exactly once for each $w > w_o$. The $E_y = 0$ curve in L is also shown in Figure 5. To the right of this curve, $E_y < 0$ and to the left, $E_y > 0$.

C. Growth and the Changes in Trade Patterns

The $E_y = 0$ curve derived above can now be superimposed on the phase diagram of the economy as in Figure 6. Throughout this section, we shall assume that the three profitability conditions are satisfied and $\hat{K}_x \geqq \tilde{K}_x$, leaving other cases to interested readers.

Let $(\hat{K}_x, \hat{K}_y)$ be the unique intersection of the $R_x = \mu$ and $R_y = \mu$ curves. Then, either $E_y(\hat{K}_x, \hat{K}_y) > 0$ (as in Figure 6), or $E_y(\hat{K}_x, \hat{K}_y) \leqq 0$. In the former case, consider a path starting from point A in S. Since $E_y > 0$ at A, the economy is initially exporting agricultural goods and importing manufactured goods. However, the growth path eventually crosses the $E_y = 0$ curve, so that E_y becomes negative. That is, when the growth path crosses the $E_y = 0$ curve, the trade pattern is reversed and the economy will become an exporter of manufactured goods and importer of agricultural goods. Thus, in the present model, if the economy starts from an initial point like A, urbanization is accompanied by a shift in export patterns from primary to manufactured goods.

A trade pattern reversal, however, is not an attribute of urbanization. For example, the path starting from B leads to urban concentration but the economy exports X and imports Y throughout. In turn, the path starting from C leads to rural expansion but the trade pattern reversal takes place, here in the opposite direction. It is important to note that, except for the critical path, any path will move toward complete specialization in one of the two goods.

The case where $E_y(\hat{K}_x, \hat{K}_y) \leqq 0$ can be analyzed similarly. The major differences are that urbanization proceeds without experiencing trade pattern reversals, and that, if the economy is exporting agricultural goods initially, the economy will not move toward specialization in manufactured goods, unless the demand or technological conditions are significantly altered.

VI. Summary and Conclusion

In this paper, we used a two-sector growth model to examine how far the urbanization trend is to continue and what implication it has on rural growth. Postulating increasing returns to scale in the urban production, we showed that urban concentration, once started, has a strong tendency to continue until the rural sector becomes very small. Since labor is pulled by accumulated capital, the model suggests that any regional labor redistribution policy must be accompanied by an appropriate policy for regional redistribution of private investments. For the paths starting from the initial condition of unlimited labor supply, we showed the transition of the economy from unlimited to limited labor supply. In turn, the development possibility of the economy is seen to hinge crucially on the initial profitability of the two sectors, especially that of the rural sector, which points to the importance of the rural sector in an early stage of economic development. Finally, the analysis of trade patterns entailed a tendency for specialization in the products of the sector that becomes dominant.

In the present model, we have not incorporated negative effects of urbanization on production, as our focus is on the process of urbanization that generates large urban areas. An incorporation of the negative effects would reduce economies of scale in city size at a later stage of urbanization and probably result in a stationary state with large but finite size of the urban sector with a small finite size of the rural sector. However, even with such a modification, the continuing trend for urban concentration would still hold until the agglomeration diseconomies begin to outweigh the agglomeration economies.

The small country assumption postulated in the present paper had an effect of suppressing the product market out of consideration. If this assumption is relaxed, the relative price of the two goods becomes endogenous and the whole analysis must be done within a general equilibrium framework. Some preliminary attempt along this line is under way, but it remains to be seen where the economy will lead to if such a modification is incorporated.

Notes

1/ Moomaw (1981) critically reviews Segal's and Sveikauskas' results for accuracy of measurement. Nonetheless, his revised estimates for Segal's results still bear out a substantial productivity gain in city size.

2/ See, for example, Fei and Ranis (1964), Jorgenson (1961, 1967) Hayami and Ruttan (1971), and Johnston (1970) among others.

3/ If the labor market imperfection is due to a proportional wage differential, the subsequent analysis can be extended without affecting the major results.

4/ The Inada derivative conditions hold for Cobb-Douglas functions but not for most of CES production functions. See Burmeister and Dobell (1970), pp.30-31. However, we retain these assumptions for the sake of brevity.

5/ The finiteness of the right-hand and left-hand derivatives at zero combined with the continuity assumption implies that the investors of either sector do not shift their investment too drastically when the profit rate in one sector becomes slightly higher than that in the other sector. Thus, these assumptions ensure moderate reaction of investors to the change in $R_x - R_y$.

6/ Different but constant rates of depreciation in the two sectors can easily be accommodated without changing the results of the subsequent analysis.

7/ The present probelm is a maximization of a strictly concave function subject to a nonnegativity constraint. Hence, the Kuhn-Tucker quasi-saddle point condition is necessary and sufficient for a global optimum. See Takayama (1974), Chapter 1.

8/ Lewis (1954) states that "Whether marginal productivity is zero or negligible is not, however, of fundamental importance to our analysis. The price of labor, in these economies, is a wage at the subsistence level (...). The supply of labor is therefore 'unlimited' so long as the supply of labor at this price exceeds the demand." (p.142).

9/ See Samuelson (1947), p.34; Afriat (1971), pp.355-357; and Takayama (1974), pp.160-165.

10/ The solution to (9) and (10) is unique if the functions on the right-hand side are locally Lipschitzian in the interior of R_+^2. This condition is satisfied if, for example, the fraction functions, ξ_i and ζ_i $(i = x,y)$, have finite right-hand and left-hand derivatives at each point in R.

11/ The figure corresponds to the case where $\hat{K}_x \geqq \tilde{K}_x$. The subsequent analysis is limited to this case unless otherwise noted. The case of $K_x^* < \tilde{K}_x < \hat{K}_x$ can be similarly analyzed without significantly affecting the qualitative results.

12/ When $\hat{K}_x > \tilde{K}_x$, the critical path is actually bounded by $K_x = \tilde{K}_x$ and $K_x = K_x^{**}$, so that we can refine the argument by replacing max $\{\hat{K}_x, \tilde{K}_x\}$ by max $\{K_x^{**}, \tilde{K}_x\}$. Same remark holds for the subsequent discussion.

13/ Lewis (1958) does not consider capital accumulation in the rural sector, so that the "turning point" here is slightly different from that of Lewis', but the idea is essentially the same.

14/ Note that $\lim_{K_x \to 0} E_y(K_x, K_y) = \frac{1}{p} K_y[pf_y - p\alpha_y w_o g_y(w_o)]$. Since $0 < p\alpha < 1$ by (30) and (31) and since $pf_y - w_o g_y > 0$ by the homogeneity of f_y, $pf_y - p\alpha_y w_o g_y(w_o) > 0$. Hence, $\lim_{K_x \to 0} E_y(K_x, K_y) > 0$.

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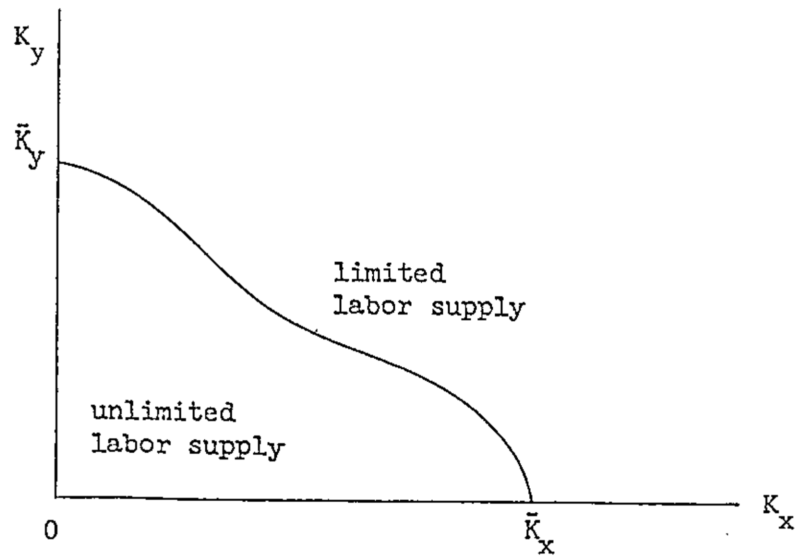

Figure 1: The Boundary Locus

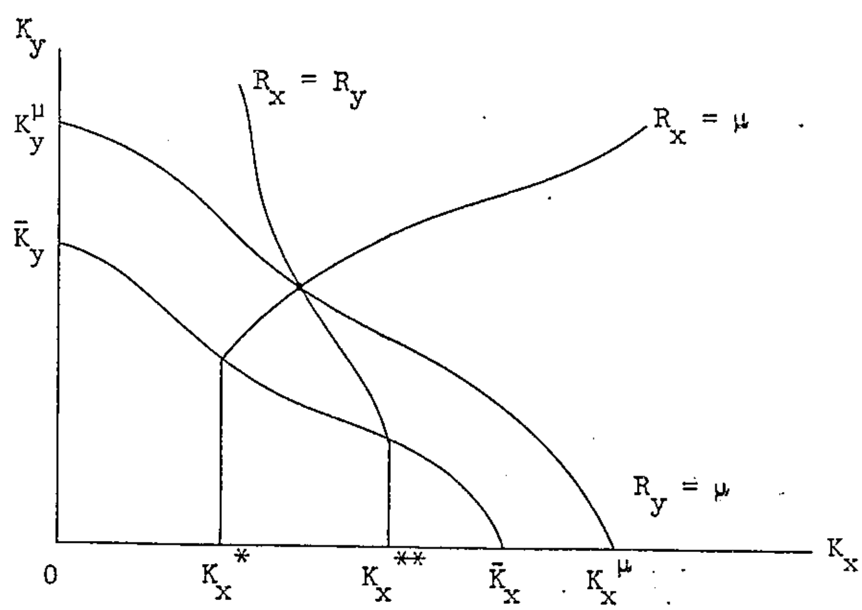

Figure 2: The $R_x = \mu$, $R_y = \mu$, and $R_x = R_y$ Curves

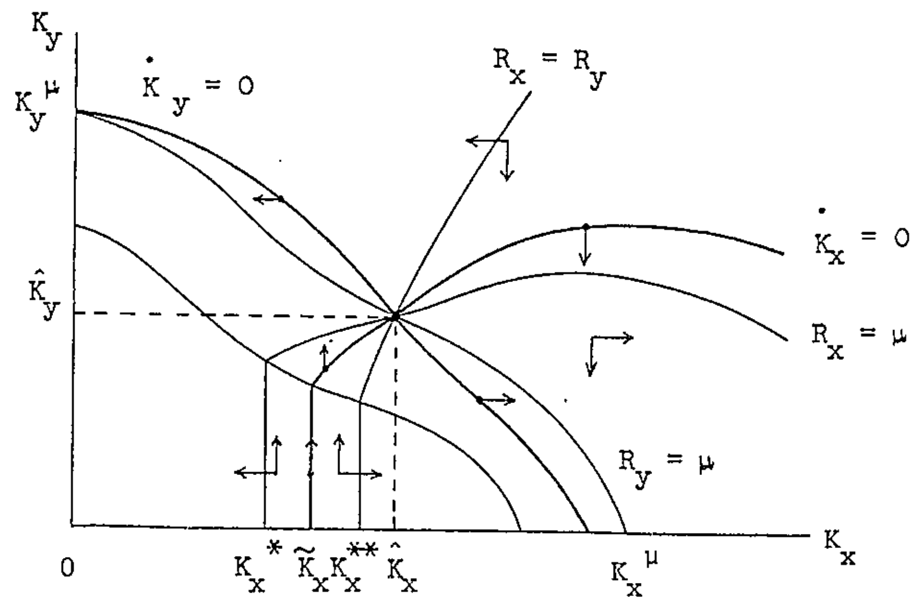

Figure 3: Arrow Diagram for $\hat{K}_x \geq \tilde{K}_x$

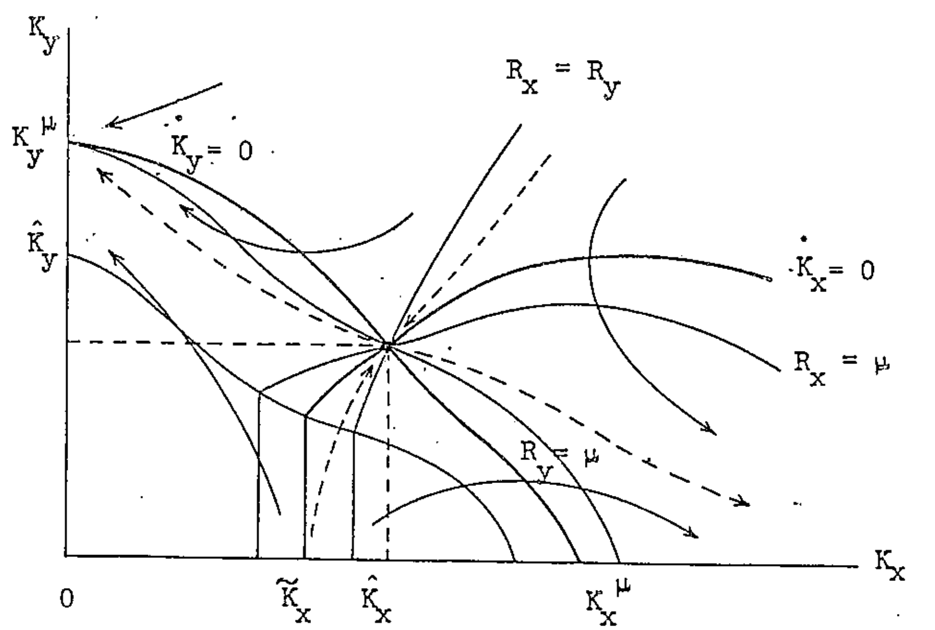

Figure 4: Growth Paths of the Economy

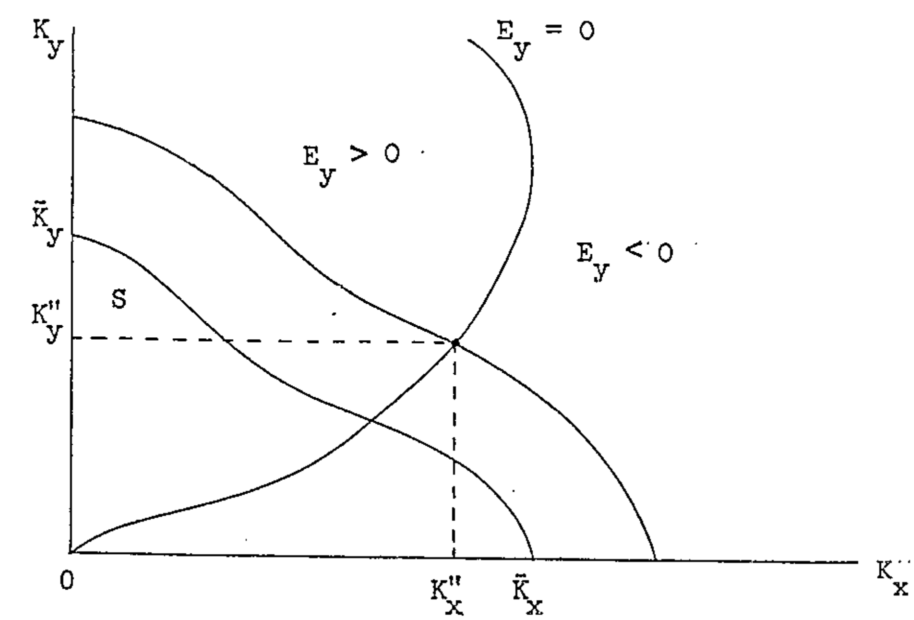

Figure 5: The $E_y = 0$ Curve.

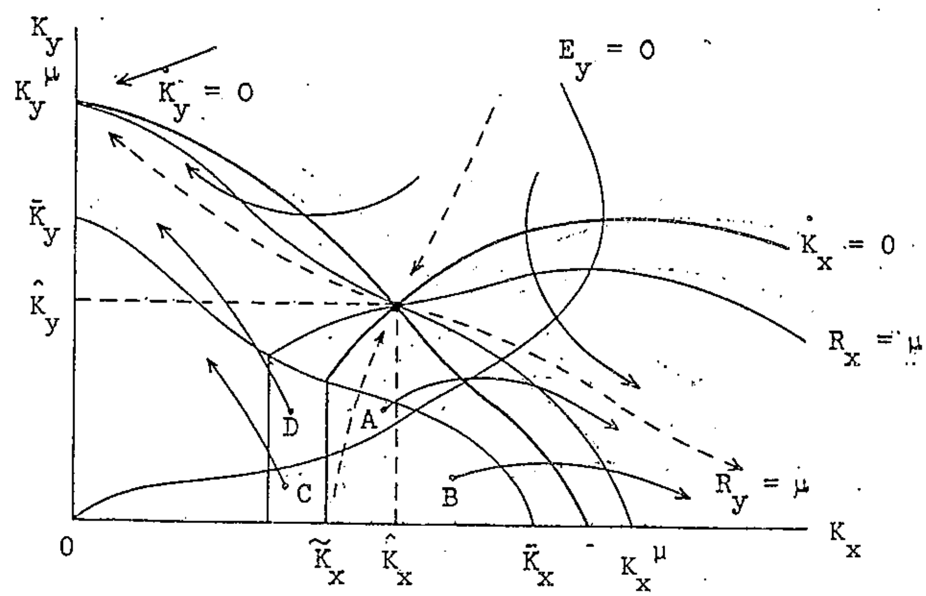

Figure 6: Growth and Trade with $E_y(K_x,K_y) > 0$

Appendix

1. Proof of Theorem 1

By (A-3), there exists a $(K_x', K_y') \in M$ such that $R_x(K_x', K_y') \geqq R_y(K_x', K_y')$. Let $\beta = R_y(K_x', K_y')$. Since $(K_x', K_y') \in M$, $\beta \leqq \mu$. Let $\bar{w} > w_o$ be the unique wage rate such that the $h(K_x, K_y) = \bar{w}$ curve coincides with the $R_y(K_x, K_y) = \beta$ curve. Since R_x is strictly monotone increasing in K_x along the iso-wage rate curve, $R_x(K_x', K_y') \geqq \beta$ implies that there exists a unique (K_x'', K_y'') on the $h(K_x, K_y) = \bar{w}$ curve such that $R_x(K_x'', K_y'') = \beta = R_y(K_x'', K_y'')$.

Let K_x^{**} be as defined by Lemma 1(c), and let K_y^{**} be the K_y-coordinate of the intersection of $K_x = K_x^{**}$ and the boundary locus. By Lemma 1(c), $R_x(K_x^{**}, K_y^{**}) = R_y(K_x^{**}, K_y^{**})$. Hence, the $R_x = R_y$ curve intersects with both the $h(K_x, K_y) = w_o$ and $h(K_x, K_y) = \bar{w}$ curves. Now, let $\hat{w}$ be the unique wage rate associated with the $R_y = \mu$ curve. Since $w_o < \hat{w} \leqq \bar{w}$ and since the $R_x = R_y$ curve is continuous, the $R_x = R_y$ curve also intersects with the $h(K_x, K_y) = \hat{w}$ curve in L_o. Denote this intersection by $(\hat{K}_x, \hat{K}_y)$. Then, $R_x(\hat{K}_x, \hat{K}_y) = R_y(\hat{K}_x, \hat{K}_y) = \mu$. Clearly, $\hat{K}_x > K_x^*$ and $\hat{K}_x < K_x^{\mu}$. The uniqueness of $(\hat{K}_x, \hat{K}_y)$ follows from Lemma 2(a).

Q.E.D.

2. Proof of Theorem 2

For simplicity, the proof is given assuming differentiability of ξ_x, ξ_y, ζ_x, and ζ_y at zero, although it can also be proved using their right-hand and left-hand derivatives.

First define

$$\dot{K}_x = P(K_x, K_y) \equiv R_x K_x \xi_x + R_y K_y \zeta_x - \mu K_x,$$

and

$$\dot{K}_y = Q(K_x, K_y) \equiv R_x K_x \xi_y + R_y K_y \zeta_y - \mu K_y .$$

Linearizing P and Q around the stationary point $(\hat{K}_x, \hat{K}_y)$, we obtain

$$(*) \qquad \begin{cases} \dot{K}_x = P_x(K_x - \hat{K}_x) + P_y(K_y - \hat{K}_y), \\ \dot{K}_y = Q_x(K_x - \hat{K}_x) + Q_y(K_y - \hat{K}_y), \end{cases}$$

where P_x, P_y, Q_x, and Q_y are partial derivatives of P and Q with respect to K_x and K_y evaluated at $(\hat{K}_x, \hat{K}_y)$. The characteristic equation of the system (*) is given by

$$\begin{vmatrix} P_x - \lambda & P_y \\ Q_x & Q_y - \lambda \end{vmatrix} = \lambda^2 - (P_x + Q_x)\lambda + (P_x Q_y - P_y Q_x) = 0 .$$

Note that $\xi_x' = -\xi_y' \geqq 0$ and $\zeta_x' = -\zeta_y' \geqq 0$. Moreover, at $(\hat{K}_x, \hat{K}_y)$, $\xi_x = 1$, $\xi_y = 0$, $\zeta_x = 0$, and $\zeta_y = 1$. Then, after some manipulation, we obtain

$$P_x Q_y - P_y Q_x = \hat{K}_x \hat{K}_y \left(\frac{\partial R_x}{\partial K_x} \frac{\partial R_y}{\partial K_y} - \frac{\partial R_x}{\partial K_y} \frac{\partial R_y}{\partial K_x} \right)$$

$$+ (R_x \hat{K}_x \xi_x' + R_y K_y \zeta_x') \left(\frac{\partial R_x}{\partial K_x} \frac{\partial R_y}{\partial K_y} - \frac{\partial R_x}{\partial K_y} \frac{\partial R_y}{\partial K_x} \right) (\hat{K}_x + \hat{K}_y).$$

Now, substituting from (20)-(23), we find that

$$\frac{\partial R_x}{\partial K_x} \frac{\partial R_y}{\partial K_y} - \frac{\partial R_x}{\partial K_y} \frac{\partial R_y}{\partial K_x} = -(r-1) K_x^{r-2} f_x g_y \frac{\partial h}{\partial K_y} < 0 .$$

Hence, we have $P_x Q_y - P_y Q_x < 0$ at $(\hat{K}_x, \hat{K}_y)$, which implies that the stationary point is a saddle point.

Q.E.D.

Remark: (A-1), (A-2), and (A-3) are needed only to guarantee existence of the unique intersection of the $R_x = \mu$ and $R_y = \mu$ curves. Hence, the theorem is valid as long as a unique intersection of the two curves exist in L.