Department of Policy and Planning Sciences

Discussion Paper Series

No.1400

The Integrated Theory of Selling Problems and Buying Problems Based on the Concepts of Symmetry and Analogy (ver.006)

by

Seizo IKUTA and Byung-Kook, KANG

December 12, 2025

UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305-8573
JAPAN

Alice's Adventures in Wonderland*

The Integrated Theory of Selling Problems and Buying Problems Based on Concepts of Symmetry and Analogy

— ver006 —

Dr. (Eng.) Seizo Ikuta §
Retired from University of Tsukuba

Dr. (Mgt. Sci.&Eng.) Byung-Kook, Kang ‡
Nanzan University

December 12, 2025

Keywords

selling problem, buying problem, symmetry, analogy, recognizing time, staring time, initiating time, null-time-zone, deadline-engulfing, market restriction

Abstract

Problems of trading assets, commodities, goods, etc. form the backbones of management and/or economic behavior. Each of these problems is classified into a selling problem and a buying problem, and each problem, whether a selling problem or a buying problem, is furthermore categorized into a problem where the counter-trader proposes the trading price and a problem where the leading-trader proposes the trading price. In this paper let us refer to the group consisting of the above four problems as the *quadruple-asset-trading-problems*. The main objective of this paper is twofold: (1) to construct a general theory integrating the quadruple-asset-trading-problems and (2) to analyze these problems by using this theory. To achieve the two objectives, several novel concepts are introduced: symmetry, analogy, optimal initiating time, market restriction, etc. The most notable findings resulting from the analyses of these problems are the following two: (i) the significant breakdown of symmetry between the selling problem and the buying problem and (ii) the existence of the *null-time-zone* caused by the optimal initiating time, during which any decision-making activity is entirely senseless. Particularly, the latter discovery challenges us to re-examine and rewrite the entire discussions that have been conventionally conducted for decision-making processes. Moreover interestingly, when this time zone encompasses all points in time over a given planning horizon except the deadline, it follows that all decision-making activities scheduled throughout the entire planning horizon are *engulfed* in the deadline and then the process terminates there. This phenomenon is reminiscent of all matter, even light, falling into a black hole in physics. The concepts and methodologies presented above leads us to quite a novel perspective for conventional discussions of trading problems.

Version History

- ver000 (2004/07) https://commons.sk.tsukuba.ac.jp/discussion → Discussion Paper Series DP 1084 [8,Ikuta].
  — An Integration of the optimal stopping problem and the optimal pricing problem (model with no recall)
- ver001 (2023/05) https://commons.sk.tsukuba.ac.jp/discussion → Discussion Paper Series DP 1386 [9,Ikuta].
  — The integration theory of selling problem and buying problem based on the concepts of symmetry and analogy
- ver002 (2024/09) https://commons.sk.tsukuba.ac.jp/discussion → Discussion Paper Series DP 1393 [11,Ikuta & Kang].
  — The integrated theory of selling problem and buying problem based on the concepts of symmetry and analogy
- ver003 (2024/12) https://commons.sk.tsukuba.ac.jp/discussion → Discussion Paper Series DP 1394 [10,Ikuta & Kang].
  — The integrated theory of selling problem and buying problem based on the concepts of symmetry and analogy
- ver004 (2025/05) https://commons.sk.tsukuba.ac.jp/discussion → Discussion Paper Series DP 1396 [12,Ikuta & Kang].
  — The integrated theory of selling problem and buying problem based on the concepts of symmetry and analogy
- ver005 (2025/08) https://commons.sk.tsukuba.ac.jp/discussion → Discussion Paper Series DP 1399 [13,Ikuta & Kang].
  — The integrated theory of selling problem and buying problem based on the concepts of symmetry and analogy

*Readers will be bewildered by the indications in Alice 1(p.15), Alice 2(p.42), and Alice 3(p.44).

§osp19411@outlook.jp
‡kang_758@yahoo.co.jp

Contents

Abbreviations

$\mathbb{R}$-mechanism	$\mathbb{R}$eservation price mechanism (see Section 1.1(p.3))
$\mathbb{P}$-mechanism	$\mathbb{P}$osted price mechanism (see Section 1.1(p.3))
$\mathbb{R}$-model	model with $\mathbb{R}$-mechanism
$\mathbb{P}$-model	model with $\mathbb{P}$-mechanism
sE-model	search-Enforced-model (see A5a(p.11))
sA-model	search-Allowed-model (see A5b(p.12))
ATP[$\mathbb{R}$]	Asset Trading Problem with $\mathbb{R}$-mechanism (see (1.4.1(p.5)))
ATP[$\mathbb{P}$]	Asset Trading Problem with $\mathbb{P}$-mechanism (see (1.4.1(p.5)))
ASP[$\mathbb{R}$]	Asset Selling Problem with $\mathbb{R}$-mechanism (see (1.4.2(p.5)))
ASP[$\mathbb{P}$]	Asset Selling Problem with $\mathbb{P}$-mechanism (see (1.4.2(p.5)))
ABP[$\mathbb{R}$]	Asset Buying Problem with $\mathbb{R}$-mechanism (see (1.4.3(p.6)))
ABP[$\mathbb{P}$]	Asset Buying Problem with $\mathbb{P}$-mechanism (see (1.4.3(p.6)))
M:x[$\mathbb{X}$][X]	Model of asset selling problem ($x = 1, 2, 3, \mathbb{X} = \mathbb{R}$, $\mathbb{P}$, X = A, E)
$\tilde{\text{M}}$:x[$\mathbb{X}$][X]	$\tilde{\text{M}}$odel of asset buying problem ($x = 1, 2, 3, \mathbb{X} = \mathbb{R}$, $\mathbb{P}$, X = A, E)
opt-$\mathbb{R}$-price	optimal reservation price
opt-$\mathbb{P}$-price	optimal posted price
Tom	Lemma on Total market ($\mathscr{F}$) (see (1.4.1(p.5)))
Pom	Lemma on Positive market ($\mathscr{F}^+$) (see (1.4.1(p.5)))
Mim	Lemma on Mixed market ($\mathscr{F}^\pm$) (see (1.4.1(p.5)))
Nem	Lemma on Negative market ($\mathscr{F}^-$) (see (1.4.1(p.5)))
A	Assertion
$\mathscr{A}$	$\mathscr{A}$ssertion system
SOE	System of Optimality Equations
OIT	Optimal Initiating Time
C⇝S	switch from "Conduct-search" to "Skip-search" (see Def. 2.2.1(p.12))
S⇝C	switch from "Skip-search" to "Conduct-search" (see Def. 2.2.1(p.12))

Symbols

$\mapsto$	reduction
$\twoheadrightarrow$	running-back
↬	migration
$\neq$	"not equal"
≑̸	"not always equal"

Part 1

Introduction

In Chap. 1(p.3) we present the overview of this paper and the two motives which served as the impetus for this study and deliver our philosophical background which underpins the writing of this paper. After that, in order to promote the reader's interest for this paper consisting of more than 250 pages, we provide the list of highlights of this paper. In Chap. 2(p.11) our basic perspective for this study is expressed, and then different assumptions which will become necessary to attain this perspective are listed. In addition, here, the managerial and/or economic implications of interest rate r and discount factor β not only for profit and but also for cost is reconsidered. In Chap. 3(p.17) we propose the structuration of asset trading problems treated in this paper and in Chap. 4(p.19) the strict definitions for models of the above structured problems are specified. In Chap. 5(p.23) the underlying functions which will become requisite to move on discussions that follows are defined and in Chap. 6(p.27) the system of optimality equations of each model are described by using the above underlying functions. In Chap. 7(p.41) the optimal decision rules for each problem is presented and in Chap. 8(p.47) the conclusion of this part is summarized.

Chapter 1

Preface

1.1 Overview

Problems of trading assets (house, car, a lot of land, $\cdots$), commodities (wheat, copper, gasoline, $\cdots$), goods (fruit, fish, clothes, $\cdots$), etc. form one of the backbones of management and/or economic behavior. Each of these problems is classified into a selling problem and a buying problem and each problem, whether selling problem or buying problem, is furthermore categorized into a problem with a reservation price mechanism (where the counter-trader proposes the trading price ($\mathbb{R}$-mechanism[†])) and a problem with a posted price mechanism (where the leading-trader proposes the trading price ($\mathbb{P}$-mechanism[‡])). In this paper let us refer to the group consisting of the above four problems ($N_1 = 4$ problems) as the *quadruple-asset-trading-problems* (see Section 1.4.5(p.7)). In addition, taking into account the two events, "whether or not to conduct the search for counter-trader (see A5(p.11))" ($N_2 = 2$ problems) and "the presence or absence of quitting penalty price (see A7(p.12))" ($N_3 = 3$ problems)[§], we can make up the six types of quadruple-asset-trading-problems in all ($6 = N_2 \times N_3 = 2 \times 3$). Then, the whole of problems included there is called the "*structured-unit-of-problems* (see Section 3.3(p.18))". Thus it follows that the total number of problems included in the structured-unit-of problems is $24 = N_1 \times N_2 \times N_3 = 4 \times 2 \times 3$ (see Section 1.4(p.5) for more details of the structuration of asset trading problems). The main objective of this paper is twofold:

(1) To construct a general theory that integrates the quadruple-asset-trading-problems (called the *integrated theory*) (see Part 2(p.49)),

(2) To analyze these problems by using the theory (see Part 3(p.129)).

To achieve this objectives, several novel concepts are introduced: *symmetry*, *analogy*, *initiating time*, *quitting penalty price*, *market restriction*, etc. These concepts will lead us to quite a novel horizon for conventional discussions of decision problems that has not been explored by any researchers. Now, the most notable findings resulting from the analyses of these problems are the following two,

(a) The breakdown of symmetry between selling problem and buying problem,

(b) The existence of the *null-time-zone* and *deadline-engulfing* (see Sections 7.2.4.4(p.44) and 7.2.4.5(p.44)).[¶]

Particularly, the latter discovery challenges us to re-examine and rewrite the entire discussions that have been conventionally conducted for decision-making processes.

Finally, let us here emphasize that the above concepts are *all* first in this paper; For this reason, the references which we directly cited in the writing of this paper are *merely* the two of [15,Sakaguchi] and [17,You].

[†]A pricing mechanism under which the leading trader (whether selling problem or buying problem) determines whether or not to accept a proposed price. This determination is made on the basis of a given value, referred to as the *reservation value*.

[‡]A pricing mechanism under which the leading trader (whether selling problem or buying problem) proposes (post) his/her trading price

[§]A price at which a leading trader can quit the trading process in exchange for paying a given price (whether selling problem or buying problem and whether trading problem with $\mathbb{R}$-mechanism or trading problem with $\mathbb{P}$-mechanism), called the *quitting penalty price*. Now, the price is classified into two kinds (see A7(p.12)): *terminal quitting penalty price* and *intervening quitting penalty price*. Accordingly, it follows that the problem is categorized into the following three: one without any quitting penalty price, one with only terminal quitting penalty price, and one with both terminal quitting penalty price and intervening quitting penalty price. Therefore, it eventually follows that we have the three cases ($N_3 = 3$).

[¶]It is especially interesting that when the null-time-zone encompasses all points in time on the planning horizon except the deadline, it follows that all decision-making activities scheduled throughout the entire planning horizon are *engulfed* in the deadline, which is reminiscent of all matter, even light, falling into a *black hole* in physics.

1.2 Two Motives

The writing of the present paper was triggered by the following two simple motivations:

Motive 1 **Is a buying problem always symmetrical to a selling problem ?**

Long before the inception of this study, we held the naivete question above. At that time we considered that if a certain nature of a selling problem is known, then its corresponding nature of a buying problem can be immediately known by merely altering the signs of variables, parameters, constants, etc. defined in the selling problem. However, in this paper we will demonstrate that the ultimate response to this question is No!

Motive 2 **Does a general theory integrating quadruple-asset-trading-problems exist ?**

Before beginning to write this paper, we extensively reviewed numerous papers related to selling problems and buying problems and unconsciously developed a *preliminary expectation* that there could potentially exist a "*common denominator*" underlying all discussions presented therein. A while later, this intuition guided us to the realization that this common denominator is closely connected to a function known as the T-function defined by (5.1.1(p.23)) (see [15,Sakaguchi]). Urged by this realization, we soon developed a *faint anticipation* that a general theory integrating the quadruple-asset-trading-problems might exist. As we delve into this exploration, *a ray of hope* emerged that constructing such a theory might indeed be possible. This hope was buoyed by introducing the concepts of symmetry (see Chap. 12(p.67)) and analogy (see Chap. 13(p.87)), and fortunately our attempt over more than half a century led to the successful construction of this theory (see Chap. 16(p.113)).

1.3 Philosophical Background

Before proceeding with our discussions, below let us outline our philosophical background that underpins the entire writing of this paper.

1.3.1 Chalk Talk of Physics in High School

When I (Ikuta) was a high-school student (1958), in a physics lesson, the teacher placed one cotton ball and one iron ball in a glass tube of one-meter length, setting it upright. Not surprisingly, the iron ball fell with a thud and the cotton ball fell slowly as if chasing the iron ball. Afterward, the air in the tube was evacuated with a turn of the motor switch, and the tube was again set upright. This time, both balls fell alongside. Why? A surprise passed through my mind. The teacher then drew a picture on the blackboard and explained the rationality of this phenomenon; it was my first introduction to the power of *real experiment* and *thought experiment* in physics. After an interval, he mentioned that Galileo (1564-1642) conducted an experiment of a free fall in the Tower of Pisa and harked back that it took several thousand years to recognize the shift from the earth-centered theory to the sun-centered theory (the Copernican revolution). Shortly afterwards, the teacher tossed a sponge ball from the platform toward us (students) and explained that the trajectory of an object tossed over forms a parabola expressed by the quadratic curve. Without air, a speed at which an object thrown horizontally will loop back around the earth, drawing a circular orbit, is 7.9 kilometers per second, and the speed at which it flies out of the orbit is 11.2 kilometers per second.

Even now, the sound of the chalk sliding on the blackboard echoes in the depth of my both ears.

1.3.2 Suggestion of Professor

After graduating from high-school, I enrolled in the engineering department of Keio University (Japan), where I learned high-level physics.

It was a spring afternoon in March 31, 1965. I was in the office of my academic supervisor Dr. (Eng.) Shizuo Senju. The professor silently rose from the chair and drew a picture of one apple on the blackboard. He turned to me and said "*Would you take this apple? If you do, you can eat it and that will be the end of it. However, if you choose not to, this apple will disappear, and another one may appear — either greater or smaller than the one that vanished. In considering this situation, how should you decide whether or not to take this apple ?*" After a few moments of contemplation, the professor softly continued "*Many decision-making problems in corporate management have a similar structure ···. This is the subject of your master's thesis!*". With that, he left the room.

The above scene became the *springhead* in my whole research life.

1.3.3 Decision Theory as Physics

It is the nature of things that the every behavior of human beings is influenced by their underlying philosophical background, hence it is also natural that the authors (Ikuta & Kang, both holding Dr.Engineering) consistently approached the research with a deep-rooted *physical* perspective (see C2(p.233)). Since physics is a research discipline free from preconceived premises, assumptions, hypotheses, or preconceptions, it requires researchers to actively engage both ears and eyes in observing the research object and to calmly listen to every sound from its depth and carefully monitor every light emanating from within. Of course we (authors) should be open to employing concepts, knowledge, and techniques from business administration, economics, and mathematics as necessary; however, we should not be swayed by putting too much faith in these principles. Our core viewpoint is that also a decision process is inherently one of *physical phenomena.* Accordingly, for us who are both natural scientists, the decision theory discussed in this paper is *a decision theory as physics.*

1.3.4 Aphorism of Einstein

What should be kept in mind here is that excessive mathematics-oriented researchers who get fixated on the conviction of "the truth of this world is completely included within the truth of mathematics" exist at considerable ratio. Those familiar with physics will quickly grasp the essence of the aphorism of Albert Einstein "*As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality*". However, for those without this experience (physics), the essential understanding of his apothegm may require significant time or might be impossible forever.

Here we are like to emphasize is that "The decision theory is not a mathematical theory in any way but a physical theory in every sense !"

1.4 Structuration of Asset Trading Problems

The section tries to clarify the structure of asset trading problems which were presented in Section 1.1(p.3) .

1.4.1 Leading-Trader and Counter-Trader

Before moving on, here let us establish definitions for some key terms that will be used in our upcoming discussions.

- We refer to the subject matter of transaction (assets, commodities, or goods) as the *asset* in a general term.
- We refer to the decision-making problem related to the trading of asset as the *asset trading problem,* `ATP` for short, consisting of *asset selling problem* and *asset buying problem*, simply `ASP` and `ABP` respectively.
- For the parts involved in a trading, we use the terms "*leading-trader*" and "*counter-trader*" to distinguish between the part leading the trading and its counterpart. Accordingly, in `ASP` (`ABP`), the seller (buyer) is a *leading-trader* and the buyer (seller) is an *counter-trader.*

1.4.2 Asset Trading Problem (ATP)

Below, let us conceptualize the asset trading problem as a drama involving a *leading-trader* and a *counter-trader* on the two scenes below:

- `Scene` $\mathbb{R}$ in which
  - first, a counter-trader appears and posts his trading price,
  - then, a leading-trader appears and answers whether or not to accept it based on his $\underline{\mathbb{R}}$eservation price.†
- `Scene` $\mathbb{P}$ in which
  - first, a leading-trader appears and $\underline{\mathbb{P}}$osts his trading price,
  - then, a counter-trader appears and answers whether or not to accept it based on his reservation price.

Let us refer to the trading in `Scene` $\mathbb{R}$ (`Scene` $\mathbb{P}$) as the *asset trading problem with the reservation price mechanism* (*posted price mechanism*), simply `ATP` with $\mathbb{R}$-mechanism (`ATP` with $\mathbb{P}$-mechanism), further abbreviated as

$$\texttt{ATP}[\mathbb{R}] \ (\texttt{ATP}[\mathbb{P}]). \tag{1.4.1}$$

As presented in the two sections that follows, the above asset *trading* problem (`ATP`) can be translated into the asset *selling* problem (`ASP`) and the asset *buying* problem (`ABP`).

1.4.3 Asset Selling Problem (ASP)

In the asset selling problem (`ASP`), a leading-trader is a seller and its counter-trader is a buyer, hence the drama of the asset trading problem (`ATP`) presented in the previous section can be translated into as below:

`Scene` $\mathbb{R}$ in which
- first, a buyer (counter-trader) appears and posts his buying price,
- then, a seller (leading-trader) appears and answers whether or not to accept it based on his $\underline{\mathbb{R}}$eservation price.

`Scene` $\mathbb{P}$ in which
- first, a seller (leading-trader) appears and $\underline{\mathbb{P}}$osts his selling price,
- then, a buyer (counter-trader) appears and answers whether or not to accept it based on his reservation price.

Let us refer to the asset selling problem in `Scene` $\mathbb{R}$ (`Scene` $\mathbb{P}$) as the *asset selling problem with reservation price mechanism* (*posted price mechanism*), simply `ASP` with $\mathbb{R}$-mechanism ($\mathbb{P}$-mechanism), further abbreviated as

$$\texttt{ASP}[\mathbb{R}] \ (\texttt{ASP}[\mathbb{P}]). \tag{1.4.2}$$

The following two examples convey a flavor of the above asset selling problems, which mirror a "mental conflict" (see Section 7.3(p.45)) of a seller (leading-trader).

† A threshold based on which it is judged whether or not to accept it.

□ *Example* **1.4.1** **(Scene $\mathbb{R}$)** Suppose you (seller (leading-trader)) have to sell your car by a specified deadline due to a compelling reason, for example, such as being required to suddenly return to your mother country by order of the head office when you are stationed in a foreign country. Then, suppose a potential buyer (counter-trader) has just appeared. In this situation, if the buyer offers a high buying price, you would likely sell the car. However, if the offered price is very low, you might hesitate. In either case, you are faced with a decision that involves the following risks. Selling the car carries the risk of missing out a higher-paying buyer that may appear in the future. On the other hand, not selling the car carries the risk that a higher-paying buyer may not appear before the deadline, or even worse, no buyers may appear at all up to the deadline, leading to the necessity of selling the car at a very low price (a giveaway price) or incurring costs to dispose of it. Considering these risks, you must decide whether or not to sell your car to each successive buyer. This perspective implies that, as the deadline approaches, it is necessary to gradually lower *the minimum permissible selling price* (*reservation price*). The above *speculation* reflects a *mental conflict* that more and more you must become "selling spree" as the deadline approaches. □

The above example is what has been defined and investigated under the name "optimal stopping problem". To the best of the authors' knowledge, the earliest papers related to the problem can be traced back to 1960's (see [15,Sakaguchi]).

□ *Example* **1.4.2** **(Scene $\mathbb{P}$)** In the same example as mentioned above, let us suppose that you (seller (leading-trader)) set a selling price for your car to buyers who appear successively. In this situation, if you set the price too low, a buyer will buy the car, conversely, if it is excessively high, the buyer will leave (walk away). This indicates that a low posted price carries the risk of missing an opportunity that a potential buyer who is willing to pay a higher price appears in the future. On the other hand, setting a high posted price carries the risk of no buyer who buys for such a price appearing before the deadline; if so, then you are compelled to sell your car at a significantly reduced price (a rock-bottom price) or dispose of it at a cost. Considering these risks, you must decide whether or not to sell your car to each successive buyer. Similarly to in *Example* 1.4.1(p.6) , this perspective implies that, as the deadline approaches, it is necessary to gradually lower *the selling price to propose* (*posted price*). The above *speculation* reflects a *mental conflict* that more and more you must become "selling spree" as the deadline approaches. □

1.4.4 Asset Buying Problem (ABP)

In the asset buying problem (ABP), a leading-trader is a buyer and its counter-trader is a seller, hence the drama of the asset trading problem (ATP) presented in Section 1.4.2(p.5) can be translated into as below:

Scene $\mathbb{R}$ in which

- first, a seller (counter-trader) appears and posts his selling price,
- then, a buyer (leading-trader) appears and answers whether or not to accept it based on his $\mathbb{R}$eservation price.

Scene $\mathbb{P}$ in which

- first, a buyer (leading-trader) appears and $\mathbb{P}$osts his buying price,
- then, a seller (counter-trader) appears and answers whether or not to accept it based on his reservation price.

Let us refer to the asset buying problem in Scene $\mathbb{R}$ (Scene $\mathbb{P}$) as the *asset buying problem with reservation price mechanism* (*posted price mechasnism*), simply ABP with $\mathbb{R}$-mechanism (ABP with $\mathbb{P}$-mechanism), further abbreviated as

$$\text{ABP}[\mathbb{R}]\ (\text{ABP}[\mathbb{P}]). \tag{1.4.3}$$

Anyone may say that each of the above two asset buying problems seem to be *mere inverses* of the asset selling problem. However, it will be shown later on that a *fine difference* between both problems produces a *significant difference.* The following two examples convey a flavor of the asset buying problem, which mirror a "mental conflict" (see Section 7.3(p.45)) of a buyer (leading-trader) in the above drama.

□ *Example* **1.4.3** **(Scene $\mathbb{R}$)** Suppose you (buyer (leading-trader)) have to buy a car by a specified date (deadline), and then you find a potential seller. If the price offered by the seller is low enough, you will buy the car from the seller. However, if it is very high, you will hesitate. Buying the car carries the risk of missing an opportunity that you can find a potential seller offering a lower price in the future. On the other hand, not buying a car carries the risk that a lower-offering seller may not appears before the deadline. Considering these risks, you must decide whether or not to buy a car from each successive seller. This perspective implies that, as the deadline approaches, it is necessary to gradually raise *the maximum permissible buying price* (*reservation price*). The above *speculation* reflects the *mental conflict* that more and more you must become "buying spree" as the deadline approaches. □

□ *Example* **1.4.4** **(Scene $\mathbb{P}$)** Suppose that you (buyer (leading-trader)) propose your buying price to a potential seller. Then, if your proposed price is high enough, the seller will sell the car, conversely, if it is very low, the seller will reject the offer. Buying the car carries the risk that a seller offering a lower price may appear in the future. On the other hand, not buying a car carries the risk that a lower-offering seller may not appear before the deadline. Considering these risks, you must determine your buying price to propose. Similarly to in *Example* 1.4.3(p.6) , this perspective implies that, as the deadline approaches, it is necessary to gradually raise *the buying price to propose* (*proposed price*). The above *speculation* reflects the *mental conflict* that more and more you must more and more become "buying spree" as the deadline approaches. □

1.4.5 Quadruple-Asset-Trading-Problems

Let us refer to the set of the four asset trading problems ASP[$\mathbb{R}$], ABP[$\mathbb{R}$], ASP[$\mathbb{P}$], and ABP[$\mathbb{P}$] (see (1.4.2(p.5)) and (1.4.3(p.6))) as the *quadruple-asset-trading-problems*, represented as

$$\{\texttt{ASP}[\mathbb{R}], \texttt{ABP}[\mathbb{R}], \texttt{ASP}[\mathbb{P}], \texttt{ABP}[\mathbb{P}]\}. \tag{1.4.4}$$

The interconnectedness among these problems are somewhat akin to a drama played across the *looking glass*, depicted as in Figure 1.4.1(p.7) below.

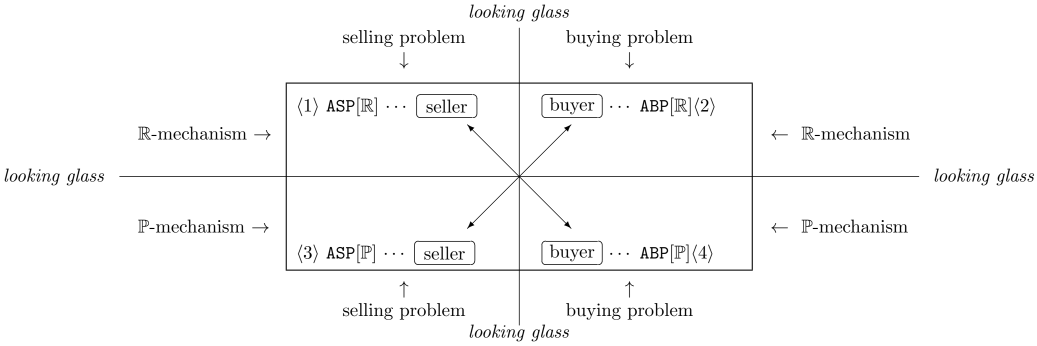


Figure 1.4.1: Interconnectedness among the quadruple-asset-trading-problems

The aslant arrows ⤧ in the above figure symbolizes a drama which revolves between a leading-trader in ASP and a leading-trader in ABP, i.e.,

- ↘ The leading-trader seller in ⟨1⟩ ASP[$\mathbb{R}$] faces, across the *looking glass*, the leading-trader buyer in ⟨4⟩ ABP[$\mathbb{P}$],
- ↖ The leading-trader buyer in ⟨4⟩ ABP[$\mathbb{P}$] faces, across the *looking glass*, the leading-trader seller in ⟨1⟩ ASP[$\mathbb{R}$],
- ↙ The leading-trader buyer in ⟨2⟩ ABP[$\mathbb{R}$] faces, across the *looking glass*, the leading-trader seller in ⟨3⟩ ASP[$\mathbb{P}$],
- ↗ The leading-trader seller in ⟨3⟩ ASP[$\mathbb{P}$] faces, across the *looking glass*, the leading-trader buyer in ⟨2⟩ ABP[$\mathbb{R}$].

1.4.6 Symmetry and Analogy

The concepts of symmetry (see H3(p.9)) and analogy (see H4(p.9)) play pivotal role in the construction of the integrated theory (see Motive 2(p.4)). The strict definitions of symmetry and analogy will be given in Chaps. 12(p.67), 13(p.87), 14(p.99), and 15(p.109), which is schematized as in Figure 1.4.2(p.7) below.

- A symmetry is observed between ⟨1⟩ ASP[$\mathbb{R}$] and ⟨2⟩ ABP[$\mathbb{R}$] and between ⟨3⟩ ASP[$\mathbb{P}$] and ⟨4⟩ ABP[$\mathbb{P}$],
- An analogy is observed between ⟨1⟩ ASP[$\mathbb{R}$] and ⟨3⟩ ASP[$\mathbb{P}$] and between ⟨2⟩ ABP[$\mathbb{R}$] and ⟨4⟩ ABP[$\mathbb{P}$].

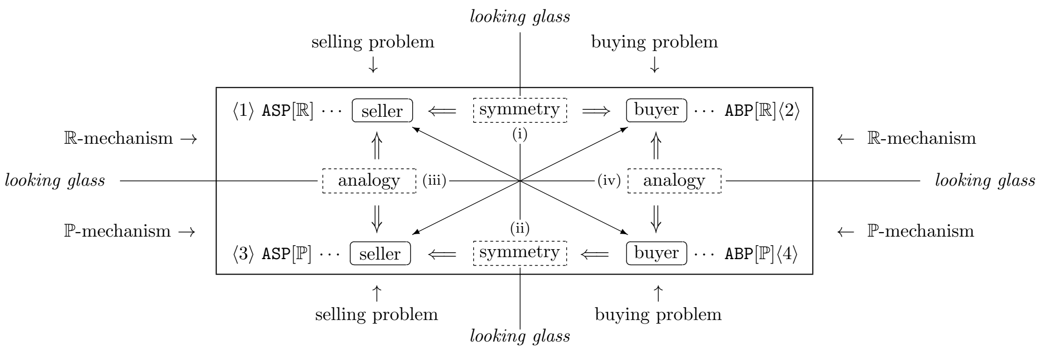


Figure 1.4.2: Symmetry and analogy among the quadruple-asset-trading-problems

Roughly speaking;

- The symmetry relation in (i) and (ii) implies that, for $\mathbb{X} = \mathbb{R}, \mathbb{P}$, a given *transformation* of some variables in an assertion on ASP[$\mathbb{X}$] yields its corresponding assertion on ABP[$\mathbb{X}$] and vice versa.
- The analogy relation in (iii) and (iv) implies that a given *replacement* of some variables in an assertion on ATP[$\mathbb{R}$] yields its corresponding assertion on ATP[$\mathbb{P}$] and vice versa.

For further details, see Chaps. 12(p.67) - 15(p.109).

1.5 Highlights of This Paper

In this section we outline the key points of this paper.

H1. Four points in time

All physical phenomena are not alien to a time concept, or equivalently there do not exist physical phenomena alien to the time concept. This fact inescapably led us to the perspective that all asset trading problems in the real world should be defined under the following four points in time (see Section 7.1(p.41)).

a. Recognizing time t_r

To begin with, let us focus on the grim reality that since a decision is an action that is made by human-beings; accordingly, it follows that a behaviour of "decision" materializes only when it was recognized in the bottom of heart of a person. Let us refer to the time point of this recognition as the *recognizing time* t_r, denoted by 0, i.e., $t_r = 0$ (see Figure 1.5.1(p.8)).

b. Starting time τ

Some amount of time will be needed to start tackling the problem for reasons of making budgets, arranging staffs, etc. After a preparation period, suppose it arrives at the time when the decision-maker can *start* to *initiate* the attack of the decision-making problem.† Let us refer to this time point as the *starting time* $\tau \geq t_r = 0$.

c. Deadline δ

Not to change the subject, but, in this paper, from a physical viewpoint, we consider that a decision process with an *infinite* planning horizon is a *fictitious product* of mathematical imagination beyond the real world; in fact, considering a planning horizon spanning with more than 100 hundreds millions years is nonsensical and futile. Therefore, in this paper we focus on only models with *finite* planning horizons. Let us refer to the terminal (final) point in time of the decision process as *deadline* $\delta \geq \tau$.

d. Initiating time t_i

Almost all decision-makers will try

(1) to *initiate* the attack of the decision problem at the starting time τ.

In fact, also authors in the past believed that it is quite natural, i.e., the attack of the problem should start at the starting time τ. However, one time, we happened to wake up to a thought that there may exist the option of *postponing* the initiation of its attack, i.e., we have the following options:

(2) to initiate the attack of the problem at the time $\tau + 1$,‡

(3) to initiate the attack of the problem at the time $\tau + 2$, ‡

⋮

(4) to initiate the attack of the problem at the deadline δ.‡

If these options were introduced in the discussion of decision processes, then what will occur in the world? Here let us suppose that it was determined to initiate the attack of the problem at time t ($\tau \leq t \leq \delta$). Then let us refer to this time point as the *initiating time* t_i ($\tau \leq t_i \leq \delta$). In this case, it is naturally questioned when to initiate the attack of the problem; let us refer to its time point as the *optimal initiating time*, denoted by t^*_τ ($\tau \leq t^*_\tau \leq \delta$) (see Section 7.2.4.1(p.43)). Here let us turn our attention to the fact that the time interval between the initiating time t_i and the deadline δ is the length of the planning horizon of a decision process. Then, as the planning horizon becomes longer and longer, we rencounter better and better chances, so the profit attained becomes larger and larger. If so, the optimal initiating time t^*_τ should always become the starting time τ (i.e., $t^*_\tau = \tau$), hence it is quite impossible that it becomes the deadline δ (i.e., $t^*_\tau \neq \delta$). However, surprisingly enough, this paper will demonstrate that the event with $t^*_\tau = \delta$ is theoretically possible in fact.

The flow of the above four points in time can be depicted as below.

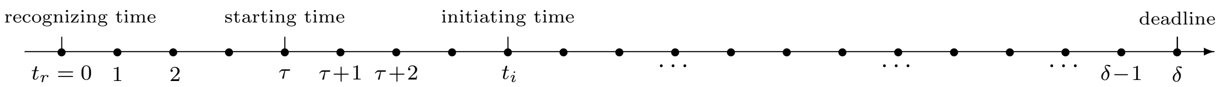


Figure 1.5.1: The flow of the four points in time

As seen from the above figure we have

$$0 = t_r \leq \tau \leq t_i \leq \delta. \tag{1.5.1}$$

In other words, letting $t_r = 0$, we number the above four points in time as $t_\tau = 0, 1, 2, \cdots, \delta$ toward the deadline δ.

†Note here that the synonymous words "start" and "initiate" are differently used for then purpose.
‡The starting time τ and the initiating time t_i are identical in (1) and separated in (2,3,$\cdots$,4).

H2. Null-time-zone and Deadline-engulfing

Before delving into the implication of the two terms in the above title, let us recall here the definitions of the starting time τ (see H1b(p.8)) and the initiating time t_i (see H1d(p.8)). First, note that the case of $\tau < t_\tau^*$ indicates that no action is taken at every point in time $t \in \{\tau, \tau-1, \cdots, t_\tau^*-1\}$. We will refer to this period of time as the *null-time-zone* (see Section 7.2.4.4(p.44)). Here let us consider the case of $t_\tau^* = \delta$, i.e., the optimal initiating time t_τ^* coincides with the deadline δ. This event ultimately implies that any actions prior to the deadline are rendered meaningless, suggesting "Don't do anything until the deadline." Using a metaphorical comparison, it is akin to "All actions are engulfed by the deadline", much like that all forms of matter, including light, are absorbed into a black hole. Taking this into consideration, we refer to this phenomenon as *deadline-engulfing* (see Section 7.2.4.5(p.44)). Then, if we regard a decision process with the *infinite* planning horizon as the limiting process of the finite planning horizon process, the existence of "deadline-engulfing" implies that the decision process with the *finite* planning horizon fades away in time toward the infinite future. What are presented above can be said to be one of the most remarkable discoveries in this paper, compelling us to undertake a comprehensive re-examination and rewriting of the entire theory of decision processes that have been conventionally explored so far without taken into account the phenomenon of "deadline-engulfing".

H3. Symmetry

The notion of the adjective "symmetrical" used in the description of Motive 1(p.4) was what is initially sparked by a vague intuition. This notion was theorized in the form that transforming some of variables and constants related to the asset selling problem with $\mathbb{R}$-mechanism (ASP[$\mathbb{R}$]) produces its corresponding asset buying problem with $\mathbb{R}$-mechanism ABP[$\mathbb{R}$] (see Chap. 12(p.67)).

H4. Analogy

At the earlier stage of this study we could not absolutely imagine that there will exist a relationship between the asset selling problem with $\mathbb{R}$-mechanism (ASP[$\mathbb{R}$]) and the asset selling problem with $\mathbb{P}$-mechanism (ASP[$\mathbb{P}$]). However, in the process of delving into discussions, we observed certain similarity between the two problems. This insight led us, before long, to a procedure, called the *analogy replacement operation*, replacing the two parameters a and $\mu^\dagger$ included within ASP[$\mathbb{R}$] by $a^{\star\ddagger}$ and a respectively yields ASP[$\mathbb{P}$] (see Chap. 13(p.87)).

H5. Integrated Theory

One of the most important results obtained in this paper is the successful construction of the theory integrating all problems in the structured-unit-of-problems (see Table 3.2.1(p.17)) based on concepts of symmetry and analogy. The two concepts were derived through a highly complicated discussion in Chaps. 12(p.67), 13(p.87), 14(p.99), and 15(p.109). The full spectrum of this theory can be schematized by Figure 16.2.1(p.113).

H6. Collapse of symmetry

It will be known later on (see Chap. 12) that the symmetry in H3(p.9) is the concept which is discussed under the premise that a price ξ is defined on the interval $(-\infty, \infty)$, which allows for the possibility of negative values. However, in a real-world, the price is always positive, i.e., $\xi \in (0, \infty)$. Consequently, if $(-\infty, \infty)$ is constrained to $(0, \infty)$, then a natural question arises: "Is the symmetry inherited under this restriction ?" (see Motive 1(p.4)). It will be observed later that it is not inherited.

H7. Underlying functions

The introduction of the underlying functions T, L, K, and $\mathcal{L}$ (see Chap. 5(p.23)) stands as one of significant highlights in this paper. While T-function has been widely recognized thus far in fields of statistics, operational research, and economics (see [1,DeGroot]), the remaining underlying functions L, K, and $\mathcal{L}$ are all what are first defined in the present paper. It will be known later on that the properties of these functions (see Chap. 10(p.53)) play a central role in the analyses of all the models dealt with in the present paper. Without properties of these functions, not only could we challenge systematic analysis of these models but also the successful construction of the integrated theory would have been nearly impossible.

H8. Structuration of models

In Section 3.2(p.17) we will define the 24 models. In this paper we refer to the whole of these models as the *structured-unit-of-model* (see Section 3.3(p.18)). Now, these models are not what were *capriciously* defined but what were *inevitably* established based on the principles of "with or without quitting penalty price ρ (see (B(p.17)))", "$\mathbb{R}$-mechanism and $\mathbb{P}$-mechanism (see (C(p.17)))", and "search enforced or allowed-case (see (D(p.17)))". In this paper, through treating the entirety of these 24 models as a cohesive unit, we endeavored to comprehensively analyze all of them. Although so many models of asset trading problems have been posed so far, all of them have been *one-by-one* and *independently* treated thus far without touching upon any relationships each other. On the other hand, the present paper aims to clarify the *interconnectedness* among all models included in the structured-unit-of-model.

---

† The lower bound a and the expectation μ of the distribution function of ξ (see A9(p.13))

‡ See (5.1.26(p.24))

Chapter 2

Preliminaries

2.1 Experimental Study vs Theoretical Study

In addressing a given real-world problem, two distinct approaches emerge. One is the construction of a model that faithfully represents its research object to the extent possible. The other involves building the simplest model conceivable where further simplification risks the loss of its existence itself. Here, we label research based on the former viewpoint as *experimental study* and on the latter viewpoint as *theoretical study*. While there is no substantive superiority between the two approaches, our overall stance in the present paper aligns with the latter. The methodology classification into the above two categories acts as a *dividing ridge*, causing a study to bifurcate into counter directions. The first drop of water from the former follows the east wall, and the first drop of water from the latter follows the west wall. Eventually, both converge in a lake with a common bottom, and shortly thereafter, a flower blooms. This amalgamation of results from both methodologies leads us to a genuine understanding of the reality in question.

2.2 Assumptions

The following assumptions are what were configured in order to realize the simplification of models based on the latter viewpoint.

A1 Points in time

The asset trading process occurs intermittently at points in time equally spaced along a finite length of the time axis as depicted in Figure 2.2.1(p.11) below. We shall backward label each point in time from the final point in time, denoted as time 0 (deadline), as 0, 1, and so forth. Accordingly, when the *present* point in time is designated as time t, the two adjacent points in time, $t+1$ and $t-1$, are the *previous* and *next* points in time respectively.

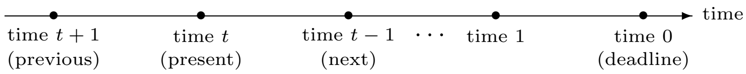


Figure 2.2.1: Points in time

A2 Absolutely necessary condition

In `ASP` (`ABP`), the leading-trader acting as a seller (buyer) must sell (buy), by all means, the trading asset to a buyer (from a seller) till the deadline. To rephrase, the seller (buyer) is not allowed to quit the selling (buying) process without completing the sale (purchase) of the asset.

A3 Stop of process

In the asset trading problem with $\mathbb{R}$-mechanism (`ATP`[$\mathbb{R}$]) the process *stops* when a leading-trader accepts a price proposed by an counter-trader and in the asset trading problem with $\mathbb{P}$-mechanism (`ATP`[$\mathbb{P}$]) it *stops* when a counter-trader accepts a price proposed by the leading-trader.

A4 Search cost

A cost $s \geq 0$ (*search cost*) must be paid to search for counter-traders, which includes expenses for advertising, communication, transfer, and so on.

A5 Whether or not to conduct the search

The existence of the search cost s inevitably leads us to a question "Always conducting the search activity might not turn a profit". In the paper we consider the following two cases.

a. Search Enforced model (sE-model): This refers to the case in which, once the process has initiated, conducting the search is mandatory at every subsequent point in time. Then, a decision-maker must continue to conduct the search until the process stops (see Figure 2.2.2(p.12)) below. In this case, the above question loses its meaning.

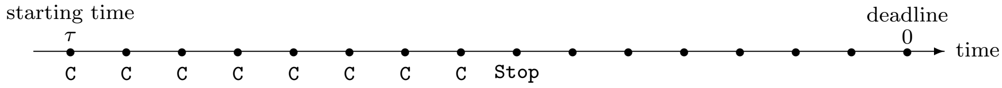


Figure 2.2.2: Flow of search-Conducts in the search-Enforced-model

b. search-Allowed-model (sA-model): This refers to the case in which, after the process has initiated, it is *permissible* to skip the search at every subsequent point in time. In other words, a leading-trader has the option whether to conduct the search or to skip at every point in time as long as the process does not stop. Then, we can consider different types of flows for search-Conduct and search-Skip (see Figure 2.2.3(p.12) below) where "⇝" represents the shift from search-Skip to search-Conduct or from search-Conduct to search-Skip.

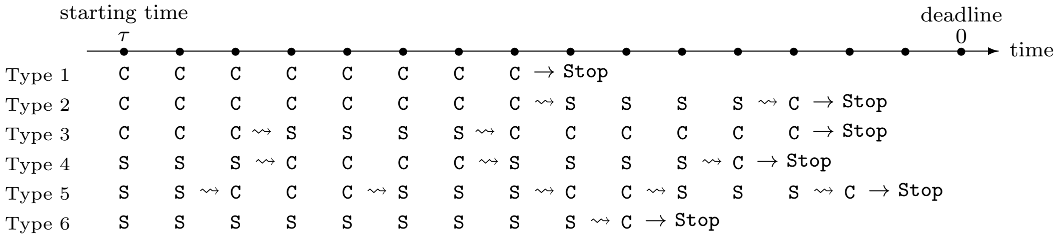


Figure 2.2.3: Different flows of search-Conduct and search-Skip

Definition 2.2.1 By C⇝S (S⇝C) let us denote the shift from search-Conduct to search-Skip (search-Skip to search-Conduct). □

A6 Counter-trader's appearance probability λ

In this paper, it is assumed that when the search is conducted at a certain point in time, a counter-trader appears at the next point in time with a known probability λ $(0 < \lambda \leq 1)$.

A7 Quitting penalty price

Suppose that a counter-trader appearing probability λ is less than 1, i.e., $0 < \lambda < 1$. Then it is possible that no counter-trader appears in the subsequent points in time even if conducting the search. This situation can lead to the risk that a leading-trader potentially has to quit the process at the final point in time point (deadline) without executing the trade for the asset, which contradicts the requirement of A2. When facing with such a circumstance, the leading-trader will take the following actions at the deadline:

- In ASP, the seller (leading-trader) will attempt to find ways to sell the asset by proposing a giveaway price ρ to any available buyer (counter-trader).
- In ABP, the buyer (leading-trader) will strive to acquire the asset by presenting a notably high-price ρ to any available seller (counter-trader).

Let us refer to such a price ρ as the *terminal quitting penalty price* ρ, implying that, at the deadline, the leading-trader can quit the process in exchange for paying the cost ρ. Then, we can consider also the case that such a ρ is available at every point in time including the deadline. Then let us refer to it as the *intervening quitting penalty price*. In the explanation above, the ρ is implicitly assumed to be positive $\rho \in (0, \infty)$; however, to generalize discussions that follows, we define it to be $\xi \in (-\infty, \infty)$.

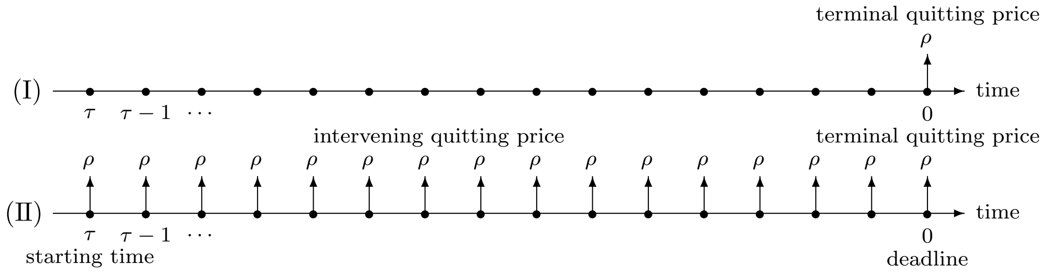


Figure 2.2.4: Intervening quitting price ρ and terminal quitting price ρ

A8 Range of price

Whether a price ξ proposed by an appearing counter-trader in ATP[$\mathbb{R}$] or the reservation price ξ of an appearing counter-trader in ATP[$\mathbb{P}$], the price ξ should be defined on $(0, \infty)$ in the normal market of the real-world. However, in this paper, to successfully construct the integrated theory in Part 2 (p.49) we dare to define it on $(-\infty, \infty)$.

A9 Distribution function

In ATP[$\mathbb{R}$] (ATP[$\mathbb{P}$]) we assume that the prices proposed by successively appearing counter-trader, $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\cdots$ (the reservation prices of successively appearing counter-trader, $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\cdots$) are independent identically distributed random variables having a *continuous* distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\}$ with a finite expectation μ where

$$\begin{aligned} F(\xi) &= 0 & \cdots(1) \quad & \xi \le a, \\ 0 < F(\xi) &< 1 & \cdots(2) \quad & a < \xi < b, \\ F(\xi) &= 1 & \cdots(3) \quad & b \le \xi, \end{aligned} \tag{2.2.1}$$

for given constants a and b such that

$$-\infty < a < \mu < b < \infty. \tag{2.2.2}$$

Furthermore, for its probability density function $f(\xi)$ let us assume

$$\begin{aligned} f(\xi) &= 0 & \cdots(1) \quad & \xi < a, \\ 0 < f(\xi) &< 1 & \cdots(2) \quad & a \le \xi \le b, \\ f(\xi) &= 0 & \cdots(3) \quad & b < \xi. \end{aligned} \tag{2.2.3}$$

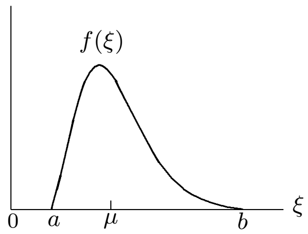


Figure 2.2.5: Probability density function $f(\xi)$

Here assume that there exits $\underline{f}$ such that

$$\underline{f} = \inf_{a \le \xi \le b} f(\xi) d\xi > 0. \tag{2.2.4}$$

Let us represent the set consisting of all possible distribution functions with (2.2.2(p.13)) by $\mathscr{F}$, i.e.,

$$\mathscr{F} = \{F \mid -\infty < a < \mu < b < \infty\}, \tag{2.2.5}$$

called the *total distribution function space*, simply the Total-dF-Space.

A10 Recallability of once rejected counter-trader

Whether model with $\mathbb{R}$-mechanism or model with $\mathbb{P}$-mechanism, if a once-rejected counter-trader can be *recalled* later and accepted at the discretion of the leading-trader, then it is referred to as the *recall-model* or *model-with-recall.* Conversely, if such recallability is not allowed, then it is referred to as the *no-recall-model*, *model-with-no-recall*, or *model-without-recall.* In this paper we treat only models-with-no-recall; discussions for models-with-recall are left as subjects as future studies. For problems specified by these models let us use the terminologies of *recall-problem*, *problem-with-recall*, *no-recall-problem*, *problem-with-no-recall*, and *problem-without-recall.*

A11 Finiteness of planning horizon

In the present paper we consider only models with the *finite* planning horizon (see H1c(p.8)). Our basic standpoint over the whole of this paper lies in a *grim reality* that a process with the *infinite* planning horizon is a *mere product of fantasy* created by mathematics, which does not exist in the real world at all; in fact, it is an inanity to consider a model with the planning horizon of more than 135 hundred millions years. However, we can have the two reasons for which it becomes still meaningful to discuss the model with the *infinite* planning horizon. One is that it can become an approximation for the process with an enough long (finite) planning horizon, the other is that results obtained from it can provide a meaningful information for the analyses of models with the *finite* planning horizon.

2.3 Discount Factor

This section presents the managerial and/or economic implication of the discount factor β which will be used in describing the systems of optimality equations for all decision processes treated in this paper (see Chap. 6(p.27))

2.3.1 Definitions

To start with, we provide the following four definitions.

⟪a⟫ *Fund F*: We refer to the total amount of available money on hand as the *fund F*, which can be always and freely invested.

《b》 *Interest rate* r: We denote the interest rate per period by $r \geq 0$, implying that the today's fund of one unit increases to the $1+r$ units tomorrow. Here let us define $\beta = (1+r)^{-1}$ $(0 < \beta \leq 1)$, called the *discount factor*. Then $1 + r = \beta^{-1}$.

《c》 *Profit* P: Suppose that an amount of fund F has been yielded for a reason. Then, let us call the yielded amount of the fund F the *incremented fund* F^i and let us refer to the *incremented fund* F^i as *profit* P, schematized as

$$\text{incremented fund } F^i \to \text{profit } P \cdots (1^\bullet)$$

where F^i and P are numerically equivalent, i.e., $F^i = P \cdots (2^\bullet)$.

《d》 *Cost* C:

1. Suppose that an amount of fund F on hand has been paid away (lost) for a reason. Then, let us call the amount of the fund F the *decremented fund* $F^d \cdots (3^\bullet)$.
2. Although the *decremented fund* F^d is what has been <u>*already paid away*</u>, *temporarily let us assume here that the decremented fund* F^d <u>*were not paid away*</u>. Then, the *decremented fund* F^d that were not paid away is backed to the fund F and is stored as a *savings* on hand. Let us refer to this savings as the *conditional savings* S^c, schematized as

$$\text{decremented fund } F^d \to \text{conditional savings } S^c \cdots (4^\bullet)$$

where F^d and S^c are numerically equivalent, i.e., $F^d = S^c \cdots (5^\bullet)$.

3. However, since the *conditional savings* S^c is what will be *eventually* paid away, it will be lost *in the end*. For this reason, we refer to the *conditional savings* S^c as the *loss of conditional savings*, denoted by S^{lc}, schematized as

$$\text{conditional savings } S^c \to \text{loss of conditional savings } S^{lc} \cdots (6^\bullet)$$

where S^c and S^{lc} are numerically equivalent, i.e., $S^c = S^{lc} \cdots (7^\bullet)$.

4. Finally, we define the *loss of conditional savings* S^{lc} as *cost* C, schematized as

$$\text{lost of conditional savings } S^{lc} \to \text{cost } C \cdots (8^\bullet)$$

where S^{lc} and C are numerically equivalent, i.e., $S^{lc} = C \cdots (9^\bullet)$.

5. From $(4^\bullet)$, $(6^\bullet)$, $(8^\bullet)$ we have

$$\text{decremented fund } F^d \to \text{conditional savings } S^c \to \text{loss of conditional saving } S^{lc} \to \text{cost } C \cdots (10^\bullet).$$

where F^d, S^c, S^{lc}, and C are numerically equivalent, i.e., $F^d = S^c = S^{lc} = C \cdots (11^\bullet)$.

What should be especially noted in the above definitions 《a-d》 is that while the *profit* P is <u>directly</u> defined by F^i (see $(1^\bullet)$), the *cost* C is <u>indirectly</u> defined via S^c and S^{lc} in addition to F^d. Roughly speaking, it follows that the cost C is the loss that is yielded by losing what were saved if not having been paid away

Remark 2.3.1 (thought experiment) The flow of $(11^\bullet)$ is what is called the *thought experiment* in physics. Although this thinking way may seem to be *periphrastic* at a glance, it will be seen in Section 2.3.4(p.15) that this flow $(11^\bullet)$ will play a *decisively essential* role when trying to introduce the interest rate r to the evaluation of cost on the time axis. □

2.3.2 Discount Factor for Fund

Suppose you have the fund F today. Then, since it can be invested at a given interest rate r, the today's fund F increases to $(1+r)^n F = \beta^{-n} F$ after n days, i.e., $F \to \beta^{-n} F$. Multiplying this relation by β^n leads to $\beta^n F \to F$. This implies that if you have the fund $\beta^n F$ today, it increase to F after n days. Accordingly, denoting the fund of n days later by F_n, $n = 0, 1, \cdots$, we have

$$\beta^n F_n \to F_n, \quad n = 0, 1, \cdots \quad \cdots (12^\bullet).$$

In other words, it follows that the fund F_n of n days later is equivalent to the fund $\beta^n F_n$ of today (as an economic value). In this sense, $\beta^n F_n$ is usually called the *present* (*today*) *value* of the fund F_n of n days later.

2.3.3 Discount Factor for Profit

(a) Consider an action A with F_0^i, F_1^i, F_2^i, $\cdots$ where F_n^i denotes incremented funds of n days later. Then the present discounted values of F_0^i, F_1^i, F_2^i, $\cdots$ are given by F_0^i, βF_1^i, $\beta^2 F_2^i$, $\cdots$ (see $(12^\bullet)$).

(b) Since $F_n^i = P_n$ due to $(2^\bullet)$, the above flow F_0^i, βF_1^i, $\beta^2 F_2^i$, $\cdots$ can be rewritten as P_0, βP_1, $\beta^2 P_2$, $\cdots$.

(c) Consequently, it follows that the *total present discounted value of profits* for the action A is given by $V_0 = P_0 + \beta P_1 + \beta^2 P_2 + \cdots$. Letting $V_1 = P_1 + \beta P_2 + \beta^2 P_3 + \cdots$, we have $V_0 = P_0 + \beta V_1$. Therefore, as seen in Chap. 6(p.27), it follows that the discount factor β can be introduced in describing the system of optimality equations for the selling problem with *profit maximization*.

2.3.4 Discount Factor for Cost

♡ Alice 1 *Here Alice wandered round with the following question. "In the asset buying problem with the cost minimization, a buying price is what has been already paid away, hence it does not remain on hand, so it cannot invest!. But, but —, if so, the concept of the discount factor cannot be applied to the asset buying problem! Then what will happen ?". Then, Dr. Rabbit clad in the waistcoat-pocket suddenly appeared in front of her and told "Well it's puzzled $\cdots$.". And, after looking dead at her for a while, taking a watch out of its waistcoat-pocket and then murmuring "Oh dear! Oh dear!, I shall be too late for the faculty meeting", he disappeared down the hole.* □

Several days later, Alice got the following letter from Dr. Rabbit:

" To Miss. Alice.

(a) *Consider an action A with decremented funds* F^d_0, F^d_1, F^d_2, $\cdots$ (see 《d1(p.14)》 and (3•)).

(b) *Let the conditional savings corresponding to* F^d_n *be denoted by* S^c_n (see 《d2(p.14)》 and (4•)), *i.e.,* S^c_0, S^c_1, S^c_2, $\cdots$. *Thus the present discounted values of which are* S^c_0, βS^c_1, $\beta^2 S^c_2$, $\cdots$.

(c) *Let the loss of conditional savings corresponding to* S^c_n *be denoted by* S^{lc}_n, *i.e.,* S^{lc}_0, S^{lc}_1, S^{lc}_2, $\cdots$ (see 《d3(p.14)》 and (6•)), *the present discounted value of which are* S^{lc}_0, βS^{lc}_1, $\beta^2 S^{lc}_2$, $\cdots$.

(d) *Let the loss of conditional saving* S^{lc}_n *be defined as* C_n *in* 《d4(p.14)》 and (8•), *i.e.,* $S^{lc}_n = C_n$ *(see* (8•)*). Then, the above flow* S^{lc}_0, βS^{lc}_1, $\beta^2 S^{lc}_2$, $\cdots$ *can be rewritten as* C_0, βC_1, $\beta^2 C_2$, $\cdots$

(e) *Consequently, it follows that the total present discounted value of costs for the action A is given by* $V_0 = C_0 + \beta C_1 + \beta^2 C_2 + \cdots$. *Letting* $V_1 = C_1 + \beta C_2 + \beta^2 C_3 + \cdots$, *we have* $V_0 = C_0 + \beta V_1$. *Therefore, as seen in* Chap. 6(p.27), *it follows that the discount factor* β *can be introduced in describing the system of optimality equations for the buying problem with cost minimization.*

Kind regards."

Chapter 3

Structuration of Models

3.1 Model Classification Factors

The paper categorizes models based on the following four factors:

(A) The first factor is "whether selling model or buying model", represented as:

- Selling model → M.
- Buying model → $\tilde{\mathsf{M}}$.[‡]

(B) The second factor is "with or without the quitting penalty price ρ" (see A7(p.12)), classified as:

- Model 1 in which the quitting penalty price ρ is not available.
- Model 2 in which only the *terminal* quitting penalty price ρ is available (see Figure 2.2.4(p.12) (I)).
- Model 3 in which both *terminal* quitting penalty price ρ and *intervening* quitting penalty ρ are available (see Figure 2.2.4(p.12) (II)).

(C) The third factor is "whether $\mathbb{R}$-mechanism or $\mathbb{P}$-mechanism" (see Section 1.1(p.3)), denoted as:

- $\mathbb{R}$-mechanism-model ($\mathbb{R}$-`model`) → $[\mathbb{R}]$.
- $\mathbb{P}$-mechanism-model ($\mathbb{P}$-`model`) → $[\mathbb{P}]$.

(D) The last factor is "whether search-Enforced-model or search-Allowed-model" (see A5(p.11)), symbolized as:

- search-Enforced-model (sE-model) → $[\mathtt{E}]$.
- search-Allowed-model (sA-model) → $[\mathtt{A}]$.

3.2 Quadruple-Asset-Trading-Problems

Let us designate the models treated in this paper by

$$\mathsf{M}{:}x[\mathbb{X}][\mathtt{X}] \quad (\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{X}]) \quad x = 1,2,3 \text{ (B)}, \quad \mathbb{X} = \mathbb{R}, \mathbb{P} \text{ (C)}, \quad \mathtt{X} = \mathtt{E}, \mathtt{A} \text{ (D)}^{\ddagger}$$

Then let us define the set

$$\mathcal{Q}\langle \mathsf{M}:x[\mathtt{X}]\rangle \overset{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}], \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}], \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}], \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]\}, \quad x = 1,2,3, \ \ \mathtt{X} = \mathtt{E}, \mathtt{A},$$

called the *quadruple-asset-trading-models*, consisting of the 24 models in the table below:

Table 3.2.1: Twenty four models

		ASP[ℝ]	ABP[ℝ]	ASP[ℙ]	ABP[ℙ]	
$\mathcal{Q}\{\mathsf{M}{:}1[\mathtt{E}]\}$	= {	$\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]$,	$\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]$	}
$\mathcal{Q}\{\mathsf{M}{:}1[\mathtt{A}]\}$	= {	$\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$,	$\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$	}
$\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{E}]\}$	= {	$\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]$,	$\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]$	}
$\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{A}]\}$	= {	$\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]$,	$\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$	}
$\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{E}]\}$	= {	$\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}]$,	$\mathsf{M}{:}3[\mathbb{P}][\mathtt{E}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{E}]$	}
$\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{A}]\}$	= {	$\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]$,	$\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]$,	$\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]$	}

Throughout this paper, we use the terminology of *quadruple-asset-trading-problems* for the problems specified by these model.

[‡] Throughout the paper, the model of the asset *buying* problem (ABP) is represented by the symbol upon which the tilde "˜" is capped like $\tilde{\mathsf{M}}$.

3.3 Structured-Unit-of-Problems

Let us refer to the set of the 24 models defined in Tables 3.2.1(p.17) as the *structured-unit-of-models*. Here note that all models within this structured-unit-of-models are not what are *blindly* defined but what are *systematically* and *inevitably* defined according to the four factors in Section 3.1(p.17). The big difference of this study from all other ones that have been conventionally made by many researchers lies in clarifying the *interconnectedness* among these models. Throughout this paper, we use the terminology of *structured-unit-of-problems* for the problems specified by these model.

3.4 Decisions

What a leading-trader should determine in models defined in Table 3.2.1(p.17) are as follows:

⟨1⟩ Whether or not to accept the price proposed by a counter-trader (only for $\mathbb{R}$-model) (see Section 7.2.1(p.42)),

⟨2⟩ What price to propose (post) (only for $\mathbb{P}$-model) (see Section 7.2.2(p.42)),

⟨3⟩ Whether or not to conduct the search (only for sA-model) (see Section 7.2.3(p.42)),

⟨4⟩ When to initiate the process (for all models) (see Section 7.2.4(p.43)).

3.5 Asset Trading Problem with Negative Trading Price

In A8(p.12) we defined a price ξ on $(-\infty, \infty)$. However, this seemingly unrealistic assumption can be justified for the following reason. First let us note here that "sell" and "buy" mean "deliver" and "receive" respectively; more precisely speaking:

- In a selling problem, a seller (leading-trader) *delivers* the asset to a buyer (counter-trader), who *receives* it from the seller.
- In a buying problem, a buyer (leading-trader) *receives* the asset from a seller (counter-trader), who *delivers* it to the buyer.

The above two scenarios can be schematized as below.

	leading-trader				counter-trader
	↓				↓
selling problem:	seller	(delivering-side)	↔	(recieving-side)	buyer
buying problem:	buyer	(recieving-side)	↔	(delivering-side)	seller

In other words, "selling problem" and "buying problem" can be said to be "delivering problem" and "receiving problem" respectively. Now let us consider here a transaction in which the asset traded there is a worthless debris such as surplus soil, concrete blocks and so on which are disposed of when a building is broken up. In this case, a receiving-side (buyer), in whether selling problem or buying problem, rightly requires some amount of money as a disposal cost although being a buyer. Seeing the problem from the standpoint of the seller (delivering-side), the seller gives some amount of money to the buyer (receiving-side) although being a seller. This interpretation implies that the trading problem stated above can be regarded as "a trading problem with a *negative* trading price". To discuss the trading problem more generally for the above reason, expanding the range of the trading price to $(-\infty, \infty)$ cannot be always said to be preposterous from a practical viewpoint (see Section A 7.6(p.264)).

3.6 Simplified Notations of Models

In the paper we will sometimes use the following notations.

- By M:x[$\mathbb{R}$/$\mathbb{P}$][X] let us denote M:x[$\mathbb{R}$][X] and M:x[$\mathbb{P}$][X].
- By $\tilde{\text{M}}$:x[$\mathbb{R}$/$\mathbb{P}$][X] let us denote $\tilde{\text{M}}$:x[$\mathbb{R}$][X] and $\tilde{\text{M}}$:x[$\mathbb{P}$][X].
- By M/$\tilde{\text{M}}$:x[$\mathbb{R}$/$\mathbb{P}$][X] let us denote M:x[$\mathbb{R}$/$\mathbb{P}$][X] and $\tilde{\text{M}}$:x[$\mathbb{R}$/$\mathbb{P}$][X].
- By M:1/2/3[$\mathbb{X}$][X] let us denote M:1[$\mathbb{X}$][X], M:2[$\mathbb{X}$][X], and M:3[$\mathbb{X}$][X].
- By M:x[$\mathbb{X}$][E/A] let us denote M:x[$\mathbb{X}$][E] and M:x[$\mathbb{X}$][A].
- By $\tilde{\text{M}}$:1/2/3[$\mathbb{X}$][X] let us denote $\tilde{\text{M}}$:1[$\mathbb{X}$][X], $\tilde{\text{M}}$:2[$\mathbb{X}$][X], and $\tilde{\text{M}}$:3[$\mathbb{X}$][X].
- By $\tilde{\text{M}}$:x[$\mathbb{X}$][E/A] let us denote $\tilde{\text{M}}$:x[$\mathbb{X}$][E] and $\tilde{\text{M}}$:x[$\mathbb{X}$][A].

Chapter 4

Definitions of Models

4.1 Model 1

4.1.1 Search-Enforced-Model 1: $\mathcal{Q}\langle \mathsf{M}\!:\!1[\mathrm{E}]\rangle = \{\mathsf{M}\!:\!1[\mathbb{R}][\mathrm{E}],\ \mathsf{M}\!:\!1[\mathbb{P}][\mathrm{E}],\ \tilde{\mathsf{M}}\!:\!1[\mathbb{R}][\mathrm{E}], \tilde{\mathsf{M}}\!:\!1[\mathbb{P}][\mathrm{E}]\}$

4.1.1.1 M:1[$\mathbb{R}$][E] **and** M:1[$\mathbb{P}$][E]

The two are the most basic models of the asset selling problem [15,Sakaguchi][16,You], which are defined by the following assumptions:

*A*1. Once the process initiates, at every point in time after that it is enforced to conduct the search for buyers (see (A5a(p.11))), hence the search cost $s \geq 0$ is paid at every point in time (see A4(p.11)).

*A*2. After the search has been conducted at a point in time $t > 0$, a buyer certainly appears at time $t-1$ (next point in time), i.e., the buyer (counter-trader) appearing probability $\lambda = 1$ (see A6(p.12)).

*A*3. The prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ proposed by successively appearing buyers in M:1[$\mathbb{R}$][E] and the reservation prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ of successively appearing buyers in M:1[$\mathbb{P}$][E] are both assumed to be independent identically distributed random variables having a known *continuous* probability distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \leq \xi\}$ (see A9(p.13)).[†]

*A*4. Any once rejected price cannot be recalled in the future (see A10(p.13)).

*A*5. Both terminal quitting penalty price ρ and intervening quitting penalty price ρ are not available (see A7(p.12)).

*A*6. The selling process stops at the point in time when the asset is sold to an appearing buyer (see A3(p.11)).

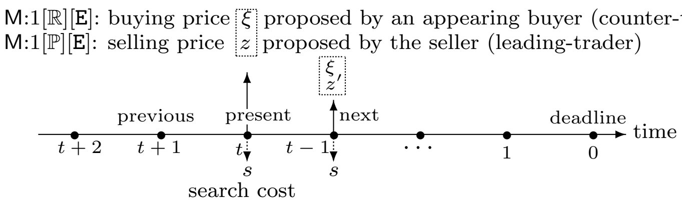


Figure 4.1.1: M:1[$\mathbb{R}$][E] and M:1[$\mathbb{P}$][E]

The objective is to maximize the total expected present discounted *profit*, i.e., the expected present discounted value of the price for which the asset is sold, *minus* the total expected present discounted value of the search costs which will be paid until the process stops with selling the asset.

Remark 4.1.1 Suppose the process has proceeded up to time 1. Then, since the search is conducted at that time due to *A*1(p.19), a buyer certainly appears at time 0 (deadline) due to *A*2(p.19).

(a) In M:1[$\mathbb{R}$][E], due to A2(p.11) the seller must sell the asset to the buyer however small the price proposed by the buyer may be.

(b) In M:1[$\mathbb{P}$][E], the seller must propose the price a to the buyer where a is the lower bound of the distribution function F for the reservation price $\boldsymbol{\xi}$ of the buyer (see Figure 2.2.5(p.13)). Then, the buyer certainly buys the asset. □

4.1.1.2 $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E] **and** $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E]

The two are both the models of the asset *buying* problem, defined by the following assumptions:

*A*1. Once the process initiates, at every point in time after that it is enforced to conduct the search for sellers, hence the search cost $s \geq 0$ is paid at every point in time.

[†] $\boldsymbol{\xi}$ and ξ represent a random variable and a realized value respectively.

*A*2. After the search has been conducted at a point in time $t > 0$, a seller (counter-trader) certainly appears at time $t-1$ (next point in time), i.e., the seller appearing probability $\lambda = 1$.

*A*3. The prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ proposed by successively appearing sellers in M̃:1[ℝ][E] and the reservation prices $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, $\boldsymbol{\xi}''$, $\cdots$ of successively appearing sellers in M̃:1[ℙ][E] are both assumed to be independent identically distributed random variables having a known *continuous* probability distribution function $F(\xi) = \Pr\{\boldsymbol{\xi} \leq \xi\}$.

*A*4. Any once rejected price cannot be recalled in the future.

*A*5. Both terminal quitting penalty price ρ and intervening quitting penalty price ρ are not available.

*A*6. The buying process stops at the point in time when the asset is bought by an appearing seller.

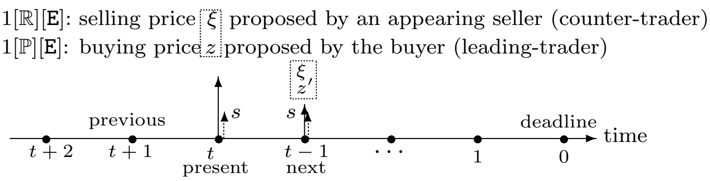


Figure 4.1.2: M̃:1[ℝ][E] and M̃:1[ℙ][E]

The objective is to minimize the total expected present discounted *cost*, i.e., the expected present discounted value of the price for which the asset is bought, *plus* the total expected present discounted value of the search costs which will be paid until the process stops with buying the asset.

Remark 4.1.2 Here it should be noted that although in Figure 4.1.2(p.20) (buying model), ξ, z, and s are all in an upward direction, in Figure 4.1.1(p.19) (selling model) only s is in a downward direction. □

4.1.2 Search-Allowed-Model 1: $\mathcal{Q}\langle$M:1[A]$\rangle$ = {M:1[ℝ][A], M:1[ℙ][A], M̃:1[ℝ][A], M̃:1[ℙ][A]}

4.1.2.1 M:1[ℝ][A] and M:1[ℙ][A]

The two are the same as M:1[ℝ][E] and M:1[ℙ][E] in Section 4.1.1.1(p.19) only except that *A*1(p.19) is changed into as follows:

*A*1. At every point in time $t > 0$, it is allowed to skip the search (see (A5b(p.12))); in other words, the seller has an option whether to conduct the search or to skip.

Remark 4.1.3 Suppose the process has proceeded up to time 1. Then, if the search is skipped at that time, no buyer appears at time 0, hence the seller is faced with the situation of having to quit the process without selling the asset, which contradicts A2(p.11). Accordingly, the search must be necessarily conducted at time $t = 1$. □

4.1.2.2 M̃:1[ℝ][A] and M̃:1[ℙ][A]

The two are the same as M̃:1[ℝ][E] and M̃:1[ℙ][E] in Section 4.1.1.2(p.19) only except that after the process has initiated, it is allowed to skip the search.

4.2 Model 2 (see Figure 2.2.4(p.12) (I))

4.2.1 Search-Enforced-Model 2: $\mathcal{Q}\langle$M:2[E]$\rangle$ = {M:2[ℝ][E], M:2[ℙ][E], M̃:2[ℝ][E], M̃:2[ℙ][E]}

The quadruple models indicated within the above brace are the same as in Section 4.1.1(p.19) only except that the assumptions *A*2(p.19) and *A*5(p.19) are changed into as follows:

*A*2. After the search has been conducted at time $t > 0$, a buyer appears at the next point in time with a probability $\lambda \leq 1$.
*A*4. The terminal quitting penalty price ρ is available.

Remark 4.2.1 In the two models it is possible to stop the process by accepting the terminal quitting penalty price ρ at time 0 (deadline), hence the starting time $\tau = 0$ is permitted since the leading-trader can quit the process with accepting the ρ at time 0. Accordingly, in the two models it follows that the starting time τ is greater than or equal to 0, i.e., $\tau \geq 0$. □

4.2.2 Search-Allowed-Model 2: $\mathcal{Q}\langle$M:2[A]$\rangle$ = {M:2[ℝ][A], M:2[ℙ][A], M̃:2[ℝ][A], M̃:2[ℙ][A]}

The quadruple models indicated in the above brace are the same as in Section 4.2.1(p.20) only except that *A*1(p.19) is changed as follows:

*A*1. After the process has initiated, it is allowed to skip the search at every point in time $t > 0$.

4.3 Model 3 (see Figure 2.2.4(p.12) (II))

4.3.1 Search-Enforced-Model 3: $\mathcal{Q}\langle$M:3[E]$\rangle$ = {M:3[ℝ][E], M:3[ℙ][E], M̃:3[ℝ][E], M̃:3[ℙ][E]}

The quadruple models are the same as in Section 4.2.1(p.20) only except that the assumption *A*4(p.20) is changed as follows:

*A*4. In addition to the terminal quitting penalty price ρ, the intervening quitting penalty price ρ is also available.

4.3.2 Search-Allowed-Model 3: $\mathcal{Q}\langle \mathsf{M}:3[\mathtt{A}]\rangle = \{\mathsf{M}:3[\mathbb{R}][\mathtt{A}],\ \mathsf{M}:3[\mathbb{P}][\mathtt{A}],\ \tilde{\mathsf{M}}:3[\mathbb{R}][\mathtt{A}],\ \tilde{\mathsf{M}}:3[\mathbb{P}][\mathtt{A}]\}$

The quadruple models are the same as in Section 4.2.2(p.20) only except that after the process has initiated, it is allowed to skip the search.

4.4 Total-pdF-Space

Let us refer to $\lambda \in (0,1]$, $\beta \in (0,1]$, $s \in [0,\infty)$, and $\rho \in (-\infty,\infty)$ as the *parameter* of models, all of which are independent of the distribution function F. Then, let $\boldsymbol{p} = (\lambda, \beta, s)$ for Model 1 and $\boldsymbol{p} = (\lambda, \beta, s, \rho)$ for Models 2/3, which are called the *parameter vector*. We represent the set of all possible $\boldsymbol{p}$'s by

$$\mathscr{P} = \{\boldsymbol{p} \mid \lambda = 1,\ 0 < \beta \leq 1,\ 0 \leq s\} \qquad \text{for Model 1,} \tag{4.4.1}$$

$$\mathscr{P} = \{\boldsymbol{p} \mid 0 < \lambda \leq 1,\ 0 < \beta \leq 1,\ 0 \leq s,\ -\infty < \rho < \infty\} \qquad \text{for Models 2/3,} \tag{4.4.2}$$

called the *total parameter space*, simply Total-p-Space. Then, the direct product (Cartesian product) of Total-p-Space $\mathscr{P}$ and Total-dF-Space $\mathscr{F}$ (see (2.2.5(p.13))) can be defined by

$$\mathscr{P} \times \mathscr{F} = \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}, F \in \mathscr{F}\} \tag{4.4.3}$$

called the Total-pdF-Space.

Chapter 5

Underlying Functions

5.1 Definition

This section defines some functions, called the *underlying function*, which will be used to derive the system of optimality equations of the 24 model in Table 3.2.1(p.17).

5.1.1 T, L, K, and $\mathcal{L}$ of Type $\mathbb{R}$

For any $F \in \mathscr{F}$ let us define

$$\begin{aligned} T(x) &= \mathbf{E}[\max\{\boldsymbol{\xi} - x, 0\}] && (5.1.1)\\ &= \textstyle\int_{-\infty}^{\infty} \max\{\xi - x, 0\} f(\xi) d\xi,^{\dagger} && (5.1.2) \end{aligned}$$

and then define

$$\begin{aligned} L(x) &= \lambda\beta T(x) - s, && (5.1.3)\\ K(x) &= \lambda\beta T(x) - (1-\beta)x - s,^{\S} && (5.1.4)\\ \mathcal{L}(s) &= L(\lambda\beta\mu - s), && (5.1.5)\\ \kappa &= \lambda\beta T(0) - s && (5.1.6)\\ &= L(0) = K(0) = \lambda\beta T(0) - s && (5.1.7) \end{aligned}$$

Let us refer to each of T, L, K, and $\mathcal{L}$ as the *underlying function* of Type $\mathbb{R}$ and to κ as the *κ-value* of Type $\mathbb{R}$. The formula below will be sometimes used in the rest of the paper.

$$\begin{aligned} K(x) + (1-\beta)x &= L(x), && (5.1.8)\\ K(x) + x &= L(x) + \beta x, && (5.1.9)\\ \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}, x\}] + (1-\lambda)\beta x - s &= K(x) + x && (5.1.10) \end{aligned}$$

5.1.2 $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ of Type $\mathbb{R}$

For any $F \in \mathscr{F}$ let us define

$$\begin{aligned} \tilde{T}(x) &= \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}] && (5.1.11)\\ &= \textstyle\int_{-\infty}^{\infty} \min\{\xi - x, 0\} f(\xi) d\xi, && (5.1.12) \end{aligned}$$

and then define

$$\begin{aligned} \tilde{L}(x) &= \lambda\beta\tilde{T}(x) + s, && (5.1.13)\\ \tilde{K}(x) &= \lambda\beta\tilde{T}(x) - (1-\beta)x + s, && (5.1.14)\\ \tilde{\mathcal{L}}(s) &= \tilde{L}(\lambda\beta\mu + s), && (5.1.15)\\ \tilde{\kappa} &= \lambda\beta\tilde{T}(0) + s && (5.1.16)\\ &= \tilde{L}(0) = \tilde{K}(0). && (5.1.17) \end{aligned}$$

Let us refer to each of $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ as the *underlying function* of $\tilde{\text{T}}$ype $\mathbb{R}$ and to $\tilde{\kappa}$ as the *$\tilde{\kappa}$-value* of $\tilde{\text{T}}$ype $\mathbb{R}$.

† See [1,DeGroot].
§ See Figure A 7.3(p.262) (II) ,

5.1.3 T, L, K, and $\mathcal{L}$ of Type $\mathbb{P}$

For any $F \in \mathscr{F}$ let us define

$$p(z) = \Pr\{z \le \boldsymbol{\xi}\}, \quad (5.1.18)$$
$$T(x) = \max_z p(z)(z-x)^\dagger \quad (5.1.19)$$

and then define

$$L(x) = \lambda\beta T(x) - s, \quad (5.1.20)$$
$$K(x) = \lambda\beta T(x) - (1-\beta)x - s, \quad (5.1.21)$$
$$\mathcal{L}(s) = L(\lambda\beta a - s), \quad (5.1.22)$$
$$\kappa = \lambda\beta T(0) - s \quad (5.1.23)$$
$$= L(0) = K(0) \quad (5.1.24)$$

Let us refer to each of T, L, K, and $\mathcal{L}$ as the *underlying function* of Type $\mathbb{P}$ and to κ as the *κ-value* of Type $\mathbb{P}$. Let us denote z maximizing $p(z)(z-x)$ by $z(x)$ if it exists, i.e.,

$$T(x) = p(z(x))(z(x) - x). \quad (5.1.25)$$

Definition 5.1.1 If there exists multiple $z(x)$, let us define the *smallest* of them as $z(x)$. □

Furthermore, for convenience of later discussions, let us define

$$a^\star = \inf\{x \mid T(x) + x > a\} = \inf\{x \mid T(x) > a - x\}, \quad (5.1.26)$$
$$x^\star = \inf\{x \mid z(x) > a\}. \quad (5.1.27)$$

Noting that (5.1.18(p.24)) can be rewritten as $p(z) = 1 - \Pr\{\boldsymbol{\xi} < z\} = 1 - \Pr\{\boldsymbol{\xi} \le z\}$ due to the assumption of F being continuous (see A9(p.13)), we have $p(z) = 1 - F(z)$. Accordingly, it can be immediately seen that

$$p(z)\begin{cases} = 1, & z \le a \quad \cdots(1) \quad \text{due to (2.2.1 (1) (p.13))}, \\ < 1, & a < z \quad \cdots(2) \quad \text{due to (2.2.1 (2,3) (p.13))}, \end{cases} \quad (5.1.28)$$

$$p(z)\begin{cases} > 0, & z < b \quad \cdots(1), \quad \text{due to (2.2.1 (1,2) (p.13))}, \\ = 0, & b \le z \quad \cdots(2), \quad \text{due to (2.2.1 (3) (p.13))}. \end{cases} \quad (5.1.29)$$

In general, $p(z)(z-x)$ can be depicted as below.

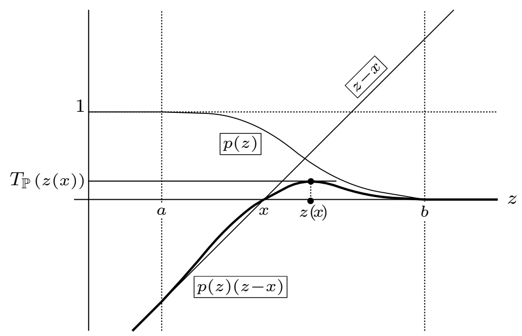


Figure 5.1.1: Graph of $p(z)(z-x)$

When F is the uniform distribution function on $[a, b]$, we have

$$a^\star = x^\star = 2a - b \quad \text{(see (A 7.3(p.262)) and (A 7.4(p.262)))}. \quad (5.1.30)$$

5.1.4 $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ of $\tilde{\text{T}}$ype $\mathbb{P}$

For any $F \in \mathscr{F}$ let us define

$$\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \le z\}, \quad (5.1.31)$$
$$\tilde{T}(x) = \min_z \tilde{p}(z)(z-x), \quad (5.1.32)$$

and then define

† See Figure A 7.4(p.262) .

$$\begin{aligned}\tilde{L}(x) &= \lambda\beta\tilde{T}(x) + s, &(5.1.33)\\ \tilde{K}(x) &= \lambda\beta\tilde{T}(x) - (1-\beta)x + s, &(5.1.34)\\ \tilde{\mathcal{L}}(s) &= \tilde{L}(\lambda\beta b + s), &(5.1.35)\\ \tilde{\kappa} &= \lambda\beta\tilde{T}(0) + s &(5.1.36)\\ &= \tilde{L}(0) = \tilde{K}(0). &(5.1.37)\end{aligned}$$

Let us refer to each of $\tilde{T}$, $\tilde{L}$, $\tilde{K}$, and $\tilde{\mathcal{L}}$ as the *underlying function* of $\tilde{\text{T}}$ype $\mathbb{P}$ and to $\tilde{\kappa}$ as the *$\tilde{\kappa}$-value* of $\tilde{\text{T}}$ype $\mathbb{P}$. Let us denote z minimizing $\tilde{p}(z)(z-x)$ by $z(x)$ if it exists, i.e.,

$$\tilde{T}(x) = \tilde{p}(z(x))(z(x)-x). \tag{5.1.38}$$

Definition 5.1.2 If there exists multiple $z(x)$, let us define the *largest* of them as $z(x)$. □

Furthermore, for convenience of later discussions, let us define

$$b^\star = \sup\{x \mid \tilde{T}(x) + x < b\} = \sup\{x \mid \tilde{T}(x) < b - x\}, \tag{5.1.39}$$

$$\tilde{x}^\star = \sup\{x \,|\, z(x) < b\}. \tag{5.1.40}$$

Noting that (5.1.31(p.24)) can be rewritten as $\tilde{p}(z) = F(z)$, we can immediately see that

$$\tilde{p}(z)\begin{cases} = 0, & z \le a \quad \cdots (1) \quad \text{due to (2.2.1 (1) (p.13))},\\ > 0, & a < z \quad \cdots (2) \quad \text{due to (2.2.1 (2.3) (p.13))},\end{cases} \tag{5.1.41}$$

$$\tilde{p}(z)\begin{cases} < 1, & z < b \quad \cdots (1) \quad \text{due to (2.2.1 (1,2) (p.13))},\\ = 1, & b \le z \quad \cdots (2) \quad \text{due to (2.2.1 (3) (p.13))}.\end{cases} \tag{5.1.42}$$

5.2 Solutions

The solutions defined below are used in the analyses of all models in this paper.

(a) Let us define the solutions of $L(x) = 0$, $K(x) = 0$, and $\mathcal{L}(s) = 0$ (whether Type $\mathbb{R}$ or Type $\mathbb{P}$) by x_L, x_K, and $s_{\mathcal{L}}$ respectively if they exist, i.e.,

$$L(x_L) = 0 \cdots (1), \qquad K(x_K) = 0 \cdots (2), \qquad \mathcal{L}(s_{\mathcal{L}}) = 0 \cdots (1). \tag{5.2.1}$$

If multiple solutions exist for each of the above three equations, we employ the *smallest* as its solution.

(b) Let us define the solutions of $\tilde{L}(x) = 0$, $\tilde{K}(x) = 0$, and $\tilde{\mathcal{L}}(s) = 0$ (whether $\tilde{\text{T}}$ype $\mathbb{R}$ or $\tilde{\text{T}}$ype $\mathbb{P}$) by $x_{\tilde{L}}$, $x_{\tilde{K}}$, and $s_{\tilde{\mathcal{L}}}$ respectively if they exist.

$$\tilde{L}(x_{\tilde{L}}) = 0 \cdots (1), \qquad \tilde{K}(x_{\tilde{K}}) = 0 \cdots (2), \qquad \tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0 \cdots (1). \tag{5.2.2}$$

If multiple solutions exist for each of the above three equations, we employ the *largest* as its solution.

5.3 Primitive Underlying Functions and Derivative Underlying Functions

Sometimes let us refer to each of T- and $\tilde{T}$-functions as the *primitive underlying function* and to each of L-, K-, $\mathcal{L}$-, $\tilde{L}$-, $\tilde{L}$-, and $\tilde{\mathcal{L}}$-functions as the *derivative underlying function*, which are defined by use of primitive underlying functions T and $\tilde{T}$.

5.4 Identical Representation and Explicit Representation

In the rest of the paper, when we need to distinguish

$$T,\, L,\, K,\, \mathcal{L},\, \kappa,\; x_L,\; x_K,\; s_{\mathcal{L}},\, \tilde{T},\, \tilde{L},\, \tilde{K},\, \tilde{\mathcal{L}},\, \tilde{\kappa},\; x_{\tilde{L}},\; x_{\tilde{K}},\; s_{\tilde{\mathcal{L}}} \tag{5.4.1}$$

between Type $\mathbb{R}$ and Type $\mathbb{P}$, let us denote them by

$$T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}, x_{L_{\mathbb{R}}}, x_{K_{\mathbb{R}}}, s_{\mathcal{L}_{\mathbb{R}}}, \tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}, x_{\tilde{L}_{\mathbb{R}}}, x_{\tilde{K}_{\mathbb{R}}}, s_{\tilde{\mathcal{L}}_{\mathbb{R}}}, \tag{5.4.2}$$

$$T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}, x_{L_{\mathbb{P}}}, x_{K_{\mathbb{P}}}, s_{\mathcal{L}_{\mathbb{P}}}, \tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}, x_{\tilde{L}_{\mathbb{P}}}, x_{\tilde{K}_{\mathbb{P}}}, s_{\tilde{\mathcal{L}}_{\mathbb{P}}}. \tag{5.4.3}$$

Let us refer to (5.4.1) as the *identical representation* and to (5.4.2) and (5.4.3) as the *explicit representation.*

5.5 Characteristic Vector and Characteristic Element

Let us here define the two vectors, $\boldsymbol{C}_{\mathbb{R}}$ consisting of (5.1.3(p.23))-(5.1.6(p.23)) and $\tilde{\boldsymbol{C}}_{\mathbb{R}}$ consisting of (5.1.13(p.23))-(5.1.16(p.23)), i.e,

$$\boldsymbol{C}_{\mathbb{R}} = (L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}), \qquad \tilde{\boldsymbol{C}}_{\mathbb{R}} = (\tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}).$$

Likewise, let us define the two vectors, $\boldsymbol{C}_{\mathbb{P}}$ consisting of (5.1.20(p.24))-(5.1.23(p.24)) and $\tilde{\boldsymbol{C}}_{\mathbb{P}}$ consisting of (5.1.33(p.25))-(5.1.36(p.25)), i.e.,

$$\boldsymbol{C}_{\mathbb{P}} = (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}), \qquad \tilde{\boldsymbol{C}}_{\mathbb{P}} = (\tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}).$$

Furthermore, adding T- and $\tilde{T}$-functions to the above vectors, let us define

$$\boldsymbol{C}_{\mathbb{R}}^{T} = (T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}), \quad \tilde{\boldsymbol{C}}_{\mathbb{R}}^{T} = (\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}),$$
$$\boldsymbol{C}_{\mathbb{P}}^{T} = (T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}), \quad \tilde{\boldsymbol{C}}_{\mathbb{P}}^{T} = (\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}).$$

Let us call each of the vectors defined above the *characteristic vector* and its element the *characteristic element.* In the identical representation, the above vectors are all represented by $\boldsymbol{C}$, $\tilde{\boldsymbol{C}}$, $\boldsymbol{C}^{T}$, and $\tilde{\boldsymbol{C}}^{T}$ respectively.

Chapter 6

Systems of Optimality Equations

6.1 Preliminary

This section provides a definition used to describe the system of optimality equations (`SOE`) for each models in Table 3.2.1(p.17).

Definition 6.1.1 Let us represent the action "Conduct the search at time t (Skip the search at time t)" as

$$\texttt{Conduct}_t\ (\texttt{Skip}_t). \tag{6.1.1}$$

When this action is *simply* optimal, *indifferently* optimal, or *strictly* optimal, let us represent it as respectively

$$\texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}),\quad \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}),\quad \text{or}\quad \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}).\ \square$$

Remark 6.1.1 (relationship between `SOE` and assertion) In general, a model M of a decision process has the system of optimality equations, denoted by `SOE`{M}, which should be said to be a mirror exhaustively reflecting the entire aspect of the model M. In other words, `SOE`{M} involves the exhaustive information of the model M as if a gene has the exhaustive information of a life. This implies that any assertion which is characterized by the sequence $\{V_t\}$ generated from `SOE`{M} can be regarded as an assertion on the model M; conversely, an assertion which is not characterized by the sequence $\{V_t\}$ cannot be said to be an assertion on the M. □

Below let us represent "buyer (seller) proposing a price ξ" by "buyer (seller) w" for short.

6.2 Search-Allowed-Model

6.2.1 Model 1

Let us note here that $\lambda = 1$ is assumed in this model (see $A2$(p.19)).

6.2.1.1 M:1[ℝ][A]

By $v_t(\xi)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer ξ and with no buyer respectively. Then, we have

$$v_0(\xi) = w, \tag{6.2.1}$$
$$v_t(\xi) = \max\{\xi, V_t\}, \quad t > 0, \tag{6.2.2}$$

where V_t is the maximum of the total expected present discounted profit from rejecting the proposed price w. Then, we have

$$V_1 = \beta\,\mathbf{E}[v_0(\boldsymbol{\xi})] - s = \beta\,\mathbf{E}[\boldsymbol{\xi}] - s = \beta\mu - s \quad \text{(see Remark 4.1.3(p.20))}, \tag{6.2.3}$$
$$V_t = \max\{\texttt{C} : \beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - s,\ \texttt{S} : \beta V_{t-1}\}, \quad t > 1, \tag{6.2.4}$$

where `C` and `S` represent the actions of `C`onducting the search and `S`kipping the search respectively. Hence, since $v_{t-1}(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, V_{t-1}\} = \max\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1}$, we have $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = T(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (5.1.1(p.23))), hence (6.2.4(p.27)) can be written as

$$\begin{aligned} V_t &= \max\{\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.4(p.23)) with } \lambda = 1) \qquad (6.2.5) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.8(p.23)))}. \qquad (6.2.6) \end{aligned}$$

□ `SOE`{M:1[ℝ][A]} is given by the set of (6.2.1(p.27)) – (6.2.4(p.27)). However, since the sequence $\{V_t\}$ is generated from the two expressions (6.2.3(p.27)) and (6.2.5(p.27)), due to Remark 6.1.1(p.27) it can be reduced to only the two in Table 6.5.1(p.39) (I). □

Now, let us here define

$$\mathbb{S}_t \;=\; \beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 1. \tag{6.2.7}$$

Then, (6.2.4(p.27)) can be rewritten as

$$\begin{aligned} V_t \;&=\; \max\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \beta V_{t-1} - s, 0\} + \beta V_{t-1} \\ &=\; \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{6.2.8}$$

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \tag{6.2.9}$$

which can be rewritten as, due to Def. 6.1.1(p.27) ,

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.10}$$

$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.11}$$

$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.12}$$

Then, from (6.2.2(p.27)) we can rewrite (6.2.7(p.28)) as

$$\mathbb{S}_t = \beta(\,\mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) - s = \beta\,\mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] - s.$$

Accordingly, from (5.1.1(p.23)) and (5.1.3(p.23)) with $\lambda = 1$ we have

$$\mathbb{S}_t \;=\; \beta T(V_{t-1}) - s \tag{6.2.13}$$

$$\;=\; L(V_{t-1}), \quad t > 1. \tag{6.2.14}$$

6.2.1.2 $\tilde{\mathsf{M}}$:1[ℝ][A]

By $v_t(\xi)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller ξ and with no seller respectively. Then, we have

$$v_0(\xi) = \xi, \tag{6.2.15}$$

$$v_t(\xi) = \min\{\xi, V_t\}, \quad t > 0, \tag{6.2.16}$$

where V_t is the minimum of the total expected present discounted cost from rejecting the proposed price ξ. Then, we have

$$V_1 \;=\; \beta\,\mathbf{E}[v_0(\boldsymbol{\xi})] + s = \beta\,\mathbf{E}[\boldsymbol{\xi}] + s = \beta\mu + s, \tag{6.2.17}$$

$$V_t \;=\; \min\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + s,\ \beta V_{t-1}\}, \quad t > 1. \tag{6.2.18}$$

Hence, since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \min\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1}$, we have $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ (see (5.1.11(p.23))), hence (6.2.18(p.28)) can be written as

$$\begin{aligned} V_t &= \min\{\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14(p.23)) with } \lambda = 1) \end{aligned} \tag{6.2.19}$$

$$\begin{aligned} &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.14(p.23)) and (5.1.13(p.23)) with } \lambda = 1). \end{aligned} \tag{6.2.20}$$

□ SOE{$\tilde{\mathsf{M}}$:1[ℝ][A]} can be reduced to (6.2.17(p.28)) and (6.2.19(p.28)), listed in Table 6.5.1(p.39) (II). ∎

Remark 6.2.1 Note here that the same notations $v_t(\xi)$ and V_t are used for both M:1[ℝ][A] and $\tilde{\mathsf{M}}$:1[ℝ][A]. For explanatory convenience, later on we sometimes represent the $v_t(\xi)$ and V_t for $\tilde{\mathsf{M}}$:1[ℝ][A] by $\tilde{v}_t(\xi)$ and $\tilde{V}_t$ respectively. Then (6.2.15(p.28))-(6.2.18(p.28)) are written as respectively

$$\tilde{v}_0(\xi) = \xi,$$

$$\tilde{v}_t(\xi) = \min\{\xi, \tilde{V}_t\},$$

$$\tilde{V}_1 = \beta\mu + s,$$

$$\tilde{V}_t = \min\{\beta\,\mathbf{E}[\tilde{v}_{t-1}(\boldsymbol{\xi})] + s, \beta\tilde{V}_{t-1}\}. \quad ∎$$

Now, let us here define

$$\tilde{\mathbb{S}}_t \;=\; \beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 1. \tag{6.2.21}$$

Then, (6.2.18(p.28)) can be rewritten as

$$\begin{aligned} V_t \;&=\; \min\{\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \beta V_{t-1} + s, 0\} + \beta V_{t-1} \\ &=\; \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{6.2.22}$$

which can be rewritten as

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t) \quad (\text{see Def. 6.1.1(p.27)}), \tag{6.2.23}$$

which can be rewritten as

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.24}$$

$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.25}$$

$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.26}$$

Then, from (6.2.16(p.28)) we can rewrite (6.2.21(p.28)) as

$$\tilde{\mathbb{S}}_t = \beta(\mathbf{E}[\min\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) + s = \beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + s.$$

Accordingly, from (5.1.11(p.23)) and (5.1.13(p.23)) with $\lambda = 1$ we have

$$\tilde{\mathbb{S}}_t = \beta\tilde{T}(V_{t-1}) + s \tag{6.2.27}$$

$$= \tilde{L}(V_{t-1}), \quad t > 1. \tag{6.2.28}$$

6.2.1.3 M:1[$\mathbb{P}$][A]

By v_t ($t \ge 0$) and V_t ($t > 0$) let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \ge 0$ by z_t. In this model, since the search must be necessarily conducted at time 1 (see Remark 4.1.3(p.20)), there exists a buyer at time 0. Suppose the process has proceeded up to time 0. Then, since the seller must necessarily sell the asset at that time, he must propose the price a[†] to a buyer appearing at that time (Remark 4.1.1(p.19) (b) is applicable also to this model as a sA-model), thus we have

$$z_0 = a. \tag{6.2.29}$$

Hence, the profit that the seller obtains at time 0 becomes a, i.e.,

$$v_0 = a. \tag{6.2.30}$$

Now, since the search is conducted at time 1 (see Remark 4.1.3(p.20)), we have

$$V_1 = \beta v_0 - s = \beta a - s. \tag{6.2.31}$$

In addition, we have

$$V_t = \max\{\beta v_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 1. \tag{6.2.32}$$

If the seller proposes a price z, the probability of a buyer buying the asset is given by $p(z) = \Pr\{z \le \boldsymbol{\xi}\}$ (see (5.1.18(p.24))), hence we have

$$v_t = \max_z\{p(z)z + (1 - p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0, \tag{6.2.33}$$

due to (5.1.19(p.24)), implying that the optimal price z_t which the seller should propose is given by

$$z_t = z(V_t), \quad t > 0, \quad (\text{see (5.1.25(p.24))}). \tag{6.2.34}$$

Now, since $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.33(p.29)), we can rearrange (6.2.32(p.29)) as follows

$$V_t = \max\{\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\}$$

$$= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad (\text{see (5.1.21(p.24)) with } \lambda = 1) \tag{6.2.35}$$

$$= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1}$$

$$= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1, \quad (\text{see (5.1.21(p.24)) and (5.1.20(p.24)) with } \lambda = 1) \tag{6.2.36}$$

□ SOE{M:1[$\mathbb{P}$][A]} is given by (6.2.31(p.29)) and (6.2.35(p.29)), listed in Table 6.5.1(p.39) (III). ■

Now, let us here define

$$\mathbb{S}_t = \beta(v_{t-1} - V_{t-1}) - s, \quad t > 1. \tag{6.2.37}$$

Then, (6.2.32(p.29)) can be rewritten as

$$V_t = \max\{\beta v_{t-1} - \beta V_{t-1} - s, 0\} + \beta V_{t-1}$$

$$= \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 1, \tag{6.2.38}$$

[†]The lower bound of the distribution function for the reservation price (maximum permissible buying price) of the buyer.

implying that

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t) \quad \text{(see Def. 6.1.1(p.27))}, \tag{6.2.39}$$

which can be rewritten as

$$\mathbb{S}_t \geq (\leq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.40}$$

$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.41}$$

$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.42}$$

Then, from (6.2.33(p.29)) with $t-1$ we have $v_{t-1} = T(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = T(V_{t-1})$, thus, noting (5.1.20(p.24)), we can rewrite (6.2.37(p.29)) as below

$$\mathbb{S}_t \ = \ \beta T(V_{t-1}) - s \tag{6.2.43}$$

$$= \ L(V_{t-1}), \quad t > 1. \tag{6.2.44}$$

6.2.1.4 $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][$\mathtt{A}$]

By v_t $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. In this model, since the search must be necessarily conducted at time 1, there exists a seller at time 0 for the same reason as in Section 6.2.1.3(p.29). Suppose the process has proceeded up to time 0. Then, since the buyer must necessarily buy the asset at that time, he must propose the price $b^\dagger$ to a seller appearing at that time, thus we have

$$z_0 = b. \tag{6.2.45}$$

Hence, the cost that the buyer pays at time 0 becomes b, i.e.,

$$v_0 = b. \tag{6.2.46}$$

Now, since the search is conducted at time 1, we have

$$V_1 = \beta v_0 + s = \beta b + s. \tag{6.2.47}$$

In addition, we have

$$V_t \ = \ \min\{\beta v_{t-1} + s,\ \beta V_{t-1}\}, \quad t > 1. \tag{6.2.48}$$

If the buyer proposes a price z, the probability of a seller selling the asset is given by $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \leq z\}$ (see (5.1.31(p.24))), hence we have

$$v_t \ = \ \min_z\{\tilde{p}(z)z + (1-\tilde{p}(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.49}$$

due to (5.1.32(p.24)), implying that the optimal price z_t which the buyer should propose is given by

$$z_t = z(V_t), \quad t > 0, \quad \text{(see (5.1.38(p.25)))}. \tag{6.2.50}$$

Now, since $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.49(p.30)), we can rearrange (6.2.48(p.30)) as

$$\begin{aligned} V_t &= \min\{\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.25)) with } \lambda = 1) \end{aligned} \tag{6.2.51}$$

$$\begin{aligned} &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1. \quad \text{(see (5.1.34(p.25)) and (5.1.33(p.25)) with } \lambda = 1) \end{aligned} \tag{6.2.52}$$

□ $\mathtt{SOE}$\{$\tilde{\mathsf{M}}$:1[$\mathbb{P}$][$\mathtt{A}$]\} is given by (6.2.47(p.30)) and (6.2.51(p.30)), listed in Table 6.5.1(p.39) (IV). ∎

Now, let us here define

$$\tilde{\mathbb{S}}_t \ = \ \beta(v_{t-1} - V_{t-1}) + s, \quad t > 1. \tag{6.2.53}$$

Then, (6.2.48(p.30)) can be rewritten as

$$\begin{aligned} V_t \ &= \ \min\{\beta v_{t-1} - \beta V_{t-1} + s, 0\} + \beta V_{t-1} \\ &= \ \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{6.2.54}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t) \quad \text{(see Def. 6.1.1(p.27))}. \tag{6.2.55}$$

which can be rewritten as

---

† The upper bound of the distribution function for the reservation price (minimum permissible selling price) of the seller

$$\tilde{\mathbb{S}}_t \le (\ge)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.56}$$
$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.57}$$
$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.58}$$

Then, from (6.2.49(p.30)) with $t-1$ we have $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = \tilde{T}(V_{t-1})$, thus, noting (5.1.33(p.25)), we can rewrite (6.2.53(p.30)) as below

$$\mathbb{S}_t = \beta\tilde{T}(V_{t-1}) + s \tag{6.2.59}$$
$$= \tilde{L}(V_{t-1}), \quad t > 1. \tag{6.2.60}$$

6.2.2 Model 2

6.2.2.1 M:2[$\mathbb{R}$][A]

By $v_t(\xi)$ $(t \ge 0)$ and V_t $(t > 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer ξ and with no buyer respectively. Then we have

$$v_0(\xi) = \max\{\xi, \rho\}, \tag{6.2.61}$$
$$v_t(\xi) = \max\{\xi, V_t\}, \quad t > 0, \tag{6.2.62}$$

where

$$V_t = \max\{\lambda\beta\, \mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.63}$$

Here letting

$$V_0 = \rho, \tag{6.2.64}$$

we see that (6.2.62(p.31)) holds for $t \ge 0$ instead of $t > 0$, i.e.,

$$v_t(\xi) = \max\{\xi, V_t\}, \quad t \ge 0. \tag{6.2.65}$$

Since $v_{t-1}(\boldsymbol{\xi}) = \max\{\boldsymbol{\xi}, V_{t-1}\} = \max\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 0$ (see (5.1.1(p.23))), from (6.2.63(p.31)) we have

$$\begin{aligned} V_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad (\text{see } (5.1.4(\text{p.23}))) \qquad (6.2.66) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad (\text{see } (5.1.8(\text{p.23}))). \qquad (6.2.67) \end{aligned}$$

□ SOE{M:2[$\mathbb{R}$][A]} is given by (6.2.64(p.31)) and (6.2.66(p.31)), listed in Table 6.5.3(p.39) (I). ■

Let us here define

$$\mathbb{S}_t = \lambda\beta(\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) - s, \quad t > 0. \tag{6.2.68}$$

Then, (6.2.63(p.31)) can be rewritten as

$$\begin{aligned} V_t &= \max\{\lambda\beta\, \mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \lambda\beta V_{t-1} - s, 0\} + \beta V_{t-1} \\ &= \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \qquad (6.2.69) \end{aligned}$$

implying that

$$\mathbb{S}_t \ge (\le)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t), \quad t > 0 \quad (\text{see Def. 6.1.1(p.27)}). \tag{6.2.70}$$

which can be rewritten as, due to Def. 6.1.1(p.27),

$$\mathbb{S}_t \ge (\le)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.71}$$
$$\mathbb{S}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.72}$$
$$\mathbb{S}_t > (<)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.73}$$

Then, from (6.2.62(p.31)) we can rewrite (6.2.68(p.31)) as

$$\mathbb{S}_t = \beta(\mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) - s = \beta\, \mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] - s.$$

Accordingly, from (5.1.1(p.23)) and (5.1.3(p.23)) we have

$$\mathbb{S}_t = \beta T(V_{t-1}) - s \tag{6.2.74}$$
$$= L(V_{t-1}), \quad t > 0. \tag{6.2.75}$$

6.2.2.2 $\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]

By $v_t(\xi)$ $(t \geq 0)$ and V_t $(t > 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller ξ and with no seller respectively. Then, we have

$$v_0(\xi) = \min\{\xi, \rho\}, \tag{6.2.76}$$

$$v_t(\xi) = \min\{\xi, V_t\}, \quad t > 0, \tag{6.2.77}$$

where

$$V_t = \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.78}$$

Here letting

$$V_0 = \rho, \tag{6.2.79}$$

we see that (6.2.77(p.32)) holds for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(\xi) = \min\{\xi, V_t\}, \quad t \geq 0. \tag{6.2.80}$$

Since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \min\{\boldsymbol{\xi} - V_{t-1}, 0\} + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 0$ (see (5.1.11(p.23))), from (6.2.78(p.32)) we have

$$\begin{aligned} V_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14(p.23)))} \qquad (6.2.81) \\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0. \quad \text{(see (5.1.14(p.23)) and (5.1.13(p.23)))} \qquad (6.2.82) \end{aligned}$$

□ SOE{$\tilde{\mathsf{M}}$:2[$\mathbb{R}$][A]} is given by (6.2.79(p.32)) and (6.2.81(p.32)), listed in Table 6.5.3(p.39) (II). ■

Let us here define

$$\tilde{\mathbb{S}}_t = \lambda\beta(\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - V_{t-1}) + s, \quad t > 0. \tag{6.2.83}$$

Then, (6.2.78(p.32)) can be rewritten as

$$\begin{aligned} V_t &= \min\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] - \lambda\beta V_{t-1} + s, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{6.2.84}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t) \quad \text{(see Def. 6.1.1(p.27))}, \tag{6.2.85}$$

which can be rewritten as

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.86}$$

$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.87}$$

$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.88}$$

Then, from (6.2.77(p.32)) we can rewrite (6.2.83(p.32)) as

$$\tilde{\mathbb{S}}_t = \beta(\,\mathbf{E}[\min\{\boldsymbol{\xi}, V_{t-1}\}] - V_{t-1}) + s = \beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + s.$$

Accordingly, from (5.1.11(p.23)) and (5.1.13(p.23)) we have

$$\tilde{\mathbb{S}}_t = \beta\tilde{T}(V_{t-1}) + s \tag{6.2.89}$$

$$= \tilde{L}(V_{t-1}), \quad t > 1. \tag{6.2.90}$$

6.2.2.3 M:2[$\mathbb{P}$][A]

By v_t $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a buyer at time $t = 0$ (deadline). Then, the seller must determine whether to accept the terminal quitting penalty ρ or to sell the asset to the buyer. Let the ρ is accepted. Then the profit which the seller can obtain is ρ. On the other hand, let the asset be sold to the buyer. Then, since the seller must necessarily sell the asset to the buyer due to A2(p.11), the price a[†] must be proposed to the buyer; in other words, the optimal price to propose at time $t = 0$ is given by

$$z_0 = a, \tag{6.2.91}$$

[†]The lower bound of the distribution function for the reservation price (the maximum permissible buying price) of the buyer.

hence the profit which the seller can obtain at that time is a. Accordingly, it follows that the profit that the seller can obtain at time 0 is given by

$$v_0 = \max\{\rho, a\}. \tag{6.2.92}$$

Suppose there exists a buyer at a time $t > 0$. Then, since the reservation price (maximum permissible buying price) of the buyer is $\boldsymbol{\xi}$, if the seller proposes a price z, the probability of the buyer buying the asset is given by $p(z) = \Pr\{z \le \boldsymbol{\xi}\}$ (see (5.1.18(p.24))). Hence, we have

$$v_t = \max_z\{p(z)z + (1-p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0, \tag{6.2.93}$$

due to (5.1.19(p.24)), implying that the optimal selling price z_t which the seller should propose is given by

$$z_t = z(V_t), \qquad t > 0, \tag{6.2.94}$$

due to (5.1.25(p.24)). Finally V_t can be expressed as follows.

$$V_0 = \rho, \tag{6.2.95}$$

$$V_t = \max\{\lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\}, \quad t > 0. \tag{6.2.96}$$

For $t = 1$ we have

$$\begin{aligned} V_1 &= \max\{\lambda\beta v_0 + (1-\lambda)\beta V_0 - s, \beta V_0\} \\ &= \max\{\lambda\beta\max\{\rho, a\} + (1-\lambda)\beta\rho - s, \beta\rho\} \\ &= \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}. \end{aligned} \tag{6.2.97}$$

Since $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.93(p.33)), we can rearrange (6.2.96(p.33)) as follows.

$$\begin{aligned} V_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.21(p.24)))} \qquad (6.2.98) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad \text{(see (5.1.21(p.24)) and (5.1.20(p.24)))}. \qquad (6.2.99) \end{aligned}$$

□ **SOE**{M:1[ℙ][A]} is given by (6.2.95(p.33)), (6.2.97(p.33)), and (6.2.98(p.33)), listed in Table 6.5.3(p.39) (III). ■
Now, let us here define

$$\mathbb{S}_t = \lambda\beta(v_{t-1} - V_{t-1}) - s, \quad t > 0. \tag{6.2.100}$$

Then (6.2.96(p.33)) can be rewritten as

$$\begin{aligned} V_t &= \max\{\lambda\beta v_{t-1} - \lambda\beta V_{t-1} - s, 0\} - \beta V_{t-1} \\ &= \max\{\mathbb{S}_t, 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{6.2.101}$$

implying that

$$\mathbb{S}_t \ge (\le)\, 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t) \quad \text{(see Def. 6.1.1(p.27))}, \tag{6.2.102}$$

which can be rewritten as

$$\mathbb{S}_t \ge (\le)\, 0 \Rightarrow \texttt{Conduct}_{t\vartriangle}\ (\texttt{Skip}_{t\vartriangle}). \tag{6.2.103}$$

$$\mathbb{S}_t = (=)\, 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.104}$$

$$\mathbb{S}_t > (<)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{6.2.105}$$

Then, from (6.2.93(p.33)) with $t-1$ we have $v_{t-1} = T(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = T(V_{t-1})$, thus, noting (5.1.20(p.24)), we can rewrite (6.2.100(p.33)) as below

$$\mathbb{S}_t = \beta T(V_{t-1}) - s \tag{6.2.106}$$

$$= L(V_{t-1}), \quad t > 0. \tag{6.2.107}$$

6.2.2.4 $\tilde{\mathsf{M}}$:2[ℙ][A]

By v_t ($t \ge 0$) and V_t ($t \ge 0$) let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \ge 0$ by z_t. Suppose there exists a seller at time $t = 0$ (deadline). Then, the buyer must determine whether to accept the terminal quitting penalty ρ or to buy the asset from the seller. Let the ρ is accepted. Then the cost which the buyer pays is ρ. On the other hand, let an asset be bought from the seller. Then, since the buyer must necessarily buy the asset from the seller due to A2(p.11), the price b[†] must be proposed to the seller; in other words, the optimal price to propose at time $t = 0$ is given by

[†] The upper bound of the distribution function for the reservation price (the minimum permissible selling price) of the seller.

$$z_0 = b, \tag{6.2.108}$$

hence the cost which the buyer pays at that time is b. Accordingly, the cost that the buyer pays at time 0 becomes

$$v_0 = \min\{\rho, b\}. \tag{6.2.109}$$

Suppose there exists a seller at a time $t > 0$. Then, since the reservation price (minimum permissible selling price) of the seller is $\boldsymbol{\xi}$, if the buyer proposes a price z, the probability of the seller selling the asset is given by $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \leq z\}$ (see (5.1.31(p.24))). Hence, we have

$$v_t = \min_z\{\tilde{p}(z)z + (1 - p(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.110}$$

due to (5.1.32(p.24)), implying that the optimal buying price z_t which the buyer should propose is given by

$$z_t = z(V_t), \qquad t > 0, \tag{6.2.111}$$

due to (5.1.38(p.25)). Finally V_t can be expressed as follows.

$$V_0 = \rho, \tag{6.2.112}$$

$$V_t = \min\{\lambda\beta v_{t-1} + (1 - \lambda)\beta V_{t-1} + s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.113}$$

For $t = 1$ we have

$$\begin{aligned} V_1 &= \min\{\lambda\beta v_0 + (1 - \lambda)\beta V_0 + s, \beta V_0\} \\ &= \min\{\lambda\beta \min\{\rho, b\} + (1 - \lambda)\beta\rho + s, \beta\rho\} \\ &= \min\{\lambda\beta \min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}. \end{aligned} \tag{6.2.114}$$

Since $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$ from (6.2.110(p.34)), we can rearrange (6.2.113(p.34)) as follows.

$$\begin{aligned} V_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1 - \lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.25)))} \end{aligned} \tag{6.2.115}$$

$$\begin{aligned} &= \min\{\tilde{K}(V_{t-1}) + (1 - \beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \min\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1. \quad \text{(see (5.1.34(p.25)) and (5.1.33(p.25)))} \end{aligned} \tag{6.2.116}$$

□ SOE{$\tilde{\mathsf{M}}$:2[ℙ][A]} can be reduced to (6.2.112(p.34)), (6.2.114(p.34)), and (6.2.115(p.34)), listed in Table 6.5.3(p.39) (IV). ∎

Now, let us here define

$$\tilde{\mathbb{S}}_t = \lambda\beta(v_{t-1} - V_{t-1}) + s, \quad t > 0. \tag{6.2.117}$$

Then, (6.2.113(p.34)) can be rewritten as

$$\begin{aligned} V_t &= \min\{\lambda\beta v_{t-1} - \lambda\beta V_{t-1} + s, 0\} - \beta V_{t-1} \\ &= \min\{\tilde{\mathbb{S}}_t, 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{6.2.118}$$

implying that

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_t\ (\texttt{Skip}_t) \quad \text{(see Def. 6.1.1(p.27))}, \tag{6.2.119}$$

which can be rewritten as

$$\tilde{\mathbb{S}}_t \leq (\geq)\ 0 \Rightarrow \texttt{Conduct}_{t^\vartriangle}\ (\texttt{Skip}_{t^\vartriangle}). \tag{6.2.120}$$

$$\tilde{\mathbb{S}}_t = (=)\ 0 \Rightarrow \texttt{Conduct}_{t\parallel}\ (\texttt{Skip}_{t\parallel}). \tag{6.2.121}$$

$$\tilde{\mathbb{S}}_t < (>)\ 0 \Rightarrow \texttt{Conduct}_{t^\blacktriangle}\ (\texttt{Skip}_{t^\blacktriangle}). \tag{6.2.122}$$

Then, from (6.2.110(p.34)) with $t - 1$ we have $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$, hence $v_{t-1} - V_{t-1} = \tilde{T}(V_{t-1})$, thus, noting (5.1.33(p.25)), we can rewrite (6.2.117(p.34)) as below

$$\mathbb{S}_t = \beta\tilde{T}(V_{t-1}) + s \quad t > 0. \tag{6.2.123}$$

$$= \tilde{L}(V_{t-1}), \quad t > 0. \tag{6.2.124}$$

6.2.3 Model 3

6.2.3.1 M:3[ℝ][A]

By $v_t(\xi)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer ξ and with no buyer respectively, expressed as

$$v_0(\xi) = \max\{\xi, \rho\}, \tag{6.2.125}$$

$$v_t(\xi) = \max\{\xi, \rho, U_t\}, \quad t > 0, \tag{6.2.126}$$

$$V_0 = \rho, \tag{6.2.127}$$

$$V_t = \max\{\rho, U_t\}, \qquad t > 0, \tag{6.2.128}$$

where U_t is the maximum of the total expected present discounted *profit* from rejecting both the price w and intervening quitting penalty ρ in (6.2.126(p.34)) and from rejecting the intervening quitting penalty ρ in (6.2.128(p.34)). Then, U_t can be expressed as

$$U_t = \max\{\lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} - s,\ \beta V_{t-1}\}, \quad t > 0. \tag{6.2.129}$$

Here, letting $U_0 = \rho$, from (6.2.127(p.34)) we have

$$V_0 = U_0 = \rho, \tag{6.2.130}$$

hence, both (6.2.126(p.34)) and (6.2.128(p.34)) hold true for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(\xi) = \max\{\xi, \rho, U_t\}, \quad t \geq 0, \tag{6.2.131}$$

$$V_t = \max\{\rho, U_t\}, \qquad t \geq 0, \tag{6.2.132}$$

thus (6.2.131(p.35)) can be expressed as

$$v_t(\xi) = \max\{\xi, V_t\}, \quad t \geq 0. \tag{6.2.133}$$

Accordingly, since $\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] = \mathbf{E}[\max\{\boldsymbol{\xi}, V_{t-1}\}] = \mathbf{E}[\max\{\boldsymbol{\xi} - V_{t-1}, 0\}] + V_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 0$ from (5.1.1(p.23)), we can rewrite (6.2.129(p.35)) as

$$\begin{aligned} U_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.4(p.23)))} \qquad (6.2.134) \\ &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad \text{(see (5.1.8(p.23)))}. \qquad (6.2.135) \end{aligned}$$

□ SOE{M:3[ℝ][A]} can be reduced to (6.2.130(p.35)), (6.2.132(p.35)), and (6.2.134(p.35)), listed in Table 6.5.5(p.39) (I). □

6.2.3.2 $\tilde{\text{M}}$:3[ℝ][A]

By $v_t(\xi)$ $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time $t \geq 0$ with a seller ξ and with no seller respectively, expressed as

$$v_0(\xi) = \min\{\xi, \rho\}, \tag{6.2.136}$$

$$v_t(\xi) = \min\{\xi, \rho, U_t\}, \quad t > 0, \tag{6.2.137}$$

$$V_0 = \rho, \tag{6.2.138}$$

$$V_t = \min\{\rho, U_t\}, \qquad t > 0, \tag{6.2.139}$$

where U_t is the minimum of the total expected present discounted *cost* from rejecting both the price w and intervening quitting penalty ρ in (6.2.137(p.35)) and from rejecting the intervening quitting penalty ρ in (6.2.139(p.35)). Then, U_t can be expressed as

$$U_t = \min\{\mathtt{C}: \lambda\beta\,\mathbf{E}[v_{t-1}(\boldsymbol{\xi})] + (1-\lambda)\beta V_{t-1} + s,\ \mathtt{S}: \beta V_{t-1}\}, \quad t > 0. \tag{6.2.140}$$

Here, letting $U_0 = \rho$, from (6.2.138(p.35)) we have

$$V_0 = U_0 = \rho, \tag{6.2.141}$$

hence, both (6.2.137(p.35)) and (6.2.139(p.35)) hold true for $t \geq 0$ instead of $t > 0$, i.e.,

$$v_t(\xi) = \min\{\xi, \rho, U_t\}, \quad t \geq 0, \tag{6.2.142}$$

$$V_t = \min\{\rho, U_t\}, \qquad t \geq 0, \tag{6.2.143}$$

thus (6.2.137(p.35)) can be expressed as

$$v_t(\xi) = \min\{\xi, V_t\}, \quad t \geq 0. \tag{6.2.144}$$

Accordingly, since $v_{t-1}(\boldsymbol{\xi}) = \min\{\boldsymbol{\xi}, V_{t-1}\} = \mathbf{E}[\min\{\boldsymbol{\xi} - V_{t-1}, 0\}] + V_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 0$ from (5.1.11(p.23)), we can rewrite (6.2.140(p.35)) as follows.

$$\begin{aligned} U_t &= \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\lambda\beta\tilde{T}(V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &= \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.14(p.23)))} \qquad (6.2.145) \\ &= \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{\tilde{L}(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0 \quad \text{(see (5.1.14(p.23)) and (5.1.13(p.23)))}. \qquad (6.2.146) \end{aligned}$$

□ SOE{$\tilde{\text{M}}$:3[ℝ][A]} can be reduced to (6.2.141(p.35)), (6.2.143(p.35)), and (6.2.145(p.35)), listed in Table 6.5.5(p.39) (II). □

6.2.3.3 M:3[ℙ][A]

By v_t $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the maximums of the total expected present discounted *profit* from initiating the process at time t with a buyer and with no buyer respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a buyer at time $t = 0$ (deadline). Then, the seller must determine whether to accept the terminal quitting penalty ρ or to sell the asset to the buyer. Let the ρ be accepted. Then the profit which the seller can obtain is ρ. On the other hand, let the asset be sold to the buyer. Then, since the seller must sell the asset to the buyer due to A2(p.11), the price $a^{\dagger}$ must be proposed to the buyer, in other words, the optimal price to propose at time $t = 0$ is given by

$$z_0 = a, \tag{6.2.147}$$

hence the profit which the seller obtains at that time is a. Accordingly, the profit that the seller obtains at time 0 becomes

$$v_0 = \max\{\rho, a\}. \tag{6.2.148}$$

Next we have

$$v_t = \max\{\rho, H_t\}, \quad t > 0, \tag{6.2.149}$$

$$V_0 = \rho, \tag{6.2.150}$$

$$V_t = \max\{\rho, U_t\}, \quad t > 0, \tag{6.2.151}$$

where H_t and U_t are defined as follows. Firstly H_t is the maximum of the total expected present discounted *profit* from rejecting the intervening quitting penalty ρ. Since a buyer exists due to the above definition of v_t and since the reservation price (maximum permissible buying price) of the buyer is $\boldsymbol{\xi}$, if the seller proposes a price z, the probability of the buyer buying the asset is given by $p(z) = \Pr\{z \leq \boldsymbol{\xi}\}$ (see (5.1.18(p.24))). Hence we have

$$H_t \;=\; \max_z\{p(z)z + (1-p(z))V_t\} = \max_z p(z)(z - V_t) + V_t = T(V_t) + V_t, \quad t > 0 \tag{6.2.152}$$

due to (5.1.19(p.24)), implying that the optimal selling price z_t which the seller should propose is given by

$$z_t = z(V_t), \qquad t > 0, \tag{6.2.153}$$

due to (5.1.25(p.24)). Finally U_t is the maximum of the total expected present discounted *profit* from rejecting the intervening quitting penalty ρ. Since no buyer exists due to the above definition of V_t, it can be expressed as follows.

$$U_t = \max\{\,\texttt{C}\colon \lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} - s,\ \texttt{S}\colon \beta V_{t-1}\}, \quad t > 0. \tag{6.2.154}$$

For $t = 1$ we have

$$\begin{aligned} U_1 \;&=\; \max\{\lambda\beta v_0 + (1-\lambda)\beta V_0 - s, \beta V_0\} \\ &=\; \max\{\lambda\beta\max\{\rho, a\} + (1-\lambda)\beta\rho - s, \beta\rho\} \\ &=\; \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}. \end{aligned} \tag{6.2.155}$$

Now, from (6.2.152(p.36)) we have $H_t - V_t = T(V_t)$ for $t > 0$, hence from (6.2.149(p.36)) we have $v_t - V_t = \max\{\rho - V_t, H_t - V_t\} = \max\{\rho - V_t, T(V_t)\} \cdots (1)$ for $t > 0$. Since $V_t \geq \rho$ for $t > 0$ from (6.2.151(p.36)), we have $\rho - V_t \leq 0$ for $t > 0$. In addition, since $p(b) = 0$ due to (5.1.29 (2) (p.24)), from (5.1.19(p.24)) we have $T(V_t) \geq p(b)(b - V_t) = 0$. Therefore, since $\rho - V_t \leq 0 \leq T(V_t)$, from (1) we have $v_t - V_t = T(V_t)$ for $t > 0$, i.e., $v_t = T(V_t) + V_t$ for $t > 0$, hence $v_{t-1} = T(V_{t-1}) + V_{t-1}$ for $t > 1$. Accordingly (6.2.154(p.36)) with $t > 1^{\ddagger}$ can be rearranged as

$$\begin{aligned} U_t &= \max\{\lambda\beta(T(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{\lambda\beta T(V_{t-1}) + \beta V_{t-1} - s, \beta V_{t-1}\} \\ &= \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad (\text{see } (5.1.21(\text{p.}24))) \end{aligned} \tag{6.2.156}$$

$$\begin{aligned} &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1 \quad (\text{see } (5.1.21(\text{p.}24)) \text{ and } (5.1.20(\text{p.}24))). \end{aligned} \tag{6.2.157}$$

Here, letting $U_0 = \rho$, due to (6.2.150(p.36)) we have

$$V_0 = U_0 = \rho, \tag{6.2.158}$$

hence (6.2.151(p.36)) holds true for $t \geq 0$ instead of $t > 0$, i.e.,

$$V_t = \max\{\rho, U_t\}, \quad t \geq 0. \tag{6.2.159}$$

□ SOE{M:3[ℙ][A]} is given by (6.2.158(p.36)), (6.2.159(p.36)), (6.2.155(p.36)), and (6.2.156(p.36)), listed in Table 6.5.5(p.39) (III). ■

---

† The lower bound of the distribution function for the reservation price (the maximum permissible buying price) of the buyer
‡ Instead of $t > 0$.

6.2.3.4 $\tilde{\mathsf{M}}$:3[$\mathbb{P}$][A]

By v_t $(t \geq 0)$ and V_t $(t \geq 0)$ let us denote the minimums of the total expected present discounted *cost* from initiating the process at time t with a seller and with no seller respectively. In addition, let us denote the optimal price to propose at time $t \geq 0$ by z_t. Suppose there exists a seller at time $t = 0$ (deadline). Then, the buyer must determine whether to accept the terminal quitting penalty ρ or to buy the asset from the seller. Let the ρ be accepted. Then, the cost which the buyer pays at time 0 is ρ. On the other hand, let the asset be bought for the buyer. Then, since the buyer must buy the asset from the seller due to A2(p.11), the price b† must be is proposed to the seller; in other words, the optimal price to propose is given by

$$z_0 = b, \tag{6.2.160}$$

hence the cost which the buyer pays at that time is b. Accordingly, the buyer pays at time 0 becomes

$$v_0 = \min\{\rho, b\}. \tag{6.2.161}$$

Next we have

$$v_t = \min\{\rho, H_t\}, \quad t > 0. \tag{6.2.162}$$

$$V_0 = \rho, \tag{6.2.163}$$

$$V_t = \min\{\rho, U_t\}, \quad t > 0, \tag{6.2.164}$$

where H_t and U_t are defined as follows. Firstly H_t is the minimum of the total expected present discounted *cost* from rejecting the intervening quitting penalty ρ. Since a seller exists due to the above definition of v_t and since the reservation price (minimum permissible selling price) of the seller is $\boldsymbol{\xi}$, if the buyer proposes the price z to an appearing seller, the probability of the seller selling the asset for the price z is $\tilde{p}(z) = \Pr\{\boldsymbol{\xi} \leq z\}$ (see (5.1.31(p.24))). Hence we have

$$H_t \;=\; \min_z\{\tilde{p}(z)z + (1-\tilde{p}(z))V_t\} = \min_z \tilde{p}(z)(z - V_t) + V_t = \tilde{T}(V_t) + V_t, \quad t > 0, \tag{6.2.165}$$

due to (5.1.32(p.24)), implying that the optimal buying price which the buyer should pay is given by

$$z_t = z(V_t), \qquad t \geq 0, \tag{6.2.166}$$

due to (5.1.38(p.25)). Finally U_t is the minimum of the total expected present discounted *cost* from rejecting the intervening quitting penalty ρ. Since no seller exists due to the above definition of V_t, it can be expressed as follows.

$$U_t \;=\; \min\{\texttt{C}\colon \lambda\beta v_{t-1} + (1-\lambda)\beta V_{t-1} + s,\ \texttt{S}\colon \beta V_{t-1}\}, \quad t > 0. \tag{6.2.167}$$

For $t = 1$ we have

$$\begin{aligned} U_1 \;&=\; \min\{\lambda\beta v_0 + (1-\lambda)\beta V_0 + s, \beta V_0\} \\ &=\; \min\{\lambda\beta \min\{\rho, b\} + (1-\lambda)\beta\rho + s, \beta\rho\} \\ &=\; \min\{\lambda\beta \min\{0, b-\rho\} + \beta\rho + s, \beta\rho\}. \end{aligned} \tag{6.2.168}$$

Now, from (6.2.165(p.37)) we have $H_t - V_t = \tilde{T}(V_t)$ for $t > 0$, hence from (6.2.162(p.37)) we have $v_t - V_t = \min\{\rho - V_t, H_t - V_t\} = \min\{\rho - V_t, \tilde{T}(V_t)\} \cdots (2)$ for $t > 0$. Since $V_t \leq \rho$ for $t > 0$ from (6.2.164(p.37)), we have $\rho - V_t \geq 0$ for $t > 0$. In addition, since $\tilde{p}(a) = 0$ due to (5.1.41 (1) (p.25)), from (5.1.32(p.24)) we have $\tilde{T}(V_t) \leq \tilde{p}(a)(a - V_t) = 0$. Therefore, since $\rho - V_t \geq 0 \geq \tilde{T}(V_t)$, from (2) we have $v_t - V_t = \tilde{T}(V_t)$ for $t > 0$, i.e., $v_t = \tilde{T}(V_t) + V_t$ for $t > 0$, hence $v_{t-1} = \tilde{T}(V_{t-1}) + V_{t-1}$ for $t > 1$. Accordingly (6.2.167(p.37)) with $t > 1$ can be rearranged as

$$\begin{aligned} U_t \;&=\; \min\{\lambda\beta(\tilde{T}(V_{t-1}) + V_{t-1}) + (1-\lambda)\beta V_{t-1} + s, \beta V_{t-1}\} \\ &=\; \min\{\lambda\beta\tilde{T}(V_{t-1}) + V_{t-1}) + \beta V_{t-1} + s, \beta V_{t-1}\} \\ &=\; \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.25)))} \end{aligned} \tag{6.2.169}$$

$$\begin{aligned} &=\; \min\{\tilde{K}(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1}, \quad t > 1 \\ &=\; \max\{\tilde{L}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} \quad \text{(see (5.1.34(p.25)) and (5.1.33(p.25)))} \end{aligned} \tag{6.2.170}$$

Here, letting $U_0 = \rho$, due to (6.2.163(p.37)) we have

$$V_0 = U_0 = \rho, \tag{6.2.171}$$

hence (6.2.164(p.37)) holds true for $t \geq 0$ instead of $t > 0$, i.e.,

$$V_t = \min\{\rho, U_t\}, \quad t \geq 0. \tag{6.2.172}$$

□ SOE{$\tilde{\mathsf{M}}$:3[$\mathbb{R}$][A]} is given by (6.2.171(p.37)), (6.2.172(p.37)), (6.2.168(p.37)), and (6.2.169(p.37)), listed in Table 6.5.5(p.39) (IV). ■

---

† The upper bound of the distribution function for the reservation price (the minimum permissible selling price) of the seller.

6.3 Search-Enforced-Model

In search-**E**nforced-model ($\mathsf{M}{:}x[\mathbb{X}][\mathtt{E}]$ and $\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{E}]$ with $x = 1, 2, 3$ and $\mathbb{X} = \mathbb{R}, \mathbb{P}$), a leading-trader needs to take no decision activity regarding whether or not to conduct the search. This implies that eliminating the terms related to this decision from the systems of optimality equations in search-**A**llowed-model ($\mathtt{SOE}\{\mathsf{M}{:}x[\mathbb{X}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{A}]\}$) produces the systems of optimality equations in search-**E**nforced-model ($\mathtt{SOE}\{\mathsf{M}{:}x[\mathbb{X}][\mathtt{E}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}x[\mathbb{X}][\mathtt{E}]\}$). Noting this, from Tables 6.5.1(p.39), 6.5.3(p.39), and 6.5.5(p.39) we can immediately obtain the systems of optimality equations for search-**E**nforced-model, which are given by Tables 6.5.2(p.39), 6.5.4(p.39), and 6.5.6(p.39).

6.4 Assertion and Assertion System

In general, let us refer to a description on whether or not a given statement is true as the *assertion*, denoted by A, and as a set consisting of some assertions as the *assertion system*, denoted by $\mathscr{A}$. In addition, let us denote an assertion and an assertion system for a given Model by respectively $A\{\mathsf{Model}\}$ and $\mathscr{A}\{\mathsf{Model}\}$.

6.5 Summary of the System of Optimality Equations (SOE)

Model 1

Table 6.5.1: **Search-Allowed-Model** 1

(I) $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

$$V_1 = \beta\mu - s, \tag{6.5.1}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.2}$$

(II) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

$$V_1 = \beta\mu + s, \tag{6.5.3}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.4}$$

(III) $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$

$$V_1 = \beta a - s, \tag{6.5.5}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.6}$$

(IV) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

$$V_1 = \beta b + s, \tag{6.5.7}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.8}$$

Table 6.5.2: **Search-Enforced-Model** 1

(I) $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}$

$$V_1 = \beta\mu - s, \tag{6.5.9}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.10}$$

(II) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]\}$

$$V_1 = \beta\mu + s, \tag{6.5.11}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.12}$$

(III) $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}$

$$V_1 = \beta a - s, \tag{6.5.13}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.14}$$

(IV) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]\}$

$$V_1 = \beta b + s, \tag{6.5.15}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.16}$$

Model 2

Table 6.5.3: **Search-Allowed-Model** 2

(I) $\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$

$$V_0 = \rho, \tag{6.5.17}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.18}$$

(II) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$

$$V_0 = \rho, \tag{6.5.19}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.20}$$

(III) $\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$

$$V_0 = \rho, \tag{6.5.21}$$
$$V_1 = \max\{\lambda\beta\max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \tag{6.5.22}$$
$$V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.23}$$

(IV) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$

$$V_0 = \rho, \tag{6.5.24}$$
$$V_1 = \min\{\lambda\beta\min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \tag{6.5.25}$$
$$V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.26}$$

Table 6.5.4: **Search-Enforced-Model** 2

(I) $\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{E}]\}$

$$V_0 = \rho, \tag{6.5.27}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 0, \tag{6.5.28}$$

(II) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{E}]\}$

$$V_0 = \rho, \tag{6.5.29}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 0, \tag{6.5.30}$$

(III) $\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{E}]\}$

$$V_0 = \rho, \tag{6.5.31}$$
$$V_1 = \lambda\beta\max\{0, a - \rho\} + \beta\rho - s, \tag{6.5.32}$$
$$V_t = K(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.33}$$

(IV) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{E}]\}$

$$V_0 = \rho, \tag{6.5.34}$$
$$V_1 = \lambda\beta\min\{0, b - \rho\} + \beta\rho + s, \tag{6.5.35}$$
$$V_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1, \tag{6.5.36}$$

Model 3

Table 6.5.5: **Search-Allowed-Model** 3

(I) $\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.37}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.38}$$
$$U_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.39}$$

(II) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.40}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.41}$$
$$U_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 0. \tag{6.5.42}$$

(III) $\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.43}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.44}$$
$$U_1 = \max\{\lambda\beta\max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \tag{6.5.45}$$
$$U_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.46}$$

(IV) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.47}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.48}$$
$$U_1 = \min\{\lambda\beta\min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \tag{6.5.49}$$
$$U_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1. \tag{6.5.50}$$

Table 6.5.6: **Search-Enforced-Model** 3

(I) $\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.51}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.52}$$
$$U_t = K(V_{t-1}) + V_{t-1},\ t > 0. \tag{6.5.53}$$

(II) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.54}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.55}$$
$$U_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 0. \tag{6.5.56}$$

(III) $\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{E}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.57}$$
$$V_t = \max\{\rho, U_t\},\ t \geq 0, \tag{6.5.58}$$
$$U_1 = \lambda\beta\max\{0, a - \rho\} + \beta\rho - s, \tag{6.5.59}$$
$$U_t = K(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.60}$$

(IV) $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{E}]\}$

$$V_0 = U_0 = \rho, \tag{6.5.61}$$
$$V_t = \min\{\rho, U_t\},\ t \geq 0, \tag{6.5.62}$$
$$U_1 = \lambda\beta\min\{0, b - \rho\} + \beta\rho + s, \tag{6.5.63}$$
$$U_t = \tilde{K}(V_{t-1}) + V_{t-1},\ t > 1. \tag{6.5.64}$$

Chapter 7

Optimal Decision Rules

7.1 Points in Time

This section presents the structure of the optimal decision rules for the 24 models in Table 3.2.1(p.17). Before that, let us note here that the optimal decision rules for these models are closely related to the following four points in time (see H1(p.8)).

(1) *Recognizing time* t_r,
(2) *Starting time* τ,
(3) *Initiating time* t_i,
(4) *Deadline* δ, the terminal (final) time point of the asset trading problem (see H1c(p.8)).

In H1(p.8), letting $t_r = 0$, we numbered the time points as $t_r = 0, 1, 2, \cdots, \delta$ toward the deadline δ (see Figure 1.5.1(p.8)). On the contrary, in the whole discussions that follows throughout this paper, as shown in Figure 7.1.1(p.41) below, letting $\delta = 0$, we renumber the time points as t_r, $t_r - 1$, $\cdots$, 2, 1, $0 = \delta$ toward the starting time t_τ. It will be known later that this numbering is convenient in describing the system of optimality equations and analyzing problems.

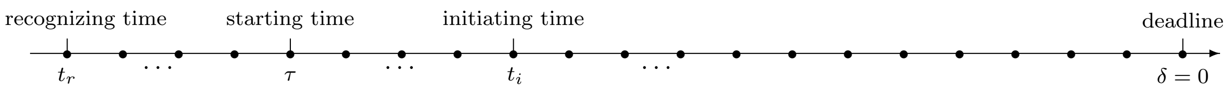


Figure 7.1.1: Four points in time

As seen from the above figure we have

$$t_r \geq \tau \geq t_i \geq \delta = 0. \tag{7.1.1}$$

In order to further move on, we must define the following point in time.

(5) *Quasi deadline* δ_q.

a. In Model 1, if the initiating time $t_i = 0$, then since there exists no buyer at time 0, the process must stop without selling the asset at that time, which contradicts A2(p.11), hence this is not is permissible, thus we have $t_i > 0$. Then, let us define $\delta_q = \min\{t_i \mid t_i > 0\} = 1$.

b. In Models 2/3, if $t_i = 0$, there exists no buyer at time 0; however, the process can stop by accepting the terminal quitting penalty price ρ at that time, hence this is permissible, thus we have $t_i \geq 0$. Then let us define $\delta_q = \min\{t_i \mid t_i \geq 0\} = 0$.

The δ_q defined above is the smallest of all conceivable initiation times t_i's, called the *quasi deadline*. Then we have

$$\tau \geq t_i \geq \delta_q. \tag{7.1.2}$$

The above five points times (1-5) can be schematized as the figures below.

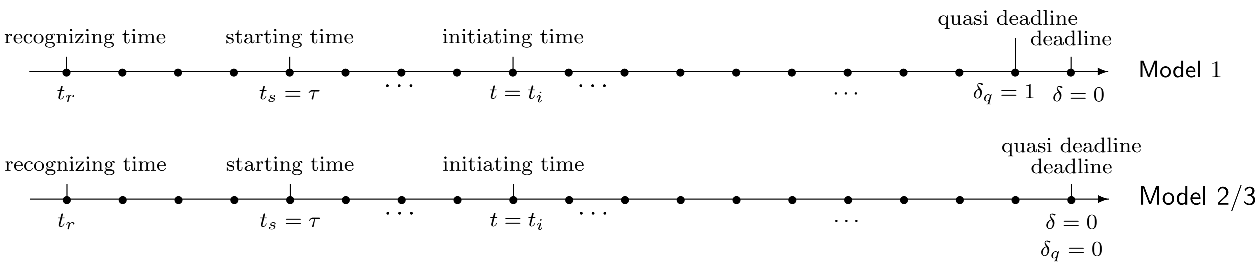


Figure 7.1.2: Six time points in the asset trading process

7.2 Four Types of Decisions Rules

Below let us provide the strict definitions for the five decision rules prescribed in Section 3.4(p.18).

7.2.1 Whether or Not to Accept the Proposed Price

This is the decision only for $\mathbb{R}$-mechanism-model. Below let us represent

$$\text{Accept a price } \xi \text{ at time } t \text{ by } \texttt{Accept}_t\langle\xi\rangle, \tag{7.2.1}$$

$$\text{Reject a price } \xi \text{ at time } t \text{ by } \texttt{Reject}_t\langle\xi\rangle. \tag{7.2.2}$$

First, in the selling model, if a buyer appearing at a time t has proposed a buying price ξ, then from (6.2.2(p.27)) we have

$$\xi \geq (\leq)\, V_t \Rightarrow \texttt{Accept}_t\langle\xi\rangle \;\; (\texttt{Reject}_t\langle\xi\rangle). \tag{7.2.3}$$

Similarly, in the buying model, if a seller appearing at a time t has proposed a selling price ξ, then from (6.2.16(p.28)) and (6.2.77(p.32)) we have

$$\xi \leq (\geq)\, V_t \Rightarrow \texttt{Accept}_t\langle\xi\rangle \;\; (\texttt{Reject}_t\langle\xi\rangle).$$

Then, we refer to the V_t as the `optimal-reservation-price`, opt-$\mathbb{R}$-price for short.

7.2.2 What Price to Propose

This is the decision only for $\mathbb{P}$-mechanism-model. In the selling model, the optimal selling price which a seller (leading-trader) should propose at a time t is given by

$$z_t = z(V_t) \quad \text{(see (6.2.34(p.29)) and (6.2.94(p.33)))}.$$

Similarly, in the buying model, the optimal buying price which a buyer (leading-trader) should propose at a time t is given by

$$z_t = z(V_t) \quad \text{(see (6.2.50(p.30)) and (6.2.111(p.34)))}.$$

Then, we refer to the z_t as the `optimal-posted-price`, opt-$\mathbb{P}$-price for short.

7.2.3 Whether or not to Conduct the Search

This is the decision only for the search-**A**llowed-model (see (A5b(p.12))). Then, its decision rule is given by (6.2.9(p.28)), (6.2.23(p.29)), (6.2.39(p.30)), (6.2.55(p.30)), (6.2.70(p.31)), (6.2.85(p.32)), (6.2.102(p.33)), and (6.2.119(p.34)).

Remark 7.2.1 (`Conduct⇝Skip` (`C⇝S`) (see Figure 2.2.3(p.12))) Consider Model 1. Figure 7.2.1(p.42) (I) below sketches the case that the search-Conduct starts at the optimal initiating time t^*_τ and continue up to the quasi-deadline $\delta_q = 1$ so long as the process does not stop. Contrary to this, Figure 7.2.1(p.42) (II) schematizes the case that the search-Conduct starts at the optimal initiating time t^*_τ, continues for a while, and switches to the search-Skip at a certain point in time $t' > \delta_q = 1$; it will be known later on that this is possible in fact although being a very rare case. Let us represent the case as `Conduct⇝Skip`, simply `C⇝S` (Def. 2.2.1(p.12)). □

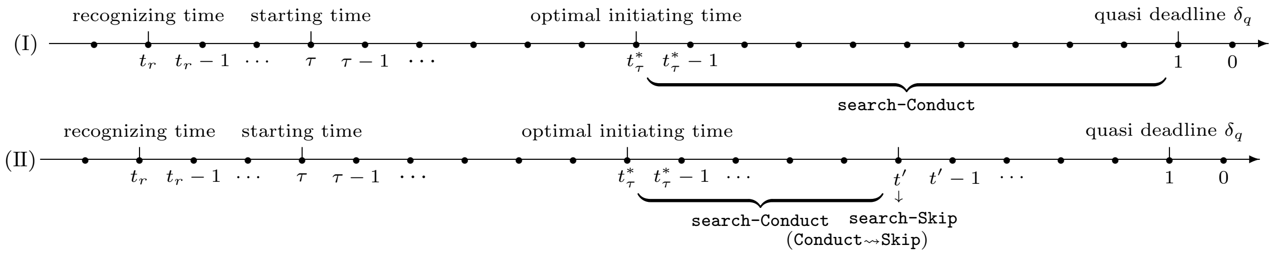


Figure 7.2.1: `Conduct⇝Skip` (`C⇝S`)

♡ Alice **2 (jumble of intuition and theory)** *Herein, Alice was hit by the following question. Suppose that* $\mathbb{S}_t < 0$ *at a time* t (see (6.2.12(p.28))), *i.e., the search-skip becomes strictly optimal at that time* ($\texttt{Skip}_{t\blacktriangle}$). *Then, since* $\max\{\mathbb{S}_t, 0\} = 0$, *we have* $V_t = \beta V_{t-1}$ *from* (6.2.8(p.28)), *implying that initiating the process at time t becomes indifferent to initiating the process at time* $t-1$, *nevertheless, the search skip is strictly optimal! After having mumbled, letting out a strange noise "Is this a little bit funny ?", she gave a shout "Such a laughable affair !". Then, Dr. Rabbit again appeared and pedantically told to Alice "The above two results are both ones based on a theory of mathematics, but your confusion is caused by an intuition; there does not exist any logical relationship between the two! Well··· your confusion is what is caused by a jumble of intuition and theory!!", and he again disappeared down the hole as murmuring "Oh dear! Oh dear! I shall be too late for the faculty meeting!".* ▮

7.2.4 When to Initiate the Process (Optimal Initiating Time)

Below let us consider only a selling problem (For a buying problem it suffices to change the direction of related inequalities and to change "max" into "min").

7.2.4.1 Optimal Initiating Time

Suppose that the process has reached the starting time τ and the seller (leading-trader) has determined to *initiate* the process at a given initiating time t_i after that ($\tau \geq t_i \geq \delta_q$), i.e., $\tau - t_i$ periods hence. Then, the total expected present discounted profit at the *starting time* τ is given by

$$I_\tau^{t_i} \overset{\text{def}}{=} \beta^{\tau-t_i} V_{t_i}, \quad \tau \geq t_i \geq \delta_q. \tag{7.2.4}$$

See (6.2.3(p.27)) and (6.2.4(p.27)) for the definition of V_{t_i}. Then, by t_τ^* let us denote t maximizing $I_\tau^{t_i}$ on $\tau \geq t_i \geq \delta_q$, i.e.,

$$I_\tau^{t_\tau^*} = \max_{\tau \geq t_i \geq \delta_q} I_\tau^{t_i} \quad \text{or equivalently} \quad I_\tau^{t_\tau^*} \geq I_\tau^{t_i}, \quad \tau \geq t \geq \delta_q. \tag{7.2.5}$$

Let us call the t_τ^* the *optimal initiating time*, denoted by $\texttt{OIT}_\tau\langle t_\tau^*\rangle_\vartriangle$. If

$$I_\tau^{t_\tau^*} > I_\tau^{t_i} \text{ for } t_i \neq t_\tau^*, \tag{7.2.6}$$

then it is called the <u>*strictly*</u> *optimal initiating time*, denoted by $\texttt{OIT}_\tau\langle t_\tau^*\rangle_\blacktriangle$.

Remark 7.2.2 (strict optimality ▲) Suppose that the initiating time t_τ^* is <u>strictly</u> optimal. Then, since $I_\tau^{t_\tau^*} > I_\tau^{t_\tau^*-1}$, we have $\beta^{\tau-t_\tau^*} V_{t_\tau^*} > \beta^{\tau-t_\tau^*+1} V_{t_\tau^*-1}$, hence $V_{t_\tau^*} > \beta V_{t_\tau^*-1}$. Now, since $V_{t_\tau^*} = \max\{\mathbb{S}_{t_\tau^*}, 0\} + \beta V_{t_\tau^*-1}$ from (6.2.8(p.28)) with $t = t_\tau^*$, we have $\max\{\mathbb{S}_{t_\tau^*}, 0\} > 0$, hence $\mathbb{S}_{t_\tau^*} > 0$. Accordingly, we have $\texttt{Conduct}_{t\blacktriangle}$ due to (6.2.12(p.28)), i.e., it becomes <u>*strictly*</u> optimal to conduct the search, or equivalently, it follows that it is not allowed to skip the search. □

Throughout the paper, let us employ the following preference rule.

Preference Rule **7.2.1** *Let $I_\tau^t = I_\tau^{t-1}$ for a given t. Then, the seller (leading-trader) prefers $t-1$ to t as the initiating time, implying that "Postpone the initiation of the process so long as it is not unprofitable to do so."* □

7.2.4.2 β-adjusted sequence $V_{\beta[\tau]}$

First, let us denote the sequence consisting of V_τ, $V_{\tau-1}$, $V_{\tau-2}$, $\cdots$, V_{δ_q} by

$$V_{[\tau]} \overset{\text{def}}{=} \{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{\delta_q}\}, \tag{7.2.7}$$

called the *original sequence* and let

$$t_\tau^{*\prime} = \arg\max V_{[\tau]} = \arg\max\{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{\delta_q}\}. \tag{7.2.8}$$

Next, let us denote the sequence

$$V_{\beta[\tau]} \overset{\text{def}}{=} \{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^{\tau-\delta_q} V_{\delta_q}\} = \{I_\tau^\tau, I_\tau^{\tau-1}, I_\tau^{\tau-2}, \cdots, I_\tau^{\delta_q}\}, \tag{7.2.9}$$

called the *β-adjusted sequence* of $V_{[\tau]}$. By definition, the optimal initiating time t_τ^* is given by t attaining the maximum of elements within *β-adjusted sequence* $V_{\beta[\tau]}$, i.e.,

$$t_\tau^* = \arg\max V_{\beta[\tau]} = \arg\max\{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^{\tau-\delta_q} V_{\delta_q}\}. \tag{7.2.10}$$

Note here that the monotonicity of the original sequence $V_{[\tau]}$ is not always inherited to the β-adjusted sequence $V_{\beta[\tau]}$ (see Section A 5.2.2(p.257)).

7.2.4.3 Three Possibilities

Below let us define the three types of the optimal initiating time ($\texttt{OIT}$).

(A) **Degeneration to the starting time τ**

Let $t_\tau^* = \tau$, i.e., it is optimal to initiate the process at the starting time τ, denoted by Ⓢ. Then, the optimal initiating time t_τ^* is said to *degenerate* to the *starting time* τ, represented by $Ⓢ_\tau\langle\tau\rangle_\vartriangle$ ($Ⓢ_\vartriangle$ for short). If the optimal initiating time t_τ^* is *strict* (see (7.2.6(p.43))), it is called the *strictly degenerate* $\texttt{OIT}$, represented by $Ⓢ_\tau\langle\tau\rangle_\blacktriangle$ ($Ⓢ_\blacktriangle$ for short).

(B) **Non-degeneration ($\tau > t_\tau^* > \delta_q$)**

Let $\tau > t_\tau^* > \delta_q$, i.e., the optimal initiating time is between the starting time τ and the quasi deadline δ_q, denoted by ◎. Then, the optimal initiating time t_τ^* is said to be the *non-degenerate* $\texttt{OIT}$, represented by $◎_\tau\langle t_\tau^*\rangle_\vartriangle$ ($◎_\vartriangle$ for short). If

$$I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^{t_\tau^*} \geq I_\tau^{\delta_q}, \tag{7.2.11}$$

then it is said to be the *indifferently* non-degenerate $\texttt{OIT}$ (see Preference Rule 7.2.1(p.43)), represented by $◎_\tau\langle t_\tau^*\rangle_\parallel$ ($◎_\parallel$ for short). If $I_\tau^{t_\tau^*} > I_\tau^{t_i}$ for all $t_i \neq t_\tau^*$, then it is said to be the *strictly* non-degenerate $\texttt{OIT}$, represented by $◎_\tau\langle t_\tau^*\rangle_\blacktriangle$ ($◎_\blacktriangle$ for short).

(C) **Degeneration to the deadline δ_q**

Let $t^*_\tau = \delta_q = 1\,(0)$ for Model 1 (Model 2/3), i.e., the optimal initiating time is the quasi deadline δ_q, denoted by ⓓ. Then, the optimal initiating time t^*_τ is said to *degenerate* to the *quasi deadline* δ_q, represented by ⓓ$_\tau\langle\delta_q\rangle_\triangle$ (ⓓ$_\triangle$ for short). If its optimality is *strict*, then it is called the *strictly degenerate* OIT, represented by ⓓ$_\tau\langle\delta_q\rangle_\blacktriangle$ (ⓓ$_\blacktriangle$ for short). If

$$I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^{\delta_q}_\tau, \cdots (1)$$

then the degeneration is said to be *indifferent*, represented by ⓓ$_\tau\langle\delta_q\rangle_\parallel$ (ⓓ$_\parallel$ for short).

7.2.4.4 Null-Time-Zone

In this section let us raise a *perplexing* question caused by the optimal initiating time t^*_τ. Here, let $\tau > t^*_\tau$, i.e., the optimal initiating time t^*_τ is not the starting time τ (see Figure 7.2.2(p.44) below), implying that no decision-making action is taken at every point in time $t = \tau, \tau-1, \cdots, t^*_\tau+1$. Let us refer to each of $\tau, \tau-1, \cdots, t^*_\tau+1$ as the *null time point* and the whole of these time points as the *null-time-zone*, denoted as `Null-TZ`.

$$\texttt{Null-TZ} \stackrel{\text{def}}{=} \langle \tau, \tau-1, \cdots, t^*_\tau+1\rangle.$$

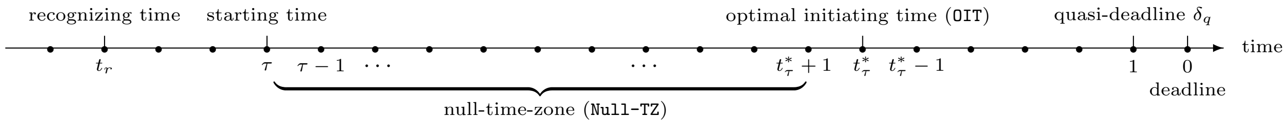


Figure 7.2.2: Null-time-zone in Model 1 with $\delta_q = 1$ (Null-TZ)

The above event implies that, if not noticing the existence of `Null-TZ`, we *unwittingly* or *unconsciously* might have continued to fall into the senselessness of engaging in *unnecessary decision-making activities* over these points in time.

7.2.4.5 Deadline-Engulfing

♡ Alice **3 (black hole)** *Hereupon, Alice supposed "If the optimal initiating time t^*_τ degenerates to the deadline (time 0), then what will ever happen ?", and screamed out "If so, it follows that don't conduct any decision-making activity up to the deadline !; If that happens, the whole of decision-making activities which are scheduled at the starting time τ come to nought as if being engulfed in the deadline !". Alice was heavily nonplused and cried "It ···, it is the same as that black hole into which all physical matters, even light, are squeezed into ! If so, ···, a decision process with an infinite planning horizon vanishes away in time toward an infinite future ! Oh dear!! Oh dear !!! ··· " She hunkered down, and then buried her head in her hands. Then, Dr. Rabbit again appeared and told to her a little bit ungraciously "This is an undeniable conclusion that is theoretically derived !," and he disappeared down the hole as murmuring "Oh dear! Oh dear! I shall be too late for the faculty meeting !"* ▮

□ *Example* **7.2.1** We showed in Tom 20.2.4(p.193) (d2i) that ⓓ$_{\tau>0}\langle 0\rangle_\blacktriangle$ (ⓓ$_\blacktriangle$) occurs in fact under the condition of "$\beta < 1$, $s > 0$, $\rho > x_K$, $\rho > x_L$, $\kappa \le 0$". □

□ *Example* **7.2.2** We demonstrated in Pom 20.2.1(p.194) (b) that ⓓ$_{\tau>0}\langle 0\rangle_\parallel$ (ⓓ$_\parallel$) occurs in fact under the condition of "$a > 0$, $\beta = 1$, $s = 0$, and $\rho \ge b$". What should be noted here is that this event is possible even on the simplest condition of $\beta = 1$ and $s = 0$. □

In this paper, let us refer to "the event of being engulfed in the deadline" as "*deadline engulfing*", represented by ⓓ-engulfing. This situation can be depicted as the two figures below.

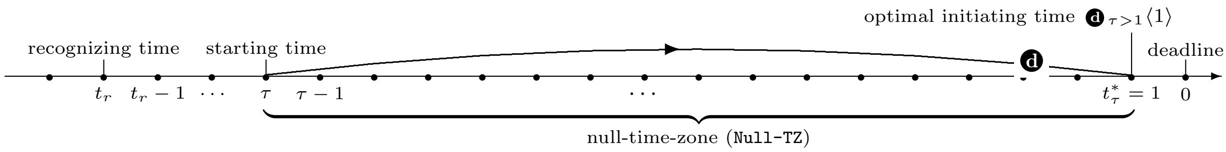


Figure 7.2.3: Deadline engulfing (ⓓ) for Model 1

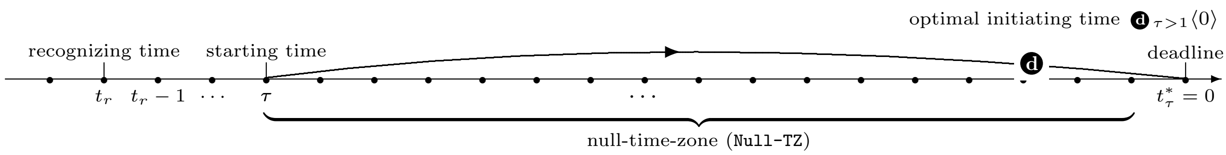


Figure 7.2.4: Deadline engulfing (ⓓ) for Model 2/3

Later on we will demonstrate that the ⓓ-engulfing is not a rare event but a phenomenon which is very often possible; amazingly, it can occur even in the simplest case "$\beta = 1$ and $s = 0$" (see Pom's 20.2.1(p.194), 20.2.5(p.197), 20.2.9(p.207), and 20.2.17(p.213)). Taking this fact into consideration, we will inevitably be led to the re-examination and rewriting of the whole discussion that have been conventionally made so far for all decision processes, including Markovian decision processes [5,Howard] (see Section A 5(p.256)).

7.3 Mental Conflicts

Below consider only a selling problem (For a buying problem, it suffices to change the direction of its monotonicity). In addition, below, by opt-$\mathbb{R}$/$\mathbb{P}$-price (V_t/z_t) let us represent the collective term of

opt-$\mathbb{R}$-price (V_t) (`optimal-reservation-price` (see Section 7.2.1(p.42))) (7.3.1)
opt-$\mathbb{P}$-price (z_t) (`optimal-posted-price` (see Section 7.2.2(p.42))). (7.3.2)

One of our main concerns on the opt-$\mathbb{R}$/$\mathbb{P}$-price (V_t/z_t) is its monotonicity.

7.3.1 Normality

Suppose that the monotonicity over the entire planning horizon is

- nondecreasing in t (see Figure 7.3.1(p.45) (I)) or
- nonincreasing in t (see Figure 7.3.1(p.45) (II)).

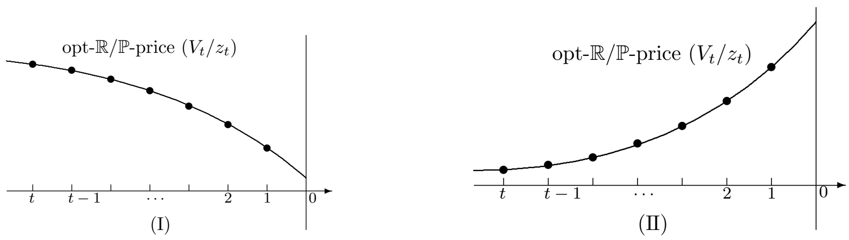


Figure 7.3.1: Normal Mental Conflict

Remark 7.3.1 (normal mental conflict) The above monotonicity of the opt-$\mathbb{R}$/$\mathbb{P}$-price reflects the mental conflict of decision-maker that was presented within *Example*s 1.4.1(p.6) - 1.4.4(p.6). This mental conflict can be restated as follows. As the deadline approaches,

- a seller becomes "selling spree" in the selling problem.
- a buyer becomes "buying spree" in the buying problem.

Let us refer to this mental conflict as the *normal mental conflict*. □

7.3.2 Abnormality

Suppose that the monotonicity over the entire planning horizon shifts

- from "nondecreasing" to "nonincreasing" in t (see Figure 7.3.2(p.45) (I)) or
- from "nonincreasing" to "nondecreasing "in t (see Figure 7.3.2(p.45) (I)).

Remark 7.3.2 (abnormal mental conflict) The above monotonicity of the opt-$\mathbb{R}$/$\mathbb{P}$-price reflects the mental conflict stated below. As the deadline approaches

- A seller shift from "selling spree" to "buying spree"in the selling problem.
- A buyer shift from "buying spree" to "selling spree" in the buying problem.

Let us refer to this mental conflict as the *abnormal mental conflict*. □

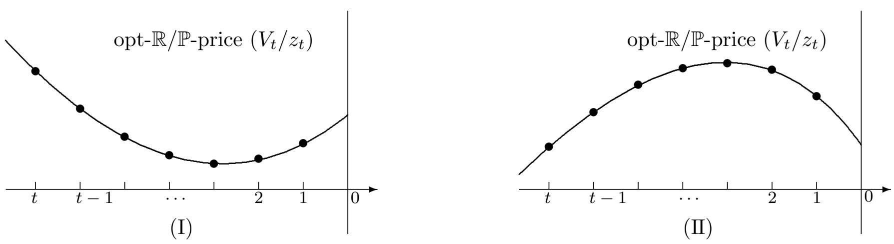


Figure 7.3.2: Abnormal Mental Conflict

Chapter 8

Conclusions of Part 1 (Introduction)

The whole discussions over Chaps. 1(p.3) - 7(p.41) are summarized as below.

C1. Two motives

Behavior of human-beings, whether a little action or a significant one, often starts with subtle motives. Also in an early stage of this study, the authors observed similarities in definitions between selling problem and buying problem as well as resemblances in logics between methodologies used to analyze the two problems. This observation led us, before long, to the motives with the following two questions (see Section 1.2(p.4)): (1) *Is a buying problem always symmetrical to a selling problem ?* and (2) *Does a general theory integrating quadruple-asset-trading-problems exist ?* This study, spanning over near half a century, was inspired by the desire to answer the above two questions. Our final conclusions are "No" for (1) and "Yes" for (2).

C2. Philosophical background

Refer to Section 1.3(p.4) for the philosophical background of "how and why we came to perceive *a decision theory as physics*". For further deep implication of this philosophy, see C2(p.233).

C3. Structuration of problems

Before delving into the core of the study, we clarified the general structure of asset trading problems (see Section 1.4(p.5)), which gave rise to the concepts of the quadruple-asset-trading-problems (see Section 1.4.5(p.7)) and the structured-unit-of-problems (see Section 3.3(p.18)). One of the key points in this paper is not to *discretely* and *individualistically* analyze problems included in the structured-unit-of-problems but to *systematically* and *comprehensively* examine the *interconnectedness* among these problems by using the integrated theory in Part 2(p.49).

C4. Assumptions

In Section 2.2(p.11) we presented the eleven assumptions, A1(p.11) - A11(p.13), which become necessary for providing strict definitions of all models for asset trading problems included in the structured-unit-of-problems Table 3.2.1(p.17). The two of them, A5(p.11) (search-Enforced-model and search-Allowed-model) and A7(p.12) (quitting penalty price), are all what are first introduced in this paper. The former, A5(p.11), is introduced from the realistic requirement and the latter, A7(p.12), is inevitably configured from the assumption $\lambda < 1$.

C5. Underlying functions.

The systems of optimality equations (see Chap. 6(p.27)) for all models in Table 3.2.1(p.17) are expressed by using the underlying functions T, L, K, and $\mathcal{L}$ (see Chap. 5(p.23)). Now, the function T has been often defined and used thus far in the fields of mathematical statistics, operational research, and economics (see [1,DeGroot]). However, the introduction of the remaining functions L, K, and $\mathcal{L}$ (see (5.1.3(p.23)) - (5.1.5(p.23)))) is presumably first in this paper. The properties of these functions are consistently utilized in the analyses of these models. All properties of them (see Lemmas 10.1.1(p.53) - 10.3.1(p.57)) were derived through the repeated arrangement and rearrangement, as if solving a jigsaw puzzle, of many results that were obtained from the mathematical analyses of various models over more than ten years.

C6. Optimal initiating time

Guided by the philosophical background in C2(p.47), we came to regard *human beings* as real entities that scientists study as their research objects, and an unconscious recognition that there is no physical existence devoid of the time concept led us to the four time points: *recognizing time*, *starting time*, *initiating time*, and *deadline* (see H1(p.8) and Section 7.1(p.41)). Especially noteworthy one among the above four time point is the *initiating time*, which leads us to the *optimal initiating time* (`OIT`) (see Section 7.2.4.1(p.43)). This yields three kinds of time points: starting time (Ⓢ), non-degenerate time (◎), and deadline (ⓓ) (see Section 7.2.4.3(p.43)).

C7. Null-time-zone and deadline-engulfing

The optimal initiating times ◎ and ⓓ inevitably gives rise to the events of *null-time-zone* (see Section 7.2.4.4(p.44)) and *deadline-engulfing* (see Sections 7.2.4.5(p.44)). What is furthermore remarkable is that the existence of the two optimal initiating times are not rare but rather frequent (see 22.2% and 33.4% in Table 22.0.1(p.229)). Moreover, it should be also emphasized that ◎$_{\blacktriangle}$ and ⓓ$_{\blacktriangle}$ (strictly optimal) can occur although at the very small rates of 2.6% and 3.2% respectively (see Table 22.0.1(p.229)). Lastly, note that the existence of the above two events suggests the need for a comprehensive re-examination and rewriting of all results derived in the conventional investigations of decision processes without incorporating the concept of the optimal initiating time.

C8. Mental conflict

As illustrated in *Examples* 1.4.1(p.6) - 1.4.4(p.6) , although the *normal* mental conflict experienced by a leading-trader (see Remark 7.3.1(p.45)) can be intuitively understood, the *abnormal* mental conflict (see Remark 7.3.2(p.45)) is hard to immediately grasp, which is possible (see C1b2(p.229)).

C9. **Discount factor for cost**

While the interpretation concerning the economic implication of the interest rate r and the discount factor β for *profit* is very easy and simple, surprisingly enough, to the best of the authors' knowledge, we have no reference which gives a persuasive explanation for the implications of the interest rate r and the discount factor β for *cost*. We provided a clear interpretation for this issue in Section 2.3(p.13) .

Part 2

Integrated Theory

In Chap. 9(p.51) , we provide the bird's-eye view of the flow for constructing the general theory that integraties selling problems and buying problems based on concepts of symmetry and analogy. As its preliminary step, in Chap. 10(p.53) the properties of underlying functions is clarify and in Chaps. 11(p.59) - 15(p.109) the five steps which become necessary for constructing the integrated theory are presented. In Chap. 16(p.113) , the above flow is schematized by a graphic chart. In Chap. 17(p.115) the concept of market restriction is introduced and in Chap. 18(p.127) the essential points in this part are summarized.

Chapter 9

Overview

9.1 Bird's-Eye View of Integrated Theory

Figure 9.1.1(p.51) below provides a bird's-eye view of the flow of discussions which constructs the integrated theory in this Part.

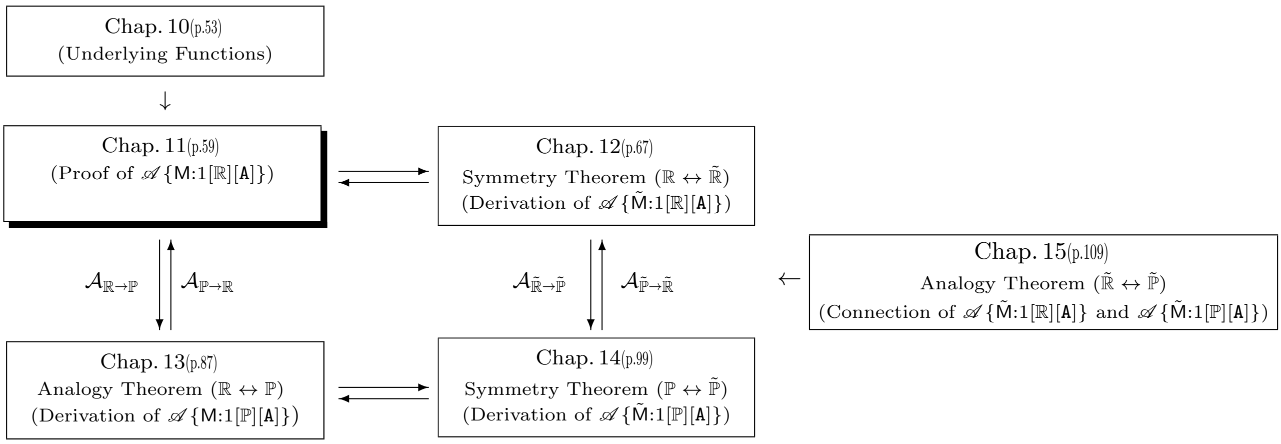


Figure 9.1.1: The flow of the construction of the integrated theory

The above figure presents the following:

- In Chap. 10(p.53), lemmas and corollaries for underlying functions are proven.
- In Chap. 11(p.59), $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is proven by using the results in Chap. 10(p.53).
- In Chap. 12(p.67), the symmetry theorem ($\mathbb{R} \leftrightarrow \tilde{\mathbb{R}}$) is proven, by which $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.
- In Chap. 13(p.87), the analogy theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$) is proven, by which $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.
- In Chap. 14(p.99), the symmetry theorem ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$) is proven, by which $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is derived form $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$.
- In Chap. 15(p.109), the analogy theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$) is proven, by which $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ are connected.

9.2 Connection with Both Directions

In the flow of Figure 9.1.1(p.51) above we should note the following:

- It is only $\boxed{\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}}$ that is directly proven.
- The remaining three $\boxed{\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}}$, $\boxed{\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}}$, and $\boxed{\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}}$ are derived by applying operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, and $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\boxed{\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}}$.
- The above four boxes are connected with both directions ($\leftrightarrow \updownarrow$). This interrelationship implies that any given box can be derived from any other box by applying operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$, $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$, $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$, $\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, $\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$, $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$, and $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128)) - (18.0.8(p.128))).

Chapter 10

Properties of Underlying Functions

This chapter examines the properties of underlying functions $T_{\mathbb{R}}$, $L_{\mathbb{R}}$, $K_{\mathbb{R}}$, and $\mathcal{L}_{\mathbb{R}}$ and the $\kappa_{\mathbb{R}}$-value (see (5.1.1(p.23))-(5.1.6(p.23))), which are used to clarify the properties of the optimal decision rules for M:1[ℝ][A] (see Chap. 11(p.59)).

Definition 10.0.1 ($A\{X_{\mathbb{R}}\}$and $\mathscr{A}\{X_{\mathbb{R}}\}$) Let us denote an assertion on $X_{\mathbb{R}} = T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}$ by $A\{X_{\mathbb{R}}\}$ and an assertion system consisting of some assertions $A\{X_{\mathbb{R}}\}$'s by $\mathscr{A}\{X_{\mathbb{R}}\}$. □

10.1 Primitive Underlying Function $T_{\mathbb{R}}$

To begin with, let us prove the following lemma for the assertion system $\mathscr{A}\{T_{\mathbb{R}}\}$.

Lemma 10.1.1 ($\mathscr{A}\{T_{\mathbb{R}}\}$) *For any $F \in \mathscr{F}$:*

(a) $T(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $T(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $T(x)$ *is strictly decreasing on* $(-\infty, b]$.
(d) $T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $T(x) + x$ *is strictly increasing on* $[a, \infty)$.
(f) $T(x) = \mu - x$ *on* $(-\infty, a]$ *and* $T(x) > \mu - x$ *on* (a, ∞).
(g) $T(x) > 0$ *on* $(-\infty, b)$ *and* $T(x) = 0$ *on* $[b, \infty)$.
(h) $T(x) \geq \max\{0, \mu - x\}$ *on* $(-\infty, \infty)$.
(i) $T(0) = \mu$ *if* $a > 0$ *and* $T(0) = 0$ *if* $b < 0$.
(j) $\beta T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta T(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x < y$ *and* $a < y$, *then* $T(x) + x < T(y) + y$.
(m) $\lambda\beta T(\lambda\beta\mu - s) - s$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$.
(n) $a < \mu$.[†] □

● *Proof* First, for any x and y let us prove the following two inequalities:

$$-(x - y)(1 - F(y)) \leq T(x) - T(y) \leq -(x - y)(1 - F(x)) \cdots \mathbf{(1)}, \tag{10.1.1}$$

$$(x - y)F(y) \leq T(x) + x - T(y) - y \leq (x - y)F(x) \cdots \mathbf{(2)}. \tag{10.1.2}$$

Then, let $T(x, y) \overset{\text{def}}{=} \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)]$ for any x and y.[‡] Since $1 \geq I(\boldsymbol{\xi} > y) \geq 0$ and since $\max\{\boldsymbol{\xi} - x, 0\} \geq 0$ and $\max\{\boldsymbol{\xi} - x, 0\} \geq \boldsymbol{\xi} - x$, we have

$$\max\{\boldsymbol{\xi} - x, 0\} \geq \max\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} > y) \geq (\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y),$$

hence from (5.1.1(p.23)) we get $T(x) \geq \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)] = T(x, y)$. Accordingly, for any x and y we have

$$T(x) - T(y) \geq T(x, y) - T(y) = \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > y)] - \mathbf{E}[(\boldsymbol{\xi} - y)I(\boldsymbol{\xi} > y)] = -(x - y)\,\mathbf{E}[I(\boldsymbol{\xi} > y)].$$

Since $I(\boldsymbol{\xi} \leq y) + I(\boldsymbol{\xi} > y) = 1$, we have

$$T(x) - T(y) \geq -(x - y)(\mathbf{E}[1 - I(\boldsymbol{\xi} \leq y)]) = -(x - y)(1 - \mathbf{E}[I(\boldsymbol{\xi} \leq y)]).$$

Then, since

$$\mathbf{E}[I(\boldsymbol{\xi} \leq y)] = \int_{-\infty}^{\infty} I(\xi \leq y)f(\xi)d\xi = \int_{-\infty}^{y} 1 \times f(\xi)d\xi = \int_{-\infty}^{y} f(\xi)d\xi = \Pr\{\boldsymbol{\xi} \leq y\} = F(y),$$

we have $T(x) - T(y) \geq -(x - y)(1 - F(y))$, hence the far left inequality of (1) holds. Multiplying both sides of the inequality by -1 leads to $-T(x) + T(y) \leq (x - y)(1 - F(y))$ or equivalently $T(y) - T(x) \leq -(y - x)(1 - F(y))$. Then, interchanging the

[†] The self-evident assertion is intentionally added here in order to keep the consistency with Lemma 13.2.1(p.91)(n).
[‡] If a given statement S is true, then $I(S) = 1$, or else $I(S) = 0$.

notations x and y yields $T(x)-T(y) \le -(x-y)(1-F(x))$, hence the far right inequality of (1) holds. (2) is immediate from adding $x-y$ to (1). Let us note here that $T(x)$ defined by (5.1.1(p.23)) can be rewritten as

$$T(x) = \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(a \le \boldsymbol{\xi})] + \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} < a) \cdots (3), \tag{10.1.3}$$

$$= \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(b < \boldsymbol{\xi})] + \mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} \le b)] \cdots (4). \tag{10.1.4}$$

(a,b) Immediate from (5.1.1(p.23)) and from the fact that $\max\{\boldsymbol{\xi}-x,0\}$ is continuous and nonincreasing in $x \in (-\infty,\infty)$ for any given $\boldsymbol{\xi}$.

(c) Let $y < x < b$, hence $x-y > 0$. Then, since $F(x) < 1$ due to (2.2.1 (1,2) (p.13)), we have $-(x-y)(1-F(x)) < 0$, hence $T(x)-T(y) < 0$ due to (1), so $T(x) < T(y)$, i.e., $T(x)$ is *strictly* decreasing on $x < b \cdots$ (5). Let us assume $T(x) = T(b)$ on $x < b$. Then, for any sufficiently small $\varepsilon > 0$ such that $b-x > 2\varepsilon$ we have $b > b-\varepsilon > x+\varepsilon > x$, hence $T(b) = T(x) > T(b-\varepsilon) \ge T(b)$ due to the *strict* decreasingness shown above and the nonincreasingness in (b), which is a contradiction. Thus, it must be that $T(x) \ne T(b)$ on $x < b$, so $T(x) > T(b)$ or $T(x) < T(b)$ on $x < b$. However, the latter is impossible due to (b), hence it must follow that $T(x) > T(b)$ on $x < b$. From this fact and (5) it follows that $T(x)$ is strictly decreasing on $x \le b$, or equivalently $T(x)$ is strictly decreasing on $(-\infty, b]$.

(d) Evident from the fact that $T(x)+x = \mathbf{E}[\max\{\boldsymbol{\xi},x\}]$ from (5.1.1(p.23)) and $\max\{\boldsymbol{\xi},x\}$ is nondecreasing in x for any $\boldsymbol{\xi}$.

(e) Let $a < y < x$, hence $F(y) > 0$ due to (2.2.1 (2,3) (p.13)). Then, since $(x-y)F(y) > 0$, we have $0 < T(x)+x-T(y)+y$ from (2), hence $T(y)+y < T(x)+x$, i.e., $T(x)+x$ is *strictly* increasing on $a < x \cdots$ (6). Let us assume $T(a)+a = T(x)+x$ on $a < x$. Then, for any sufficiently small $\varepsilon > 0$ such that $x-a > \varepsilon$ we have $a < a+\varepsilon < x$, hence $T(a)+a = T(x)+x > T(a+\varepsilon)+a+\varepsilon \ge T(a)+a$ due to the *strict* increasingness shown above and the nondecreasing in (d), which is a contradiction. Thus, it must be that $T(x)+x \ne T(a)+a$ on $a < x$, so we have $T(x)+x > T(a)+a$ or $T(x)+x < T(a)+a$ on $a < x$. However, the latter is impossible due to (d), hence it must follow that $T(x)+x > T(a)+a$ on $a < x$. From this fact and (6) it inevitably follows that $T(x)+$ is strictly increasing on $a \le x$, i.e., $T(x)+x$ is strictly increasing on $[a,\infty)$.

(f) Let $x \le a$. If $a \le \boldsymbol{\xi}$, then $x \le \boldsymbol{\xi}$, hence $\max\{\boldsymbol{\xi}-x,0\} = \boldsymbol{\xi}-x$ and if $\boldsymbol{\xi} < a$, then $f(\boldsymbol{\xi}) = 0 \cdots$ (7) due to (2.2.3 (1) (p.13)). Thus, from (3) we have $T(x) = \mathbf{E}[(\boldsymbol{\xi}-x)I(a \le \boldsymbol{\xi})]+0$. Then, since $\mathbf{E}[(\boldsymbol{\xi}-x)I(\boldsymbol{\xi} < a)] = \int_{\infty}^{a}(\xi-x)f(\xi)d\xi = 0$ due to (7), we have

$$T(x) = \mathbf{E}[(\boldsymbol{\xi}-x)I(a \le \boldsymbol{\xi})] + \mathbf{E}[(\boldsymbol{\xi}-x)I(\boldsymbol{\xi} < a)] = \mathbf{E}[(\boldsymbol{\xi}-x)(I(a \le \boldsymbol{\xi})+I(\boldsymbol{\xi} < a)] = \mathbf{E}[\boldsymbol{\xi}-x] = \mu - x,$$

hence the former half is true. Then, since $T(a) = \mu - a$ or equivalently $T(a)+a = \mu$, if $a < x$, from (e) we have $T(x)+x > T(a)+a = \mu$, hence $T(x) > \mu - x$, thus the latter half is true.

(g) Let $b \le x$. If $b < \boldsymbol{\xi}$, then $f(\boldsymbol{\xi}) = 0$ due to (2.2.3 (3) (p.13)), hence $\mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(b < \boldsymbol{\xi})] = 0$ and if $\boldsymbol{\xi} \le b$, then $\boldsymbol{\xi} \le x$, hence $\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} \le b) = 0$, so $\mathbf{E}[\max\{\boldsymbol{\xi}-x,0\}I(\boldsymbol{\xi} \le b)] = 0$. Accordingly, from (4) we have $T(x) = 0 \cdots$ (8), so the latter half is true. Let $x < b$. Then, since $T(x) > T(b)$ from (c) and $T(b) = 0$ from (8), we have $T(x) > 0$, hence the former half is true.

(h) Since $T(x) \ge \mu - x$ on $(-\infty,\infty)$ from (f) and $T(x) \ge 0$ on $(-\infty,\infty)$ from (g), it follows that $T(x) \ge \max\{0, \mu - x\}$ on $(-\infty,\infty)$.

(i) From (5.1.1(p.23)) and (2.2.3 (1,3) (p.13)) we have $T(0) = \mathbf{E}[\max\{\boldsymbol{\xi},0\}] = \mathbf{E}[\max\{\boldsymbol{\xi},0\}I(a \le \boldsymbol{\xi} \le b)]$. Hence, if $a > 0$, then $T(0) = \mathbf{E}[\boldsymbol{\xi}I(a \le \boldsymbol{\xi} \le b)] = \mathbf{E}[\boldsymbol{\xi}] = \mu$ and if $b < 0$, then $T(0) = \mathbf{E}[0I(a \le \boldsymbol{\xi} \le b)] = 0$.

(j) If $\beta = 1$, then $\beta T(x)+x = T(x)+x$, hence the assertion is true from (d).

(k) Since $\beta T(x)+x = \beta(T(x)+x)+(1-\beta)x$, if $\beta < 1$, then $(1-\beta)x$ is strictly increasing in x, hence the assertion is true from (d).

(l) Let $x < y$ and $a < y$. If $x \le a$, then $T(x)+x \le T(a)+a < T(y)+y$ due to (d,e), and if $a < x$, then $a \le x < y$, hence $K(x)+x < K(y)+y$ due to (e). Thus, whether $x \le a$ or $a < x$, we have $T(x)+x < T(y)+y$

(m) From (5.1.1(p.23)) we have

$$\begin{aligned} \lambda\beta T(\lambda\beta\mu - s) - s &= \lambda\beta\,\mathbf{E}[\max\{\boldsymbol{\xi}-\lambda\beta\mu+s,0\}] - s \\ &= \mathbf{E}[\max\{\lambda\beta\boldsymbol{\xi}-(\lambda\beta)^2\mu+\lambda\beta s,0\}] - s \\ &= \mathbf{E}[\max\{\lambda\beta\boldsymbol{\xi}-(\lambda\beta)^2\mu-(1-\lambda\beta)s,-s\}], \end{aligned}$$

which is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.

(n) Evident. ∎

10.2 Derivative Underlying Functions

First let us define

$$\delta = 1-(1-\lambda)\beta. \tag{10.2.1}$$

Then, since $0 < \beta \le 1$ and $1 \ge \lambda > 0$, we have

$$\delta \ge 1-(1-\lambda)\times 1 = \lambda > 0 \cdots (1), \qquad \delta \le 1-(1-\lambda)\times 0 = 1 \cdots (2). \tag{10.2.2}$$

Now, from (5.1.3(p.23)) and (5.1.4(p.23)) and from Lemma 10.1.1(p.53) (f) we obtain

$$L(x)\begin{cases} = \lambda\beta\mu - s - \lambda\beta x \text{ on } (-\infty, a] & \cdots(1), \\ > \lambda\beta\mu - s - \lambda\beta x \text{ on } (a, \infty) & \cdots(2), \end{cases} \tag{10.2.3}$$

$$K(x)\begin{cases} = \lambda\beta\mu - s - \delta x \quad \text{on} \quad (-\infty, a] & \cdots(1), \\ > \lambda\beta\mu - s - \delta x \quad \text{on} \quad (a, \infty) & \cdots(2). \end{cases} \tag{10.2.4}$$

In addition, from (5.1.4(p.23)) and Lemma 10.1.1(p.53) (g) we have

$$K(x)\begin{cases} > -(1-\beta)x - s \text{ on } \quad (-\infty, b) & \cdots(1), \\ = -(1-\beta)x - s \text{ on } \quad [b, \infty) & \cdots(2), \end{cases} \tag{10.2.5}$$

from which we obtain

$$K(x) + x \geq \beta x - s \quad \text{on} \quad (-\infty, \infty). \tag{10.2.6}$$

Then, from (10.2.4 (1) (p.55)) and (10.2.5 (2) (p.55)) we get

$$K(x) + x = \begin{cases} \lambda\beta\mu - s + (1-\lambda)\beta x \text{ on } \quad (-\infty, a] & \cdots(1), \\ \beta x - s \qquad\qquad\quad \text{ on } \quad [b, \infty) & \cdots(2). \end{cases} \tag{10.2.7}$$

From (5.1.8(p.23)) we have $K(x) = L(x) - (1-\beta)x$ and $L(x) = K(x) + (1-\beta)x$. Accordingly, if x_L and x_K exist, then we get

$$K(x_L) = -(1-\beta)\, x_L \cdots(1), \qquad L(x_K) = (1-\beta)\, x_K \cdots(2). \tag{10.2.8}$$

Lemma 10.2.1 ($\mathscr{A}\{L_{\mathbb{R}}\}$)

(a) *$L(x)$ is continuous.*
(b) *$L(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *Let $s = 0$. Then $x_L = b$ where $x_L > (\leq)\, x \Leftrightarrow L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.*
(e) *Let $s > 0$.*

1. *x_L uniquely exists with $x_L < b$ where $x_L > (= (<))\, x \Leftrightarrow L(x) > (= (<))\, 0$.*
2. *$(\lambda\beta\mu - s)/\lambda\beta \leq (>)\, a \Leftrightarrow x_L = (>)\, (\lambda\beta\mu - s)/\lambda\beta$.* □

● *Proof* (a-c) Immediate from (5.1.3(p.23)) and Lemma 10.1.1(p.53) (a-c).

(d) Let $s = 0$. Then, since $L(x) = \lambda\beta T(x)$, from Lemma 10.1.1(p.53) (g) we have $L(x) > 0$ for $b > x$ and $L(x) = 0$ for $b \leq x$, hence $x_L = b$ by the definition of x_L (see Section 5.2(p.25) (a)), thus $x_L > (\leq)\, x \Rightarrow L(x) > (=)\, 0$. The inverse is true by contraposition. In addition, since $L(x) = 0 \Rightarrow L(x) \leq 0$, we have $L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.

(e) Let $s > 0$.

(e1) From (10.2.3 (1) (p.55)) and from $\lambda > 0$ and $\beta > 0$ we have $L(x) > 0$ for a sufficiently small $x < 0$ such that $x \leq a$. In addition, we have $L(b) = \lambda\beta T(b) - s = -s < 0$ due to Lemma 10.1.1(p.53) (g). Hence, from (a,c) it follows that x_L uniquely exists. The inequality $x_L < b$ is immediate from $L(b) < 0$. The latter half is evident.

(e2) If $(\lambda\beta\mu - s)/\lambda\beta \leq (>)\, a$, from (10.2.3(p.55)) we have

$$L((\lambda\beta\mu - s)/\lambda\beta) = (>)\ \lambda\beta\mu - s - \lambda\beta(\lambda\beta\mu - s)/\lambda\beta = 0,$$

hence $x_L = (>)\, (\lambda\beta\mu - s)/\lambda\beta$ from (e1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

Corollary 10.2.1 ($\mathscr{A}\{L_{\mathbb{R}}\}$)

(a) $x_L > (\leq)\, x \Leftrightarrow L(x) > (\leq)\, 0$.
(b) $x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. □

● *Proof* (a) "⇒" is immediate from Lemma 10.2.1(p.55) (d,e1). "⇐" is evident by contraposition.

(b) Since $x_L > (\leq)\, x \Rightarrow L(x) > (\leq)\, 0$ due to (a) and since $L(x) > (\leq)\, 0 \Rightarrow L(x) \geq (\leq)\, 0$, we have $x_L > (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. In addition, if $x_L = x$, then $L(x) = L(x_L) = 0$ or equivalently $x_L = x \Rightarrow L(x) = 0$, hence $x_L = x \Rightarrow L(x) \geq 0$. Accordingly, it follows that $x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. ∎

Lemma 10.2.2 ($\mathscr{A}\{K_{\mathbb{R}}\}$)

(a) *$K(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$K(x) + x$ is nondecreasing on $(-\infty, \infty)$.*

(f) *$K(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\lambda<1$.*
(g) *$K(x)+x$ is strictly increasing on $[a,\infty)$.*
(h) *If $x<y$ and $a<y$, then $K(x)+x<K(y)+y$.*
(i) *Let $\beta=1$ and $s=0$. Then $x_K=b$ where $x_K>(\le)\,x \Leftrightarrow K(x)>(=)\,0 \Rightarrow K(x)>(\le)\,0$.*
(j) *Let $\beta<1$ or $s>0$.*

1. *There uniquely exists x_K where $x_K>(=(<))\,x \Leftrightarrow K(x)>(=(<))\,0$.*
2. *$(\lambda\beta\mu-s)/\delta\le(>)\,a \Leftrightarrow x_K=(>)\,(\lambda\beta\mu-s)/\delta$.*
3. *Let $\kappa>(=(<))\,0$. Then $x_K>(=(<))\,0$.* □

• *Proof* (a-c) Immediate from (5.1.4(p.23)) and Lemma 10.1.1(p.53) (a-c).

(d) Immediate from (5.1.4(p.23)) and Lemma 10.1.1(p.53) (b).

(e) From (5.1.4(p.23)) we have

$$K(x)+x=\lambda\beta T(x)+\beta x-s=\lambda\beta(T(x)+x)+(1-\lambda)\beta x-s\cdots\mathbf{(1)},$$

hence the assertion holds from Lemma 10.1.1(p.53) (d).

(f) Obvious from (1) and Lemma 10.1.1(p.53) (d).

(g) Clearly from (1) and Lemma 10.1.1(p.53) (e).

(h) Let $x<y$ and $a<y$. If $x\le a$, then $K(x)+x\le K(a)+a<K(y)+y$ due to (e,g), and if $a<x$, then $a<x<y$, hence $K(x)+x<K(y)+y$ due to (g). Thus, whether $x\le a$ or $a<x$, we have $K(x)+x<K(y)+y$

(i) Let $\beta=1$ and $s=0$. Then, since $K(x)=\lambda T(x)$ due to (5.1.4(p.23)), from Lemma 10.1.1(p.53) (g) we have $K(x)>0$ for $x<b$ and $K(x)=0$ for $b\le x$, hence $x_K=b$ by the definition of x_K (see Section 5.2(p.25) (a)). Thus $x_K>(\le)\,x\Rightarrow K(x)>(=)\,0$. The inverse holds by contraposition. In addition, since $K(x)=0\Rightarrow K(x)\le 0$, we have $K(x)>(=)\,0\Rightarrow K(x)>(\le)\,0$.

(j) Let $\beta<1$ or $s>0$.

(j1) This proof consists of the following six steps:

- First note (10.2.5 (2) (p.55)). If $\beta<1$, then $K(x)<0$ for any sufficiently large $x>0$ with $x\ge b$ and if $s>0$, then, whether $\beta<1$ or $\beta=1$, we have $K(x)<0$ for any sufficiently large $x>0$ with $x\ge b$. Hence, whether $\beta<1$ or $s>0$, we have $K(x)<0$ for any sufficiently large $x>0$ with $x\ge b$.
- Next note (10.2.4 (1) (p.55)). Then, since $\delta>0$ from (10.2.2 (1) (p.54)), whether $\beta<1$ or $s>0$ we have $K(x)>0$ for any sufficiently small $x<0$ with $x\le a$.
- Hence, whether $\beta<1$ or $s>0$, it follows that there exists the solution x_K.

∘ Let $\beta<1$. Then, the solution x_K is unique from (d).

∘ Let $s>0$. If $\beta<1$, the solution x_K is unique for the reason just above. If $\beta=1$, we have $K(b)=-s<0$ from (10.2.5 (2) (p.55)), hence $x_K<b$ due to (c), so $K(x)$ is strictly decreasing on the neighbourhood of $x=x_K$ due to (c), hence the solution x_K is unique. Therefore, whether $\beta<1$ or $\beta=1$, it follows that the solution x_K is unique.

∘ Accordingly, whether $\beta<1$ or $s>0$, it follows that the solution x_K is unique.

From all the above, whether $\beta<1$ or $s>0$, it follows that the solution x_K uniquely exists and hence that the latter half becomes true.

(j2) Let $(\lambda\beta\mu-s)/\delta\le(>)\,a$. Then, from (10.2.4 (1(2)) (p.55)) we have

$$K((\lambda\beta\mu-s)/\delta)=(>)\,\lambda\beta\mu-s-\delta(\lambda\beta\mu-s)/\delta=0,$$

hence $x_K=(>)\,(\lambda\beta\mu-s)/\delta$ due to (j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition.

(j3) If $\kappa>(=(<))\,0$, then $K(0)>(=(<))\,0$ from (5.1.7(p.23)), hence $x_K>(=(<))\,0$ from (j1). ∎

Corollary 10.2.2 ($\mathscr{A}\{K_{\mathbb{R}}\}$)

(a) $x_K>(\le)\,x \Leftrightarrow K(x)>(\le)\,0$.
(b) $x_K\ge(\le)\,x \Rightarrow K(x)\ge(\le)\,0$. □

• *Proof* (a) "⇒" is immediate from Lemma 10.2.2(p.55) (i,j1). "⇐" is evident by contraposition.

(b) Since $x_K>(\le)\,x\Rightarrow K(x)>(\le)\,0$ due to (a) and since $K(x)>(\le)\,0\Rightarrow K(x)\ge(\le)\,0$, we have $x_K>(\le)\,x\Rightarrow K(x)\ge(\le)\,0$. In addition, if $x_K=x$, then $K(x)=K(x_K)=0$ or equivalently $x_K=x\Rightarrow K(x)=0$, hence $x_K=x\Rightarrow K(x)\ge 0$. Accordingly, it follows that $x_K\ge(\le)\,x\Rightarrow K(x)\ge(\le)\,0$. ∎

Lemma 10.2.3 ($\mathscr{A}\{L_{\mathbb{R}}/K_{\mathbb{R}}\}$)

(a) *Let $\beta=1$ and $s=0$. Then $x_L=x_K=b$.*
(b) *Let $\beta=1$ and $s>0$. Then $x_L=x_K$.*
(c) *Let $\beta<1$ and $s=0$. Then $b>(=(<))\,0 \Leftrightarrow x_L>(=(<))\,x_K \Rightarrow x_K>(=(=))\,0$.*
(d) *Let $\beta<1$ and $s>0$. Then $\kappa>(=(<))\,0 \Leftrightarrow x_L>(=(<))\,x_K \Rightarrow x_K>(=(<))\,0$.* □

● *Proof* (a) If $\beta = 1$ and $s = 0$, then $x_L = b$ from Lemma 10.2.1(p.55) (d) and $x_K = b$ from Lemma 10.2.2(p.55) (i), hence $x_L = x_K = b$.

(b) Let $\beta = 1$ and $s > 0$. Then $K(x_L) = 0$ from (10.2.8 (1) (p.55)), hence $x_K = x_L$ from Lemma 10.2.2(p.55) (j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_L = b \cdots$ **(1)** from Lemma 10.2.1(p.55) (d).

∘ If $b > 0$, then $x_L > 0$, hence $K(x_L) < 0$ from (10.2.8 (1) (p.55)), so $x_L > x_K$ from Lemma 10.2.2(p.55) (j1). If $b = (<)\ 0$, then $x_L = (<)\ 0$, hence $K(x_L) = (>)\ 0$ from (10.2.8 (1) (p.55)), so $x_L = (<)\ x_K$ from Lemma 10.2.2(p.55) (j1). Accordingly, we have "⇒" holds and its inverse "⇐" is immediate by contraposition. Thus the *first relation* "⇔" holds.

∘ If $b > 0$, from (5.1.7(p.23)) we have $K(0) = \lambda\beta T(0) > 0$ due to Lemma 10.1.1(p.53) (g), hence $x_K > 0 \cdots$ **(2)** from Lemma 10.2.2(p.55) (j1). If $b = (<)\ 0$, from (5.1.7(p.23)) we have $K(0) = \lambda\beta T(0) = 0$ due to Lemma 10.1.1(p.53) (g), hence $x_K = (<)\ 0$ from Lemma 10.2.2(p.55) (j1). Accordingly, we have the *second relation* "⇒".

(d) Let $\beta < 1$ and $s > 0$. Now, since $\kappa = K(0)$ from (5.1.7(p.23)), if $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$, thus $x_K > (= (<))\ 0 \cdots$ **(3)** from Lemma 10.2.2(p.55) (j1). Accordingly $L(x_K) > (= (<))\ 0$ from (10.2.8 (2) (p.55)), hence $x_L > (= (<))\ x_K$ from Lemma 10.2.1(p.55) (e1). Thus, "⇒" in the *first relation* "⇔" holds and its inverse "⇐" is immediate by contraposition. Finally, the *first relation* "⇒" is immediate from (3). ∎

Lemma 10.2.4 ($\mathcal{L}_{\mathbb{R}}$)

(a) *$\mathcal{L}(s)$ is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.*

(b) *Let $\lambda\beta\mu \geq b$.*

1. *$x_L \leq \lambda\beta\mu - s$.*
2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_L < \lambda\beta\mu - s$.*

(c) *Let $\lambda\beta\mu < b$. Then, there exists a $s_{\mathcal{L}} > 0$ such that if $s_{\mathcal{L}} > (\leq)\ s$, then $x_L > (\leq)\ \lambda\beta\mu - s$.* □

● *Proof* (a) From (5.1.5(p.23)) and (5.1.3(p.23)) we have

$$\mathcal{L}(s) = L(\lambda\beta\mu - s) = \lambda\beta T(\lambda\beta\mu - s) - s \cdots \textbf{(1)},$$

hence the assertion holds from Lemma 10.1.1(p.53) (m).

(b) Let $\lambda\beta\mu \geq b$. Then, from (1) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta\mu) = 0 \cdots$ **(2)** due to Lemma 10.1.1(p.53) (g).

(b1) Since $s \geq 0$, from (a) we have $\mathcal{L}(s) \leq \mathcal{L}(0) = 0$ due to (2) or equivalently $L(\lambda\beta\mu - s) \leq 0$ due to (1), hence $x_L \leq \lambda\beta\mu - s$ from Corollary 10.2.1(p.55) (a).

(b2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (a) we have $\mathcal{L}(s) < \mathcal{L}(0) = 0 \cdots$ **(3)** due to (2) or equivalently $L(\lambda\beta\mu - s) < 0$ due to (1), hence $x_L < \lambda\beta\mu - s$ from Lemma 10.2.1(p.55) (e1).

(c) Let $\lambda\beta\mu < b$. From (1) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta\mu) > 0 \cdots$ **(4)** due to Lemma 10.1.1(p.53) (g). Note (10.2.3 (1) (p.55)). Then, for any sufficiently large $s > 0$ such that $\lambda\beta\mu - s \leq a$ and $\lambda\beta\mu - s < 0$ we have

$$\mathcal{L}(s) = L(\lambda\beta\mu - s) = \lambda\beta\mu - s - \lambda\beta(\lambda\beta\mu - s) = (1 - \lambda\beta)(\lambda\beta\mu - s) \leq 0.$$

Accordingly, due to (a) it follows that there exists the solution $s_{\mathcal{L}}$ of $\mathcal{L}(s) = 0$ where $s_{\mathcal{L}} > 0$ due to (4). Then, since $\mathcal{L}(s) > 0$ for $s < s_{\mathcal{L}}$ and $\mathcal{L}(s) \leq 0$ for $s \geq s_{\mathcal{L}}$ or equivalently $L(\lambda\beta\mu - s) > 0$ for $s < s_{\mathcal{L}}$ and $L(\lambda\beta\mu - s) \leq 0$ for $s \geq s_{\mathcal{L}}$, from Corollary 10.2.1(p.55) (a) we get $x_L > \lambda\beta\mu - s$ for $s < s_{\mathcal{L}}$ and $x_L \leq \lambda\beta\mu - s$ for $s \geq s_{\mathcal{L}}$. ∎

10.3 $\kappa_{\mathbb{R}}$-value

Lemma 10.3.1 ($\mathscr{A}\{\kappa_{\mathbb{R}}\}$)

(a) *$\kappa = \lambda\beta\mu - s$ if $a > 0$ and $\kappa = -s$ if $b < 0$.*

(b) *Let $\beta < 1$ or $s > 0$, Then $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$.* □

● *Proof* (a) Immediate from (5.1.6(p.23)) and Lemma 10.1.1(p.53) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\kappa > (= (<))\ 0$, we have $K(0) > (= (<))\ 0$ from (5.1.7(p.23)), hence $x_K > (= (<))\ 0$ from Lemma 10.2.2(p.55) (j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

Chapter 11

First Step: Proof of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

The first step for constructing the integrated theory is to prove the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model with $\mathbb{R}$-mechanism).

11.1 Preliminary

From (6.2.8(p.28)) and (6.2.14(p.28)) we have

$$V_t - \beta V_{t-1} = \max\{\mathbb{S}_t, 0\}$$
$$= \max\{L(V_{t-1}), 0\}, \quad t > 1. \tag{11.1.1}$$

Accordingly:

1. If $L(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = L(V_{t-1})$, hence from (5.1.9(p.23)) we have
$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{11.1.2}$$
2. If $L(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = 0$, hence
$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{11.1.3}$$

Now, from (6.5.2(p.39)) with $t = 2$ we have

$$V_2 - V_1 = \max\{K(V_1), -(1-\beta)V_1\}. \tag{11.1.4}$$

Finally, from (6.2.14(p.28)) and (6.2.12(p.28)) we have

$$\mathbb{S}_t = L(V_{t-1}) > (<)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}), \quad t > 1.. \tag{11.1.5}$$

11.2 Proof of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

Definition 11.2.1 (assertion and assertion system) By $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ let us represent an *assertion* included in each of Tom's 11.2.1(p.59) and 11.2.2(p.60) below and by $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ the *assertion system* consisting of all assertions included in each Tom (see Def. 10.0.1(p.53)). □

Definition 11.2.2 (primitive Tom (▣) and derivative Tom (⊡)) Let us refer to a Tom the all assertions which are *directly proven* as the *primitive* Tom (▣) and to a Tom the all assertions which are *indirectly derived* by transforming assertions included in a primitive Tom (▣) as the *derivative* Tom (⊡). □

Below, note that $\lambda = 1$ is assume in the model (See Section 4.1.2.1(p.20) for the meaning of symbol ▣ which is used below).

□ **Tom 11.2.1 (▣ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$)** *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ →* → $Ⓢ_\blacktriangle$

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.4(p.23)) we have $K(x) = T(x) \geq 0 \cdots$ **(1)** for any x due to Lemma 10.1.1(p.53) (g), hence from (6.5.2(p.39)) and (1) we have

$$V_t = \max\{T(V_{t-1}) + V_{t-1}, V_{t-1}\} = \max\{T(V_{t-1}), 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1} \cdots \textbf{(2)}, \quad t > 1.$$

(a) Since $V_2 = T(V_1) + V_1$, we have $V_2 \geq V_1$ due to (1). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 10.1.1(p.53) (d) we have $V_t \leq T(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Since $V_1 = \mu$ from (6.5.1(p.39)), we have $V_1 < b$. Suppose $V_{t-1} < b$. Then, from (2) we have $V_t < T(b) + b = b$ due to Lemma 10.1.1(p.53) (l,g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 10.2.1(p.55) (d); accordingly, $L(V_{t-1}) > 0 \cdots$ **(3)** for $\tau \geq t > 1$. Thus, from (11.1.1(p.59)) we obtain $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$

for $\tau \geq t > 1$. Accordingly, since $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1}V_1$, we have $t^*_\tau = \tau$ for $\tau > 1$, i.e., $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$, hence we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (3) and (11.1.5(p.59)). ▮

Let us define

$$\mathtt{S}_1\boxed{Ⓢ\blacktriangle}\boxed{◎\|} = \left\{\begin{array}{ll}\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that} \\ (1) \quad Ⓢ_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_\blacktriangle \text{ where } \mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}, \\ (2) \quad ◎_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\| \text{ where } \mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}. \end{array}\right\}$$

□ **Tom 11.2.2** (▣$\mathscr{A}$\{M:1[ℝ][A]\}) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\|$ → → ⓓ$_\|$

(c) *Let $\beta\mu < b$.*

  1. *Let $\beta = 1$.*
     i. *Let $\mu - s \leq a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\|$ → → ⓓ$_\|$
     ii. *Let $\mu - s > a$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → Ⓢ$_\blacktriangle$
  2. *Let $\beta < 1$ and $s = 0$ ⟅$s > 0$⟆ .*
     i. *Let $b > 0$ ⟅$\kappa > 0$⟆ . Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → Ⓢ$_\blacktriangle$
     ii. *Let $b = 0$ ⟅$\kappa = 0$⟆ .*
        1. *Let $\beta\mu - s \leq a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\|$ → → ⓓ$_\|$
        2. *Let $\beta\mu - s > a$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$ → → Ⓢ$_\blacktriangle$
     iii. *Let $b < 0$ ⟅$\kappa < 0$⟆ .*
        1. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\|$ → → ⓓ$_\|$
        2. *Let $\beta\mu - s > a$ and $s < s_{\mathcal{L}}$. Then* $\mathtt{S}_1$(p.60)$\boxed{Ⓢ\blacktriangle}\boxed{◎\|}$ *is true* → → Ⓢ$_\blacktriangle$/◎$_\|$

● *Proof* Let $\beta < 1$ or $s > 0$. In this model, note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.20)) and that $\delta = 1 \cdots$ **(1)** (see (10.2.1(p.54))) due to the assumption $\lambda = 1 \cdots$ **(2)**.

(a) Since $x_K \geq \beta\mu - s = V_1$ due to Lemma 10.2.2(p.55) (j2) and (6.5.1(p.39)), we have $K(V_1) \geq 0$ due to Lemma 10.2.2(p.55) (j1), hence $V_2 - V_1 \geq 0$ from (11.1.4(p.59)), i.e., $V_1 \leq V_2$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.2(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Consider a sufficiently large $M > 0$ with $\beta\mu - s \leq M$ and $b \leq M$, hence $V_1 \leq M$ from (6.5.1(p.39)). Suppose $V_{t-1} \leq M$. Then, from (6.5.2(p.39)), Lemma 10.2.2(p.55) (e), and (10.2.7 (2) (p.55)) we have $V_t \leq \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\} \leq \max\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Hence, by induction $V_t \leq M$ for $t > 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.2(p.39)) we have $V = \max\{K(V) + V, \beta V\}$, hence $0 = \max\{K(V), -(1-\beta)\beta V\}$. Thus, since $K(V) \leq 0$, we have $V \geq x_K$ from Lemma 10.2.2(p.55) (j1).

(b) Let $\beta\mu \geq b$. Then $x_L \leq \beta\mu - s = V_1$ from Lemma 10.2.4(p.57) (b1) with $\lambda = 1$, hence $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ due to Corollary 10.2.1(p.55) (a), we have $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Hence, from (11.1.3(p.59)) we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., ⓓ$_{\tau>1}\langle 1\rangle_\|$ (see Preference Rule 7.2.1(p.43)).

(c) Let $\beta\mu < b$.

(c1) Let $\beta = 1 \cdots$ **(3)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then, from (3), (1), (2) we have $(\lambda\beta\mu - s)/\delta = \mu - s \cdots$ **(4)**. In addition, since $x_L = x_K \cdots$ **(5)** from Lemma 10.2.3(p.56) (b), we have $K(x_L) = K(x_K) = 0 \cdots$ **(6)**.

(c1i) Let $\mu - s \leq a$. Then $x_L = x_K = \mu - s = V_1$ from (5), Lemma 10.2.2(p.55) (j2), (4), and (6.5.1(p.39)). Accordingly, since $x_L \leq V_{t-1}$ for $t > 1$ from (a), we have $L(V_{t-1}) \leq 0$ for $t > 1$ due to Lemma 10.2.1(p.55) (e1). Hence, for the same reason as in the proof of (b) we obtain ⓓ$_{\tau>1}\langle 1\rangle_\|$.

(c1ii) Let $\mu - s > a$. Then $x_L = x_K > \mu - s = V_1 > a$ from (5) and Lemma 10.2.2(p.55) (j2), hence $a < V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} < x_L$, hence $L(V_{t-1}) > 0$ from Lemma 10.2.1(p.55) (e1). Then, from (11.1.2(p.59)), Lemma 10.2.2(p.55) (g), and (5) we have $V_t < K(x_L) + x_L = K(x_K) + x_L = x_L$. Accordingly, by induction $V_{t-1} < x_L$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.55) (a). Thus, for the same reason as in the proof of $\mathtt{Tom}$ 11.2.1(p.59) (b) we have Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2) Let $\beta < 1$ and $s = 0$ ⟅$s > 0$⟆ .

(c2i) Let $b > 0$ ⟅$\kappa > 0$⟆ . Then $x_L > x_K > 0 \cdots$ **(7)** from Lemma 10.2.3(p.56) (c ⟅d⟆). Now, since $x_K \geq \beta\mu - s$ due to Lemma 10.2.2(p.55) (j2), (1), and (2), we have $x_K \geq V_1$ from (6.5.1(p.39)). Suppose $x_K \geq V_{t-1}$. Then, from (6.5.2(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to (7). Accordingly, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (7), thus $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.55) (a). Hence, for the same reason as in the proof of $\mathtt{Tom}$ 11.2.1(p.59) (b) we have Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2ii) Let $b = 0$ ⟅$\kappa = 0$⟆ . Then $x_L = x_K \cdots$ **(8)** from Lemma 10.2.3(p.56) (c ⟅d⟆).

(c2ii1) Let $\beta\mu - s \leq a$. Then, $x_K = \beta\mu - s = V_1$ from Lemma 10.2.2(p.55) (j2). Suppose $V_{t-1} = x_K$, hence $V_{t-1} = x_L$ from (8), so $L(V_{t-1}) = L(x_L) = 0$. Then, from (11.1.2(p.59)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$

for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (8). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (11.1.3(p.59)), hence, for the same reason as in the proof of (b) we obtain ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.

(c2ii2) Let $\beta\mu - s > a$. Then, since $V_1 > a$ from (6.5.1(p.39)), we have $V_{t-1} > a$ for $t > 1$ due to (a). In addition, we have $x_K > \beta\mu - s = V_1$ from Lemma 10.2.2(p.55) (j2). Suppose $x_K > V_{t-1}$, hence $x_L > V_{t-1}$ from (8). Then, since $L(V_{t-1}) > 0$ due to Corollary 10.2.1(p.55) (a), from (11.1.2(p.59)) and Lemma 10.2.2(p.55) (g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $x_K > V_{t-1}$ for $t > 1$, so $x_L > V_{t-1}$ for $t > 1$ due to (8). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.55) (a), for the same reason as in the proof of (c1ii) we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

(c2iii) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ (9) from Lemma 10.2.3(p.56) (c ⟪d⟫).

(c2iii1) Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. First let $\beta\mu - s \leq a$. Then, since $x_K = \beta\mu - s = V_1$ from Lemma 10.2.2(p.55) (j2), we have $x_L < V_1$ from (9), hence $x_L \leq V_1$. Next, let $s_{\mathcal{L}} \leq s$. Then, since $x_L \leq \beta\mu - s$ due to Lemma 10.2.4(p.57) (c), we have $x_L \leq V_1$. Accordingly, whether $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, thus $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Hence, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_\parallel$ for $\tau > 1$.

(c2iii2) Let $\beta\mu - s > a \cdots$ (10) and $s < s_{\mathcal{L}}$. Then, from (9) and Lemma 10.2.4(p.57) (c) we have $x_K > x_L > \beta\mu - s = V_1 \cdots$ (11), hence $K(V_1) > 0 \cdots$ (12) from Lemma 10.2.2(p.55) (j1). In addition, since $V_1 > a$ due to (10), we have $V_{t-1} > a$ for $t > 0$ from (a). Now, from (11.1.4(p.59)) and (12) we have $V_2 - V_1 > 0$, i.e., $V_2 > V_1$. Suppose $V_{t-1} < V_t$. Then, from Lemma 10.2.2(p.55) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (11). Assume that $V_{t-1} < x_L$ for *all* $t > 1$, hence $V \leq x_L$ due to (a). Then, from (9) and from $V \geq x_K$ due to (a) we have the contradiction of $V \geq x_K > x_L \geq V$. Hence, it is impossible that $V_{t-1} < x_L$ for all $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 < V_2 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < V_{t^\bullet_\tau + 2} < \cdots,$$

from which

$$V_{t-1} < x_L, \quad t^\bullet_\tau \geq t > 1, \qquad x_L \leq V_{t-1}, \quad t > t^\bullet_\tau. \tag{11.2.1}$$

Therefore, from Corollary 10.2.1(p.55) (a) we have

$$L(V_{t-1}) > 0 \cdots (13), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \leq 0 \cdots (14), \quad t > t^\bullet_\tau.$$

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $L(V_{t-1}) > 0 \cdots$ (15) for $\tau \geq t > 1$ from (13), for the same reason as in the proof of (c1ii) we have ⓢ$_{t^\bullet_\tau\geq\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. Hence $\texttt{S}_1(1)$ is true.
2. Let $\tau > t^\bullet_\tau$. First, let $\tau \geq t > t^\bullet_\tau$. Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t^\bullet_\tau$ from (14), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (11.1.3(p.59)), thus

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (16).$$

Next let $t^\bullet_\tau \geq t > 1$. Then, from (13) and (11.1.1(p.59)) we have $V_t - \beta V_{t-1} > 0$ for $t^\bullet_\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, hence

$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \beta^2 V_{t^\bullet_\tau - 2} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots (17).$$

From (16) and (17) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} > \cdots > \beta^{\tau-1} V_1,$$

hence we obtain $t^*_\tau = t^\bullet_\tau$, i.e., ◎$_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\parallel$ due to Preference Rule 7.2.1(p.43). In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (13) and (11.1.5(p.59)). Hence $\texttt{S}_1(2)$ is true. ▮

Definition 11.2.3 (model-migration) If "ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$" holds in M:1[ℝ][A], then the search is conducted over $\tau \geq t > 1$, implying that the model M:1[ℝ][A] is substantively reduced to the model in which the search is enforced over $\tau \geq t > 1$, i.e. , M:1[ℝ][E]. We refer to this event as " M:1[ℝ][A] *migrates* over to M:1[ℝ][E] ", represented as

M:1[ℝ][A] ↬ M:1[ℝ][E]. □

Definition 11.2.4 (the occurrence frequency (rate) for each of ⓢ, ◎, and ⓓ) Let us refer to the frequency (rate) for each of ⓢ, ◎, and ⓓ appearing in the primitive Tom's (▣) (Tom's 11.2.1(p.59) and 11.2.2(p.60)) as the *occurrence frequency* (*rate*) for each of ⓢ, ◎, and ⓓ. Then we have the occurrence frequency = 5 for ⓢ, 1 for ◎, and 4 for ⓓ, hence the occurrence rate = 0.5 for ⓢ, 0.1 for ◎, and 0.4 for ⓓ. □

11.3 Structure of Assertion System $\mathscr{A}$\{M:1[ℝ][A]\}

From Tom's 11.2.1(p.59) and 11.2.1(p.59) we can *roughly* see the structure of the assertion system $\mathscr{A}$\{M:1[ℝ][A]\} (see Def. 11.2.1(p.59)). In this section we try to more determinably explain its structure. It will be known later on that this structure will play an essential role in the discussions in Step 6 (p.76).

11.3.1 Breakdown and Aggregation

Before moving on, let us define the following two perspectives (see Figure 11.3.1(p.62) below ($k = 3$)).

(I) The breakdown of a given set $\mathscr{X}$ into k mutually disjoint subsets $\mathscr{X}_1$, $\mathscr{X}_2$, $\cdots$, and $\mathscr{X}_k$ ($k > 0$), i.e.,

$$\mathscr{X} = \mathscr{X}_1 \cup \mathscr{X}_2 \cup \cdots \cup \mathscr{X}_k \text{ where } \mathscr{X}_i \cap \mathscr{X}_j = \emptyset \text{ for any } i \neq j,$$

called the *breakdown scenario*, represented as $\mathscr{X} \Rightarrow \{\mathscr{X}_1, \mathscr{X}_2, \cdots, \mathscr{X}_k\}$.

(II) The aggregation of k mutually disjoint subsets $\mathscr{X}'_1$, $\mathscr{X}'_2$, $\cdots$, and $\mathscr{X}'_k$ ($k > 0$) of a given set $\mathscr{X}$, i.e.,

$$\mathscr{X}' \overset{\text{def}}{=} \mathscr{X}'_1 \cup \mathscr{X}'_2 \cup \cdots \cup \mathscr{X}'_k \subseteq \mathscr{X} \text{ where } \mathscr{X}'_i \cap \mathscr{X}'_j = \emptyset \text{ for any } i \neq j,$$

called the *aggregation scenario*, represented as $\{\mathscr{X}'_1, \mathscr{X}'_2, \cdots, \mathscr{X}'_k\} \Rightarrow \mathscr{X}' \subseteq \mathscr{X}$. □

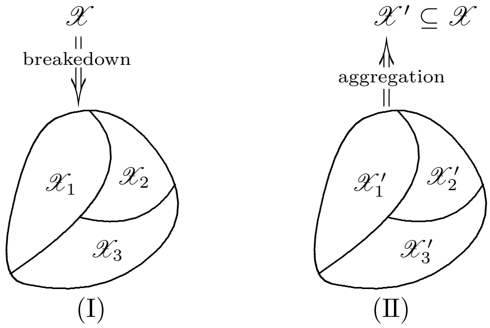


Figure 11.3.1: Breakdown and aggregation

11.3.2 Structure of Assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

11.3.2.1 Condition Space $\mathscr{C}\langle A\rangle$ of an Assertion A

In general, any given assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ consists of two terms: a *statement* S and a *condition expression* CE, schematized as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds if } \mathsf{CE} \text{ is satisfied}\}. \qquad (11.3.1)$$

□ *Example* **11.3.1** The assertion given by Tom 11.2.2(p.60) (b) can be rewritten as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{🅓_{\tau>1}\langle 1\rangle_{\shortparallel} \text{ holds if } \beta\mu \geq b \text{ is satisfied}\}$$

where $\mathsf{S} = \{🅓_{\tau>1}\langle 1\rangle_{\shortparallel}\}$ and $\mathsf{CE} = \{\beta\mu \geq b\}$. □

More strictly, for a given parameter space $\mathscr{P}_{\!A} \subseteq \mathscr{P}$ (Total-p-Space: see (4.4.1(p.21)) and (4.4.2(p.21))) and a given distribution function space $\mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}$ (Total-dF-Space: see (2.2.5(p.13))) related to a given parameter $\boldsymbol{p} \in \mathscr{P}_{\!A}$, the condition expression CE is given as a *conditional* on a parameter vector $\boldsymbol{p}$ and a distribution function F where

$$\boldsymbol{p} \in \mathscr{P}_{\!A} \subseteq \mathscr{P}, \qquad (11.3.2)$$

$$F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}. \qquad (11.3.3)$$

Then (11.3.1(p.62)) can be rewritten as

$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ holds for } \mathsf{CE} \text{ with } \boldsymbol{p} \in \mathscr{P}_{\!A} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A|\boldsymbol{p}} \subseteq \mathscr{F}\}. \qquad (11.3.4)$$

□ *Example* **11.3.2** For the assertion A given by Tom 11.2.2(p.60) (c1i) we have

$$\mathscr{P}_{\!A} = \{\boldsymbol{p} \mid \lambda = 1 \cap \beta = 1 \cap s > 0\},^{\dagger}$$
$$\mathscr{F}_{A|\boldsymbol{p}} = \{F \mid \beta\mu < b \cap \mu - s \leq a\}. \ □$$

□ *Example* **11.3.3** For the assertion A given by Tom 11.2.2(p.60) (c2iii2) we have

$$\mathscr{P}_{\!A} = \{\boldsymbol{p} \mid \lambda = 1 \cap \beta < 1 \cap \ s = 0 ⦅s > 0⦆\},$$
$$\mathscr{F}_{A|\boldsymbol{p}} = \{F \mid \beta\mu < b \cap \ b < 0 ⦅\kappa < 0⦆ \cap \beta\mu - s > a \cap s < \ s_{\mathcal{L}}\}. \ □$$

---

† When $\beta = 1$, we have $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$".

Here let us define
$$\mathscr{C}\langle A\rangle \overset{\text{def}}{=} \{(\boldsymbol{p},F)\;\big|\;\boldsymbol{p}\in\mathscr{P}_A\subseteq\mathscr{P}, F\in\mathscr{F}_{A|\boldsymbol{p}}\subseteq\mathscr{F}\}, \tag{11.3.5}$$

called the *condition space* of a given assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. Then, (11.3.4(p.62)) can be rewritten as
$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \{\mathsf{S}\text{ holds for }\mathsf{CE}\text{ on }\mathscr{C}\langle A\rangle\;\}. \tag{11.3.6}$$

Throughout the rest of the paper, let us *alternatively* express the whole of (11.3.6(p.63)) as
$$A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}\text{ holds for }\mathsf{CE}\text{ on }\mathscr{C}\langle A\rangle \tag{11.3.7}$$
for short

11.3.2.2 Structure of `Tom`

Definition 11.3.1

(a) We represent `Tom` 11.2.1(p.59) and `Tom` 11.2.2(p.60) by "`Tom`" for short, removing "11.2.1" and "11.2.2".

(b) For multiple `Tom`'s we use terms $\mathtt{Tom}_1$, $\mathtt{Tom}_2$, $\cdots$. For example, $\mathtt{Tom}_1 = \mathtt{Tom}$ 11.2.1(p.59) and $\mathtt{Tom}_2 = \mathtt{Tom}$ 11.2.2(p.60).

(c) In order to stress that an assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is included in a given `Tom` (i.e., $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}\in\mathtt{Tom}$), let us represent it as $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and an assertion system consisting of all $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$'s as $\mathscr{A}_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.

(d) We represent $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ included in $\mathscr{A}_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ by $A_{\mathtt{Tom}}$ for short or $A^1_{\mathtt{Tom}}$, $A^2_{\mathtt{Tom}}$, $\cdots$. □

Then (11.3.4(p.62)) - (11.3.7(p.63)) can be rewritten as respectively
$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \{\mathsf{S}\text{ holds for }\boldsymbol{p}\in\mathscr{P}_{A_{\mathtt{Tom}}}\subseteq\mathscr{P}\text{ and }F\in\mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}\}, \tag{11.3.8}$$
$$\mathscr{C}\langle A_{\mathtt{Tom}}\rangle \;\overset{\text{def}}{=}\; \{(\boldsymbol{p},F)\;\big|\;\boldsymbol{p}\in\mathscr{P}_{A_{\mathtt{Tom}}}\subseteq\mathscr{P}, F\in\mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}\}, \tag{11.3.9}$$
$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \{\mathsf{S}\text{ holds for }\mathsf{CE}\text{ on }\mathscr{C}\langle A_{\mathtt{Tom}}\rangle\;\}, \tag{11.3.10}$$
$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}\text{ holds for }\mathsf{CE}\text{ on }\mathscr{C}\langle A_{\mathtt{Tom}}\rangle. \tag{11.3.11}$$

Closely looking into the structure of `Tom`'s 11.2.1(p.59) and 11.2.2(p.60), in general we see that a given `Tom` consists of two items; a *basic premise* $\mathsf{BP}_{\mathtt{Tom}}$ and some *assertions* $A^1_{\mathtt{Tom}}$, $A^2_{\mathtt{Tom}}$, $\cdots$, i.e.,
$$\mathtt{Tom} = \{\text{Let }\mathsf{BP}\text{ be true. Then assertions }A^1_{\mathtt{Tom}},\, A^2_{\mathtt{Tom}},\,\cdots\text{ hold.}\}$$
or equivalently
$$\mathtt{Tom} = \{\text{Assertions }A^1_{\mathtt{Tom}},\, A^2_{\mathtt{Tom}},\,\cdots\text{ hold if }\mathsf{BP}_{\mathtt{Tom}}\text{ be true.}\}. \tag{11.3.12}$$
Here let us define
$$\mathscr{P}_{\mathtt{Tom}} \;\overset{\text{def}}{=}\; \mathscr{P}_{A^1_{\mathtt{Tom}}}\cup\mathscr{P}_{A^2_{\mathtt{Tom}}}\cup\cdots\subseteq\mathscr{P}, \tag{11.3.13}$$
$$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} \;\overset{\text{def}}{=}\; \mathscr{F}_{A^1_{\mathtt{Tom}|\boldsymbol{p}}}\cup\mathscr{F}_{A^2_{\mathtt{Tom}|\boldsymbol{p}}}\cup\cdots\subseteq\mathscr{F} \tag{11.3.14}$$
where $\mathscr{P}_{A^i_{\mathtt{Tom}}}\subseteq\mathscr{P}$ and $\mathscr{F}_{A^i_{\mathtt{Tom}|\boldsymbol{p}}}\subseteq\mathscr{F}$ for $i=1,2,\cdots$ (see (11.3.2(p.62)) and (11.3.3(p.62))). Then the basic premise $\mathsf{BP}_{\mathtt{Tom}}$ can be written as
$$\mathsf{BP}_{\mathtt{Tom}} \;=\; \{\text{a condition on }\boldsymbol{p}\in\mathscr{P}_{\mathtt{Tom}}\subseteq\mathscr{P}\text{ and }F\in\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}}\subseteq\mathscr{F}\}. \tag{11.3.15}$$

□ *Example* **11.3.4** For $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ in Section 11.2(p.59) we have

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p}\;|\;\lambda=1\cap\beta=1\cap s=0\}$ for `Tom` 11.2.1(p.59)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p}\;|\;\lambda=1\cap(\beta<1\cup s>0)\}$ for `Tom` 11.2.2(p.60)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \mathscr{F}$ for `Tom` 11.2.1(p.59)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \mathscr{F}$ for `Tom` 11.2.2(p.60)

For $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$ in Section 20.1.3(p.154) we have

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p}\;|\;\lambda\le1\cap\beta=1\cap s=0\cap-\infty<\rho<\infty\}$ for `Tom` 20.1.1(p.154)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p}\;|\;\lambda\le1\cap(\beta<1\cup s>0)\cap-\infty<\rho<\infty\}$ for `Tom` 20.1.2(p.154)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p}\;|\;\lambda\le1\cap(\beta<1\cup s>0)\cap-\infty<\rho<\infty\}$ for `Tom` 20.1.3(p.157)

$\mathscr{P}_{\mathtt{Tom}} = \{\boldsymbol{p}\;|\;\lambda\le1\cap(\beta<1\cup s>0)\cap-\infty<\rho<\infty\}$ for `Tom` 20.1.4(p.157)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F\;|\;-\infty<a<\mu<b<\infty\}=\mathscr{F}$ for `Tom` 20.1.1(p.154)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F\;|\;F\in\mathscr{F}\cap\rho<x_K\}$ for `Tom` 20.1.2(p.154)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F\;|\;F\in\mathscr{F}\cap\rho=x_K\}$ for `Tom` 20.1.3(p.157)

$\mathscr{F}_{\mathtt{Tom}|\boldsymbol{p}} = \{F\;|\;F\in\mathscr{F}\cap\rho>x_K\}$ for `Tom` 20.1.4(p.157) □

11.3.2.3 Condition Space $\mathscr{C}\langle\texttt{Tom}\rangle$

For a given Tom let us define

$$\mathscr{C}\langle\texttt{Tom}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{\texttt{Tom}} \subseteq \mathscr{P}, F \in \mathscr{F}_{\texttt{Tom}|\boldsymbol{p}} \subseteq \mathscr{F}\}, \tag{11.3.16}$$

called the *condition space* $\mathscr{C}\langle\texttt{Tom}\rangle$ of Tom. Then (11.3.15(p.63)) can be rewritten as

$$\mathsf{BP}_{\texttt{Tom}} = \{\text{a condition on } \mathscr{C}\langle\texttt{Tom}\rangle\}. \tag{11.3.17}$$

Then (11.3.12(p.63)) can be rewritten as

$$\texttt{Tom} = \{\text{Assertions } A^1_{\texttt{Tom}},\, A^2_{\texttt{Tom}}.\ \cdots \text{ hold on } \mathsf{BP}_{\texttt{Tom}}\}, \tag{11.3.18}$$

alternatively as

$$\texttt{Tom} = \{\text{Assertions } A^1_{\texttt{Tom}},\, A^2_{\texttt{Tom}},\ \cdots \text{ hold on } \mathscr{C}\langle\texttt{Tom}\rangle\}. \tag{11.3.19}$$

For explanatory convenience, we will sometimes express "$A^j_{\texttt{Tom}}$ is included in Tom" as "$A^j_{\texttt{Tom}} \in \texttt{Tom}$" or sometimes as "$A_{\texttt{Tom}} \in \texttt{Tom}$" removing the superscript "^{j}".

11.3.3 Assertion System $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

breakdown scenario

↓

11.3.3.1 Breakdown of $\mathscr{C}\langle\texttt{Tom}\rangle$

Here consider the breakdown of the condition space $\mathscr{C}\langle\texttt{Tom}\rangle$ to the condition spaces $\mathscr{C}\langle A^1_{\texttt{Tom}}\rangle$, $\mathscr{C}\langle A^2_{\texttt{Tom}}\rangle$, $\cdots$, i.e.,

$$\mathscr{C}\langle\texttt{Tom}\rangle = \cup_{j=1,2,\cdots}\mathscr{C}\langle A^j_{\texttt{Tom}}\rangle = \cup_{A_{\texttt{Tom}}\in\texttt{Tom}}\mathscr{C}\langle A_{\texttt{Tom}}\rangle, \tag{11.3.20}$$

depicted as in Figure 11.3.2(p.64) $(k = 3)$ below.

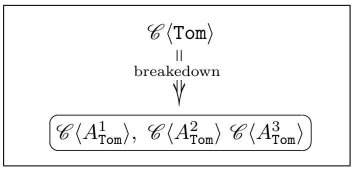


Figure 11.3.2: Breakedown of $\mathscr{C}\langle\texttt{Tom}\rangle$ to $\mathscr{C}\langle A^1_{\texttt{Tom}}\rangle$, $\mathscr{C}\langle A^2_{\texttt{Tom}}\rangle$, $\mathscr{C}\langle A^3_{\texttt{Tom}}\rangle$ $(k = 3)$

11.3.3.2 Construction of $\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

Consider the list of (11.3.11(p.63)) over Tom, i.e., $A^1_{\texttt{Tom}}, A^2_{\texttt{Tom}}, \cdots \in \texttt{Tom}$, or equivalently

$$\text{“ } A^1_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds for } \mathsf{CE} \text{ on } \mathscr{C}\langle A^1_{\texttt{Tom}}\rangle \text{ ”},$$
$$\text{“ } A^2_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds for } \mathsf{CE} \text{ on } \mathscr{C}\langle A^2_{\texttt{Tom}}\rangle \text{ ”},$$
$$\vdots$$

Then, gathering the above list with noting (11.3.20(p.64)), we get

$$\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds for } \mathsf{CE} \text{ on } \mathscr{C}\langle\texttt{Tom}\rangle \tag{11.3.21}$$

where

$$\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \overset{\text{def}}{=} \{A^1_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, A^2_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, \cdots\}. \tag{11.3.22}$$

11.3.3.3 Condition Space $\mathscr{C}\langle\mathcal{T}\texttt{om}\rangle$

For explanatory convenience, let us represent Tom 11.2.1(p.59) and Tom 11.2.2(p.60) by $\texttt{Tom}_1$ and $\texttt{Tom}_2$ respectively; in general, let $\texttt{Tom}_1, \texttt{Tom}_2, \cdots$. Then, let us define

$$\mathcal{T}\texttt{om} \overset{\text{def}}{=} \{\texttt{Tom}_1, \texttt{Tom}_2, , \cdots\} = \{\texttt{Tom}\}.$$

□ *Example* **11.3.5** For example we have

$$\mathcal{T}\texttt{om} = \{\texttt{Tom}_1 = \texttt{Tom } 11.2.1(\text{p.59}), \texttt{Tom}_2 = \texttt{Tom } 11.2.2(\text{p.60})\},$$
$$\mathcal{T}\texttt{om} = \{\texttt{Tom}_1 = \texttt{Tom } 20.1.1(\text{p.154}), \texttt{Tom}_2 = \texttt{Tom } 20.1.2(\text{p.154}), \texttt{Tom}_3 = \texttt{Tom } 20.1.3(\text{p.157}), \texttt{Tom}_4 = \texttt{Tom } 20.1.4(\text{p.157})\}. \quad \square$$

Here let us define

$$\mathscr{C}\langle\mathcal{T}\texttt{om}\rangle \overset{\text{def}}{=} \cup_{i=1,2,\cdots}\mathscr{C}\langle\texttt{Tom}_i\rangle = \cup_{\texttt{Tom}\in\mathcal{T}\texttt{om}}\mathscr{C}\langle\texttt{Tom}\rangle, \tag{11.3.23}$$

called the *condition space* of $\mathcal{T}\texttt{om}$, schematized as in Figure 11.3.3(p.65) below.

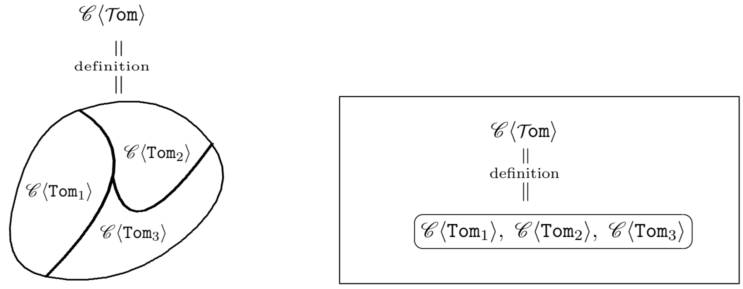


Figure 11.3.3: Condition space $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$

11.3.3.4 Construction of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$

Using (11.3.20(p.64)), we can express (11.3.23(p.64)) as below

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle \;=\; \cup_{i=1,2,\cdots}\cup_{j=1,2,\cdots}\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle \tag{11.3.24}$$

$$= \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}}\cup_{j=1,2,\cdots}\mathscr{C}\langle A^j_{\mathtt{Tom}}\rangle \tag{11.3.25}$$

$$= \cup_{\mathtt{Tom}\in\mathcal{T}\mathtt{om}}\cup_{A_{\mathtt{Tom}}\in\mathtt{Tom}}\mathscr{C}\langle A_{\mathtt{Tom}}\rangle \tag{11.3.26}$$

This relation implies the *breakdown* of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$ with $i = 1,2,\cdots$ and $j = 1,2,\cdots$, into $\mathscr{C}\langle A^j_{\mathtt{Tom}}\rangle$ with $\mathtt{Tom}\in\mathcal{T}\mathtt{om}$ and $j = 1,2,\cdots$, and into $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ with $\mathtt{Tom}\in\mathcal{T}\mathtt{om}$ and $A_{\mathtt{Tom}}\in\mathtt{Tom}$.

□ *Example* **11.3.6** As an example let us consider $\mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1,\mathtt{Tom}_2,\mathtt{Tom}_3\}$ where $\mathtt{Tom}_1 = \{A^1_{\mathtt{Tom}_1}, A^2_{\mathtt{Tom}_1}, A^3_{\mathtt{Tom}_1}\}$, $\mathtt{Tom}_2 = \{A^1_{\mathtt{Tom}_2}, A^2_{\mathtt{Tom}_2}, A^3_{\mathtt{Tom}_2}\}$, and $\mathtt{Tom}_3 = \{A^1_{\mathtt{Tom}_3}, A^2_{\mathtt{Tom}_3}, A^3_{\mathtt{Tom}_3}\}$. ▯

Then, fetching Figure 11.3.2(p.64) in Figure 11.3.3(p.65), we can depict (11.3.24(p.65)) as Figure 11.3.4(p.65) below, demonstrating the *breakdown* of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$.

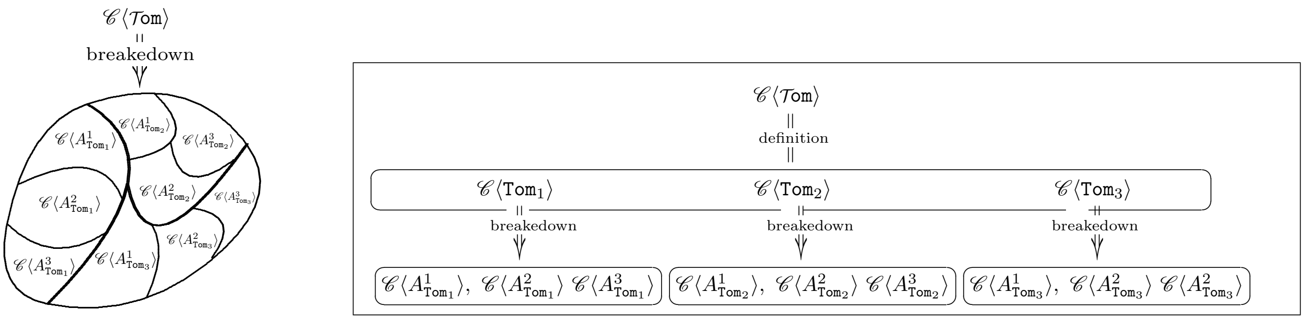


Figure 11.3.4: Breakdown of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ into $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$, $i,j = 1,2,3$

Figure 11.3.4(p.65) above implies that first

“ $\boxed{\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle}$ is broken down to $\boxed{\mathscr{C}\langle\mathtt{Tom}_i\rangle}$, $i = 1,2,3$ ”,

and then

“ each $\boxed{\mathscr{C}\langle\mathtt{Tom}_i\rangle}$, $i = 1,2,3$ is broken down to $\boxed{\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle}$, $i,j = 1,2,3.$ ”

The above two successive breakdown procedures eventually yields

“ $\boxed{\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle}$ is broken down to $\boxed{\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle}$ for $i,j = 1,2,3$ ”,

more generally

“ $\boxed{\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle}$ is broken down to $\boxed{\mathscr{C}\langle A_{\mathtt{Tom}}\rangle}$ with $\mathtt{Tom}\in\mathcal{T}\mathtt{om}$ ”

Here, consider the list of (11.3.21(p.64)) over $\mathtt{Tom}_1, \mathtt{Tom}_2, , \cdots \in \mathcal{T}\mathtt{om} = \{\mathtt{Tom}_1, \mathtt{Tom}_2, \cdots\}$, i.e.,

“ $\mathscr{A}_{\mathtt{Tom}_1}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle\mathtt{Tom}_1\rangle$ ”.

“ $\mathscr{A}_{\mathtt{Tom}_2}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{C}\langle\mathtt{Tom}_2\rangle$ ”.

⋮

Then, gathering the above list with noting (11.3.24(p.65)), we obtain

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle \tag{11.3.27}$$

where

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \overset{\text{def}}{=} \{\mathscr{A}_{\mathtt{Tom}_1}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, \mathscr{A}_{\mathtt{Tom}_2}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}, \cdots\}.$$

11.3.3.5 Completeness of $\mathcal{T}\mathtt{om}$ on $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}$

Closely looking at the contents of **Tom**'s 11.2.1(p.59) and 11.2.2(p.60), we see that the whole of assertions on $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ examined there is over all possible $(\boldsymbol{p}, F) \in \mathscr{P}\times\mathscr{F}$. In other words, it follows that at least one assertion $A^j_{\mathtt{Tom}_i}$ is defined and discussed for any given $(\boldsymbol{p}, F) \in \mathscr{P}\times\mathscr{F}$; in other words, there does not exist $(\boldsymbol{p}, F) \in \mathscr{P}\times\mathscr{F}$ for which any assertion is not be treated. This implies that we have

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F} \tag{11.3.28}$$

$$= \{(\boldsymbol{p}, F) \mid \boldsymbol{p}\in\mathscr{P}, F\in\mathscr{F}\} \quad (\text{see } (4.4.3(\text{p.21}))). \tag{11.3.29}$$

Remark 11.3.1 (a necessary requirement) What should be especially noted here is that the above equality (11.3.28(p.66)) is not *what should be proven* but a necessary condition that must be satisfied in the process of moving on the breakdown scenario. ☐

Let us refer to this requirement as the *completeness of* $\mathcal{T}\mathtt{om}$ on $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}$. We will require this completeness for the analyses of all models dealt with in the present paper. The above perspective can be depicted as in Figure 11.3.4(p.65) as below. In fact we can directly confirm that this requirement is satisfied in all discussions made in the present paper.

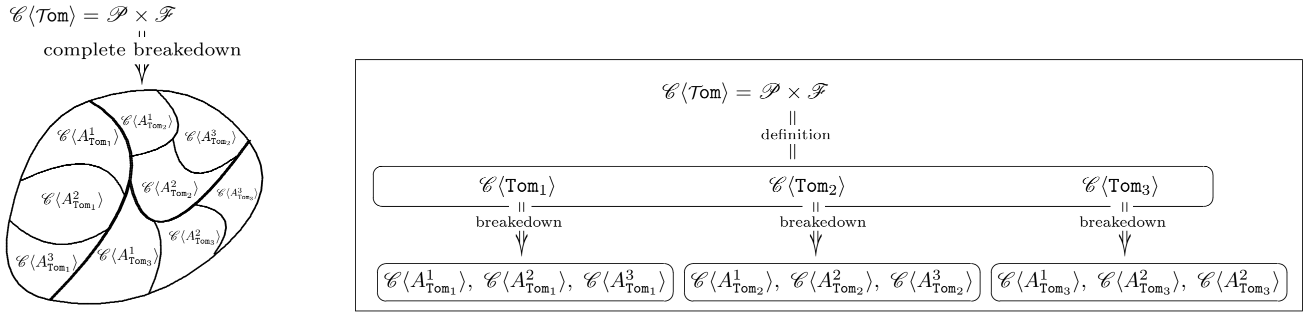


Figure 11.3.5: The completeness of $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ to $\mathscr{C}\langle A^j_{\mathtt{Tom}_i}\rangle$, $i, j = 1, 2, 3$

↑

breakdown scenario

Chapter 12

Second Step: Symmetry Theorem ($\mathbb{R} \leftrightarrow \tilde{\mathbb{R}}$)

The second step for constructing the integrated theory is to provide the theorem which derives the assertion system $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (buying model with $\mathbb{R}$-mechanism) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model with $\mathbb{R}$-mechanism) that was derived in Chap. 11(p.59) .

12.1 Two Kinds of Equality

12.1.1 Correspondence Equality

For $\boldsymbol{\xi}$, a, μ, b, $T(x)$, $\cdots$, which are all dependent on a given distribution function $F \in \mathscr{F}$ (see (2.2.5(p.13))), let us define $\hat{\boldsymbol{\xi}} = -\boldsymbol{\xi}$, $\hat{a} = -a$, $\hat{\mu} = -\mu$, $\hat{b} = -b$, $\hat{T}(x) = -T(x)$, $\cdots$ respectively, called the *reverse operation* $\mathcal{R}$. Then, for any given distribution function $F \in \mathscr{F}$, i.e.,

$$F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} \subseteq \mathscr{F}, \tag{12.1.1}$$

let us define the distribution function of $\hat{\boldsymbol{\xi}}$ by $\check{F}$, i.e.,

$$\check{F}(\xi) \overset{\text{def}}{=} \Pr\{\hat{\boldsymbol{\xi}} \le \xi\}, \tag{12.1.2}$$

where its probability density function is represented by $\check{f}$ and the set of all possible $\check{F}$ is denoted by $\check{\mathscr{F}}$, i.e.,

$$\check{\mathscr{F}} \overset{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}\}. \tag{12.1.3}$$

Now, since $\check{\check{F}}(\xi) = \Pr\{\hat{\hat{\boldsymbol{\xi}}} \le \xi\}$ for any ξ due to the definition (12.1.2(p.67)) and since

$$\hat{\hat{\boldsymbol{\xi}}} = \widehat{-\boldsymbol{\xi}} = -(-\boldsymbol{\xi}) = \boldsymbol{\xi}, \tag{12.1.4}$$

we have $\check{\check{F}}(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} = F(\xi)$ for any ξ due to (12.1.1(p.67)), i.e.,

$$\check{\check{F}} \equiv F. \tag{12.1.5}$$

For any subset $\mathscr{F}' \subseteq \mathscr{F}$ let us define

$$\check{\mathscr{F}}' \overset{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}'\}. \tag{12.1.6}$$

Then we have

$$\check{\check{\mathscr{F}}}' = \{\check{\check{F}} \mid \check{F} \in \check{\mathscr{F}}'\} = \{F \mid \check{F} \in \check{\mathscr{F}}'\} \tag{12.1.7}$$

due to (12.1.5(p.67)). If $F \in \mathscr{F}'$, then $\check{F} \in \check{\mathscr{F}}'$ from (12.1.6(p.67)), hence $F \in \check{\check{\mathscr{F}}}'$ due to (12.1.7(p.67)); accordingly, we have $\mathscr{F}' \subseteq \check{\check{\mathscr{F}}}' \cdots (*)$. If $F \in \check{\check{\mathscr{F}}}'$, then $\check{F} \in \check{\mathscr{F}}'$ due to (12.1.7(p.67)), hence $F \in \mathscr{F}'$ from (12.1.6(p.67)); therefore, we have $\check{\check{\mathscr{F}}}' \subseteq \mathscr{F}'$. From this and $(*)$ it follows that

$$\check{\check{\mathscr{F}}}' = \mathscr{F}'. \tag{12.1.8}$$

By $\check{a}$, $\check{\mu}$, and $\check{b}$ let us denote the lower bound, expectation, and upper bound of $\check{F} \in \check{\mathscr{F}}$ corresponding to any given $F \in \mathscr{F}$ with the lower bound a, expectation μ, and upper bound b. Then, from Figure 12.1.1(p.67) just below we clearly have, for any ξ,

$$f(\xi) = \check{f}(\hat{\xi}), \tag{12.1.9}$$

called the <u>*correspondence equality*</u>, where

$$\hat{a} = \check{b}, \quad \hat{\mu} = \check{\mu}, \quad \hat{b} = \check{a}. \tag{12.1.10}$$

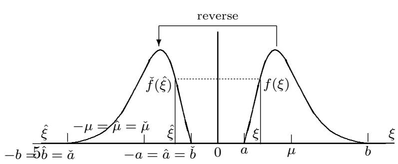


Figure 12.1.1: Relationship between probability density functions f and $\check{f}$

12.1.2 Identity Equality

Lemma 12.1.1

(a) *$\mathscr{F}$ and $\check{\mathscr{F}}$ are one-to-one correspondent where $\mathscr{F} = \check{\mathscr{F}}$.*

(b) *For any $\check{F} \in \check{\mathscr{F}}$ there exists a $F \in \mathscr{F}$ which is identical to the $\check{F}$, i.e., $F \equiv \check{F}$.*[†]

(c) *For any $F \in \mathscr{F}$ there exists a $\check{F} \in \check{\mathscr{F}}$ which is identical to the F, i.e., $\check{F} \equiv F$.*

● *Proof* If $F \in \mathscr{F}$, then $\check{F} \in \check{\mathscr{F}}$ from (12.1.3(p.67)), hence $F \in \mathscr{F} \Rightarrow \check{F} \in \check{\mathscr{F}} \cdots$ **(1)**. Conversely, if $\check{F} \in \check{\mathscr{F}}$, then F from which $\check{F} \in \check{\mathscr{F}}$ is defined is clearly an element of $\mathscr{F}$ due to (12.1.3(p.67)), i.e., $F \in \mathscr{F}$, hence $\check{F} \in \check{\mathscr{F}} \Rightarrow F \in \mathscr{F} \cdots$ **(2)**.

(a) First, for any $F \in \mathscr{F}$ and for the $\check{F} \in \check{\mathscr{F}}$ corresponding to the F we have

$$\begin{aligned}\check{F}(\xi) &= \Pr\{\check{\boldsymbol{\xi}} \le \xi\} = \Pr\{-\hat{\check{\boldsymbol{\xi}}} \le -\hat{\xi}\} = \Pr\{\hat{\check{\boldsymbol{\xi}}} \ge \hat{\xi}\} = \Pr\{\boldsymbol{\xi} \ge \hat{\xi}\} \quad \text{(due to (12.1.4(p.67)))} \\ &= 1 - \Pr\{\boldsymbol{\xi} < \hat{\xi}\} = 1 - \Pr\{\boldsymbol{\xi} \le \hat{\xi}\}^{\ddagger} = 1 - F(\hat{\xi}) \cdots \textbf{(3)}.\end{aligned}$$

Suppose any $F \in \mathscr{F}$ yields the two different $\check{F}_1 \in \check{\mathscr{F}}$ and $\check{F}_2 \in \check{\mathscr{F}}$, meaning that there exists at least one ξ' such that $\check{F}_1(\xi') \ne \check{F}_2(\xi')$. Then, since $\check{F}_1(\xi') = 1 - F(\hat{\xi}')$ and $\check{F}_2(\xi') = 1 - F(\hat{\xi}')$ due to (3), we have the contradiction of $\check{F}_1(\xi') = \check{F}_2(\xi')$, hence the $F \in \mathscr{F}$ must correspond to a *unique* $\check{F} \in \check{\mathscr{F}}$.

Next, for any $\check{F} \in \check{\mathscr{F}}$ and for $F \in \mathscr{F}$ from which $\check{F} \in \check{\mathscr{F}}$ is defined we have

$$F(\xi) = \Pr\{\boldsymbol{\xi} \le \xi\} = \Pr\{-\hat{\boldsymbol{\xi}} \le -\hat{\xi}\} = \Pr\{\check{\boldsymbol{\xi}} \ge \hat{\xi}\} = 1 - \Pr\{\check{\boldsymbol{\xi}} < \hat{\xi}\} = 1 - \Pr\{\check{\boldsymbol{\xi}} \le \hat{\xi}\}^{\ddagger} = 1 - \check{F}(\hat{\xi}) \cdots \textbf{(4)}.$$

Suppose any $\check{F} \in \check{\mathscr{F}}$ is yielded from the two different $F_1 \in \mathscr{F}$ and $F_2 \in \mathscr{F}$, meaning that there exists at least one ξ' such that $F_1(\xi') \ne F_2(\xi')$. Then, since $F_1(\xi') = 1 - \check{F}(\hat{\xi}')$ and $F_2(\xi') = 1 - \check{F}(\hat{\xi}')$ due to (4), we have the contradiction of $F_1(\xi') = F_2(\xi')$, hence the $\check{F} \in \check{\mathscr{F}}$ must correspond to a unique $F \in \mathscr{F}$. Thus, the former half of the assertion is true.

The latter half can be proven as follows. First, consider any $F \in \check{\mathscr{F}}$. Then, since $F \in \mathscr{F}$ by definition, we have $\check{\mathscr{F}} \subseteq \mathscr{F} \cdots$ **(5)**. Next, consider any $F \in \mathscr{F}$. Then, since $\check{F} \in \check{\mathscr{F}}$ due to (1), we have $\check{F} \in \mathscr{F}$ due to (5). Hence $\check{\check{F}} \in \check{\mathscr{F}}$ due to (1(p.68)), so $F \in \check{\mathscr{F}}$ due to (12.1.5(p.67)), thus we have $\mathscr{F} \subseteq \check{\mathscr{F}}$. From this and (5) we have $\check{\mathscr{F}} = \mathscr{F} \cdots$ **(6)**.

(b) Consider any $\check{F} \in \check{\mathscr{F}}$, hence $\check{F} \in \mathscr{F} \cdots$ **(7)** due to (6). Suppose every $F \in \mathscr{F}$ is not identical to the $\check{F}$, i.e., $F \not\equiv \check{F}$, implying that the $\check{F}$ lies outside $\mathscr{F}$,[§] hence cannot become an element of $\mathscr{F}$, i.e., $\check{F} \notin \mathscr{F}$, which contradicts (7). Hence, it follows that there must exist at least one F such that $F \equiv \check{F}$, thus the assertion holds.

(c) Consider any $F \in \mathscr{F}$, hence $F \in \check{\mathscr{F}} \cdots$ **(8)** due to (6). Suppose every $\check{F} \in \check{\mathscr{F}}$ is not identical to the F, i.e., $\check{F} \not\equiv F$, implying that the F lies outside $\check{\mathscr{F}}$[‖], hence cannot become an element of $\check{\mathscr{F}}$, i.e., $F \notin \check{\mathscr{F}}$, which contradicts (8). Hence, it follows that there must exist at least one $\check{F}$ such that $\check{F} \equiv F$, thus the assertion holds. ∎

Lemma 12.1.1(p.68) (b,c) implies that there always exist F and $\check{F}$ such that $F \equiv \check{F}$ holds; in other words, there always exist f and $\check{f}$ such that $f \equiv \check{f}$ or equivalently

$$f(\xi) \equiv \check{f}(\xi), \qquad (12.1.11)$$

called the *identity equality*.

12.2 Definitions of Modified Underlying Functions

The functions defined in the successive two sections are all the variations of ones that were defined in Sections 5.1.1(p.23) and 5.1.2(p.23).

12.2.1 $\check{T}$, $\check{L}$, $\check{K}$, $\check{\mathcal{L}}$, and $\check{\kappa}$ of Type $\mathbb{R}$

Let us define the underlying functions of Type $\mathbb{R}$ (see Section 5.1.1(p.23)) *for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$* as follows.

$$\check{T}(x) = \check{\mathbf{E}}[\max\{\boldsymbol{\xi} - x, 0\}] = \int_{-\infty}^{\infty} \max\{\xi - x, 0\}\check{f}(\xi)d\xi, \qquad (12.2.1)$$

$$\check{L}(x) = \lambda\beta\check{T}(x) - s, \qquad (12.2.2)$$

$$\check{K}(x) = \lambda\beta\check{T}(x) - (1-\beta)x - s, \qquad (12.2.3)$$

$$\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\check{\mu} - s). \qquad (12.2.4)$$

Let the solutions of $\check{L}(x) = 0$, $\check{K}(x) = 0$, and $\check{\mathcal{L}}(s) = 0$ be denoted by $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$ respectively if they exist. If each of the equations has the multiple solutions, let us employ the *smallest* one (see (a) of Section 5.2(p.25)). Let us define

$$\check{\kappa} = \lambda\beta\check{T}(0) - s. \qquad (12.2.5)$$

[†]This means $F(x) = \check{F}(x)$ for all $x \in (-\infty, \infty)$.

[‡]Due to the assumption of F being continuous (see A9(p.13))

[§]Note that $\mathscr{F}$ is a set consisting of al possible F's by definition.

[‖]Note that $\check{\mathscr{F}}$ is a set consisting of al possible $\check{F}$'s by definition.

By $\check{\mathsf{M}}$:1[$\mathbb{R}$][A] let us define M:1[$\mathbb{R}$][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for M:1[$\mathbb{R}$][A] we can express SOE{$\check{\mathsf{M}}$:1[$\mathbb{R}$][A]} as (see Table 6.5.1(p.39) (I))

$$\texttt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} = \{V_1 = \beta\check{\mu} - s,\, V_t = \max\{\check{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}.$$

12.2.2 $\check{\tilde{T}}$, $\check{\tilde{L}}$, $\check{\tilde{K}}$, $\check{\tilde{\mathcal{L}}}$, and $\check{\tilde{\kappa}}$ of $\tilde{\text{T}}$ype $\mathbb{R}$

Let us define the underlying functions of $\tilde{\text{T}}$ype $\mathbb{R}$ *for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to any* $F \in \mathscr{F}$ as follows.

$$\check{\tilde{T}}(x) \;=\; \check{\mathbf{E}}[\min\{\boldsymbol{\xi} - x, 0\}] = \textstyle\int_{-\infty}^{\infty} \min\{\xi - x, 0\}\check{f}(\xi)d\xi, \tag{12.2.6}$$

$$\check{\tilde{L}}(x) \;=\; \lambda\beta\check{\tilde{T}}(x) + s, \tag{12.2.7}$$

$$\check{\tilde{K}}(x) \;=\; \lambda\beta\check{\tilde{T}}(x) - (1-\beta)x + s, \tag{12.2.8}$$

$$\check{\tilde{\mathcal{L}}}(s) \;=\; \check{\tilde{L}}(\lambda\beta\check{\mu} + s). \tag{12.2.9}$$

Let the solutions of $\check{\tilde{L}}(x) = 0$, $\check{\tilde{K}}(x) = 0$, and $\check{\tilde{\mathcal{L}}}(s) = 0$ be denoted by $x_{\check{\tilde{L}}}$, $x_{\check{\tilde{K}}}$, and $s_{\check{\tilde{\mathcal{L}}}}$ respectively if they exist. If each of the equations has the multiple solutions, let us employ the *largest* one (see (b) of Section 5.2(p.25)). Let us define

$$\check{\tilde{\kappa}} \;=\; \lambda\beta\check{\tilde{T}}(0) + s. \tag{12.2.10}$$

By $\check{\tilde{\mathsf{M}}}$:1[$\mathbb{R}$][A] let us define $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] we can express SOE{$\check{\tilde{\mathsf{M}}}$:1[$\mathbb{R}$][A]} as (see Table 6.5.1(p.39) (II))

$$\texttt{SOE}\{\check{\tilde{\mathsf{M}}}{:}1[\mathbb{R}][\texttt{A}]\} = \{V_1 = \beta\check{\mu} + s,\, V_t = \min\{\check{\tilde{K}}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}.$$

12.2.3 List of Underline Functions of Type $\mathbb{R}$ and $\tilde{\text{T}}$ype $\mathbb{R}$

So far we have defined the four kinds of underlying functions, which may cause confusions. To give a clearer picture of these functions, we shall coordinate them as in Table 12.2.1(p.69).

Table 12.2.1: List of the underlying functions of Type $\mathbb{R}$ and $\tilde{\text{T}}$ype $\mathbb{R}$

Type $\mathbb{R}$	$\tilde{\text{T}}$ype $\mathbb{R}$
For any $F \in \mathscr{F}$	For $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$
$T(x) = \int_a^b \max\{\xi - x, 0\}f(\xi)d\xi$ $L(x) = \beta T(x) - s$ $K(x) = \beta T(x) - (1-\beta)x - s$ $\mathcal{L}(x) = L(\beta\mu - s)$ See Section 5.1.1(p.23)	$\check{T}(x) = \int_a^b \max\{\xi - x, 0\}\check{f}(\xi)d\xi$ $\check{L}(x) = \beta\check{T}(x) - s$ $\check{K}(x) = \beta\check{T}(x) - (1-\beta)x - s$ $\check{\mathcal{L}}(x) = \check{L}(\beta\check{\mu} - s)$ See Section 12.2.1(p.68)
$\tilde{T}(x) = \int_a^b \min\{\xi - x, 0\}f(\xi)d\xi$ $\tilde{L}(x) = \beta\tilde{T}(x) + s$ $\tilde{K}(x) = \beta\tilde{T}(x) - (1-\beta)x + s$ $\tilde{\mathcal{L}}(x) = \tilde{L}(\beta\mu + s)$ See Section 5.1.2(p.23)	$\check{\tilde{T}}(x) = \int_a^b \min\{\xi - x, 0\}\check{f}(\xi)d\xi$ $\check{\tilde{L}}(x) = \beta\check{\tilde{T}}(x) + s$ $\check{\tilde{K}}(x) = \beta\check{\tilde{T}}(x) - (1-\beta)x + s$ $\check{\tilde{\mathcal{L}}}(x) = \check{\tilde{L}}(\beta\check{\mu} + s)$ See Section 12.2.2(p.69)

12.3 Two Kinds of Replacements

12.3.1 Correspondence Replacement

Lemma 12.3.1 ($\mathcal{C}_{\mathbb{R}}$) *The left-hand side of each equality below is for* any $F \in \mathscr{F}$ *and its right-hand side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F.

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{a} = \check{b}$, $\hat{\mu} = \check{\mu}$, $\hat{b} = \check{a}$.
(c) $\hat{T}(x) = \check{\tilde{T}}(\hat{x})$.
(d) $\hat{L}(x) = \check{\tilde{L}}(\hat{x})$.
(e) $\hat{K}(x) = \check{\tilde{K}}(\hat{x})$.
(f) $\hat{\mathcal{L}}(s) = \check{\tilde{\mathcal{L}}}(s)$.
(g) $\hat{x}_L \,=\, x_{\check{\tilde{L}}}$.
(h) $\hat{x}_K \,=\, x_{\check{\tilde{K}}}$.
(i) $s_{\mathcal{L}} = s_{\check{\tilde{\mathcal{L}}}}$.
(j) $\hat{\kappa} = \check{\tilde{\kappa}}$. □

- *Proof* (a) The same as (12.1.9(p.67)).

(b) The same as (12.1.10(p.67)).

(c) The function $T(x)$ for any F (see (5.1.2(p.23))) can be rewritten as

$$\begin{aligned} T(x) &= \int_{-\infty}^{\infty} \max\{-\hat{\xi} + \hat{x}, 0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \min\{\hat{\xi} - \hat{x}, 0\} f(\xi) d\xi \\ &= -\int_{-\infty}^{\infty} \min\{\hat{\xi} - \hat{x}, 0\} \check{f}(\hat{\xi}) d\xi \quad \text{due to (a).} \end{aligned}$$

Let $\eta \stackrel{\text{def}}{=} \hat{\xi} = -\xi$, hence $d\eta = -d\xi$. Then, we have

$$\begin{aligned} T(x) &= \int_{\infty}^{-\infty} \min\{\eta - \hat{x}, 0\} \check{f}(\eta) d\eta \\ &= -\int_{-\infty}^{\infty} \min\{\eta - \hat{x}, 0\} \check{f}(\eta) d\eta \\ &= -\int_{-\infty}^{\infty} \min\{\xi - \hat{x}, 0\} \check{f}(\xi) d\xi \quad \text{(without loss of generality}^{\dagger}\text{)} \\ &= -\check{\check{T}}(\hat{x}) \quad \text{(see (12.2.6(p.69))),} \end{aligned}$$

hence $\hat{T}(x) = \check{\check{T}}(\hat{x})$.

(d) From (5.1.3(p.23)) and (c) we have $L(x) = -\lambda\beta\hat{T}(x) - s = -\lambda\beta\check{\check{T}}(\hat{x}) - s = -\check{\check{L}}(\hat{x})$ from (12.2.7(p.69)), hence $\hat{L}(x) = \check{\check{L}}(\hat{x})$.

(e) From (5.1.4(p.23)) and (c) we have $K(x) = -\lambda\beta\hat{T}(x) + (1-\beta)\hat{x} - s = -\lambda\beta\check{\check{T}}(\hat{x}) + (1-\beta)\hat{x} - s = -\check{\check{K}}(\hat{x})$ from (12.2.8(p.69)), hence $\hat{K}(x) = \check{\check{K}}(\hat{x})$.

(f) From (5.1.5(p.23)) we have $\mathcal{L}(s) = -\hat{L}(\lambda\beta\mu - s)$, hence from (d) we obtain $\mathcal{L}(s) = -\check{\check{L}}(\widehat{\lambda\beta\mu - s}) = -\check{\check{L}}(-\lambda\beta\mu + s) = -\check{\check{L}}(\lambda\beta\hat{\mu} + s) = -\check{\check{L}}(\lambda\beta\check{\mu} + s)$ due to (b). Accordingly, from (12.2.9(p.69)) we obtain $\mathcal{L}(s) = -\check{\check{\mathcal{L}}}(s)$, hence $\hat{\mathcal{L}}(s) = \check{\check{\mathcal{L}}}(s)$.

(g) Since $L(x_L) = 0$ by definition, we have $\hat{L}(x_L) = 0$, which can be rewritten as $\check{\check{L}}(\hat{x}_L) = 0$ from (d), implying that $\check{\check{L}}(x) = 0$ has the solution $x_{\check{\check{L}}} = \hat{x}_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $\hat{K}(x_K) = 0$, which can be rewritten as $\check{\check{K}}(\hat{x}_K) = 0$ from (e), implying that $\check{\check{K}}(x) = 0$ has the solution $x_{\check{\check{K}}} = \hat{x}_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, which can be rewritten as $\check{\check{\mathcal{L}}}(s_{\mathcal{L}}) = 0$ from (f), implying that $\check{\check{\mathcal{L}}}(s) = 0$ has the solution $s_{\check{\check{\mathcal{L}}}} = s_{\mathcal{L}}$ by definition.

(j) From (5.1.6(p.23)) we have $\kappa = -\lambda\beta\hat{T}(0) - s$, which can be rewritten as $\kappa = -\lambda\beta\check{\check{T}}(\hat{0}) - s$ from (c), hence $\kappa = -\lambda\beta\check{\check{T}}(0) - s = -\check{\check{\kappa}}$ from (12.2.10(p.69)), thus $\hat{\kappa} = \check{\check{\kappa}}$. ∎

Definition 12.3.1 (correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 12.3.1(p.69) by its right-hand the *correspondence replacement operation* $\mathcal{C}_{\mathbb{R}}$. □

Lemma 12.3.2 ($\tilde{\mathcal{C}}_{\mathbb{R}}$) *The left-hand side of each equality below is for* any $F \in \mathscr{F}$ *and its right-hand side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F.

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{b} = \check{a}, \quad \hat{\mu} = \check{\mu}, \quad \hat{a} = \check{b}$.
(c) $\hat{\check{\check{T}}}(x) = \check{T}(\hat{x})$.
(d) $\hat{\check{\check{L}}}(x) = \check{L}(\hat{x})$.
(e) $\hat{\check{\check{K}}}(x) = \check{K}(\hat{x})$.
(f) $\hat{\check{\check{\mathcal{L}}}}(s) = \check{\mathcal{L}}(s)$.
(g) $\hat{x}_{\tilde{L}} = x_{\check{L}}$.
(h) $\hat{x}_{\tilde{K}} = x_{\check{K}}$.
(i) $s_{\tilde{\mathcal{L}}} = s_{\check{\mathcal{L}}}$.
(j) $\hat{\check{\check{\kappa}}} = \check{\kappa}$. □

- *Proof* (a) The same as (12.1.9(p.67)).

(b) The same as (12.1.10(p.67)).

(c) The function $\tilde{T}(x)$ for any F (see (5.1.12(p.23))) can be rewritten as

[†] The mere replacement of the symbol η by ξ.

$$\begin{aligned}\tilde{T}(x) &= \int_{-\infty}^{\infty} \min\{-\hat{\xi}+\hat{x},0\}f(\xi)d\xi \\ &= -\int_{-\infty}^{\infty} \max\{\hat{\xi}-\hat{x},0\}f(\xi)d\xi \\ &= -\int_{-\infty}^{\infty} \max\{\hat{\xi}-\hat{x},0\}\check{f}(\hat{\xi})d\xi \quad \text{(due to (a(p.70)))}.\end{aligned}$$

Let $\eta=\hat{\xi}=-\xi$. Then, since $d\eta=-d\xi$, we have

$$\begin{aligned}\tilde{T}(x) &= \int_{\infty}^{-\infty} \max\{\eta-\hat{x},0\}\check{f}(\eta)d\eta \\ &= -\int_{-\infty}^{\infty} \max\{\eta-\hat{x},0\}\check{f}(\eta)d\eta \\ &= -\int_{-\infty}^{\infty} \max\{\xi-\hat{x},0\}\check{f}(\xi)d\xi \quad \text{(without loss of generality}^\dagger) \\ &= -\check{T}(\hat{x}) \quad \text{(see (12.2.1(p.68)))},\end{aligned}$$

hence $\hat{\tilde{T}}(x)=\check{T}(\hat{x})$.

(d) From (5.1.13(p.23)) and (c) we have $\tilde{L}(x)=-\lambda\beta\hat{\tilde{T}}(x)+s=-\lambda\beta\check{T}(\hat{x})+s=-\check{L}(\hat{x})$ from (12.2.2(p.68)), hence $\hat{\tilde{L}}(x)=\check{L}(\hat{x})$.

(e) From (5.1.14(p.23)) and (c) we have $\tilde{K}(x)=-\lambda\beta\hat{\tilde{T}}(x)+(1-\beta)\hat{x}+s=-\lambda\beta\check{T}(\hat{x})+(1-\beta)\hat{x}+s=-\check{K}(\hat{x})$ from (12.2.3(p.68)), hence $\hat{\tilde{K}}(x)=\check{K}(\hat{x})$.

(f) From (5.1.15(p.23)) and (d) we have $\tilde{\mathcal{L}}(s)=-\hat{\tilde{L}}(\lambda\beta\mu+s)=-\check{L}(\widehat{\lambda\beta\mu+s})=-\check{L}(-\lambda\beta\mu-s)=-\check{L}(\lambda\beta\hat{\mu}-s)=-\check{L}(\lambda\beta\check{\mu}-s)$ due to (b), hence from (12.2.4(p.68)) we obtain $\tilde{\mathcal{L}}(s)=-\check{\mathcal{L}}(s)$, hence $\hat{\tilde{\mathcal{L}}}(s)=\check{\mathcal{L}}(s)$.

(g) Since $\tilde{L}(x_{\tilde{L}})=0$ by definition, we have $\hat{\tilde{L}}(x_{\tilde{L}})=0$, which can be rewritten as $\check{L}(\hat{x}_{\tilde{L}})=0$ from (d), implying that $\check{L}(x)=0$ has the solution $x_{\check{L}}=\hat{x}_{\tilde{L}}$ by definition.

(h) Since $\tilde{K}(x_{\tilde{K}})=0$ by definition, we have $\hat{\tilde{K}}(x_{\tilde{K}})=0$, which can be rewritten as $\check{K}(\hat{x}_{\tilde{K}})=0$ from (e), implying that $\check{K}(x)=0$ has the solution $x_{\check{K}}=\hat{x}_{\tilde{K}}$ by definition.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}})=0$ by definition, we have $\hat{\tilde{\mathcal{L}}}(s_{\tilde{\mathcal{L}}})=0$, which can be rewritten as $\check{\mathcal{L}}(s_{\tilde{\mathcal{L}}})=0$ from (f), implying that $\check{\mathcal{L}}(s)=0$ has the solution $s_{\check{\mathcal{L}}}=s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (5.1.16(p.23)) we have $\tilde{\kappa}=-\lambda\beta\hat{\tilde{T}}(0)+s$, which can be rewritten as $\tilde{\kappa}=-\lambda\beta\check{T}(\hat{0})+s$ from (c), hence $\tilde{\kappa}=-\lambda\beta\check{T}(0)+s=-\check{\kappa}$ from (12.2.5(p.68)), thus $\hat{\tilde{\kappa}}=\check{\kappa}$. ∎

Definition 12.3.2 (correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 12.3.2(p.70) by its right-hand the *correspondence replacement operation* $\tilde{\mathcal{C}}_{\mathbb{R}}$. □

Definition 12.3.3 (reversible element and non-reversible element) It should be noted that the left-hand of each of the equalities in Lemmas 12.3.1(p.69) (i) and 12.3.2(p.70) (i) have not the hat symbol " ˆ ". In other words, $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ are not subjected to the reverse. For the reason, let us refer to each of $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ as the *non-reversible element* and to each of all the other elements as the *reversible element*. □

12.3.2 Identity Replacement

Lemma 12.3.3 ($\mathcal{I}_{\mathbb{R}}$) *The left-hand side of each equality below is for $\check{F}\in\check{\mathscr{F}}$ corresponding to any $F\in\mathscr{F}$ and the right-hand side is for $F\in\mathscr{F}$ such that $F\equiv\check{F}\cdots$[1*].*†

(a) $\check{F}(\xi)=F(\xi)\cdots$[2*] *and* $\check{f}(\xi)=f(\xi)\cdots$[3*] *for any* ξ.
(b) $\check{a}=a,\ \check{\mu}=\mu,\ \check{b}=b$.
(c) $\check{\tilde{T}}(x)=\tilde{T}(x)$.
(d) $\check{\tilde{L}}(x)=\tilde{L}(x)$.
(e) $\check{\tilde{K}}(x)=\tilde{K}(x)$.
(f) $\check{\tilde{\mathcal{L}}}(s)=\tilde{\mathcal{L}}(s)$.
(g) $x_{\check{\tilde{L}}}=x_{\tilde{L}}$.
(h) $x_{\check{\tilde{K}}}=x_{\tilde{K}}$.
(i) $s_{\check{\tilde{\mathcal{L}}}}=s_{\tilde{\mathcal{L}}}$.
(j) $\check{\tilde{\kappa}}=\tilde{\kappa}$. □

• *Proof* (a) Clear from [1*].

(b) Obvious from (a).

(c) Evident from (12.2.6(p.69)), (5.1.12(p.23)), and [3*].

(d) From (12.2.7(p.69)) and (c) we have $\check{\tilde{L}}(x)=\lambda\beta\tilde{T}(x)+s$, hence $\check{\tilde{L}}(x)=\tilde{L}(x)$ from (5.1.13(p.23)).

(e) From (12.2.8(p.69)) and (c) we have $\check{\tilde{K}}(x)=\lambda\beta\tilde{T}(x)-(1-\beta)x+s$, hence $\check{\tilde{K}}(x)=\tilde{K}(x)$ from (5.1.14(p.23)).

---

†The mere replacement of the symbol η by ξ.

†See Lemma 12.1.1(p.68) (b,c).

(f) From (12.2.9(p.69)) and (d) we have $\check{\tilde{\mathcal{L}}}(s) = \tilde{L}(\lambda\beta\check{\mu} + s)$, hence $\check{\tilde{\mathcal{L}}}(s) = \tilde{L}(\lambda\beta\mu + s)$ from (b), so $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$ (5.1.15(p.23)).

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $\check{\tilde{L}}(x_{\tilde{L}}) = 0$ from (d), hence $\check{\tilde{L}}(x) = 0$ has the solution $x_{\check{\tilde{L}}} = x_{\tilde{L}}$.

(h) Since $\tilde{K}(x_{\tilde{K}}) = 0$ by definition, we have $\check{\tilde{K}}(x_{\tilde{K}}) = 0$ from (e), hence $\check{\tilde{K}}(x) = 0$ has the solution $x_{\check{\tilde{K}}} = x_{\tilde{K}}$.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $\check{\tilde{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), hence $\check{\tilde{\mathcal{L}}}(x) = 0$ has the solution $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (12.2.10(p.69)) and (c) with $x = 0$ we have (5.1.16(p.23)). ∎

Definition 12.3.4 (identity replacement operation $\mathcal{I}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand side of each equality in Lemma 12.3.3(p.71) by its right-hand side the *identity replacement operation* $\mathcal{I}_{\mathbb{R}}$. □

Lemma 12.3.4 ($\tilde{\mathcal{I}}_{\mathbb{R}}$) *The left-hand side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right-hand side is for $F \in \mathscr{F}$ such that $F \equiv \check{F} \cdots [1^*]$.*†

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ.

(b) $\check{a} = a,\ \check{\mu} = \mu,\ \check{b} = b$.

(c) $\check{T}(x) = T(x)$.

(d) $\check{L}(x) = L(x)$.

(e) $\check{K}(x) = K(x)$.

(f) $\check{\mathcal{L}}(s) = \mathcal{L}(s)$.

(g) $x_{\check{L}} = x_L$.

(h) $x_{\check{K}} = x_K$.

(i) $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$.

(j) $\check{\kappa} = \kappa$. □

• *Proof* (a) Clear from [1*].

(b) Obvious from (a).

(c) Evident from (12.2.1(p.68)), (5.1.2(p.23)), and [3*].

(d) From (12.2.2(p.68)) and (c) we have $\check{L}(x) = \lambda\beta T(x) - s$, hence $\check{L}(x) = L(x)$ from (5.1.3(p.23)).

(e) From (12.2.3(p.68)) and (c) we have $\check{K}(x) = \lambda\beta T(x) - (1-\beta)x - s$, hence $\check{K}(x) = K(x)$ from (5.1.4(p.23)).

(f) From (12.2.4(p.68)) and (d) we have $\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\mu - s)$, hence $\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\mu + s)$ from (b), so $\mathcal{L}(s) = \check{L}(\lambda\beta\mu + s)$, hence $\check{\mathcal{L}}(s) = \mathcal{L}(s)$ from (5.1.5(p.23)).

(g) Since $L(x_L) = 0$ by definition, we have $\check{L}(x_L) = 0$ from (d), hence $\check{L}(x) = 0$ has the solution $x_{\check{L}} = x_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $\check{K}(x_K) = 0$ from (e), hence $\check{K}(x) = 0$ has the solution $x_{\check{K}} = x_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $\check{\mathcal{L}}(s_{\mathcal{L}}) = 0$ from (f), hence $\check{\mathcal{L}}(x) = 0$ has the solution $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$ by definition.

(j) From (12.2.5(p.68)) and (c) with $x = 0$ we have (5.1.6(p.23)). ∎

Definition 12.3.5 (identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$) Let us call the operation of replacing the left-hand of each equality in Lemma 12.3.4(p.72) by its right-hand the *identity replacement operation* $\tilde{\mathcal{I}}_{\mathbb{R}}$. □

12.4 Attribute Vector

Closely looking into the contents of all assertions $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \in \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see `Tom`'s 11.2.1(p.59) and 11.2.2(p.60)), we can immediately see that each assertion is described by using a part or all of the following twelve kinds of elements;

$$a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t$$

where V_t represents the sequence $\{V_t,\ t = 1, 2, \cdots\}$ generated from $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.5.1(p.39) (I)). Let us call each element the *attribute element* and the vector of them the *attribute vector*, denoted by

$$\boldsymbol{\theta}(A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = (a, \mu, b,\ x_L,\ x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t). \qquad (12.4.1)$$

In addition, also for the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ we can employ the similar definition, denoted by

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = (a, \mu, b,\ x_L,\ x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t). \qquad (12.4.2)$$

†See Lemma 12.1.1(p.68) (b,c).

12.5 Scenario[$\mathbb{R}$]

In this section we write up a scenario deriving an assertion on $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][A] (buying model with $\mathbb{R}$-mechanism) from a given assertion on M:1[$\mathbb{R}$][A] (selling model with $\mathbb{R}$-mechanism). Let us refer to this as the scenario of Type $\mathbb{R}$, denoted by Scenario[$\mathbb{R}$].

■ Step 1 (*opening*)

∘ The system of optimality equations for M:1[$\mathbb{R}$][A] is given by Table 6.5.1(p.39) (I), i.e.,

$$\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{V_1 = \beta\mu - s,\ V_t = \max\{K(V_{t-1}) + V_{t-1},\ \beta V_{t-1}\},\quad t > 1\}. \tag{12.5.1}$$

∘ Let us consider an assertion $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^\dagger$ included in Tom 11.2.1(p.59) or Tom 11.2.2(p.60), which can be written in general as

$$A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\mathsf{S} \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\} \quad (\text{see } (11.3.8(\text{p.63}))) \tag{12.5.2}$$

$$= \{\mathsf{S} \text{ is true on } \mathscr{C}\langle A_{\texttt{Tom}}\rangle\} \quad (\text{see } (11.3.10(\text{p.63}))). \tag{12.5.3}$$

To facilitate the understanding of the discussion that follows, let us use the following example.[‡]

$$\mathsf{S} = \langle V_t + s_{\mathcal{L}} + x_L + \kappa + a + \mu + b \ge 0,\ t > 0\rangle. \tag{12.5.4}$$

∘ The attribute vector of the assertion $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by (12.4.1(p.72)), i.e.,

$$\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = (a, \mu, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t). \tag{12.5.5}$$

■ Step 2 (*reverse operation $\mathcal{R}$*)

∘ Applying the reverse operation $\mathcal{R}$ (see Section 12.1.1(p.67)) to (12.5.1(p.73)) produces

$$\begin{aligned}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_1 = -\beta\hat{\mu} - s,\ -\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\},\ t > 1\}\\ &= \{-\hat{V}_1 = -\beta\hat{\mu} - s,\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t > 1\}\\ &= \{\hat{V}_1 = \beta\hat{\mu} + s,\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t > 1\}.\end{aligned} \tag{12.5.6}$$

∘ Applying $\mathcal{R}$ to (12.5.2(p.73)) and (12.5.3(p.73)) yields to

$$\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\} \tag{12.5.7}$$

$$= \{\mathcal{R}[\mathsf{S}] \text{ is true on } \mathscr{C}\langle A_{\texttt{Tom}}\rangle\}. \tag{12.5.8}$$

For our example we have:

$$\begin{aligned}\mathcal{R}[\mathsf{S}] &= \langle -\hat{V}_t + s_{\mathcal{L}} - \hat{x}_L - \hat{\kappa} - \hat{a} - \hat{\mu} - \hat{b} \ge 0,\ t > 0\rangle^{\S}\\ &= \langle \hat{V}_t - s_{\mathcal{L}} + \hat{x}_L + \hat{\kappa} + \hat{a} + \hat{\mu} + \hat{b} \le 0,\ t > 0\rangle.\end{aligned} \tag{12.5.9}$$

∘ The attribute vector of the assertion $\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{R}$ to (12.5.5(p.73)), i.e.,

$$\boldsymbol{\theta}(\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]) \overset{\text{def}}{=} \mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{12.5.10}$$

$$= (\hat{a}, \hat{\mu}, \hat{b}, \hat{x}_L, \hat{x}_K, s_{\mathcal{L}}, \hat{\kappa}, \hat{T}, \hat{L}, \hat{K}, \hat{\mathcal{L}}, \hat{V}_t). \tag{12.5.11}$$

■ Step 3 (*correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$*)

∘ Here let us consider the application of the correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$, i.e., the replacement of the left-hand side of each equality in Lemma 12.3.1(p.69),

$$f(\xi), \hat{a},\ \hat{\mu},\ \hat{b},\ \hat{x}_L,\ \hat{x}_K,\ s_{\mathcal{L}},\ \hat{\kappa},\ \hat{T}(x),\ \hat{L}(x),\ \hat{K}(x),\ \hat{\mathcal{L}}(s) \cdots (1^*),$$

by its right-hand,

$$\check{f}(\hat{\xi}), \check{b},\ \check{\mu},\ \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}}(\hat{x}),\ \check{\tilde{L}}(\hat{x}),\ \check{\tilde{K}}(\hat{x}),\ \check{\tilde{\mathcal{L}}}(s) \cdots (2^*),$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ corresponding to the $F \in \mathscr{F}$.

∘ Applying $\mathcal{C}_{\mathbb{R}}$ to (12.5.6(p.73)) leads to

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_1 = \beta\check{\mu} + s,\ \hat{V}_t = \min\{\check{\tilde{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t > 1\}. \tag{12.5.12}$$

[†]See Def. 11.3.1(p.63) (c) for the symbol "Tom" in $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$.
[‡]The example is a hypothetical assertion which is not contained in $\mathscr{A}_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$; It is used merely for explanatory convenience.
[§]Note Def. 12.3.3(p.71).

○ Applying $\mathcal{C}_{\mathbb{R}}$ to $\mathcal{R}[\mathsf{S}]$ in (12.5.9(p.73)), we have

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] = \langle\, \hat{V}_t - s_{\check{\tilde{\mathcal{L}}}} + x_{\check{\tilde{L}}} + \check{\tilde{\kappa}} + \check{b} + \check{\mu} + \check{a} \le 0,\ \ t > 0\,\rangle. \tag{12.5.13}$$

Now, let us note here that the application of $\mathcal{C}_{\mathbb{R}}$ inevitably transforms

$$\text{“} F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\text{”} \quad \text{in (12.5.2(p.73))}$$

into

$$\text{“} \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}} \textit{ corresponding to } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\text{”} \tag{12.5.14}$$

where

$$\check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \overset{\text{def}}{=} \{\check{F} \mid F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}\} \subseteq \{\check{F} \mid F \in \mathscr{F}\} = \check{\mathscr{F}} \qquad \text{(see (12.1.3(p.67)))}. \tag{12.5.15}$$

Hence, applying $\mathcal{C}_{\mathbb{R}}$ to (12.5.7(p.73)) produces

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ = \ \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}} \text{ corresponding to } F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}. \tag{12.5.16}$$

Now, since the phrase “$\check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}$” is *implicitly* accompanied with the phrase “*corresponding to* $F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}$”, the latter phrase becomes redundant. Accordingly, (12.5.16(p.74)) can be rewritten as

$$\begin{aligned}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ &= \ \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P} \text{ and } \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\}\\ &= \ \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\}\end{aligned} \tag{12.5.17}$$

where

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \ = \ \{(\boldsymbol{p}, \check{F}) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P}, \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\} \quad \text{(compare (11.3.5(p.63)))}. \tag{12.5.18}$$

○ The attribute vector of $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{C}_{\mathbb{R}}$ to (12.5.10(p.73)), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]) \ &= \ \mathcal{C}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})]\\ &= \ (\check{b}, \check{\mu}, \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}}, \check{\tilde{\kappa}}, \check{\tilde{T}}, \check{\tilde{L}}, \check{\tilde{K}}, \check{\tilde{\mathcal{L}}}, V_t).\end{aligned} \tag{12.5.19}$$

■ Step 4 (*identity replacement operation* $\mathcal{I}_{\mathbb{R}}$)

○ Here let us consider the application of the identity replacement operation $\mathcal{I}_{\mathbb{R}}$, i.e., the replacement of the left-hand side of each equality in Lemma 12.3.3(p.71),

$$\check{f}(\xi),\ \check{a},\ \check{\mu},\ \check{b},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}}(x),\ \check{\tilde{L}}(x),\ \check{\tilde{K}}(x),\ \check{\tilde{\mathcal{L}}}(s) \cdots (1^*),$$

by its right-hand side,

$$f(\xi),\ a,\ \mu,\ b,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa}\ \tilde{T}(x),\ \tilde{L}(x),\ \tilde{K}(x),\ \tilde{\mathcal{L}}(s) \cdots (2^*),$$

where (1^*) is for any $F \in \mathscr{F}$ and (2^*) is for $\check{F} \in \check{\mathscr{F}}$ which is identical to the $\check{F} \in \mathscr{F}$, i.e., $\check{F} \equiv F \cdots (1)$ (see Lemma 12.1.1(p.68) (c)).

○ Applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.12(p.73)) yields

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ = \ \{\hat{V}_1 = \beta\mu + s,\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ \ t > 1\}. \tag{12.5.20}$$

Now, we have $\hat{V}_1 = \beta\mu + s = V_1$ from (6.5.3(p.39)). Suppose $\hat{V}_{t-1} = V_{t-1}$. Then, since $\hat{V}_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$ from (6.5.4(p.39)), by induction $\hat{V}_t = V_t$ for $t > 0$. Thus (12.5.20(p.74)) can be rewritten as

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ = \ \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\},$$

which is the same as $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.5.1(p.39) (II)). Thus we have

$$\begin{aligned}\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \ &= \ \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\ \}] &(12.5.21)\\ &= \ \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ t > 1\}. &(12.5.22)\end{aligned}$$

○ Applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.17(p.74)) yields (note $\check{F} \equiv F$ in (1))

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ = \ \{\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\,\}. \tag{12.5.23}$$

Applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.13(p.74)) yields

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] = \langle V_t - s_{\tilde{\mathcal{L}}} + x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \le 0,\ \ t > 0\rangle. \tag{12.5.24}$$

Now V_t within $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}]$ is generated from $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$, hence (12.5.23(p.74)) can be regarded as the assertion on $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ (see Remark 6.1.1(p.27)). Thus, we have

$$\begin{aligned}A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \ &= \ \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] &(12.5.25)\\ &= \ \{\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\}. &(12.5.26)\end{aligned}$$

○ The attribute vector of $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by applying $\mathcal{I}_{\mathbb{R}}$ to (12.5.19(p.74)), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) \ &= \ \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})]\\ &= \ (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t),\end{aligned} \tag{12.5.27}$$

■ Step 5 (*symmetry transformation operation* $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$)

Lining up the four attribute vectors in Steps 1-4, we have the following:

$$\begin{array}{lll}
\text{Step 1: } \boldsymbol{\theta}(a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t) & (\leftarrow (12.5.5(\text{p.73}))) \\
\quad\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow \mathcal{R} \\
\text{Step 2: } \boldsymbol{\theta}(\hat{a},\ \hat{\mu},\ \hat{b},\ \hat{x}_L,\ \hat{x}_K,\ s_{\mathcal{L}},\ \hat{\kappa},\ \hat{T},\ \hat{L},\ \hat{K},\ \hat{\mathcal{L}},\ \hat{V}_t) & (\leftarrow (12.5.11(\text{p.73}))) \\
\quad\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow \mathcal{C}_{\mathbb{R}} \\
\text{Step 3: } \boldsymbol{\theta}(\check{b},\ \check{\mu},\ \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}},\ \check{\tilde{L}},\ \check{\tilde{K}},\ \check{\tilde{\mathcal{L}}},\ V_t) & (\leftarrow (12.5.19(\text{p.74}))) \\
\quad\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow \mathcal{I}_{\mathbb{R}} \\
\text{Step 4: } \boldsymbol{\theta}(b,\ \mu,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ V_t) & (\leftarrow (12.5.27(\text{p.74})))
\end{array} \tag{12.5.28}$$

The above flow can be eventually reduced to

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} \overset{\text{def}}{=} \left\{ \begin{array}{l} a,\ \mu,\ b,\ x_{L_{\mathbb{R}}}, x_{K_{\mathbb{R}}},\ s_{\mathcal{L}_{\mathbb{R}}}, \kappa_{\mathbb{R}}, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}},\ V_t \\ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr \\ b,\ \mu,\ a,\ x_{\tilde{L}_{\mathbb{R}}}, x_{\tilde{K}_{\mathbb{R}}},\ s_{\tilde{\mathcal{L}}_{\mathbb{R}}}, \tilde{\kappa}_{\mathbb{R}}, \tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}},\ V_t \end{array} \right\} \tag{12.5.29}$$

called the *symmetry transformation operation*, which can be regarded as the successive application of the three operations, i.e., " $\mathcal{R} \to \mathcal{C}_{\mathbb{R}} \to \mathcal{I}_{\mathbb{R}}$ ". Hence, defining

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}, \tag{12.5.30}$$

we can rewrite (12.5.25(p.74)) as

$$\begin{aligned} A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} &= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}] \\ &= \{\tilde{\mathsf{S}} \text{ holds on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\ \} \end{aligned} \tag{12.5.31}$$

where

$$\tilde{\mathsf{S}} \overset{\text{def}}{=} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathsf{S}]. \tag{12.5.32}$$

Then, from (12.5.24(p.74)) we have

$$\tilde{\mathsf{S}} = \langle V_t - s_{\tilde{\mathcal{L}}} + x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \le 0,\ \ t > 0\rangle. \tag{12.5.33}$$

Furthermore, (12.5.21(p.74)) can be rewritten as

$$\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\ \}]. \tag{12.5.34}$$

In addition, (12.5.27(p.74)) can be rewritten as

$$\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\})] \tag{12.5.35}$$

$$= (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). \tag{12.5.36}$$

From all the above we see that Scenario[$\mathbb{R}$] starting with (12.5.3(p.73)) finally ends up with (12.5.31(p.75)), which can be alternatively rewritten as respectively (see (11.3.7(p.63)))

$$A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\texttt{Tom}}\rangle \quad \text{(see (11.3.10(p.63))),} \tag{12.5.37}$$

$$A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle.$$

From the above two results and (12.5.34(p.75)) we eventually obtain the following lemma.

Lemma 12.5.1 *Let* $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\texttt{Tom}}\rangle$. *Then* $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle$ *where*

$$A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}].\ \ \square \tag{12.5.38}$$

Remark 12.5.1 (simple structure of $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$) At a glance, the symmetry transformation operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ seems to be rather complicated, however it can be simply prescribed as follows.

- Firstly, apply the reverse operation $\mathcal{R}$ to all *reversible* elements (see Defs 12.3.3(p.71)) appearing within the description of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ (see Tom's 11.2.1(p.59) and 11.2.2(p.60)).
- Next, replace each of all elements, whether resultant ones (reversible) or non-reversible ones, with the right side of its corresponding equality in Lemma 12.3.1(p.69) (correspondence replacement operation $\mathcal{C}_{\mathbb{R}}$).
- Finally, remove the check sign " ˇ " from all the *replaced* symbols (identity replacement operation $\mathcal{I}_{\mathbb{R}}$). □

■ Step 6 (*Completeness of* $\tilde{\mathcal{T}}\mathtt{om}$)

aggregation scenario

↓

★ Condition Space $\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$

Applying Lemma 12.5.1(p.75) to any assertion $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ included in Tom's 11.2.1(p.59) and 11.2.2(p.60), we have $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$, which are given by Tom's 12.7.1(p.81) and 12.7.2(p.81). Below let us define

$$\mathtt{Tom}_1 \overset{\text{def}}{=} \mathtt{Tom}\ 12.7.1\text{(p.81)} \quad \text{and} \quad \mathtt{Tom}_2 = \mathtt{Tom}\ 12.7.2\text{(p.81)}.$$

Furthermore, in general let

$$\mathtt{Tom} \overset{\text{def}}{=} \mathtt{Tom}_1, \mathtt{Tom}_2, \cdots. \tag{12.5.39}$$

Here, as one corresponding to (12.5.18(p.74)), let us define

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}_i}\rangle \;=\; \{(\boldsymbol{p}, \check{F}) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}_i}} \subseteq \mathscr{P}, \check{F} \in \check{\mathscr{F}}_{A\cdot\mathtt{Tom}_i|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\}, \quad i = 1, 2, \cdots. \tag{12.5.40}$$

In general, let

$$\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \;=\; \{(\boldsymbol{p}, \check{F}) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\mathtt{Tom}}} \subseteq \mathscr{P}, \check{F} \in \check{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}} \subseteq \check{\mathscr{F}}\}. \tag{12.5.41}$$

In addition, let us define

$$\mathtt{Tom}_i \;\overset{\text{def}}{=}\; \{A^1_{\mathtt{Tom}_i}, A^2_{\mathtt{Tom}_i}, \cdots\} = \{A_{\mathtt{Tom}_i}\},$$
$$\tilde{\mathcal{T}}\mathtt{om} \;\overset{\text{def}}{=}\; \{\mathtt{Tom}_1, \mathtt{Tom}_2, \cdots\} = \{\mathtt{Tom}\}.$$

Then, as one corresponding to (11.3.20(p.64)), let us define

$$\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle \;\overset{\text{def}}{=}\; \cup_{j=1,2,\cdots}\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle = \cup_{A_{\mathtt{Tom}_i}\in\mathtt{Tom}_i}\check{\mathscr{C}}\langle A_{\mathtt{Tom}_i}\rangle, \quad i = 1, 2, \cdots, \tag{12.5.42}$$

which is the *aggregation* of $\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle$, $j = 1, 2, \cdots$, into $\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle$. This can be depicted as in Figure 12.5.1(p.76) below:

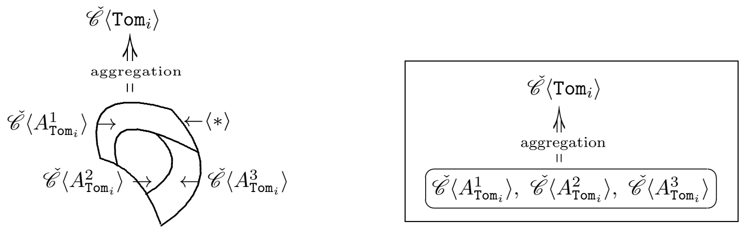


Figure 12.5.1: Aggregation of $\check{\mathscr{C}}\langle A^1_{\mathtt{Tom}_i}\rangle$, $\check{\mathscr{C}}\langle A^2_{\mathtt{Tom}_i}\rangle$,$\check{\mathscr{C}}\langle A^3_{\mathtt{Tom}_i}\rangle$ into $\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle$

★ Condition Space $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

As one corresponding to (11.3.23(p.64)), let us define

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle \;\overset{\text{def}}{=}\; \cup_{i=1,2,\cdots}\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle = \cup_{\mathtt{Tom}\in\tilde{\mathcal{T}}\mathtt{om}}\check{\mathscr{C}}\langle\mathtt{Tom}\rangle, \tag{12.5.43}$$

called the *condition space* of $\tilde{\mathcal{T}}\mathtt{om}$, which is the aggregation of $\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$, depicted as in Figure 12.5.2(p.76) below (compare Figure 11.3.3(p.65)).

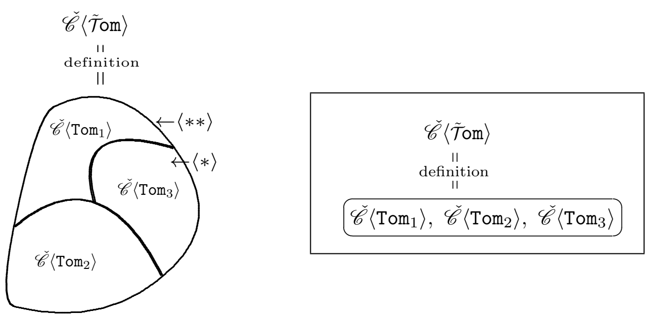


Figure 12.5.2: Condition space $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

In the above figure, the *small* deformed circle $\langle *\rangle$ is the same as the deformed circle $\langle *\rangle$ in Figure 12.5.1(p.76).

★ Construction of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Using (12.5.42(p.76)), from (12.5.43(p.76)) we have

$$\begin{aligned}\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle &= \cup_{i=1,2,\cdots}\cup_{j=1,2,\cdots}\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle & (12.5.44)\\ &= \cup_{\mathtt{Tom}\in\tilde{\mathcal{T}}\mathtt{om}}\cup_{j=1,2,\cdots}\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}}\rangle & (12.5.45)\\ &= \cup_{\mathtt{Tom}\in\tilde{\mathcal{T}}\mathtt{om}}\cup_{A_{\mathtt{Tom}}\in\mathtt{Tom}}\check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle & (12.5.46)\end{aligned}$$

Then, fetching Figure 12.5.1(p.76) in Figure 12.5.2(p.76) , we see that (12.5.44(p.77)) produces Figure 12.5.3(p.77) below, demonstrating the *aggregation* of $\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle$ to $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$.

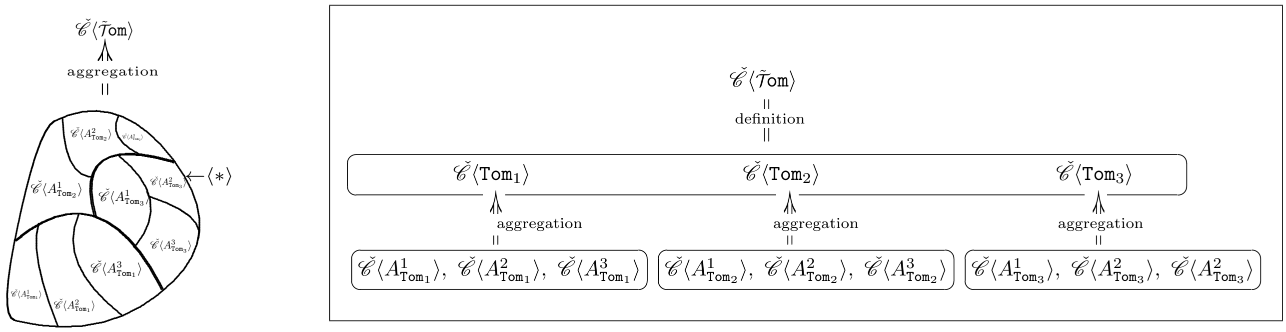


Figure 12.5.3: The aggregation of $\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle$ into $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$

Figure 12.5.3(p.77) above implies that first

$$\text{“aggregating } \boxed{\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle},\ j=1,2,3, \text{ for } i=1,2,3 \text{ produces } \boxed{\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle}\text{”}$$

and then

$$\text{“aggregating } \boxed{\check{\mathscr{C}}\langle\mathtt{Tom}_i\rangle},\ i=1,2,3, \text{ produces } \boxed{\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle}\text{”.}$$

The above two successive aggregating procedures eventually yields

$$\text{“ aggregating } \boxed{\check{\mathscr{C}}\langle A^j_{\mathtt{Tom}_i}\rangle} \text{ for } i,j=1,2,3 \text{ produces } \boxed{\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle}\text{ ”,} \tag{12.5.47}$$

Moreover, note that $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is what is aggregated over $\boxed{\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle}$, i.e.,

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle\ . \tag{12.5.48}$$

From (11.3.27(p.65)) and (12.5.48(p.77)) we see that aggregating Lemma 12.5.1(p.75) produces the following lemma.

Lemma 12.5.2 *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}].\ \square$$

★ Completeness of $\tilde{\mathcal{T}}\mathtt{om}$

Here recall that the completeness of $\mathcal{T}\mathtt{om}$ on $\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle$ in the *breakdown scenario* was set as a necessary condition in the breakdown scenario (see Remark 11.3.1(p.66)), i.e.,

$$\mathscr{C}\langle\mathcal{T}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}. \tag{12.5.49}$$

However, the question arises whether or not this completeness is inherited also to the *aggregation scenario*, or equivalently whether or not the equality below hold;

$$\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P}\times\mathscr{F}. \tag{12.5.50}$$

Below let us show that this equality holds in fact.

● *Proof* Note here that for any given $\check{F}\in\check{\mathscr{F}}$ there exists a $F\in\mathscr{F}$ such that $F\equiv\check{F}\cdots\mathbf{(1)}$ (see Lemma 12.1.1(p.68) (b)) and that for any given $F\in\mathscr{F}$ there exists a $\check{F}\in\check{\mathscr{F}}$ such that $\check{F}\equiv F\cdots\mathbf{(2)}$ (see Lemma 12.1.1(p.68) (c)).

∘ First, since $\mathscr{P}\times\mathscr{F}$ is the set of all possible $(\boldsymbol{p},F)$ due to its definition (see (4.4.3(p.21))), clearly we have $\check{\mathscr{C}}\langle\tilde{\mathcal{T}}\mathtt{om}\rangle\subseteq\mathscr{P}\times\mathscr{F}\cdots\mathbf{(3)}$.†

.

∘ Consider any $(\boldsymbol{p}, F) \in \mathscr{P} \times \mathscr{F} \cdots$ **(4)**. Then, since $(\boldsymbol{p}, F) \in \mathscr{C}\langle \mathcal{T}\mathtt{om}\rangle$ due to (12.5.49(p.77)), we have $(\boldsymbol{p}, F) \in \mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ for at least one $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ due to (11.3.26(p.65)). Hence, since $F \in \mathscr{F}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$ due to (11.3.9(p.63)), we have $\breve{F} \in \breve{\mathscr{F}}_{A_{\mathtt{Tom}}|\boldsymbol{p}}$ due to (12.1.3(p.67)), hence $(\boldsymbol{p}, \breve{F}) \in \mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ due to (12.5.18(p.74)), thus $(\boldsymbol{p}, F) \in \breve{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle$ due to (2), hence $(\boldsymbol{p}, F) \in \breve{\mathscr{C}}\langle \tilde{\mathcal{T}}\mathtt{om}\rangle$ due to (12.5.46(p.77)). Accordingly, from this fact and (4) we have $\mathscr{P} \times \mathscr{F} \subseteq \breve{\mathscr{C}}\langle \tilde{\mathcal{T}}\mathtt{om}\rangle \cdots$ **(5)**.

From (6) and (5) we obtain $\breve{\mathscr{C}}\langle \tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$. ▮

Let us refer to the equality (12.5.50(p.77)) as the <u>*completeness*</u> of $\tilde{\mathcal{T}}\mathtt{om}$ on $\breve{\mathscr{C}}\langle \tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$. Then (12.5.47(p.77)) can be rewritten as

$$\text{“aggregating } \breve{\mathscr{C}}\langle A^{j}_{\mathtt{Tom}_i}\rangle \text{ for } i, j = 1, 2, 3, \text{ produces } \breve{\mathscr{C}}\langle \tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}\text{”}, \tag{12.5.51}$$

hence Figure 12.5.3(p.77) can be rewritten as Figure 12.5.4(p.78) below.

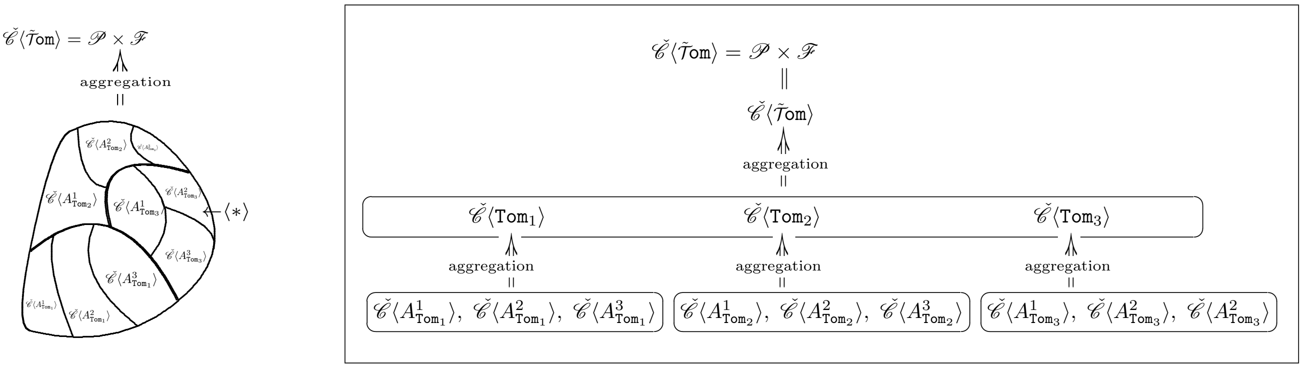


Figure 12.5.4: The aggregation of $\breve{\mathscr{C}}\langle A^{j}_{\mathtt{Tom}_i}\rangle$ into $\breve{\mathscr{C}}\langle \tilde{\mathcal{T}}\mathtt{om}\rangle = \mathscr{P} \times \mathscr{F}$

From (12.5.49(p.77)) and (12.5.50(p.77)) we can rewrite Lemma 12.5.2(p.77) as follows.

Lemma 12.5.3 *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \;\square$$

■ Step 7 (*symmetry theorem* $(\mathbb{R} \to \tilde{\mathbb{R}})$)

From (12.5.49(p.77)) and (12.5.50(p.77)), we see that Lemma 12.5.2(p.77) can be rewritten as Theorem 12.5.1(p.78) below.

Theorem 12.5.1 (symmetry theorem $(\mathbb{R} \to \tilde{\mathbb{R}})$) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P}\times\mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \;\square \tag{12.5.52}$$

Then, clearly the attribute vector of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ becomes as follows (see (12.5.35(p.75)))

$$\begin{aligned} \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) &= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] && (12.5.53)\\ &= (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) && (12.5.54) \end{aligned}$$

↑

aggregation scenario

12.6 Derivation of $\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}$, and $\tilde{\kappa}_{\mathbb{R}}$

To begin with, let us note here the fact that Scenario[$\mathbb{R}$] with $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ is applicable for an assertion $A\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ related to the attribute vector (see Section 12.4(p.72))

$$\boldsymbol{\theta} = (a, \mu, b,\ x_L,\ x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t).$$

This fact implies that the scenario can be always applied also to any assertions involved with the attribute vector $\boldsymbol{\theta}$. Accordingly, applying the scenario to any assertions on $T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}$, and $\kappa_{\mathbb{R}}$ yields the corresponding assertions on $\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}$ and $\tilde{\kappa}_{\mathbb{R}}$, i.e.,

$$\mathscr{A}\{\tilde{T}_{\mathbb{R}}, \tilde{L}_{\mathbb{R}}, \tilde{K}_{\mathbb{R}}, \tilde{\mathcal{L}}_{\mathbb{R}}, \tilde{\kappa}_{\mathbb{R}}\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}].$$

Accordingly, we have the following lemma:

---

†In fact, this can be proven as follows. From (12.5.18(p.74)) we have $\breve{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \subseteq \{(\boldsymbol{p}, \breve{F}) \mid \boldsymbol{p} \subseteq \mathscr{P}, \breve{F} \subseteq \breve{\mathscr{F}}\}$ for any $A_{\mathtt{Tom}}$, hence due to (1) we get $\breve{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle \subseteq \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}, F \in \breve{\mathscr{F}}\} = \mathscr{P} \times \breve{\mathscr{F}} = \mathscr{P} \times \mathscr{F}$ due to $\breve{\mathscr{F}} = \mathscr{F}$ from Lemma 12.1.1(p.68) (a). Accordingly, from (12.5.46(p.77)) we obtain $\breve{\mathscr{C}}\langle \tilde{\mathcal{T}}\mathtt{om}\rangle \subseteq \cup_{\mathtt{Tom}\in\tilde{\mathcal{T}}\mathtt{om}} \cup_{A_{\mathtt{Tom}}\in\mathtt{Tom}} \mathscr{P} \times \mathscr{F} = \mathscr{P} \times \mathscr{F} \cdots$ **(6)**.

Lemma 12.6.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$) *For any $F \in \mathscr{F}$:*
(a) $\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{T}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(e) $\tilde{T}(x) + x$ *strictly increasing on* $(-\infty, b]$.
(f) $\tilde{T}(x) = \mu - x$ *on* $[b, \infty)$ *and* $\tilde{T}(x) < \mu - x$ *on* $(-\infty, b)$.
(g) $\tilde{T}(x) < 0$ *on* (a, ∞) *and* $\tilde{T}(x) = 0$ *on* $(-\infty, a]$.
(h) $\tilde{T}(x) \leq \min\{0, \mu - x\}$ *on* $x \in (-\infty, \infty)$.
(i) $\tilde{T}(0) = 0$ *if* $a > 0$ *and* $\tilde{T}(0) = \mu$ *if* $b < 0$.
(j) $\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$.
(k) $\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(l) *If* $x > y$ *and* $b > y$, *then* $\tilde{T}(x) + x > \tilde{T}(y) + y$.
(m) $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta < 1$.
(n) $b > \mu$. □

● *Proof by symmetry* The lemma, excluding (a,n), can be easily obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128))) to Lemmas 10.1.1(p.53) as shown below.

(a) Evident from the fact that $\min\{\boldsymbol{\xi} - x, 0\}$ in (5.1.11(p.23)) is continuous on $(-\infty, \infty)$.

(b) Lemma 10.1.1(p.53) (b) can be rewritten as $A = \{T(x) \geq \tilde{T}(x')$ for $x < x'\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\hat{T}(x) \geq -\hat{T}(x')$ for $-\hat{x} < -\hat{x}'\} = \{\hat{T}(\hat{x}) \leq \hat{T}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\hat{T}}(\hat{x}) \leq \check{\hat{T}}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) \leq \tilde{T}(\hat{x}')$ for $\hat{x} > \hat{x}'\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A = \{\tilde{T}(x) \leq \tilde{T}(x')$ for $x > x'\}$, meaning that $\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$.

(c-e) Almost the same as the proof of (b)

(f) Let the former half of Lemma 10.1.1(p.53) (f) can by rewritten as $A = \{T(x) = \mu - x$ for $x \leq a\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\hat{T}(x) = -\hat{\mu} + \hat{x}$ for $-\hat{x} \leq -\hat{a}\} = \{\hat{T}(x) = \hat{\mu} - \hat{x}$ for $\hat{x} \geq \hat{a}\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\hat{T}}(\hat{x}) = \check{\mu} - \hat{x}$ for $\hat{x} \geq \check{b}\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this lead to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) = \mu - \hat{x}$ for $\hat{x} \geq b\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(x) = \mu - x$ for $x \geq b\} = \{\tilde{T}(x) = \mu - x$ on $[b, \infty)\}$. The proof of the latter half is almost the same as the above.

(g) The former half of Lemma 10.1.1(p.53) (g) can be rewritten by $A = \{T(x) > 0$ for $x < b\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\hat{T}(x) > 0$ for $-\hat{x} < -\hat{b}\} = \{\hat{T}(x) < 0$ for $\hat{x} > \hat{b}\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\hat{T}}(\hat{x}) < 0$ for $\hat{x} > \check{a}\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) < 0$ for $\hat{x} > a\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(x) < 0$ for $x > a\} = \{\tilde{T}(x) < 0$ on $(a, \infty)\}$. The proof of the latter half is almost the same as the above.

(h) Applying $\mathcal{R}$ to Lemma 10.1.1(p.53) (h) yields $\mathcal{R}[A] = \{-\hat{T}(x) \geq \max\{0, -\hat{\mu} + \hat{x}\}$ for $-\infty < -\hat{x} < \infty\} = \{\hat{T}(x) \leq \min\{0, \hat{\mu} - \hat{x}\}$ for $\infty > \hat{x} > -\infty\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\check{\hat{T}}(\hat{x}) \leq \min\{0, \check{\mu} - \hat{x}\}$ for $\infty > \hat{x} > -\infty\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(\hat{x}) \leq \min\{0, \mu - \hat{x}\}$ for $\infty > \hat{x} > -\infty\}$. Without loss of generality, this can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\tilde{T}(x) \leq \min\{0, \mu - x\}$ for $\infty > x > -\infty\} = \{\tilde{T}(x) \leq \min\{0, \mu - x\}$ on $(-\infty, \infty)\}$.

(i) Immediate from $\tilde{T}(0) = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}] = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}I(a \leq \boldsymbol{\xi} \leq b)]$ from (5.1.11(p.23)) and (2.2.3(p.13))).

(j,k) Almost the same as the proof of (b and c)

(l) Lemma 10.1.1(p.53) (l) can be rewritten as $A = \{$If $x < y$ and $a < y$, then $T(x) + x < T(y) + y\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{$If $-\hat{x} < -\hat{y}$ and $-\hat{a} < -\hat{y}$, then $-\hat{T}(x) - \hat{x} < -\hat{T}(y) - \hat{y}\} = \{$If $\hat{x} > \hat{y}$ and $\hat{a} > \hat{y}$, then $\hat{T}(x)\hat{x} > T(y) + \hat{y}\}$, and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{$If $\hat{x} > \hat{y}$ and $\check{b} > \hat{y}$, then $\check{\hat{T}}(\hat{x}) + \hat{x} > \check{\hat{T}}(\hat{y}) + \hat{y}\} = \{$If $x > y$ and $\check{b} > y$, then $\check{\hat{T}}(x) + x > \check{\hat{T}}(y) + y\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{$If $x > y$ and $b > y$, then $\tilde{T}(x) + x > \tilde{T}(y) + y\}$.

(m) The former half of Lemma 10.1.1(p.53) (m) can be rewritten as Let $A = \{\lambda\beta T(\lambda\beta\mu - s) - s$ is nonincreasing in $s\}$, which can be rewritten as $A = \{\lambda\beta T(\lambda\beta\mu - s) - s \geq \lambda\beta T(\lambda\beta\mu - s') - s'$ for $s < s'\}$. Applying $\mathcal{R}$ to this yields $\mathcal{R}[A] = \{-\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s) - s \geq -\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s') - s'$ for $s < s'\} = \{\lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s) + s \leq \lambda\beta\hat{T}(-\lambda\beta\hat{\mu} - s') + s'$ for $s < s'\}$,[†] and then applying $\mathcal{C}_{\mathbb{R}}$ to this produces $\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\lambda\beta\check{\hat{T}}(\widehat{-\lambda\beta\check{\mu} - s}) + s \leq \lambda\beta\check{\hat{T}}(\widehat{-\lambda\beta\check{\mu} - s'}) + s'$ for $s < s'\} = \{\lambda\beta\check{\hat{T}}(\lambda\beta\check{\mu} + s) + s \leq \lambda\beta\check{\hat{T}}(\lambda\beta\check{\mu} + s') + s'$ for $s < s'\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ to this leads to $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s \leq \lambda\beta\tilde{T}(\lambda\beta\mu + s') + s'$ for $s < s'\}$, meaning that $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ is nondecreasing in s. Similarly, the latter half of Lemma 10.1.1(p.53) (m) can be rewritten as $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A] = \{\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s < \lambda\beta\tilde{T}(\lambda\beta\mu + s') + s'$ for $s < s'\}$, meaning that $\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ is nonincreasingness in s.

(n) Clear from (2.2.2(p.13)). ▮

[†]Note Def. 12.3.3(p.71)).

• *Direct proof* See the proof of Lemma A 1.1(p.236) . ∎

We have:

$$\tilde{L}(x)\begin{cases} = \lambda\beta\mu + s - \lambda\beta x \text{ on } [b, -\infty) & \cdots(1), \\ < \lambda\beta\mu + s - \lambda\beta x \text{ on } (-\infty, b) & \cdots(2), \end{cases} \tag{12.6.1}$$

$$\tilde{K}(x)\begin{cases} = \lambda\beta\mu + s - \delta x & \text{on } [b, \infty) & \cdots(1), \\ < \lambda\beta\mu + s - \delta x & \text{on } (-\infty, b) & \cdots(2). \end{cases} \tag{12.6.2}$$

$$\tilde{K}(x)\begin{cases} < -(1-\beta)x + s & \text{on } (a, \infty) & \cdots(1), \\ = -(1-\beta)x + s & \text{on } (-\infty, a] & \cdots(2), \end{cases} \tag{12.6.3}$$

$$\tilde{K}(x) + x \le \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{12.6.4}$$

$$\tilde{K}(x) + x = \begin{cases} \lambda\beta\mu + s + (1-\lambda)\beta x \text{ on } & [b, \infty) & \cdots(1), \\ \beta x + s & \text{on } (-\infty, a] & \cdots(2). \end{cases} \tag{12.6.5}$$

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)x_{\tilde{L}} \cdots(1), \quad \tilde{L}(x_{\tilde{K}}) = (1-\beta)x_{\tilde{K}} \cdots(2). \tag{12.6.6}$$

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to (10.2.3(p.55)) - (10.2.8(p.55)). ∎

• *Direct proof* See (A 1.1(p.237))-(A 1.6(p.238)) . ∎

Lemma 12.6.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) *$\tilde{L}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{L}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{L}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *Let $s = 0$. Then $x_{\tilde{L}} = a$ where $x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\ge)\ 0$.*
(e) *Let $s > 0$.*
  1. *$x_{\tilde{L}}$ uniquely exists with $x_{\tilde{L}} > a$ where $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.*
  2. *$(\lambda\beta\mu + s)/\lambda\beta \ge (<)\ b \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta\mu + s)/\lambda\beta \ge (<)\ b$.* □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Lemmas 10.2.1(p.55) ∎

• *Direct proof* See the proof of Lemma A 1.2(p.238) . ∎

Corollary 12.6.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) $x_{\tilde{L}} < (\ge)\ x \Leftrightarrow \tilde{L}(x) < (\ge)\ 0$.
(b) $x_{\tilde{L}} \le (\ge)\ x \Rightarrow \tilde{L}(x) \le (\ge)\ 0$. □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Corollaries 10.2.1(p.55) ∎

• *Direct proof* See the proof of Corollary A 1.1(p.238) . ∎

Lemma 12.6.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) *$\tilde{K}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{K}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{K}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *$\tilde{K}(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$\tilde{K}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(f) *$K(x) + x$ is strictly increasing on $(-\infty, b]$.*
(g) *$\tilde{K}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*
(h) *If $x > y$ and $b > y$, then $\tilde{K}(x) + x > \tilde{K}(y) + y$.*
(i) *Let $\beta = 1$ and $s = 0$. Then $x_{\tilde{K}} = a$ where $x_{\tilde{K}} < (\ge)\ x \Leftrightarrow \tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\ge)\ 0$.*
(j) *Let $\beta < 1$ or $s > 0$.*
  1. *There uniquely exists $x_{\tilde{K}}$ where $x_{\tilde{K}} < (= (>))\ x \Leftrightarrow \tilde{K}(x) < (= (>))\ 0$.*
  2. *$(\lambda\beta\mu + s)/\delta \ge (<)\ b \Leftrightarrow x_{\tilde{K}} = (<)\ (\lambda\beta\mu + s)/\delta$.*
  3. *Let $\tilde{\kappa} < (= (>))\ 0$. Then $x_{\tilde{K}} < (= (>))\ 0$.* □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Lemmas 10.2.2(p.55). ∎

• *Direct proof* See the proof of Lemma A 1.3(p.238) . ∎

Corollary 12.6.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0.$
(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0.$ □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Corollaries 10.2.2(p.56). ▮

• *Direct proof* See the proof of Corollary A 1.2(p.239). ▮

Lemma 12.6.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}/\tilde{K}_{\mathbb{R}}\}$)

(a) *Let* $\beta = 1$ *and* $s = 0$*. Then* $x_{\tilde{L}} = x_{\tilde{K}} = a$.
(b) *Let* $\beta = 1$ *and* $s > 0$*. Then* $x_{\tilde{L}} = x_{\tilde{K}}$.
(c) *Let* $\beta < 1$ *and* $s = 0$*. Then* $a < (= (>))\ 0 \Leftrightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}} < (= (=))\ 0.$
(d) *Let* $\beta < 1$ *and* $s > 0$*. Then* $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}} < (= (>))\ 0.$ □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Lemmas 10.2.3(p.56). ▮

• *Direct proof* See the proof of Lemma A 1.4(p.239). ▮

Lemma 12.6.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s *and is strictly increasing in* s *if* $\lambda\beta < 1$.
(b) *Let* $\lambda\beta\mu \leq a$.
  1. $x_{\tilde{L}} \geq \lambda\beta\mu + s$.
  2. *Let* $s > 0$ *and* $\lambda\beta < 1$*. Then* $x_{\tilde{L}} > \lambda\beta\mu + s$.
(c) *Let* $\lambda\beta\mu > a$*. Then, there exists a* $s_{\tilde{\mathcal{L}}} > 0$ *such that if* $s_{\tilde{\mathcal{L}}} > (\leq)\ s$*, then* $x_{\tilde{L}} < (\geq)\ \lambda\beta\mu + s$. □

• *Proof by symmetry* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Lemmas 10.2.4(p.57). ▮

• *Direct proof* See the proof of Lemma A 1.5(p.240). ▮

Lemma 12.6.6 ($\tilde{\kappa}_{\mathbb{R}}$) *We have*:

(a) $\tilde{\kappa} = \lambda\beta\mu + s$ *if* $b < 0$ *and* $\tilde{\kappa} = s$ *if* $a > 0$.
(b) *Let* $\beta < 1$ *or* $s > 0$*. Then* $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{K}} < (= (>))\ 0.$ □

• *Proof* Obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Lemmas 10.3.1(p.57). ▮

• *Direct proof* See the proof of Lemma A 1.6(p.240). ▮

12.7 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Lemma 12.7.1 ($\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$) *The optimal initiating time* t^*_τ (OIT) *is not subject to the influence of the symmetry transformation operation* $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))). □

• *Proof* First, let us represent (7.2.5(p.43)) as $D \overset{\text{def}}{=} \{I^{t^*_\tau}_\tau \geq I^t_\tau$ for $\tau \geq t \geq \delta_q\}\cdots(1)$, which can be rewritten as $D = \{\beta^{\tau - t^*_\tau} V_{t^*_\tau} \geq \beta^{\tau-t} V_t$ for $\tau \geq t \geq \delta_q\}$. Next, applying $\mathcal{R}$ to this yields $\mathcal{R}[D]$=$\{-\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau} \geq -\beta^{\tau-t}\hat{V}_t$ for $\tau \geq t \geq \delta_q\}$=$\{\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau} \leq \beta^{\tau-t}\hat{V}_t$ for $\tau \geq t \geq \delta_q\}$. Then, even if applying $\mathcal{C}_{\mathbb{R}}$ (Lemma 12.3.1(p.69)) to this, no change occurs, i.e., $\mathcal{C}_{\mathbb{R}}\mathcal{R}[D]$ =$\{\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau} \leq \beta^{\tau-t}\hat{V}_t$ for $\tau \geq t \geq \delta_q\}$. Finally, applying $\mathcal{I}_{\mathbb{R}}$ (Lemma 12.3.3(p.71)) to this, we have $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ =$\{\beta^{\tau-t^*_\tau}\hat{V}_{t^*_\tau} \leq \beta^{\tau-t}\hat{V}_t$ for $\tau \geq t \geq \delta_q\}$. Then, since $\hat{V}_t$ changes into V_t for the same reason as been stated just below (12.5.20(p.74)), so we have $\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[A]$ =$\{\beta^{\tau-t^*_\tau}V_{t^*_\tau} \leq \beta^{\tau-t}V_t$ for $\tau \geq t \geq \delta_q\}$, i.e., $\{I^{t^*_\tau}_\tau \leq I^t_\tau$ for $\tau \geq t \geq \delta_q\}\cdots(2)$. The above result means that the optimal initiating time is t^*_τ even if $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} (= \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R})$ is applied, hence it follows that the optimal initiating time t^*_τ due to (1) is entirely inherited to t^*_τ due to (2). ▮

□ **Tom 12.7.1** (⊡$\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.
(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

• *Proof by symmetry* Immediately obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Tom 11.2.1(p.59). ▮

• *Direct proof* See the proof of Tom A 4.1(p.250). ▮

□ **Tom 12.7.2** (⊡$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.
(b) *Let* $\beta\mu \leq a$*. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
(c) *Let* $\beta\mu > a$.
  1. *Let* $\beta = 1$.
    i. *Let* $\mu + s \geq b$*. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
    ii. *Let* $\mu + s < b$*. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

2. *Let* $\beta < 1$ *and* $s = 0$ $⟅ s > 0 ⟆$.
   i. *Let* $a < 0$ $⟅\tilde{\kappa} < 0⟆$. *Then* $Ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let* $a = 0$ $⟅\tilde{\kappa} = 0⟆$.
      1. *Let* $\beta\mu + s \geq b$. *Then* $ⓓ_{\tau>1}\langle 1\rangle_{\parallel}$.
      2. *Let* $\beta\mu + s < b$. *Then* $Ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let* $a > 0$ $⟅\tilde{\kappa} > 0⟆$.
      1. *Let* $\beta\mu + s \geq b$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $ⓓ_{\tau>1}\langle 1\rangle_{\parallel}$.
      2. *Let* $\beta\mu + s < b$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\texttt{S}_1$(p.60) $\boxed{Ⓢ\blacktriangle}\boxed{◎\parallel}$ *is true.* □

- *Proof by symmetry* Immediately obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (12.5.29(p.75))) to Tom 11.2.2(p.60). ▮
- *Direct proof* See the proof of Tom A 4.2(p.250). ▮

12.8 $\tilde{\text{S}}$cenario[$\mathbb{R}$]

In this section we write up the inverse of Scenario[$\mathbb{R}$](p.73) which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 11.2.1(p.59) and 11.2.2(p.60)) from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Tom's 12.7.1(p.81) and 12.7.2(p.81)). Let us represent this scenario as $\tilde{\text{S}}$cenario[$\mathbb{R}$].

■ $\tilde{\text{S}}$**tep 1** (*opening*)

◦ The system of optimality equation of $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ is given by Table 6.5.1(p.39) (II), i.e.,

$$\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \{V_1 = \beta\mu + s,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\quad t > 1\}. \tag{12.8.1}$$

◦ Let us consider an assertion $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ in each of Tom's 12.7.1(p.81) and 12.7.2(p.81), which can be rewritten as

$$\begin{aligned} A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} &= \{\tilde{\mathsf{S}} \text{ is true for } \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P} \text{ and } F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \text{ with } \boldsymbol{p} \in \mathscr{P}_{A_{\tilde{\texttt{Tom}}}} \subseteq \mathscr{F}\} \\ &= \{\tilde{\mathsf{S}} \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\} \quad (\text{see } (12.5.31(\text{p.75}))) \end{aligned} \tag{12.8.2}$$

where

$$\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p}, F) \mid \boldsymbol{p} \in \mathscr{P}_{A_{\texttt{Tom}}} \subseteq \mathscr{P},\ F \in \mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}} \subseteq \mathscr{F}\}.$$

To facilitate the understanding of the discussion that follows let us use the following example.

$$\tilde{\mathsf{S}} = \langle V_t - s_{\tilde{\mathcal{L}}} + x_{\tilde{L}} + \tilde{\kappa} + b + \mu + a \leq 0,\ t > 0\rangle \quad (\text{see } (12.5.33(\text{p.75}))).$$

◦ The attribute vector of the assertion $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by (12.5.36(p.75)), i.e.,

$$\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). \tag{12.8.3}$$

■ $\tilde{\text{S}}$**tep 2** (*reverse operation* $\mathcal{R}$)

◦ Applying the reverse operation $\mathcal{R}$ to (12.8.1(p.82)) produces

$$\begin{aligned} \mathcal{R}[\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_1 = -\beta\hat{\mu} + s,\ -\hat{V}_t = \min\{-\hat{\tilde{K}}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\},\ t > 1\} \\ &= \{-\hat{V}_1 = -\beta\hat{\mu} + s,\ -\hat{V}_t = -\max\{\hat{\tilde{K}}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}\} \\ &= \{\hat{V}_1 = \beta\hat{\mu} - s,\ \hat{V}_t = \max\{\hat{\tilde{K}}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\},\ t > 1\}. \end{aligned} \tag{12.8.4}$$

◦ Applying $\mathcal{R}$ to (12.8.2(p.82)) yields to

$$\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true on } \check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\ \}. \tag{12.8.5}$$

For our example we have:

$$\begin{aligned} \mathcal{R}[\tilde{\mathsf{S}}] &= \langle -\hat{V}_t - s_{\tilde{\mathcal{L}}} - \hat{x}_{\tilde{L}} - \hat{\tilde{\kappa}} - \hat{b} - \hat{\mu} - \hat{a} \leq 0,\ t > 0\rangle \\ &= \langle \hat{V}_t + s_{\tilde{\mathcal{L}}} + \hat{x}_{\tilde{L}} + \hat{\tilde{\kappa}} + \hat{b} + \hat{\mu} + \hat{a} \geq 0,\ t > 0\rangle. \end{aligned} \tag{12.8.6}$$

◦ The attribute vector of the assertion $\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\mathcal{R}$ to (12.5.36(p.75)), i.e.,

$$\begin{aligned} \boldsymbol{\theta}(\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]) &\overset{\text{def}}{=} \mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \\ &= (\hat{b}, \hat{\mu}, \hat{a},\ \hat{x}_{\tilde{L}},\ \hat{x}_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \hat{\kappa}, \hat{\tilde{T}}, \hat{\tilde{L}}, \hat{\tilde{K}}, \hat{\tilde{\mathcal{L}}}, \hat{V}_t). \end{aligned} \tag{12.8.7}$$

■ $\tilde{\text{S}}$tep **3** (*correspondence replacement operation* $\tilde{\mathcal{C}}_{\mathbb{R}}$)

- Here let us consider the application of the correspondence replacement operation $\tilde{\mathcal{C}}_{\mathbb{R}}$, i.e., the replacement of the left-hand side of each equality in Lemma 12.3.2(p.70) .

$$\hat{b},\ \hat{\mu},\ \hat{a},\ \hat{x}_{\tilde{L}},\ \hat{x}_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \hat{\tilde{\kappa}},\ \hat{\tilde{T}}(x),\ \hat{\tilde{L}}(x),\ \hat{\tilde{K}}(x),\ \hat{\tilde{\mathcal{L}}}(s)\cdots(1^*)$$

by its right-hand side

$$\check{a},\ \check{\mu},\ \check{b},\ \check{x}_L,\ \check{x}_K,\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T}(\hat{x}),\ \check{L}(\hat{x}),\ \check{K}(\hat{x}),\ \check{\mathcal{L}}(s)\cdots(2^*)$$

where (1^*) is for any $F\in\mathscr{F}$ and (2^*) is for $\check{F}\in\check{\mathscr{F}}$ corresponding to the $F\in\mathscr{F}$.

- Applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to (12.8.4(p.82)) leads to

$$\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \{\hat{V}_1=\beta\check{\mu}-s,\ \hat{V}_t=\max\{\check{K}(\hat{V}_{t-1})+\hat{V}_{t-1},\beta\hat{V}_{t-1}\},\ \ t>1\}. \quad (12.8.8)$$

- Applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to $\mathcal{R}[\tilde{\mathsf{S}}]$ in (12.8.6(p.82)), we have

$$\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] = \langle\,\hat{V}_t+s_{\check{\mathcal{L}}}+\ \check{x}_L\ +\check{\kappa}+\check{a}+\check{\mu}+\check{b}\le 0,\ \ t>0\,\rangle. \quad (12.8.9)$$

Now, let us note here that the application of $\tilde{\mathcal{C}}_{\mathbb{R}}$ (see Lemma 12.3.2(p.70)) inevitably changes

"*for* $F\in\mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}$" in (12.8.5(p.82))

into

"*for* $\check{F}\in\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}$ *corresponding to any* $F\in\mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}$ *with* $\boldsymbol{p}\in\mathscr{P}_{A_{\texttt{Tom}}}\subseteq\mathscr{P}$"

where

$$\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}}=\{\check{F}\mid F\in\mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}\}\quad\text{(see (12.1.3(p.67)))}.$$

Hence, applying (12.8.5(p.82)), we have

$$\begin{aligned}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}]\text{ is true for }\boldsymbol{p}\in\mathscr{P}_{A_{\texttt{Tom}}}\subseteq\mathscr{P}\text{ and }\check{F}\in\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F} & (12.8.10)\\ &\qquad\text{corresponding to }F\in\mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}\text{ with }\boldsymbol{p}\in\mathscr{P}_{A_{\texttt{Tom}}}\subseteq\mathscr{P}\}. & (12.8.11)\end{aligned}$$

Now, since the phrase "$F\in\mathscr{F}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}$" is implicitly accompanied with the phrase "$\check{F}\in\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}$". Accordingly (12.8.11(p.83)) can be rewritten as

$$\begin{aligned}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] &= \{\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}]\text{ is true for }\boldsymbol{p}\in\mathscr{P}_{A_{\texttt{Tom}}}\subseteq\mathscr{P}\text{ and }\check{F}\in\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}\}, & (12.8.12)\\ &= \{\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathsf{S}]\text{ is true on }\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle\ \}\end{aligned}$$

where

$$\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle \overset{\text{def}}{=} \{(\boldsymbol{p},F)\mid \boldsymbol{p}\in\mathscr{P}_{A_{\texttt{Tom}}}\subseteq\mathscr{P},\check{F}\in\check{\mathscr{F}}_{A_{\texttt{Tom}}|\boldsymbol{p}}\subseteq\mathscr{F}\}. \quad (12.8.13)$$

- The attribute vector of $\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]$ is given by applying $\tilde{\mathcal{C}}_{\mathbb{R}}$ to (12.8.7(p.82)), i.e.,

$$\begin{aligned}\boldsymbol{\theta}(\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]) &= \tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})]\\ &= (\check{a},\check{\mu},\check{b},\ \check{x}_L,\ \check{x}_K,s_{\check{\mathcal{L}}}.\check{\kappa},\check{T},\check{L},\check{K},\check{\mathcal{L}},\hat{V}_t). & (12.8.14)\end{aligned}$$

■ $\tilde{\text{S}}$tep **4** (*identity replacement operation* $\tilde{\mathcal{I}}_{\mathbb{R}}$)

- Here let us consider the application of the identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{R}}$, i.e., the replacement of the left-hand of each equality in Lemma 12.3.4(p.72)

$$\check{F},\ \check{a},\ \check{\mu},\ \check{b},\ \check{x}_L,\ \check{x}_K,\ s_{\check{\mathcal{L}}},\ \check{\kappa},\ \check{T}(x),\ \check{L}(x),\ \check{K}(x),\ \check{\mathcal{L}}(s)\cdots(1^*)$$

by its right-hand side

$$F,\ a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T(x),\ L(x),\ K(x),\ \mathcal{L}(s)\cdots(2^*)$$

where (1^*) is for any $F\in\mathscr{F}$ and (2^*) is for $\check{F}\in\check{\mathscr{F}}$ which is identical to the $F\in\mathscr{F}$, i.e., $\check{F}\equiv F\cdots(1)$.

- Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.8(p.83)) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_1=\beta\mu-s,\ \hat{V}_t=\max\{K(\hat{V}_{t-1})+\hat{V}_{t-1},\beta\hat{V}_{t-1}\},\ \ t>1\}.$$

Now, we have $\hat{V}_1=\beta\mu-s=V_1$ from (6.5.5(p.39)). Suppose $\hat{V}_{t-1}=V_{t-1}$. Then, since $\hat{V}_t=\max\{\tilde{K}(V_{t-1})+V_{t-1},\beta V_{t-1}\}=V_t$ from (6.5.6(p.39)), by induction $\hat{V}_t=V_t$ for $t>0$. Thus we have

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] = \{V_1=\beta\mu-s,\ V_t=\max\{K(V_{t-1})+V_{t-1},\beta V_{t-1}\},\ \ t>1\},$$

which is the same as $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.5.1(p.39) (I)). Thus we have

$$\begin{aligned}\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} &= \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\texttt{SOE}\{\check{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] & (12.8.15)\\ &= \{V_1=\beta\mu-s,\ V_t=\max\{K(V_{t-1})+V_{t-1},\beta V_{t-1}\},\ t>1\}.\end{aligned}$$

- Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.12(p.83)) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \;=\; \{\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}. \tag{12.8.16}$$

Applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.9(p.83)) yields

$$\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] = \langle V_t +\ s_{\mathcal{L}}\ +\ x_L\ + \kappa + a + \mu + b \le 0,\ \ t > 0\rangle. \tag{12.8.17}$$

Now V_t within $\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}]$ is generated from $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, hence (12.8.16(p.84)) can be regarded as an assertion as to $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$. Thus, we have

$$\begin{aligned} A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;&=\; \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \\ &=\; \{\tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\tilde{\mathsf{S}}] \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \}. \end{aligned} \tag{12.8.18}$$

- The attribute vector of $A_{\tilde{\mathtt{Tom}}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is given by applying $\tilde{\mathcal{I}}_{\mathbb{R}}$ to (12.8.14(p.83)), i.e.,

$$\begin{aligned} \boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \;&=\; \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \\ &=\; (a, \mu, b,\ x_L\,,\ x_K\,,\ s_{\mathcal{L}}\,, \kappa, T, L, K, \mathcal{L}, V_t), \end{aligned} \tag{12.8.19}$$

■ **S̃tep 5** (*symmetry transformation operation* $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$)

Below, letting us line up the attribute vectors given in S̃tep 1 to S̃tep 4, we have the following:

$$\begin{array}{llr} \tilde{\mathsf{S}}\mathsf{tep}\ 1\text{:} & \boldsymbol{\theta}(\,\boxed{b,\ \mu,\ a,}\ x_{\tilde{L}}\,,\ x_{\tilde{K}}\,,\ s_{\tilde{\mathcal{L}}}\,, \tilde{\kappa},\ \tilde{T},\ \tilde{L}\,,\ \tilde{K},\ \tilde{\mathcal{L}}\,, V_t\,) & (\leftarrow (12.8.3\text{(p.82)})) \\ & \quad\downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow\ \mathcal{R} \\ \tilde{\mathsf{S}}\mathsf{tep}\ 2\text{:} & \boldsymbol{\theta}(\,\boxed{\hat{b},\ \hat{\mu},\ \hat{a},}\ \hat{\tilde{x}}_L\,,\ \hat{\tilde{x}}_K\,,\ s_{\tilde{\mathcal{L}}}\,, \hat{\tilde{\kappa}},\ \hat{\tilde{T}}\,,\ \hat{\tilde{L}}\,,\ \hat{\tilde{K}}\,,\ \hat{\tilde{\mathcal{L}}}\,, \hat{V}_t\,) & (\leftarrow (12.8.7\text{(p.82)})) \\ & \quad\downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow\ \tilde{\mathcal{C}}_{\mathbb{R}} \\ \tilde{\mathsf{S}}\mathsf{tep}\ 3\text{:} & \boldsymbol{\theta}(\,\boxed{\breve{a},\ \breve{\mu},\ \breve{b},}\ x_{\check{L}}\,, x_{\check{K}}\,,\ s_{\check{\mathcal{L}}}\,,\ \check{\kappa},\ \check{T},\ \check{L},\ \check{K},\ \check{\mathcal{L}}\,, V_t\,) & (\leftarrow (12.8.14\text{(p.83)})) \\ & \quad\downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow & \leftarrow\ \tilde{\mathcal{I}}_{\mathbb{R}} \\ \tilde{\mathsf{S}}\mathsf{tep}\ 4\text{:} & \boldsymbol{\theta}(\,\boxed{a,\ \mu,\ b,}\ x_L\,, x_K\,,\ s_{\mathcal{L}}\,, \kappa,\ T,\ L\,,\ K,\ \mathcal{L}\,, V_t\,) & (\leftarrow (12.8.19\text{(p.84)})) \end{array} \tag{12.8.20}$$

The above flow can be eventually reduced to

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \boxed{b,\ \mu,\ a,}\ x_{\tilde{L}}\,, x_{\tilde{K}}\,,\ s_{\tilde{\mathcal{L}}}\,, \tilde{\kappa},\ \tilde{T},\ \tilde{L}\,, \tilde{K}\,, \tilde{\mathcal{L}}\,,\ V_t \\ \ \wr\ \ \wr\ \ \wr\ \ \ \wr\ \ \ \wr\ \ \ \wr\ \ \downarrow\ \wr\ \ \wr\ \ \wr\ \ \wr\ \ \wr \\ \boxed{a,\ \mu,\ b,}\ x_L\,, x_K\,, s_{\mathcal{L}}\,, \kappa,\ T,\ L\,, K\,, \mathcal{L}\,,\ V_t \end{array} \right\} \tag{12.8.21}$$

called the *symmetry transformation operation*, which can be regarded as the successive application of the three operations, i.e., “ $\mathcal{R} \to \tilde{\mathcal{C}}_{\mathbb{R}} \to \tilde{\mathcal{I}}_{\mathbb{R}}$ ”. Here let us define

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \ \stackrel{\text{def}}{=}\ \tilde{\mathcal{I}}_{\mathbb{R}}\tilde{\mathcal{C}}_{\mathbb{R}}\mathcal{R}. \tag{12.8.22}$$

Then (12.8.18(p.84)) can be rewritten as

$$\begin{aligned} A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;&=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \\ &=\; \{\mathsf{S} \text{ is true on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle\ \} \end{aligned} \tag{12.8.23}$$

where

$$\mathsf{S} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\tilde{\mathsf{S}}]. \tag{12.8.24}$$

Then, from (12.8.17(p.84)) we have

$$\mathsf{S} = \langle V_t +\ s_{\mathcal{L}}\ +\ x_L\ + \kappa + a + \mu + b \le 0,\ \ t > 0\rangle.$$

Then, (12.8.15(p.83)) can be rewritten as

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\ \}]. \tag{12.8.25}$$

In addition, (12.5.27(p.74)) can be rewritten as

$$\begin{aligned} \boldsymbol{\theta}(A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \;&=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] && (12.8.26) \\ &=\; (b, \mu, a,\ x_{\tilde{L}}\,,\ x_{\tilde{K}}\,,\ s_{\tilde{\mathcal{L}}}\,, \tilde{\kappa}, \tilde{T}, \tilde{L}\,, \tilde{K}\,, \tilde{\mathcal{L}}\,, V_t) && (12.8.27) \end{aligned}$$

From all the above we see that S̃cenario[$\mathbb{R}$] starting with (12.8.2(p.82)) finally ends up with (12.8.23(p.84)), which can be rewritten as respectively

$$A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \mathscr{C}\langle A_{\mathtt{Tom}}\rangle, \tag{12.8.28}$$

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} \text{ holds on } \check{\mathscr{C}}\langle A_{\mathtt{Tom}}\rangle. \tag{12.8.29}$$

From the above two results and (12.8.25(p.84)) we eventually obtain the following lemma.

Lemma 12.8.1 *Let* $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\check{\mathscr{C}}\langle A_{\texttt{Tom}}\rangle$. *Then* $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\texttt{Tom}}\rangle$ *where*

$$A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}]. \quad \square \tag{12.8.30}$$

■ S̃tep 6 (*aggregation*)

We can construct quite the same procedure as in Step 6 (p.76).

■ S̃tep 7 (*symmetry theorem* $\mathbb{R} \leftarrow \tilde{\mathbb{R}}$)

Through the procedure in S̃tep 6 (p.85) we have the following theorem

Theorem 12.8.1 *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\} \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}]. \quad \square \tag{12.8.31}$$

● *Proof* Immediate for the same reason as in Theorem 12.5.1 (p.78). ▮

The attribute vector of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ is given by

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}) \;=\; \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\})] \tag{12.8.32}$$

$$\;=\; (a,\mu,b,\,x_L\,,\,x_K\,,\,s_{\mathcal{L}}\,,\kappa,T,L,K,\mathcal{L},V_t) \tag{12.8.33}$$

12.9 Definition of Symmetry

Thus far, the term of *symmetry* has been used in the rather intuitive nuance. In order to make our discussions more clear, below let us provide its strict definition.

Definition 12.9.1

(a) Let $A\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\}$ be assertions on models M_1 and M_2 respectively. Then, if $A\{\mathsf{M}_2\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[A\{\mathsf{M}_1\}]$ and $A\{\mathsf{M}_1\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[A\{\mathsf{M}_2\}]$, let $A\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\}$ be said to be *symmetrical*, denoted by $A\{\mathsf{M}_1\} \sim A\{\mathsf{M}_2\}$. Then let us employ the expression of " M_1 and M_2 are symmetrical with respect to A ".

(b) For given two assertion systems $\mathscr{A}\{\mathsf{M}_1\}$ and $\mathscr{A}\{\mathsf{M}_2\}$ which are one-to-one correspondent, if $A\{\mathsf{M}_1\} \sim A\{\mathsf{M}_2\}$ for any pair $(A\{\mathsf{M}_1\}, A\{\mathsf{M}_2\})$ where $A\{\mathsf{M}_1\} \in \mathscr{A}\{\mathsf{M}_1\}$ and $A\{\mathsf{M}_2\} \in \mathscr{A}\{\mathsf{M}_2\}$, then $\mathscr{A}\{\mathsf{M}_1\}$ and $\mathscr{A}\{\mathsf{M}_2\}$ are said to be *symmetrical*, denoted by $\mathscr{A}\{\mathsf{M}_1\} \sim \mathscr{A}\{\mathsf{M}_2\}$. Then, let us employ the expression of " M_1 and M_2 are symmetrical with respect to $\mathscr{A}$ ".

(c) Without confusion, let us remove the phrases "with respect to A" and "with respect to $\mathscr{A}$". □

Lemma 12.9.1 $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ *and* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ *are symmetrical, i.e.,*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\} \sim \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}. \quad \square \tag{12.9.1}$$

● *Proof* Immediate from (12.5.52(p.78)) and (12.8.31(p.85)). ▮

12.10 Symmetry-Operation-Free

When no change occurs even if the symmetry operation is applied to a given assertion A, the assertion is said to be *free from* the symmetry operation, called the *symmetry-operation-free assertion*.

Lemma 12.10.1 *Even if the symmetry operation is applied to the symmetry-operation-free assertion, no change occurs.* □

● *Proof* Evident. ▮

12.11 Symmetry between SOE{M:1[ℝ][A]} and SOE{M̃:1[ℝ][A]}

Here note that the symmetrical relation holds between $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ and $\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ (see (I) and (II) in Table 6.5.1(p.39)), i.e., $\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\} \sim \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$. It is an important point that, due to this very fact, the symmetry theorems (Theorems 12.5.1(p.78) and 12.8.1(p.85)) can be derived. It will be known later on that this symmetrical relation is one of the necessary conditions on which the integrated theory can be successfully constructed.

Chapter 13

Third Step: Analogy Theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$)

The third step for constructing the integrated theory is to provide a methodology which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model with $\mathbb{R}$-mechanism) that was derived in Chap. 11(p.59).

13.1 Preliminary

Lemma 13.1.1 ([16,You])
(a) *Let $x \geq b$. Then $z(x) = b$.*
(b) *Let $x < b$. Then $x < z(x) < b$.*
(c) *$z(x) \geq a$ for any x.* □

● *Proof* (a) Let $x \geq b$. If $z < b \cdots$ (I), then $z < x$, hence $p(z)(z - x) < 0$ due to (5.1.29 (1) (p.24)), and if $b \leq z \cdots$ (III), then $p(z)(z - x) = 0$ due to (5.1.29 (2) (p.24)). Hence $z(x)$ can be given by any $z \geq b$, thus $z(x) = b$ due to Def. 5.1.1(p.24).

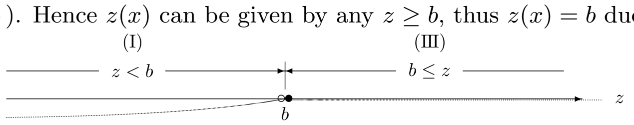


Figure 13.1.1: Case $x \geq b$

(b) Let $x < b$. If $z \leq x \cdots$ (I), then $p(z)(z - x) \leq 0$, if $x < z < b \cdots$ (II), then $p(z)(z - x) > 0$ due to (5.1.29 (1) (p.24)), and if $b \leq z \cdots$ (III), then $p(z)(z - x) = 0$ from (5.1.29 (2) (p.24)). Hence, $z(x)$ is given by z such that $x < z < b$ or equivalently $x < z(x) < b$.

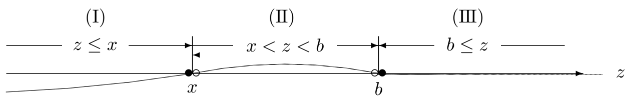


Figure 13.1.2: Case $x < b$

(c) Assume that $z(x) < a$ for a certain x. Then, since $p(z(x)) = 1 = p(a)$ due to (5.1.28 (1) (p.24)), from (5.1.25(p.24)) we have $T(x) = p(z(x))(z(x) - x) = z(x) - x < a - x = p(a)(a - x) \leq T(x)$, which is a contradiction. Hence, it must be that $z(x) \geq a$ for any x. ▮

Corollary 13.1.1 ([16,You]) *$a \leq z(x) \leq b$ for any x.* □
● *Proof* Immediate from Lemma 13.1.1(p.87). ▮

Lemma 13.1.2 ([16,You]) *$p(z)$ is nonincreasing on $(-\infty, \infty)$ and strictly decreasing in $z \in [a, b]$.* □
● *Proof* The former half is immediate from (5.1.18(p.24)). Let $a \leq z' < z \leq b$. Then $p(z') - p(z) = \Pr\{z' \leq \boldsymbol{\xi}\} - \Pr\{z \leq \boldsymbol{\xi}\} = \Pr\{z' \leq \boldsymbol{\xi} < z\} = \int_{z'}^{z} f(\xi)d\xi > 0$ (See (2.2.3 (2) (p.13))), hence $p(z') > p(z)$, i.e., $p(z)$ is strictly decreasing on $[a, b]$. ▮

Lemma 13.1.3 ([16,You]) *$z(x)$ is nondecreasing on $(-\infty, \infty)$.* □
● *Proof* From (5.1.25(p.24)), for any x and y we have

$$
\begin{aligned}
T(x) &= p(z(x))(z(x) - x) \\
&= p(z(x))(z(x) - y) - (x - y)p(z(x)) \\
&\leq T(y) - (x - y)p(z(x)) \\
&= p(z(y))(z(y) - y) - (x - y)p(z(x)) \\
&= p(z(y))\big(z(y) - x + (x - y)\big) - (x - y)p(z(x)) \\
&= p(z(y))(z(y) - x) + (x - y)(p(z(y)) - p(z(x))) \\
&\leq T(x) + (x - y)(p(z(y)) - p(z(x))).
\end{aligned}
$$

‡This is the most important property of the function T, which was proven in [14,You].

Hence $0 \leq (x-y)(p(z(y))-p(z(x)))$. Let $x > y$. Then $0 \leq p(z(y))-p(z(x))$, so $p(z(x)) \leq p(z(y)) \cdots (1)$. Since $a \leq z(x) \leq b$ and $a \leq z(y) \leq b$ from Corollary 13.1.1(p.87), if $z(x) < z(y)$, then $p(z(x)) > p(z(y))$ from Lemma 13.1.2(p.87), which contradicts (1). Hence, it must be that $z(x) \geq z(y)$, i.e., $z(x)$ is nondecreasing in $x \in (-\infty, \infty)$. ∎

Lemma 13.1.4

(a) *$T(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$T(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$T(x)$ is strictly decreasing on $(-\infty,b]$.*
(d) *$T(x)>0$ on $(-\infty,b)$ and $T(x)=0$ on $[b,\infty)$.*
(e) *$T(x)\geq a-x$ on $(-\infty,\infty)$.*
(f) *$T(x)+x$ is nondecreasing on $(-\infty,\infty)$.*
(g) *$\beta T(x)+x$ is nondecreasing on $(-\infty,\infty)$ if $\beta=1$.*
(h) *$\beta T(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\beta<1$.*
(i) *$T(x)\geq \max\{0,a-x\}$ on $(-\infty,\infty)$.*
(j) *$\lambda\beta T(\lambda\beta a-s)-s$ is nonincreasing in s and is strictly decreasing in s if $\lambda\beta<1$.* □

• *Proof* (a,b) Immediate from the fact that $p(z)(z-x)$ in (5.1.19(p.24)) is continuous and nonincreasing in $x \in (-\infty,\infty)$ for any z.

(c) Let $x' < x < b$. Then $z(x) < b$ from Lemma 13.1.1(p.87)(b). Accordingly, since $p(z(x)) > 0$ due to (5.1.29 (1) (p.24)) and since $z(x)-x < z(x)-x'$, from (5.1.25(p.24)) we have $T(x) = p(z(x))(z(x)-x) < p(z(x))(z(x)-x') \leq T(x')$, implying that $T(x)$ is *strictly* decreasing on $(-\infty,b) \cdots\cdots (1)$. Assume $T(b)=T(x)$ for a given $x<b$, so $b-x>0$. Then, for any sufficiently small $\varepsilon>0$ such that $b-x>2\varepsilon>0$ we have $b>b-\varepsilon>x+\varepsilon>x$, hence $T(b)=T(x)>T(b-\varepsilon)\geq T(b)$ due to the *strict* unceasingness shown above and the nonincreasingness in (b), which is a contradiction. Thus, since $T(x)\neq T(b)$ for any $x<b$, we have $T(x)>T(b)$ or $T(x)<T(b)$ for any $x<b$. However, the latter is impossible due to (b), hence only the former is possible. Consequently, it follows that $T(x)$ is strictly decreasing on $(-\infty,b]$ instead of $(-\infty,b)$.

(d) Let $x \geq b$. Then, since $z(x)=b$ from Lemma 13.1.1(p.87)(a), we have $p(z(x))=0$ due to (5.1.29 (2) (p.24)), hence $T(x)=p(z(x))(z(x)-x)=0$ on $[b,\infty)$. Let $x<b$. Then, from (c) we have $T(x)>T(b)=0$, i.e., $T(x)>0$ on $(-\infty,b)$.

(e) Since $p(a)=1$ from (5.1.28 (1) (p.24)), we have $T(x)\geq p(a)(a-x)=a-x$ for any x on $(-\infty,\infty)$.

(f) Let $x<x'$. Then, we have

$$\begin{aligned} T(x)+x &= p(z(x))(z(x)-x)+x \\ &= p(z(x))z(x)+(1-p(z(x)))x \\ &\leq p(z(x))z(x)+(1-p(z(x)))x' \\ &= p(z(x))(z(x)-x')+x' \leq T(x')+x', \end{aligned}$$

implying that $T(x)+x$ is nondecreasing on $(-\infty,\infty)$.

(g) If $\beta=1$, then $\beta T(x)+x=T(x)+x$, hence the assertion is true from (f).

(h) Since $\beta T(x)+x=\beta(T(x)+x)+(1-\beta)x$, if $\beta<1$, then $(1-\beta)x$ is strictly increasing in x, hence the assertion is true from (f).

(i) Immediate from the fact that $T(x)\geq a-x$ on $(-\infty,\infty)$ from (e) and $T(x)\geq 0$ on $(-\infty,\infty)$ from (d).

(j) From (5.1.19(p.24)) we have

$$\lambda\beta T(\lambda\beta a-s)-s \;=\; \lambda\beta\max_z p(z)(z-\lambda\beta a+s)-s=\max_z p(z)(\lambda\beta z-(\lambda\beta)^2 a+\lambda\beta s)-s.$$

Let $s>s'$. Then, we have

$$\begin{aligned} &\lambda\beta T(\lambda\beta a-s)-s-\lambda\beta T(\lambda\beta a-s')+s' \\ &\quad= \max_z p(z)(\lambda\beta z-(\lambda\beta)^2 a+\lambda\beta s)-\max_z p(z)(\lambda\beta z-(\lambda\beta)^2 a+\lambda\beta s')-(s-s') \\ &\quad\leq \max_z p(z)(s-s')\lambda\beta-(s-s')^\dagger \\ &\quad\leq \max_z (s-s')\lambda\beta-(s-s') \quad \text{(due to } p(z)\leq 1 \text{ and } s-s'>0) \\ &\quad= (s-s')\lambda\beta-(s-s') \\ &\quad= -(s-s')(1-\lambda\beta)\leq(<)\,0 \text{ if } \lambda\beta\leq(<)\,1. \end{aligned}$$

Hence, since $\lambda\beta T(\lambda\beta a-s)-s\leq(<)\,\lambda\beta T(\lambda\beta a-s')-s'$ if $\lambda\beta\leq(<)\,1$, it follows that $T(\lambda\beta a-s)-s$ is nonincreasing (strictly decreasing) in s if $\lambda\beta\leq(<)\,1$. ∎

Let us define

$$h(z)=p(z)(z-a)/(1-p(z)), \quad z>a,$$
$$h^\star=\sup_{a<z}h(z),$$

† $\max_x g(x)-\max_x h(x)\leq \max_x\{g(x)-h(x)\}$.

Below, for a given x let us define the following successive four assertions:

$$\begin{aligned} A_1(x) &= \langle\!\langle\, z(x) > a \,\rangle\!\rangle, \\ A_2(x) &= \langle\!\langle\, T(a,x) < T(z',x,) \text{ for at least one } z' > a \,\rangle\!\rangle, \\ A_3(x) &= \langle\!\langle\, a - h(z') < x \text{ for at least one } z' > a \,\rangle\!\rangle, \\ A_4(x) &= \langle\!\langle\, \inf_{z>a}\{a - h(z)\} < x \,\rangle\!\rangle. \end{aligned}$$

Proposition 13.1.1 For any given x we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. □

● *Proof* Letting $T(z,x) \overset{\text{def}}{=} p(z)(z-x)$, we can rewrite (5.1.19(p.24)) as $T(x) = \max_z T(z,x) = T(z(x),x)$ (see (5.1.25(p.24))).

1. Let $A_1(x)$ be true for any given x. Suppose $T(a,x) \geq T(z',x)$ for all $z' \geq a$, hence the maximum of $T(z,x)$ for all $z \geq a$ is attained at $z = a$, i.e., $z(x) = a$ (see Def. 5.1.1(p.24)), which contradicts $A_1(x)$. Hence it must be that $T(a,x) < T(z',x)$ for at least one $z' > a$, thus $A_2(x)$ becomes true. Accordingly, we have $A_1(x) \Rightarrow A_2(x)$. Suppose $A_2(x)$ is true for any given x. Then, if $z(x) = a$, we have $T(a,x) < T(z',x) \leq T(x) = T(z(x),x) = T(a,x)$, which is a contradiction, hence it must be that $z(x) > a$ due to Lemma 13.1.1(p.87) (c). Accordingly, we have $A_2(x) \Rightarrow A_1(x)$. Thus, it follows that $A_1(x) \Leftrightarrow A_2(x)$ for any given x.
2. Since $p(a) = 1$ from (5.1.28 (1) (p.24)), for $z' > a$ (hence $1 > p(z') \cdots$ (1) from (5.1.28 (2) (p.24))) we have

$$\begin{aligned} &T(a,x) - T(z',x) \\ &\quad= p(a)(a-x) - p(z')(z'-x) \\ &\quad= a - x - p(z')(z'-x) \\ &\quad= a - x - p(z')(a - x + z' - a) \\ &\quad= a - x - p(z')(a-x) - p(z')(z'-a) \\ &\quad= (1-p(z'))(a-x) - p(z')(z'-a) \\ &\quad= (1-p(z'))\big(a - x - p(z')(z'-a)/(1-p(z'))\big) \\ &\quad= (1-p(z'))(a - x - h(z')) \\ &\quad= (1-p(z'))(a - h(z') - x). \end{aligned}$$

   Accordingly, due to (1) we immediately obtain $A_2(x) \Leftrightarrow A_3(x)$ for any given x.
3. Let $A_3(x)$ be true for any given x. Then clearly $A_4(x)$ is also true, i.e., $A_3(x) \Rightarrow A_4(x)$. Let $A_4(x)$ be true for any given x. Then evidently $a - \tilde{h}(z') < x$ for at least one $z' > a$, hence $A_3(x)$ is true, so we have $A_4(x) \Rightarrow A_3(x)$. Accordingly, it follows that $A_3(x) \Leftrightarrow A_4(x)$ for any given x.

From all the above we have $A_1(x) \Leftrightarrow A_2(x) \Leftrightarrow A_3(x) \Leftrightarrow A_4(x)$. ∎

Lemma 13.1.5

(a) $0 < h^\star < \infty$.
(b) $x^\star = a - h^\star < a$.
(c) $x^\star < (\geq)\ x \Leftrightarrow z(x) > (=)\ a$.
(d) $a^\star < a$. □

● *Proof* (a) For any infinitesimal $\varepsilon > 0$ such that $a < b - \varepsilon < b \cdots$ (II) we have $0 < p(b-\varepsilon) < 1$ from (5.1.29 (1) (p.24)) and (5.1.28 (2) (p.24)), hence $h(b-\varepsilon) = p(b-\varepsilon)(b-\varepsilon-a)/(1-p(b-\varepsilon)) > 0$. If $b \leq z \cdots$ (III), then $p(z) = 0$ due to (5.1.29 (2) (p.24)), hence $h(z) = 0$ for $z \geq b$. From the above we have $h^\star > 0$ (finite) or $h^\star = \infty$.

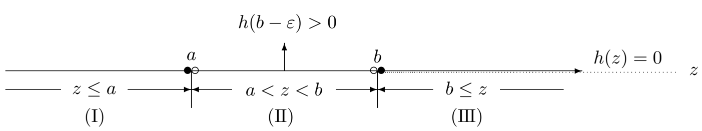


Figure 13.1.3: $h(b-\varepsilon) > 0$ and $h(z) = 0$ for $z \geq b$

Assume that $h^\star = \infty$. Then, there exists at least one z' on $a < z' < b$ such that $h(z') \geq N$ for any given $N > 0$. Hence, if the N is given by $M/\underline{f}$† with any $M > 1$, we have $h(z') \geq M/\underline{f}$ or equivalently $p(z')(z'-a)/(1-p(z')) \geq M/\underline{f}$. Hence, noting (5.1.18(p.24)), we have

$$p(z')(z'-a) \;\geq\; (1-p(z'))M/\underline{f} = (1 - \Pr\{z' \leq \boldsymbol{\xi}\})M/\underline{f} = \Pr\{\boldsymbol{\xi} < z'\}M/\underline{f} \cdots (*)$$

†See (2.2.4(p.13))

where $\Pr\{\boldsymbol{\xi} < z'\} = \int_a^{z'} f(w)dw \geq \int_a^{z'} \underline{f}dw = (z'-a)\underline{f}$. Accordingly, since $p(z')(z'-a) \geq (z'-a)\underline{f}M/\underline{f} = (z'-a)M$, we have $p(z') \geq M > 1$ due to $z'-a>0$, which is a contradiction. Hence, it must follow that $h^\star < \infty$.

(b) Since $A_1(x) \Rightarrow A_4(x)$ due to Proposition 13.1.1, we can rewritten (5.1.27(p.24)) as

$$\begin{aligned} x^\star &= \inf\{x \mid \inf_{z>a}\{a-h(z)\} < x\} \\ &= \inf_{z>a}\{a-h(z)\} \cdots \textbf{(1)} \\ &= a - \sup_{a<z} h(z) = a - h^\star < a \quad \text{(due to (a))}, \end{aligned}$$

hence (b) holds.

(c) If $x^\star < x$, then $\inf_{z>a}\{a-h(z)\} < x$ from (1), hence $z(x) > a$ due to $A_4(x) \Rightarrow A_1(x)$. If $x^\star \geq x$, then $\inf_{a<z}\{a-h(z)\} \geq x$ from (1). Now, since $\inf_{a<z}\{a-h(z)\} \geq x \Leftrightarrow z(x) \leq a$ due to a contraposition of $A_4(x) \Leftrightarrow A_1(x)$, hence we obtain $z(x) = a$ due to Lemma 13.1.1(p.87) (c).

(d) First note $T(x) \geq p(z')(z'-x)$ for any x and z'. Accordingly, for any sufficiently small $\varepsilon > 0$ such that $a+\varepsilon < b$ we have $p(a+\varepsilon) > 0$ from (5.1.29 (1) (p.24)), hence $T(a) \geq p(a+\varepsilon)(a+\varepsilon-a) = p(a+\varepsilon)\varepsilon > 0$. Adding a to the inequality yields $T(a)+a > a$. Thus, we have $T(x)+x \geq T(a)+a > a$ for *any* $x \geq a$ due to Lemma 13.1.4(p.88) (f). Accordingly, if $a^\star \geq a$, then since $T(a^\star)+a^\star \geq T(a)+a > a$, from Lemma 13.1.4(p.88) (a) we have $T(a^\star-\varepsilon)+a^\star-\varepsilon > a$ for any sufficiently small $\varepsilon > 0$ or equivalently $T(a^\star-\varepsilon) > a-(a^\star-\varepsilon)$, which contradicts the definition of $a^\star$ (see (5.1.26(p.24))). Therefore, it must be that $a^\star < a$. ∎

Lemma 13.1.6

(a) *$T(x)+x$ is strictly increasing on $[a^\star,\infty)$.*
(b) *$T(x) = a-x$ on $(-\infty,a^\star]$ and $T(x) > a-x$ on $(a^\star,\infty)$.*
(c) *$T(0) = a$ if $a^\star > 0$ and $T(0) = 0$ if $b < 0$.*
(d) *If $x < y$ and $a^\star < y$, then $T(x)+x < T(y)+y$.* □

● *Proof* (a) From (5.1.25(p.24)) we have

$$T(x)+x = p(z(x))(z(x)-x)+x = p(z(x))z(x)+(1-p(z(x)))x. \cdots \textbf{(1)}$$

◦ Let $x^\star < x$. Then $z(x) > a$ from Lemma 13.1.5(p.89) (c), hence $p(z(x)) < 1$ due to (5.1.28 (2) (p.24)), so $1-p(z(x)) > 0$. If $x < x'$, from (1) we have

$$T(x)+x = p(z(x))z(x)+(1-p(z(x)))x < p(z(x))z(x)+(1-p(z(x)))x' = p(z(x))(z(x)-x')+x' \leq T(x')+x',$$

i.e., $T(x)+x$ is strictly increasing on $(-\infty,\infty)$, hence understandably so also on $[a^\star,\infty)$.

◦ Let $x^\star \geq x$. Then $z(x) = a$ from Lemma 13.1.5(p.89) (c), hence $p(z(x)) = 1$ from (5.1.28 (1) (p.24)), so $T(x) = p(z(x))(z(x)-x) = a-x \cdots$ **(2)**. Suppose $a^\star < x^\star$. Then, since $a^\star < a^\star+2\varepsilon < x^\star$ for an infinitesimal $\varepsilon > 0$, we have $a^\star < a^\star+\varepsilon < x^\star-\varepsilon < x^\star$ or equivalently $x^\star > a^\star+\varepsilon$; accordingly, due to (2) we obtain $T(a^\star+\varepsilon) = a-(a^\star+\varepsilon) \cdots$ **(3)**. Now, due to (5.1.26(p.24)) we have $T(a^\star+\varepsilon) > a-(a^\star+\varepsilon)$, which contradicts (3). Accordingly, it must be that $x^\star \leq a^\star$. Let $x' > x > a^\star$. Then, since $x^\star < x$, we have $z(x) > a$ Lemma 13.1.5(p.89) (c), hence $p(z(x)) < 1$ due to (5.1.28 (2) (p.24)) or equivalently $1-p(z(x)) > 0$. Thus, from (1) we have

$$T(x)+x = p(z(x))z(x)+(1-p(z(x)))x < p(z(x))z(x)+(1-p(z(x)))x' = p(z(x))(z(x)-x')+x' \leq T(x')+x',$$

implying that $T(x)+x$ is *strictly increasing*$\cdots$ **(4)** *on* $(a^\star,\infty)$. Now, let us assume $T(x)+x = T(a^\star)+a^\star$ on $a^\star < x$, so $x-a^\star > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $x-a^\star > 2\varepsilon$ we have $x > x-\varepsilon > a^\star+\varepsilon > a^\star$, hence $T(x)+x = T(a^\star)+a^\star \leq T(a^\star+\varepsilon)+a^\star+\varepsilon < T(x)+x$ due to the nondecreasing in Lemma 13.1.4(p.88) (f) and the *strict increasingness* shown above, which is a contradiction. Thus, it must be that $T(x)+x \neq T(a^\star)+a^\star$ on $a^\star < x$, so we have $T(x)+x > T(a^\star)+a^\star$ or $T(x)+x < T(a^\star)+a^\star$ on $a^\star < x$; however, the latter is impossible due to the nondecreasing in Lemma 13.1.4(p.88) (f), hence it follows that $T(x)+x > T(a^\star)+a^\star$ on $a^\star < x$. From this fact and (4) it inevitably follows that $T(x)+x$ is strictly increasing on $a^\star \leq x$, i.e., $T(x)+x$ is strictly increasing on not $(a^\star,-\infty)$ but $[a^\star,-\infty)$.

Accordingly, whether $x^\star < x$ or $x^\star \geq x$, it follows that $T(x)+x$ is strictly increasing on $[a^\star,\infty)$.

(b) Due to (5.1.26(p.24)) we have $T(x) > a-x$ for $x > a^\star$, i.e., $T(x) > a-x$ on $(a^\star,\infty)$, hence the latter half is true. Since $T(x) \geq a-x$ on $(-\infty,\infty)$ due to Lemma 13.1.4(p.88) (e), we have $T(x)+x \geq a \cdots$ **(5)** on $(-\infty,\infty)$. Suppose $T(a^\star)+a^\star > a$. Then, for an infinitesimal $\varepsilon > 0$ we have $T(a^\star-\varepsilon)+a^\star-\varepsilon > a$ due to Lemma 13.1.4(p.88) (a), i.e., $T(a^\star-\varepsilon) > a-(a^\star-\varepsilon)$, which contradicts the definition of $a^\star$ (see (5.1.26(p.24))). Consequently, we have $T(a^\star)+a^\star = a \cdots$ **(6)** or equivalently $T(a^\star) = a-a^\star$. Let $x < a^\star$. Then, from Lemma 13.1.4(p.88) (f) we have $T(x)+x \leq T(a^\star)+a^\star = a$. From the result and (5) we have $T(x)+x = a$, hence $T(x) = a-x$ on $(-\infty,a^\star)$. From this and (6) it follows that $T(x) = a-x$ on $(-\infty,a^\star]$. Hence the former half is true.

(c) Let $a^\star > 0$. Then, since $0 \in (-\infty,a^\star]$, we have $T(0) = a$ from the former half of (b). We have $T(0) = \max_z p(z)z \cdots$ **(7)** from (5.1.19(p.24)). Let $b < 0$. Then, if $z \geq b$, we have $p(z)z = 0$ from (5.1.29 (2) (p.24)) and if $z < b\,(< 0)$, then $p(z)z < 0$ from (5.1.29 (1) (p.24)), hence $T(0) = 0$ due to (7).

(d) Let $x < y$ and $a^\star < y$. If $x \leq a^\star$, then $T(x)+x \leq T(a^\star)+a^\star < T(y)+y$ due to Lemma 13.1.4(p.88) (f) and (a), and if $a^\star < x$, then $a^\star \leq x < y$, hence $T(x)+x < T(y)+y$ due to (a). Thus, whether $x \leq a^\star$ or $a^\star < x$, we have $T(x)+x < \tilde{T}(y)+y$. ∎

13.2 Analogy Replacement Operation $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$

13.2.1 Three Facts

Let us focus on the three facts below.

★ Fact 1 First, the following lemma can be obtained.

Lemma 13.2.1 ($\mathscr{A}\{T_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) $T(x)$ *is continuous on* $(-\infty, \infty) \leftarrow$ ← Lemma 13.1.4(p.88) (a)
(b) $T(x)$ *is nonincreasing on* $(-\infty, \infty) \leftarrow$ ← Lemma 13.1.4(p.88) (b)
(c) $T(x)$ *is strictly decreasing on* $(-\infty, b] \leftarrow$ ← Lemma 13.1.4(p.88) (c)
(d) $T(x) + x$ *is nondecreasing on* $(-\infty, \infty) \leftarrow$ ← Lemma 13.1.4(p.88) (f)
(e) $T(x) + x$ *is strictly increasing on* $[a^\star, \infty) \leftarrow$ ← Lemma 13.1.6(p.90) (a)
(f) $T(x) = a - x$ *on* $(-\infty, a^\star]$ *and* $T(x) > a - x$ *on* $(a^\star, \infty) \leftarrow$ ← Lemma 13.1.6(p.90) (b)
(g) $T(x) > 0$ *on* $(-\infty, b)$ *and* $T(x) = 0$ *on* $[b, \infty) \leftarrow$ ← Lemma 13.1.4(p.88) (d)
(h) $T(x) \geq \max\{0, a - x\}$ *on* $(-\infty, \infty) \leftarrow$ ← Lemma 13.1.4(p.88) (i)
(i) $T(0) = a$ *if* $a^\star > 0$ *and* $T(0) = 0$ *if* $b < 0 \leftarrow$ ← Lemma 13.1.6(p.90) (c)
(j) $\beta T(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1 \leftarrow$ ← Lemma 13.1.4(p.88) (g)
(k) $\beta T(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1 \leftarrow$ ← Lemma 13.1.4(p.88) (h)
(l) *If* $x < y$ *and* $a^\star < y$, *then* $T(x) + x < T(y) + y \leftarrow$ ← Lemma 13.1.6(p.90) (d)
(m) $\lambda\beta T(\lambda\beta a - s) - s$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1 \leftarrow$ ← Lemma 13.1.4(p.88) (j)
(n) $a^\star < a \leftarrow$ ← Lemma 13.1.5(p.89) (d)

Here we shall pay attention to the fact that replacing a and μ in [Lemma 10.1.1(p.53) ($\mathscr{A}\{T_{\mathbb{R}}\}$)(p.53)] by $a^\star$ and a respectively yields [Lemma 13.2.1(p.91) ($\mathscr{A}\{T_{\mathbb{P}}\}$)]. Let us represent this replacement by

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}} = \{a \to a^\star,\ \mu \to a\}. \tag{13.2.1}$$

In other words, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the former lemma leads to the latter lemma, i.e.,

$$\text{Lemma 13.2.1(p.91)}\ (\mathscr{A}\{T_{\mathbb{P}}\}) \ = \ \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\ \text{Lemma 10.1.1(p.53)}\ (\mathscr{A}\{T_{\mathbb{R}}\})]. \tag{13.2.2}$$

Here let us focus on the following fact. The whole description proving Lemma 10.1.1(p.53) is *quite different* from that proving Lemma 13.2.1(p.91); in other words, no relation exists at all between both descriptions. Nevertheless, what is amazing here is that the whole descriptions of both lemmas are joined together by $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$. In the paper, we call $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ the *analogy replacement operation.*

★ Fact 2 Next, note that replacing μ in [$\mathcal{L}(s) = L(\lambda\beta\mu - s)$] (see (5.1.5(p.23))) by a yields [$\mathcal{L}(s) = L(\lambda\beta a - s)$] (see (5.1.22(p.24))). This means that applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the characteristic vector $(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$ (see (5.1.3(p.23)) - (5.1.6(p.23))) produces $(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$ (see (5.1.20(p.24)) - (5.1.23(p.24))), i.e.,

$$(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}) \ = \ \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})]. \tag{13.2.3}$$

★ Fact 3 Finally, note that replacing μ in [$V_1 = \beta\mu - s$] (see (6.5.1(p.39))) by a yields [$V_1 = \beta a - s$] (see (6.5.5(p.39))). This means that applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the system of optimality equations SOE{M:1[ℝ][A]} (see Table 6.5.1(p.39) (I)) leads to SOE{M:1[ℙ][A]} (see Table 6.5.1(p.39) (III)), i.e.,

$$\texttt{SOE}\{\mathsf{M{:}1}[\mathbb{P}][\mathtt{A}]\} \ = \ \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M{:}1}[\mathbb{R}][\mathtt{A}]\}]. \tag{13.2.4}$$

13.2.2 Prefiguration I

Here let us present a prefiguration through which $\mathscr{A}$\{M:1[ℙ][A]\} can be obtained *only* by replacing a and μ appearing $\mathscr{A}$\{M:1[ℝ][A]\} by $a^\star$ and a respectively.

- First, by $F_{(a,\mu,b)}$ let us denote the distribution function with the lower bound a, the expectation μ, and the upper bound b ($a < \mu < b$). For convenience of reference, below let us copy (13.2.2(p.91)) – (13.2.4(p.91)):

For $F_{(a,\mu,b)}$		For $F_{(a,\mu,b)}$	
Lemma 13.2.1(p.91) ($\mathscr{A}\{T_{\mathbb{P}}\}$)	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	Lemma 10.1.1(p.53) ($\mathscr{A}\{T_{\mathbb{R}}\}$)	]
$(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	$(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$	]
$(1^\star)^l$ SOE{M:1[ℙ][A]}	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	SOE{M:1[ℝ][A]} $(1^\star)^r$	]
Procedure[ℙ]		Procedure[ℝ]	

- Next, closely looking at the flow of the proofs of Tom's 11.2.1(p.59)-11.2.2(p.60), we see that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ was derived *only* from the procedure related to the three terms within the box $(1^\star)^r$ above; here let us denote this procedure by Procedure[ℝ]. Now, for quite the same reason as in Procedure[ℝ] we also see that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ will be derived from the procedure related to the three terms within the box $(1^\star)^l$ above, then let us denote this procedure by Procedure[ℙ]. The flow of the above two procedures can be schematized as below.

For $F_{(a,\mu,b)}$		For $F_{(a,\mu,b)}$
Lemma 13.2.1(p.91) ($\mathscr{A}\{T_{\mathbb{P}}\}$)	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	Lemma 10.1.1(p.53) ($\mathscr{A}\{T_{\mathbb{R}}\}$)]
$(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	$(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$]
$(1^\star)^l$ SOE{M:1[ℙ][A]}	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	SOE{M:1[ℝ][A]} $(1^\star)^r$]
↓		↓
Procedure[ℙ]		Procedure[ℝ]
↓		↓
$(2^\star)^l$ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$		$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ $(2^\star)^r$

- Then, since we have the relation $(1^\star)^l = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(1^\star)^r]$ due to the three Facts in the preceding section, it can be prefigured that this relation will be inherited also between Procedure[ℙ] and Procedure[ℝ], i.e.,

$$\text{Procedure}[\mathbb{P}] = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]],$$

hence also between $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, i.e.

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{13.2.5}$$

In other words, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ can be obtained by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$. From the above discussions we see that the above figure can be rewritten as below.

For $F_{(a,\mu,b)}$		For $F_{(a,\mu,b)}$
Lemma 13.2.1(p.91) ($\mathscr{A}\{T_{\mathbb{P}}\}$)	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	Lemma 10.1.1(p.53) ($\mathscr{A}\{T_{\mathbb{R}}\}$)]
$(L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}})$	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	$(L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}})$]
$(1^\star)^l$ SOE{M:1[ℙ][A]}	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	SOE{M:1[ℝ][A]} $(1^\star)^r$]
↓		↓
Procedure[ℙ]	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	Procedure[ℝ]]
↓		↓
$(2^\star)^l$ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$	$= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[$	$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ $(2^\star)^r$]

Here note that the above discussions is not *a proof* but *a prefiguration*.

13.2.3 Prefiguration II

Below is another prefiguration through which the validity of (13.2.5(p.92)) will be confirmed.

- First, let us represent the procedure proving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}_{(a,\mu,b)}$ with $F_{(a,\mu,b)}$ by $\text{Procedure}[\mathbb{R}]_{(a,\mu,b)}$ (see Section 11.2(p.59)). Now, since $a^\star < a < b$ due to Lemma 13.2.1(p.91)(n), we can express the F with the lower bound $a^\star$, the expectation a, and the upper bound b as $F_{(a^\star,a,b)}$, hence we can define $\text{Procedure}[\mathbb{R}]_{(a^\star,a,b)}$, proving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{E}]\}_{(a^\star,a,b)}$ with $F_{(a^\star,a,b)}$. Here note that $\text{Procedure}[\mathbb{R}]_{(a^\star,a,b)}$ is identical to one resulting from replacing a and μ in $\text{Procedure}[\mathbb{R}]_{(a,\mu,b)}$ by $a^\star$ and a respectively, i.e.,

$$\text{Procedure}[\mathbb{R}]_{(a^\star,a,b)} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]_{(a,\mu,b)}].$$

- Then, from the three facts in Section 13.2.1(p.91) we can regard $\text{Procedure}[\mathbb{P}]_{(a,\mu,b)}$ as *quite* the same as $\text{Procedure}[\mathbb{R}]_{(a^\star,a,b)}$ from the viewpoint of symbolic logic,† i.e.,

$$\text{Procedure}[\mathbb{P}]_{(a,\mu,b)} \overset{\text{s·logic}}{=} \text{Procedure}[\mathbb{R}]_{(a^\star,a,b)}$$

hence we have

$$\text{Procedure}[\mathbb{P}]_{(a,\mu,b)} \overset{\text{s·logic}}{=} \text{Procedure}[\mathbb{R}]_{(a^\star,a,b)} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]_{(a,\mu,b)}].$$

†A logic is regarded as reducing deduction to the process which transforms the expressions by representing propositions, the concept of logic, and so on with symbols such as +, −, >, <, ∨, ∧, ⇒, and so on (Wikipedia)

- The above relation implies that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)}$ proved by Procedure$[\mathbb{P}]_{(a,\mu,b)}$ becomes identical (in the sense of "symbolic logic") to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)}$ proved by Procedure$[\mathbb{R}]_{(a^\star,a,b)}$, i.e.,

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)} \overset{\text{s-logic}}{=} \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)}.$$

In other words, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)}$ can be given by $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)}$ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a,\mu,b)}$ or equivalently from replacing a and μ in $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a,\mu,b)}$ by $a^\star$ and a respectively, i.e.,

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}_{(a,\mu,b)} \overset{\text{s-logic}}{=} \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a^\star,a,b)} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}_{(a,\mu,b)}];$$

13.2.4 Strict Proof

In this section, by dividing the *intuitive* prefiguration in Section 13.2.2(p.91) into several stages, we shall *strictly* prove that (13.2.5(p.92)) holds also *theoretically*.

☐ First, let us note that Procedure$[\mathbb{R}]$ deriving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see Section 11.2(p.59)) can be restated as below.

- First, by applying $\mathscr{A}\{T_\mathbb{R}\}$ (see Lemma 10.1.1(p.53)) to the characteristic vector $(L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R})$ consisting of (5.1.3(p.23))-(5.1.6(p.23)), we obtain expressions (10.2.3(p.55)) - (10.2.8(p.55)); let us denote these expressions by $\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$.
- Next, by applying the $\mathscr{A}\{T_\mathbb{R}\}$ to the $\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$, we get the assertion system $\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$ (see Lemmas 10.2.1(p.55) - 10.3.1(p.57)).
- Finally, by applying the system of optimality equations $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see Table 6.5.1(p.39) (I)) to $\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}$, we get the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}$ (see Tom's 11.2.1(p.59) and 11.2.2(p.60)).

The above flow of Procedure$[\mathbb{R}]$ can be schematized as below.

$$\begin{aligned}\text{Procedure}[\mathbb{R}] \;=\; \langle\!\langle\, & \mathscr{A}\{T_\mathbb{R}\} \Rightarrow (L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}) \to \{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\},\\ & \mathscr{A}\{T_\mathbb{R}\} \Rightarrow \{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\} \to \mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\},\\ & \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\} \Rightarrow \mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\} \to \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}\,\rangle\!\rangle\end{aligned}$$

☐ Secondarily, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the above flow leads to

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_\mathbb{R}\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R})] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_\mathbb{R}\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}] \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]\,\rangle\!\rangle\end{aligned}$$

☐ Thirdly, due to (13.2.2(p.91))-(13.2.4(p.91)) we can replace

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{T_\mathbb{R}\}], \quad \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R})], \quad \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]$$

in the above flow by

$$\mathscr{A}\{T_\mathbb{P}\}, \quad (L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}), \quad \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}$$

respectively. Accordingly, the above flow can be rewritten as follows.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \underline{\mathscr{A}\{T_\mathbb{P}\}} \Rightarrow \underline{(L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{R})} \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \underline{\mathscr{A}\{T_\mathbb{P}\}} \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}],\\ & \underline{\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}} \Rightarrow \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}] \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]\,\rangle\!\rangle\end{aligned} \tag{13.2.6}$$

☐ Fourthly, let us focus our attentions on the items without underline in the above flow, i.e.,

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; \langle\!\langle\, & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow (L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}) \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]},\\ & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]},\\ & \texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_\mathbb{R}, K_\mathbb{R}, \mathcal{L}_\mathbb{R}, \kappa_\mathbb{R}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle\end{aligned} \tag{13.2.7}$$

Here $(L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P})$ can be describes as follows.

$$L(x)\begin{cases} = \lambda\beta a - s - \lambda\beta x \text{ on } (-\infty, a^\star] & \cdots(1),\\ > \lambda\beta a - s - \lambda\beta x \text{ on } (a^\star, \infty) & \cdots(2),\end{cases} \tag{13.2.8}$$

$$K(x)\begin{cases} = \lambda\beta a - s - \delta x \quad \text{on} \quad (-\infty, a^\star] & \cdots(1),\\ > \lambda\beta a - s - \delta x \quad \text{on} \quad (a^\star, \infty) & \cdots(2),\end{cases} \tag{13.2.9}$$

$$K(x)\begin{cases} > -(1-\beta)x - s \text{ on } \quad (-\infty, b) & \cdots(1),\\ = -(1-\beta)x - s \text{ on } \quad [b, \infty) & \cdots(2),\end{cases} \tag{13.2.10}$$

$$K(x)+x \geq \beta x - s \quad \text{on} \quad (-\infty,\infty), \tag{13.2.11}$$

$$K(x)+x = \begin{cases} \lambda\beta a - s + (1-\lambda)\beta x \text{ on } & (-\infty, a^\star] \quad \cdots(1), \\ \beta x - s \quad \text{on} & [b,\infty) \quad \cdots(2), \end{cases} \tag{13.2.12}$$

$$K(x_L) = -(1-\beta)\,x_L \ \cdots(1), \quad L(x_K) = (1-\beta)\,x_K \cdots(2), \tag{13.2.13}$$

- *Direct proof* See (A 2.1(p.241))-(A 2.6(p.241)) . ∎

☐ Fifthly, applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the relations $\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ (see Lemmas 10.2.1(p.55)-10.3.1(p.57)) yields the relations $\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}$, i.e.,

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}] = \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}. \tag{13.2.14}$$

☐ Finally, noting (13.2.14(p.94)), we can rewrite (13.2.7(p.93)) as below.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; & \langle\!\langle\, \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}, \\ & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]}, \\ & \texttt{SOE}\{\mathsf{M{:}1}[\mathbb{P}][\mathsf{E}]\} \Rightarrow \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle \end{aligned} \tag{13.2.15}$$

☐ Now we have

$$\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}]} = \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}. \tag{13.2.16}$$

Accordingly (13.2.15(p.94)) can be rewritten as below.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] \;=\; & \langle\!\langle\, \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow (L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{R}}) \to \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}, \\ & \mathscr{A}\{T_{\mathbb{P}}\} \Rightarrow \{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}, \\ & \texttt{SOE}\{\mathsf{M{:}1}[\mathbb{P}][\mathsf{E}]\} \Rightarrow \mathscr{A}\{L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\} \to \underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1}[\mathbb{R}][\mathsf{E}]\}]}\,\rangle\!\rangle. \end{aligned} \tag{13.2.17}$$

☐ Applying (13.2.16(p.94)) to Lemmas 10.2.1(p.55) to 10.3.1(p.57) yields the following lemmas and corollaries:

Lemma 13.2.2 **($\mathscr{A}\{L_{\mathbb{P}}\}$)**

(a) *$L(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$L(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty,b]$.*
(d) *Let $s=0$. Then $x_L = b$ where $x_L > (\leq)\ x \Leftrightarrow L(x) > (=)\ 0 \Rightarrow L(x) > (\leq)\ 0$.*
(e) *Let $s>0$.*
   1. *x_L uniquely exists with $x_L < b$ where $x_L > (= (<))\ x \Leftrightarrow L(x) > (= (<))\ 0$.*
   2. *$(\lambda\beta a - s)/\lambda\beta \leq (>)\ a^\star \Leftrightarrow x_L = (>)\ (\lambda\beta a - s)/\lambda\beta$.* □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.1(p.55). ∎
- *Direct proof* See the proof of Lemma A 2.2(p.241). ∎

Corollary 13.2.1 **($\mathscr{A}\{L_{\mathbb{P}}\}$)**

(a) *$x_L > (\leq)\ x \Leftrightarrow L(x) > (\leq)\ 0$.*
(b) *$x_L \geq (\leq)\ x \Rightarrow L(x) \geq (\leq)\ 0$.* □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Corollary 10.2.1(p.55). ∎
- *Direct proof* See the proof of Corollary A 2.1(p.241). ∎

Lemma 13.2.3 **($\mathscr{A}\{K_{\mathbb{P}}\}$)**

(a) *$K(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty,b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty,\infty)$ if $\beta<1$.*
(e) *$K(x)+x$ is nondecreasing on $(-\infty,\infty)$.*
(f) *$K(x)+x$ is strictly increasing on $[a^\star,\infty)$.*
(g) *$K(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\lambda<1$.*
(h) *If $x<y$ and $a^\star<y$, then $K(x)+x<K(y)+y$.*
(i) *Let $\beta=1$ and $s=0$. Then $x_K = b$ where $x_K > (\leq)\ x \Leftrightarrow K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.*
(j) *Let $\beta<1$ or $s>0$.*
   1. *There uniquely exists x_K where $x_K > (= (<))\ x \Leftrightarrow K(x) > (= (<))\ 0$.*

2. $(\lambda\beta a - s)/\delta \leq (>)\ a^\star \Leftrightarrow x_K = (>)\ (\lambda\beta a - s)/\delta$.
3. *Let* $\kappa > (= (<))\ 0$. *Then* $x_K > (= (<))\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.2(p.55). ▮
- *Direct proof* See the proof of Lemma A 2.3(p.241). ▮

Corollary 13.2.2 ($\mathscr{A}\{K_\mathbb{P}\}$)

(a) $x_K > (\leq)\ x \Leftrightarrow K(x) > (\leq)\ 0$.
(b) $x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Corollary 10.2.2(p.56). ▮
- *Direct proof* See the proof of Lemma A 2.2(p.242). ▮

Lemma 13.2.4 ($\mathscr{A}\{L_\mathbb{P}/K_\mathbb{P}\}$)

(a) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_L = x_K = b$.
(b) *Let* $\beta = 1$ *and* $s > 0$. *Then* $x_L = x_K$.
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $b > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (=))\ 0$.
(d) *Let* $\beta < 1$ *and* $s > 0$. *Then* $\kappa > (= (<))\ 0 \Leftrightarrow x_L > (= (<))\ x_K \Rightarrow x_K > (= (<))\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.3(p.56). ▮
- *Direct proof* See the proof of Lemma A 2.4(p.242). ▮

Lemma 13.2.5 ($\mathscr{A}\{\mathcal{L}_\mathbb{P}\}$)

(a) $\mathcal{L}(s)$ *is nonincreasing in* s *and strictly decreasing in* s *if* $\lambda\beta < 1$.
(b) *Let* $\lambda\beta a \geq b$.
 1. $x_L \leq \lambda\beta a - s$.
 2. *Let* $s > 0$ *and* $\lambda\beta < 1$. *Then* $x_L < \lambda\beta a - s$.
(c) *Let* $\lambda\beta a < b$. *Then, there exists a* $s_\mathcal{L} > 0$ *such that if* $s_\mathcal{L} > (\leq)\ s$, *then* $x_L > (\leq)\ \lambda\beta a - s$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.2.4(p.57). ▮
- *Direct proof* See the proof of Lemma A 2.5(p.243). ▮

Lemma 13.2.6 ($\kappa_\mathbb{P}$) *We have*:

(a) $\kappa = \lambda\beta a - s$ *if* $a^\star > 0$ *and* $\kappa = -s$ *if* $b < 0$.
(b) *Let* $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$. □

- *Proof by analogy* Obtained from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Lemma 10.3.1(p.57). ▮
- *Direct proof* See the proof of Lemma A 2.6(p.243). ▮

□ Since the assertion system $\underline{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1[\mathbb{R}][E]}\}}$ in (13.2.17(p.94)) is derived from $\mathtt{SOE}\{\mathsf{M{:}1[\mathbb{P}][E]}\}$, it can be regarded as an assertion system for the model $\underline{\mathsf{M{:}1[\mathbb{P}][E]}}$ (see Remark 6.1.1(p.27)), i.e., $\mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}$, hence we have

$$\mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][A]}\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M{:}1[\mathbb{R}][A]}\}] \quad \text{(the same as (13.2.5(p.92)))}. \tag{13.2.18}$$

Thus (13.2.17(p.94)) can be rewritten as follows.

$$\begin{aligned}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] = \langle\!\langle\, & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow (L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}) \to \{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\},\\ & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow \{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\} \to \mathscr{A}\{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\},\\ & \mathtt{SOE}\{\mathsf{M{:}1[\mathbb{P}][E]}\} \Rightarrow \mathscr{A}\{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\} \to \mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}\,\rangle\!\rangle\end{aligned} \tag{13.2.19}$$

□ The whole of the r.h.s. of (13.2.19(p.95)) can be regarded as the procedure deriving $\mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}$, so let us denote it by Procedure⟨ℙ⟩, i.e.,

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Procedure}[\mathbb{R}]] = \text{Procedure}[\mathbb{P}]. \tag{13.2.20}$$

Accordingly, finally it follows that we have

$$\begin{aligned}\text{Procedure}[\mathbb{P}] = \langle\!\langle\, & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow (L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}) \to \{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\},\\ & \mathscr{A}\{T_\mathbb{P}\} \Rightarrow \{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\} \to \mathscr{A}\{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\},\\ & \mathtt{SOE}\{\mathsf{M{:}1[\mathbb{P}][E]}\} \Rightarrow \mathscr{A}\{L_\mathbb{P}, K_\mathbb{P}, \mathcal{L}_\mathbb{P}, \kappa_\mathbb{P}\} \to \mathscr{A}\{\mathsf{M{:}1[\mathbb{P}][E]}\}\,\rangle\!\rangle\end{aligned}$$

13.3 Analogy Theorem ($\mathbb{R} \leftrightarrow \mathbb{P}$)

From (13.2.5(p.92)) we immediately obtain the following theorem.

Theorem 13.3.1 (analogy ($\mathbb{R} \to \mathbb{P}$)**)** *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{13.3.1}$$

Then, from the comparison of (I) and (III) of Tables 6.5.1(p.39) we also get

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{13.3.2}$$

Moreover, from (12.4.2(p.72)) we obtain the following:

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{13.3.3}$$

$$= (a^\star, a, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, V_t). \tag{13.3.4}$$

The analogy replacement operation $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is a *mere* replacement of the two symbols, $a \to a^\star$ and $\mu \to a$. Hence, defining its inverse as

$$\mathcal{A}_{\mathbb{P}\to\mathbb{R}} = \{a^\star \to a,\ a \to \mu\}, \tag{13.3.5}$$

we can immediately obtain the inverse of the above theorem becomes true as follows.

Theorem 13.3.2 (analogy ($\mathbb{P} \leftarrow \mathbb{R}$)**)** *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{13.3.6}$$

In addition, as an inverses of (13.3.2(p.96)) and (13.3.3(p.96)) we immediately obtain

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{13.3.7}$$

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{13.3.8}$$

$$= (a, \mu, b, x_L, x_K, s_{\mathcal{L}}, \kappa, T_{\mathbb{R}}, L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, V_t). \tag{13.3.9}$$

13.4 Derivation of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$

☐ **Tom 13.4.1** (⧈ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$. □

- *Proof by analogy* Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 11.2.1(p.59). ▮
- *Direct proof* See the proof of Tom A 4.3(p.252). ▮

☐ **Tom 13.4.2** (⧈ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.

(b) *Let* $\beta a \geq b$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.

(c) *Let* $\beta a < b$.

1. *Let* $\beta = 1$.
   - i. *Let* $a - s \leq a^\star$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let* $a - s > a^\star$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$ ($s > 0$).
   - i. *Let* $b > 0$ ($\kappa > 0$). *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - ii. *Let* $b = 0$ ($\kappa = 0$).
     1. *Let* $\beta a - s \leq a^\star$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let* $\beta a - s > a^\star$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - iii. *Let* $b < 0$ ($\kappa < 0$).
     1. *Let* $\beta a - s \leq a^\star$ *or* $s_{\mathcal{L}} \leq s$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\mathtt{S}_1$(p.60) [ⓢ▲ | ◎ ∥] *is true.* □

- *Proof by analogy* Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 11.2.2(p.60). ▮
- *Direct proof* See the proof of Tom A 4.4(p.252). ▮

13.5 Strict Definition of Analogy

Below let us provide the strict definition for "analogy" that we have indefinitely used so far.

Definition 13.5.1 (analogy)

(a) By $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathfrak{X}]$ ($\mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathfrak{X}]$) let us denote the assertion defined by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ ($\mathcal{A}_{\mathbb{P}\to\mathbb{R}}$) to a given $\mathfrak{X}$.

(b) If $A\{\mathfrak{X}_2\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[A\{\mathfrak{X}_1\}]$ and $A\{\mathfrak{X}_1\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[A\{\mathfrak{X}_2\}]$, then $A\{\mathfrak{X}_1\}$ and $A\{\mathfrak{X}_2\}$ is said to be *analogous*, denoted by $A\{\mathfrak{X}_1\} \bowtie A\{\mathfrak{X}_2\}$.

(c) For given two assertion systems $\mathscr{A}\{\mathfrak{X}_1\}$ and $\mathscr{A}\{\mathfrak{X}_2\}$ which are one-to-one correspondent, if $A\{\mathfrak{X}_1\} \bowtie A\{\mathfrak{X}_2\}$ for any pair $(A\{\mathfrak{X}_1\}, A\{\mathfrak{X}_2\})$ where $A\{\mathfrak{X}_1\} \in \mathscr{A}\{\mathfrak{X}_1\}$ and $A\{\mathfrak{X}_2\} \in \mathscr{A}\{\mathfrak{X}_2\}$ are correspondent each other, then $\mathscr{A}\{\mathfrak{X}_1\}$ and $\mathscr{A}\{\mathfrak{X}_2\}$ are said to be *analogous*, denoted by $\mathscr{A}\{\mathfrak{X}_1\} \bowtie \mathscr{A}\{\mathfrak{X}_2\}$. □

13.6 Analogy-Operation-Free

When no change occurs even if the analogy operation is applied to a given assertion A, the assertion is said to be *free from* the analogy operation, called the *analogy-operation-free assertion*.

Lemma 13.6.1 *Even if the analogy operation is applied to the analogy-operation-free assertion, no change occurs.* □

● *Proof* Evident. ▮

13.7 Optimal Price to Propose

Lemma 13.7.1 ($\mathscr{A}\{$M:1[$\mathbb{P}$][A]$\}$) *The optimal price z_t to propose is nondecreasing in $t > 0$.* □

● *Proof* Obvious from (6.2.34(p.29)), Tom's 13.4.1(p.96) (a) and 13.4.2(p.96) (a), and Lemma 13.1.3(p.87). ▮

13.8 Analogy between SOE{M:1[$\mathbb{R}$][A]} and SOE{M:1[$\mathbb{P}$][A]}

Here note that the analogical relation holds between SOE{M:1[$\mathbb{R}$][A]} and SOE{M:1[$\mathbb{P}$][A]} (see (I) and (III) in Table 6.5.1(p.39)), i.e., SOE{M:1[$\mathbb{P}$][A]} $\bowtie$ SOE{M:1[$\mathbb{R}$][A]}. It is an important point that, due to this very fact, the analogy theorems (Theorems 13.3.1(p.96) and 13.3.2(p.96)) can be derived. It will be known later on that the analogical relation is one of the necessary conditions on which the integrated theory can be successfully constructed.

Chapter 14

Fourth Step: Symmetry Theorem ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$)

The fourth step for constructing the integrated theory is to provide the theorem which derives the assertion system $\mathscr{A}\{\tilde{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\mathsf{M}\text{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism) that was derived in Chap. 13(p.87).

14.1 Underlying Functions $\check{T}$, $\check{L}$, $\check{K}$, and $\check{\mathcal{L}}$ of Type $\mathbb{P}$

Below let us define ones corresponding to the underlying functions that were defined in Section 5.1.3(p.24). First let us define the T-function of Type $\mathbb{P}$ *for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$* (see (5.1.19(p.24)) and (5.1.18(p.24))) by

$$\check{T}(x) = \max_z \check{p}(z)(z-x) \cdots (1), \qquad \check{p}(z) = \Pr\{z \le \hat{\boldsymbol{\xi}}\} \cdots (2). \tag{14.1.1}$$

By $\check{z}(x)$ let us define z maximizing $\check{p}(z)(z-x)$ if it exists, i.e.,

$$\check{T}(x) = \check{p}(\check{z}(x))(\check{z}(x) - x). \tag{14.1.2}$$

Furthermore, let us define

$$\check{L}(x) = \lambda\beta\check{T}(x) - s, \tag{14.1.3}$$
$$\check{K}(x) = \lambda\beta\check{T}(x) - (1-\beta)x - s, \tag{14.1.4}$$
$$\check{\mathcal{L}}(s) = \check{L}(\lambda\beta\check{a} - s), \tag{14.1.5}$$
$$\check{\kappa} = \lambda\beta\check{T}(0) - s. \tag{14.1.6}$$

Then, let the solutions of $\check{L}(x) = 0$, $\check{K}(x) = 0$, and $\check{\mathcal{L}}(s) = 0$ be denoted by respectively $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$ if they exist. If multiple solutions exist for each of $x_{\check{L}}$, $x_{\check{K}}$, and $s_{\check{\mathcal{L}}}$, let us employ the *smallest* as its solution (see Sections 5.2(p.25) (a) and 12.2.1(p.68)). Furthermore, let us define (see Figure 12.1.1(p.67) for $\check{a}$, $\check{\mu}$, and $\check{b}$)

$$\check{a}^\star = \inf\{x \mid \check{T}(x) > \check{a} - x\} \quad \text{(see (5.1.26(p.24))),} \tag{14.1.7}$$
$$\check{x}^\star = \inf\{x \mid \check{z}(x) > \check{a}\} \quad \text{(see (5.1.27(p.24))).} \tag{14.1.8}$$

By $\check{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{A}]$ let us define $\mathsf{M}\text{:}1[\mathbb{P}][\mathtt{A}]$ for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for $\mathtt{SOE}\{\mathsf{M}\text{:}1[\mathbb{P}][\mathtt{A}]\}$ (see Table 6.5.1(p.39) (III)) we can obtain

$$\mathtt{SOE}\{\check{\mathsf{M}}\text{:}1[\mathbb{P}][\mathtt{A}]\} = \{V_1 = \beta\check{a} - s,\, V_t = \max\{\check{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{14.1.9}$$

14.2 Functions $\check{\check{T}}$, $\check{\check{L}}$, $\check{\check{K}}$, and $\check{\check{\mathcal{L}}}$ of Type $\mathbb{P}$

Below let us define ones corresponding to the underlying functions that were defined in Section 5.1.4(p.24). First, let us define the $\tilde{T}$-function of $\tilde{\text{T}}$ype $\mathbb{P}$ *for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$* by (see (5.1.32(p.24))).

$$\check{\check{T}}(x) = \min_z \check{\check{p}}(z)(z-x) \cdots (1), \qquad \check{\check{p}}(z) = \Pr\{\hat{\boldsymbol{\xi}} \le z\} \cdots (2) \tag{14.2.1}$$

where by $\check{z}(x)$ let us define z minimizing $\check{\check{p}}(z)(z-x)$ if it exists, i.e.,

$$\check{\check{T}}(x) = \check{\check{p}}(\check{z}(x))(\check{z}(x) - x). \tag{14.2.2}$$

Let us define

$$\check{\check{L}}(x) = \lambda\beta\check{\check{T}}(x) + s, \tag{14.2.3}$$
$$\check{\check{K}}(x) = \lambda\beta\check{\check{T}}(x) - (1-\beta)x + s, \tag{14.2.4}$$
$$\check{\check{\mathcal{L}}}(s) = \check{\check{L}}(\lambda\beta\check{b} + s), \tag{14.2.5}$$
$$\check{\check{\kappa}} = \lambda\beta\check{\check{T}}(0) + s \tag{14.2.6}$$

where let us define the solutions of $\check{\tilde{L}}(x)=0$, $\check{\tilde{K}}(x)=0$, and $\check{\tilde{\mathcal{L}}}(x)=0$ by respectively $x_{\check{\tilde{L}}}$, $x_{\check{\tilde{K}}}$, and $s_{\check{\tilde{\mathcal{L}}}}$. If multiple solutions exist for each of $x_{\tilde{L}}$, $x_{\tilde{K}}$, and $s_{\tilde{\mathcal{L}}}$, we shall employ the *largest* as its solution (see Sections 5.2(p.25) (b)). Furthermore let us define (see Figure 12.1.1(p.67) for $\check{a}$, $\check{\mu}$, and $\check{b}$)

$$\check{b}^\star = \sup\{x \mid \check{\tilde{T}}(x) < \check{b} - x\} \quad \text{(see (5.1.39(p.25)))}, \tag{14.2.7}$$

$$\check{\tilde{x}}^\star = \sup\{x \mid \check{z}(x) < \check{b}\} \quad \text{(see (5.1.40(p.25)))}. \tag{14.2.8}$$

By $\check{\tilde{\mathsf{M}}}$:1[$\mathbb{P}$][A] let us define $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A] for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$. Then, for the same reason as for SOE{$\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]} (see Table 6.5.1(p.39) (IV)) we can obtain

$$\texttt{SOE}\{\check{\tilde{\mathsf{M}}}{:}1[\mathbb{P}][\texttt{A}]\} = \{V_1 = \beta\check{b} + s,\ V_t = \min\{\check{\tilde{K}}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\},\ \ t > 1\}. \tag{14.2.9}$$

14.3 List of Underline Functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$

The table below is the list of the four kinds of underline functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$ (see Table 12.2.1(p.69)).

Table 14.3.1: List of the underlying functions of Type $\mathbb{P}$ and $\tilde{\text{T}}$ype $\mathbb{P}$

Type $\mathbb{P}$	$\check{\text{T}}$ype $\mathbb{P}$
For any $F \in \mathscr{F}$	For $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$
$T(x) = \max_z p(z)(z-x)$ $L(x) = \beta T(x) - s$ $K(x) = \beta T(x) - (1-\beta)x - s$ $\mathcal{L}(x) = L(\beta a - s)$ See Section 5.1.3(p.24)	$\check{T}(x) = \max_z \check{p}(z)(z-x)$ $\check{L}(x) = \beta\check{T}(x) - s$ $\check{K}(x) = \beta\check{T}(x) - (1-\beta)x - s$ $\check{\mathcal{L}}(x) = \check{L}(\beta\check{a} - s)$ See Section 14.1
$\tilde{T}(x) = \min_z \tilde{p}(z)(z-x)$ $\tilde{L}(x) = \beta\tilde{T}(x) + s$ $\tilde{K}(x) = \beta\tilde{T}(x) - (1-\beta)x + s$ $\tilde{\mathcal{L}}(x) = \tilde{L}(\beta b + s)$ See Section 5.1.4(p.24)	$\check{\tilde{T}}(x) = \min_z \check{\tilde{p}}(z)(z-x)$ $\check{\tilde{L}}(x) = \beta\check{\tilde{T}}(x) + s$ $\check{\tilde{K}}(x) = \beta\check{\tilde{T}}(x) - (1-\beta)x + s$ $\check{\tilde{\mathcal{L}}}(x) = \check{\tilde{L}}(\beta\check{b} + s)$ See Section 14.2

14.4 Two Kinds of Replacements

14.4.1 Correspondence Replacement

Lemma 14.4.1 **($\mathcal{C}_\mathbb{P}$)** *The left side of each equality below is for* any $F \in \mathscr{F}$ *and its right side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to* the F. *Then*:

(a) $f(\xi) = \check{f}(\hat{\xi})$.
(b) $\hat{a} = \check{b}$, $\hat{a}^\star = \check{b}^\star$, $\hat{b} = \check{a}$.
(c) $\hat{T}(x) = \check{\tilde{T}}(\hat{x})$.
(d) $\hat{L}(x) = \check{\tilde{L}}(\hat{x})$.
(e) $\hat{K}(x) = \check{\tilde{K}}(\hat{x})$.
(f) $\hat{\mathcal{L}}(s) = \check{\tilde{\mathcal{L}}}(s)$.
(g) $\hat{x}_L = x_{\check{\tilde{L}}}$.
(h) $\hat{x}_K = x_{\check{\tilde{K}}}$.
(i) $s_\mathcal{L} = s_{\check{\tilde{\mathcal{L}}}}$.
(j) $\hat{\kappa} = \check{\tilde{\kappa}}$. □

• *Proof* (a) The same as (12.1.9(p.67)).

(The first and third equalities of (b)) The same as the first and third equalities of (12.1.10(p.67)). The second equality will be proven after the proof of (c).

(c) From (5.1.18(p.24)) and (14.2.1 (2) (p.99)), we obtain

$$p(z) = \Pr\{-\hat{z} \le -\hat{\boldsymbol{\xi}}\} = \Pr\{\hat{\boldsymbol{\xi}} \le \hat{z}\} = \check{\tilde{p}}(\hat{z}), \tag{14.4.1}$$

hence from (5.1.19(p.24)) we have $T(x) = \max_z \check{\tilde{p}}(\hat{z})(-\hat{z} + \hat{x}) = -\min_z \check{\tilde{p}}(\hat{z})(\hat{z} - \hat{x})$. Now, in general "$\min_z = \min_{-\infty<z<\infty} = \min_{-\infty<-\hat{z}<\infty} = \min_{\infty>\hat{z}>-\infty} = \min_{-\infty<\hat{z}<\infty} = \min_{\hat{z}}$", hence we have $T(x) = -\min_{\hat{z}} \check{\tilde{p}}(z)(\hat{z} - \hat{x})$. Then, without loss of generality, this can be rewritten as $T(x) = -\min_z \check{\breve{p}}(z)(z - \hat{x})$. Accordingly, since $T(x) = -\check{\breve{T}}(\hat{x})$ from (14.2.1 (1) (p.99)), we obtain $\hat{T}(x) = \check{\breve{T}}(\hat{x})$.

(The second equality of (b)) From (5.1.26(p.24)) we have $a^\star = \inf\{-\hat{x} \mid -\hat{T}(x) > -\hat{a} + \hat{x}\} = -\sup\{\hat{x} \mid \hat{T}(x) < \hat{a} - \hat{x}\} = -\sup\{\hat{x} \mid \check{\breve{T}}(\hat{x}) < \breve{b} - \hat{x}\}$ due to (c) and (b). Without loss of generality, this can be rewritten as $a^\star = -\sup\{x \mid \check{\breve{T}}(x) < \breve{b} - x\}$, hence $a^\star = -\breve{b}^\star$ due to (14.2.7(p.100)), so that $\hat{a}^\star = \breve{b}^\star$.

(d) From (5.1.20(p.24)) and (c) we have $L(x) = -\lambda\beta\hat{T}(x) - s = -\lambda\beta\check{\breve{T}}(\hat{x}) - s = -\check{\breve{L}}(\hat{x})$ from (14.2.3(p.99)), hence $\hat{L}(x) = \check{\breve{L}}(\hat{x})$.

(e) From (5.1.21(p.24)) and (c) we have $K(x) = -\lambda\beta\hat{T}(x) + (1-\beta)\hat{x} - s = -\lambda\beta\check{\breve{T}}(\hat{x}) + (1-\beta)\hat{x} - s = -\check{\breve{K}}(\hat{x})$ from (14.2.4(p.99)), hence $\hat{K}(x) = \check{\breve{K}}(\hat{x})$.

(f) From (5.1.22(p.24)) we have $\mathcal{L}(s) = -\hat{L}(\lambda\beta a - s) = -\check{\breve{L}}(\widehat{\lambda\beta a - s})$ due to (d). Then, since $\mathcal{L}(s) = -\check{\breve{L}}(-\lambda\beta a + s) = -\check{\breve{L}}(\lambda\beta\hat{a} + s) = -\check{\breve{L}}(\lambda\beta\breve{b} + s)$ due to (b), we have $\mathcal{L}(s) = -\check{\breve{\mathcal{L}}}(s)$ from (14.2.5(p.99)), hence $\hat{\mathcal{L}}(s) = \check{\breve{\mathcal{L}}}(s)$.

(g) Since $L(x_L) = 0$ by definition, we have $-\hat{L}(x_L) = 0$, i.e., $\hat{L}(x_L) = 0$, hence $\check{\breve{L}}(\hat{x}_L) = 0$ from (d), implying that $\check{\breve{L}}(x) = 0$ has the solution $x_{\check{\breve{L}}} = \hat{x}_L$ by definition.

(h) Since $K(x_K) = 0$ by definition, we have $-\hat{K}(x_K) = 0$, i.e., $\hat{K}(x_K) = 0$, hence $\check{\breve{K}}(\hat{x}_K) = 0$ from (e), implying that $\check{\breve{K}}(x) = 0$ has the solution $x_{\check{\breve{K}}} = \hat{x}_K$ by definition.

(i) Since $\mathcal{L}(s_{\mathcal{L}}) = 0$ by definition, we have $-\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, i.e., $\hat{\mathcal{L}}(s_{\mathcal{L}}) = 0$, hence $\check{\breve{\mathcal{L}}}(s_{\mathcal{L}}) = 0$ from (f), implying that $\check{\breve{\mathcal{L}}}(s) = 0$ has the solution $s_{\check{\breve{\mathcal{L}}}} = s_{\mathcal{L}}$ by definition.

(j) From (5.1.23(p.24)) we have $\kappa = -\lambda\beta\hat{T}(0) - s = -\lambda\beta\check{\breve{T}}(\hat{0}) - s$ from (c), hence $\kappa = -\lambda\beta\check{\breve{T}}(0) - s = -\check{\breve{\kappa}}$ from (14.2.6(p.99)), thus $\hat{\kappa} = \check{\breve{\kappa}}$. ∎

Definition 14.4.1 (correspondent replacement operation $\mathcal{C}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand side of each equality in Lemma 14.4.1(p.100) by its right-hand side the *correspondence replacement operation* $\mathcal{C}_{\mathbb{P}}$. □

Lemma 14.4.2 ($\tilde{\mathcal{C}}_{\mathbb{P}}$) *The left side of each equality below is for* any $F \in \mathscr{F}$ *and its right side is for* $\check{F} \in \check{\mathscr{F}}$ *corresponding to the* F. *Then*:

(a) $f(\xi) = \check{f}(\hat{\xi})$.

(b) $\hat{a} = \breve{b}$, $\hat{b}^\star = \breve{a}^\star$, $\hat{b} = \breve{a}$.

(c) $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.

(d) $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.

(e) $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.

(f) $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.

(g) $\hat{x}_{\tilde{L}} = \check{x}_L$.

(h) $\hat{x}_{\tilde{K}} = \check{x}_K$.

(i) $s_{\tilde{\mathcal{L}}} = s_{\check{\mathcal{L}}}$.

(j) $\hat{\tilde{\kappa}} = \check{\kappa}$. □

• *Proof* (a) The same as (12.1.9(p.67)).

(The first and third equalities of (b)) The same as the first and third equation of (12.1.10(p.67)). The second equality will be proven after the proof of (c).

(c) From (5.1.31(p.24)) and (14.1.1 (2) (p.99)) we obtain

$$\tilde{p}(z) = \Pr\{-\hat{\boldsymbol{\xi}} \le -\hat{z}\} = \Pr\{\hat{\boldsymbol{\xi}} \ge \hat{z}\} = \Pr\{\hat{z} \le \hat{\boldsymbol{\xi}}\} = \check{p}(\hat{z}), \tag{14.4.2}$$

hence from (5.1.32(p.24)) we have $\tilde{T}(x) = \min_z \check{p}(\hat{z})(-\hat{z} + \hat{x}) = -\max_z \check{p}(\hat{z})(\hat{z} - \hat{x})$. Now, in general "$\max_z = \max_{-\infty<z<\infty} = \max_{-\infty<-\hat{z}<\infty} = \max_{\infty>\hat{z}>-\infty} = \max_{-\infty<\hat{z}<\infty} = \max_{\hat{z}}$", hence we have $\tilde{T}(x) = -\max_{\hat{z}} \check{p}(z)(\hat{z} - \hat{x})$. Then, without loss of generality, this can be rewritten as $\tilde{T}(x) = -\max_z \check{p}(z)(z - \hat{x})$. Accordingly, since $\tilde{T}(x) = -\check{T}(\hat{x})$ from (14.1.1 (1) (p.99)), we obtain $\hat{\tilde{T}}(x) = \check{T}(\hat{x})$.

(The second equality of (b)) From (5.1.39(p.25)) we have $b^\star = \sup\{-\hat{x} \mid -\hat{\tilde{T}}(x) < -\hat{b} + \hat{x}\} = -\inf\{\hat{x} \mid \hat{\tilde{T}}(x) > \hat{b} - \hat{x}\} = -\inf\{\hat{x} \mid \check{T}(\hat{x}) > \breve{a} - \hat{x}\}$ due to (c) and (b). Without loss of generality, this can be rewritten as $b^\star = -\inf\{x \mid \check{T}(x) > \breve{a} - x\}$ we have $b^\star = -\breve{a}^\star$ due to (14.1.7(p.99)) or equivalently $-b^\star = \breve{a}^\star$, hence $\hat{b}^\star = \breve{a}^\star$.

(d) From (5.1.33(p.25)) and (c) we have $\tilde{L}(x) = -\lambda\beta\hat{\tilde{T}}(x) + s = -\lambda\beta\check{T}(\hat{x}) + s = -\check{L}(\hat{x})$ from (14.1.3(p.99)), hence $\hat{\tilde{L}}(x) = \check{L}(\hat{x})$.

(e) From (5.1.34(p.25)) and (c) we have $\tilde{K}(x) = -\lambda\beta\hat{\tilde{T}}(x) + (1-\beta)\hat{x} + s = -\lambda\beta\check{T}(\hat{x}) + (1-\beta)\hat{x} + s = -\check{K}(\hat{x})$ from (14.1.4(p.99)), hence $\hat{\tilde{K}}(x) = \check{K}(\hat{x})$.

(f) From (5.1.35(p.25)) we have $\tilde{\mathcal{L}}(s) = -\hat{\tilde{L}}(\lambda\beta b + s)$, hence from (d) we obtain $\tilde{\mathcal{L}}(s) = -\check{L}(\widehat{\lambda\beta b + s}) = -\check{L}(-\lambda\beta b - s) = -\check{L}(\lambda\beta\hat{b} - s) = -\check{L}(\lambda\beta\breve{a} - s)$ due to (b). Accordingly, from (14.1.5(p.99)) we obtain $\tilde{\mathcal{L}}(s) = -\check{\mathcal{L}}(s)$, hence $\hat{\tilde{\mathcal{L}}}(s) = \check{\mathcal{L}}(s)$.

(g) Since $\tilde{L}(x_{\tilde{L}}) = 0$ by definition, we have $-\hat{\tilde{L}}(x_{\tilde{L}}) = 0$, i.e., $\hat{\tilde{L}}(x_{\tilde{L}}) = 0$, hence $\check{L}(\hat{x}_{\tilde{L}}) = 0$ from (d), implying that $\check{L}(x) = 0$ has the solution $x_{\check{L}} = \hat{x}_{\tilde{L}}$ by definition.

(h) Since $\tilde{K}(x_{\tilde{K}}) = 0$ by definition, we have $-\hat{\check{K}}(x_{\tilde{K}}) = 0$, i.e., $\hat{\check{K}}(x_{\tilde{K}}) = 0$, hence $\check{K}(\hat{x}_{\tilde{K}}) = 0$ from (e), implying that $\check{K}(x) = 0$ has the solution $x_{\check{K}} = \hat{x}_{\tilde{K}}$ by definition.

(i) Since $\tilde{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ by definition, we have $-\hat{\check{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, i.e., $\hat{\check{\mathcal{L}}}(s_{\tilde{\mathcal{L}}}) = 0$, hence $\check{\mathcal{L}}(s_{\tilde{\mathcal{L}}}) = 0$ from (f), implying that $\check{\mathcal{L}}(s) = 0$ has the solution $s_{\check{\mathcal{L}}} = s_{\tilde{\mathcal{L}}}$ by definition.

(j) From (5.1.36(p.25)) we have $\tilde{\kappa} = -\lambda\beta\hat{\check{T}}(0) + s$, leading to $\tilde{\kappa} = -\lambda\beta\check{T}(\hat{0}) + s$ from (c), hence $\tilde{\kappa} = -\lambda\beta\check{T}(0) + s = -\check{\kappa}$ from (14.1.6(p.99)), thus $\hat{\tilde{\kappa}} = \check{\kappa}$. ∎

Remark 14.4.1 The equality $\hat{\mu} = \check{\mu}$ in Lemmas 12.3.1(p.69) (b) changes into respectively $\hat{a}^\star = \check{b}^\star$ in Lemma 14.4.1(p.100) (b) and the equality $\hat{\mu} = \check{\mu}$ in (12.1.10(p.67)) changes into $\hat{b}^\star = \check{a}^\star$ in Lemma 14.4.2(p.101) (b). □

The definition below is the same as Def. 12.3.3(p.71).

Definition 14.4.2 (reversible element and non-reversible element) It should be noted that the left side of each of the equalities in Lemmas 14.4.1(p.100) (i) and 14.4.2(p.101) (i) is respectively $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ without the hat symbol "ˆ"; in other words, $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ are not subjected to the reverse. For the reason, let us refer to each of $s_{\mathcal{L}}$ and $s_{\tilde{\mathcal{L}}}$ as the *non-reversible element* and to each of all the other elements as the *reversible element*. □

Definition 14.4.3 (correspondent replacement operation $\tilde{\mathcal{C}}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand side of each equality in the above lemma by its right-hand side the *correspondence replacement operation* $\tilde{\mathcal{C}}_{\mathbb{P}}$. □

14.4.2 Identity Replacement

Lemma 14.4.3 ($\mathcal{I}_{\mathbb{P}}$) *The left side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right side is for $F \in \mathscr{F}$ where $\check{F} \equiv F \cdots [1^*]$.*† *Then:*

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ,

(b) $\check{a} = a$, $\check{b}^\star = b^\star$, $\check{b} = b$,

(c) $\check{\tilde{T}}(x) = \tilde{T}(x)$,

(d) $\check{\tilde{L}}(x) = \tilde{L}(x)$,

(e) $\check{\tilde{K}}(x) = \tilde{K}(x)$,

(f) $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$,

(g) $x_{\check{\tilde{L}}} = x_{\tilde{L}}$,

(h) $x_{\check{\tilde{K}}} = x_{\tilde{K}}$,

(i) $s_{\check{\tilde{\mathcal{L}}}} = s_{\tilde{\mathcal{L}}}$,

(j) $\check{\tilde{\kappa}} = \tilde{\kappa}$. □

● *Proof* (a) Clear from [1*].

(the first and last equalities of (b)) Immediate from (a). The second equality will be proven after the proof of (c).

(c) From (14.2.1 (2) (p.99)) we have $\check{\tilde{p}}(z) = \Pr\{\hat{\boldsymbol{\xi}} \leq z\} = \int_{-\infty}^{z} \check{f}(\xi)d\xi$. Then, due to [3*] we have $\check{\tilde{p}}(z) = \int_{-\infty}^{z} f(\xi)d\xi = \Pr\{\boldsymbol{\xi} \leq z\} = \tilde{p}(z)$ from (5.1.31(p.24)). Hence, we have that $\check{\tilde{T}}(x)$ given by (14.2.1 (1) (p.99)) becomes $\check{\tilde{T}}(x) = \min_z \tilde{p}(z)(z - x)$, which is identical to $\tilde{T}(x)$ given by (5.1.32(p.24)), i.e., $\check{\tilde{T}}(x) = \tilde{T}(x)$ for any x.

(the second equality of (b)) From (14.2.7(p.100)) and (c) we have $\check{b}^\star = \sup\{x \mid \tilde{T}(x) < \check{b} - x\}$, hence from (b) we get $\check{b}^\star = \sup\{x \mid \tilde{T}(x) < b - x\} = b^\star$ due to (5.1.39(p.25)).

(d,e) Noting (c), from (14.2.3(p.99)) and (5.1.33(p.25)) we have $\check{\tilde{L}}(x) = \tilde{L}(x)$. Similarly, from (14.2.4(p.99)) and (5.1.34(p.25)) we have $\check{\tilde{K}}(x) = \tilde{K}(x)$.

(f) (14.2.5(p.99)) becomes $\check{\tilde{\mathcal{L}}}(s) = \check{\tilde{L}}(\lambda\beta b + s)$ due to (b). This can be rewritten as $\check{\tilde{\mathcal{L}}}(s) = \tilde{L}(\lambda\beta b + s)$ due to (d), which is the same as $\tilde{\mathcal{L}}(s)$ given by (5.1.35(p.25)), i.e., $\check{\tilde{\mathcal{L}}}(s) = \tilde{\mathcal{L}}(s)$.

(g-i) Evident from (d-f).

(j) (14.2.6(p.99)) becomes $\check{\tilde{\kappa}} = \lambda\beta\tilde{T}(0) + s$ due to (c), which is the same as $\tilde{\kappa}$ given by (5.1.36(p.25)). ∎

Definition 14.4.4 (identity replacement operation $\mathcal{I}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand of each equality in the above lemma by its right-hand the *identity replacement operation* $\mathcal{I}_{\mathbb{P}}$. □

Lemma 14.4.4 ($\tilde{\mathcal{I}}_{\mathbb{P}}$) *The left side of each equality below is for $\check{F} \in \check{\mathscr{F}}$ corresponding to any $F \in \mathscr{F}$ and the right side is for $F \in \mathscr{F}$ where $F \equiv \check{F} \cdots [1^*]$. Then:*

(a) $\check{F}(\xi) = F(\xi) \cdots [2^*]$ *and* $\check{f}(\xi) = f(\xi) \cdots [3^*]$ *for any* ξ,

(b) $\check{a} = a$, $\check{a}^\star = a^\star$, $\check{b} = b$,

(c) $\check{T}(x) = T(x)$,

(d) $\check{L}(x) = L(x)$,

---

†See Lemma 12.1.1(p.68) (b)

(e) $\check{K}(x) = K(x)$,
(f) $\check{\mathcal{L}}(s) = \mathcal{L}(s)$,
(g) $x_{\check{L}} = x_L$,
(h) $x_{\check{K}} = x_K$,
(i) $s_{\check{\mathcal{L}}} = s_{\mathcal{L}}$,
(j) $\check{\kappa} = \kappa$. □

● *Proof* (a) Clear from [1*].

(The first and last equalities of b)) Immediate form (a). The second equality will be proven after the proof of (c).

(c) From (14.1.1 (2) (p.99)) we have $\check{p}(z) = \Pr\{z \le \check{\boldsymbol{\xi}}\} = \int_z^\infty \check{f}(\xi)d\xi$. Then, due to [3*] we have $\check{p}(z) = \int_z^\infty f(\xi)d\xi = \Pr\{z \le \boldsymbol{\xi}\} = p(z)$ from (5.1.18(p.24)). Hence, we have that $\check{T}(x)$ given by (14.1.1 (1) (p.99)) becomes $\check{T}(x) = \max_z p(z)(z-x)$, which is identical to $T(x)$ given by (5.1.19(p.24)), i.e., $\check{T}(x) = T(x)$ for any x.

(the second equality of (b)) From (14.1.7 (1) (p.99)) and (c) we have $\check{a}^\star = \inf\{x \mid T(x) > \check{a} - x\}$, hence from (b) we get $\check{a}^\star = \inf\{x \mid T(x) > a - x\} = a^\star$ due to (5.1.26(p.24)). Thus, the second equality of (b) is true.

(d,e) Noting (c), from (14.1.3(p.99)) and (5.1.20(p.24)) we have $\check{L}(x) = L(x)$. Similarly, from (14.1.4(p.99)) and (5.1.21(p.24)) we have $\check{K}(x) = K(x)$.

(f) (14.1.5(p.99)) becomes $\check{\mathcal{L}}(s) = \check{L}(\lambda\beta a - s)$ due to (b). This can be rewritten as $\check{\mathcal{L}}(s) = L(\lambda\beta a - s)$ due to (d), which is the same as $\mathcal{L}(s)$ given by (5.1.22(p.24)), i.e., $\check{\mathcal{L}}(s) = \mathcal{L}(s)$.

(g-i) Evident from (d-f).

(j) (14.1.6(p.99)) becomes $\check{\kappa} = \lambda\beta T(0) - s$ due to (c), which is the same as κ given by (5.1.23(p.24)). ∎

Definition 14.4.5 (Identity replacement operation $\tilde{\mathcal{I}}_{\mathbb{P}}$) Let us call the operation of replacing the left-hand of each equality in the above lemma by its right-hand the *identity replacement operation* $\tilde{\mathcal{I}}_{\mathbb{P}}$. □

14.5 Scenario of Type $\mathbb{P}$

14.5.1 Scenario[$\mathbb{P}$]

This section provides the scenario deriving $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ (buying model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling model with $\mathbb{P}$-mechanism), denoted by Scenario[$\mathbb{P}$].

■ Before moving on, here let us carry out a review of Scenario[$\mathbb{R}$]. For convenience of reference, below let us copy the transformation process of the attribute vectors (see (12.5.28(p.75))) in Scenario[$\mathbb{R}$].

$$
\begin{array}{llcl}
\textsf{Step } 1[\mathbb{R}]: & & & \boldsymbol{\theta}(\,a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t\,) \ = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) \\
& & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 2[\mathbb{R}]: & & \mathcal{R}\to & \boldsymbol{\theta}(\,\hat{a},\ \hat{\mu},\ \hat{b},\ \hat{x}_L,\ \hat{x}_K,\ s_{\mathcal{L}},\ \hat{\kappa},\ \hat{T},\ \hat{L},\ \hat{K},\ \hat{\mathcal{L}},\ \hat{V}_t\,) \\
& & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 3[\mathbb{R}]: & \text{Lemma 12.3.1(p.69)} & \mathcal{C}_{\mathbb{R}}\to & \boldsymbol{\theta}(\,\check{b},\ \check{\mu},\ \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}},\ \check{\tilde{L}},\ \check{\tilde{K}},\ \check{\tilde{\mathcal{L}}},\ \hat{V}_t\,) \\
& & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 4[\mathbb{R}]: & \text{Lemma 12.3.3(p.71)} & \mathcal{I}_{\mathbb{R}}\to & \boldsymbol{\theta}(\,b,\ \mu,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ V_t\,) \ = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})
\end{array}
\tag{14.5.1}
$$

■ From the above flow of the attribute vectors, we see that Scenario[$\mathbb{P}$] is the same as Scenario[$\mathbb{R}$] only except that
- ○ a and μ in $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})$ is replaced $a^\star$ and a in $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})$ (see (13.2.1(p.91))) and
- ○ Lemmas 12.3.1(p.69) and 12.3.3(p.71) are changed into Lemmas 14.4.1(p.100) and 14.4.3(p.102) respectively.

Therefore the above flow of attribute vectors can be rewritten as follows.

$$
\begin{array}{llcl}
\textsf{Step } 1[\mathbb{R}]: & & & \boldsymbol{\theta}(\,a,\ \mu,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t\,) \\
& & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\hline
\textsf{Step } 1[\mathbb{P}]: & \overset{\downarrow}{\text{Scenario}}[\mathbb{P}] & & \boldsymbol{\theta}(\,a^\star,\ a,\ b,\ x_L,\ x_K,\ s_{\mathcal{L}},\ \kappa,\ T,\ L,\ K,\ \mathcal{L},\ V_t\,) \ = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} \\
& & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 2[\mathbb{P}]: & & \mathcal{R}\to & \boldsymbol{\theta}(\,\hat{a}^\star,\ \hat{a},\ \hat{b},\ \hat{x}_L,\ \hat{x}_K,\ s_{\mathcal{L}},\ \hat{\kappa},\ \hat{T},\ \hat{L},\ \hat{K},\ \hat{\mathcal{L}},\ \hat{V}_t\,) \\
& & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 3[\mathbb{P}]: & \text{Lemma 14.4.1(p.100)} & \mathcal{C}_{\mathbb{P}}\to & \boldsymbol{\theta}(\,\check{b}^\star,\ \check{b},\ \check{a},\ x_{\check{\tilde{L}}},\ x_{\check{\tilde{K}}},\ s_{\check{\tilde{\mathcal{L}}}},\ \check{\tilde{\kappa}},\ \check{\tilde{T}},\ \check{\tilde{L}},\ \check{\tilde{K}},\ \check{\tilde{\mathcal{L}}},\ \hat{V}_t\,) \\
& & & \quad\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\
\textsf{Step } 4[\mathbb{P}]: & \text{Lemma 14.4.3(p.102)} & \mathcal{I}_{\mathbb{P}}\to & \boldsymbol{\theta}(\,b^\star,\ b,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}},\ \tilde{\kappa},\ \tilde{T},\ \tilde{L},\ \tilde{K},\ \tilde{\mathcal{L}},\ V_t\,) \ = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}
\end{array}
\tag{14.5.2}
$$

Accordingly, it follows that the operation which transforms $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})$ into $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})$ can be eventually reduced to the operation below:

$$
\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} \overset{\text{def}}{=} \left\{
\begin{array}{cccccccccccc}
a^\star, & a, & b, & x_{L_{\mathbb{P}}}, & x_{K_{\mathbb{P}}}, & s_{\mathcal{L}_{\mathbb{P}}}, & \kappa_{\mathbb{P}}, & T_{\mathbb{P}}, & L_{\mathbb{P}}, & K_{\mathbb{P}}, & \mathcal{L}_{\mathbb{P}}, & V_t \\
\wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr \\
b^\star, & b, & a, & x_{\tilde{L}_{\mathbb{P}}}, & x_{\tilde{K}_{\mathbb{P}}}, & s_{\tilde{\mathcal{L}}_{\mathbb{P}}}, & \tilde{\kappa}_{\mathbb{P}}, & \tilde{T}_{\mathbb{P}}, & \tilde{L}_{\mathbb{P}}, & \tilde{K}_{\mathbb{P}}, & \tilde{\mathcal{L}}_{\mathbb{P}}, & V_t
\end{array}
\right\}.^\dagger
\tag{14.5.3}
$$

† Compare the dash box ⬚ with that in (12.5.29(p.75)).

■ Thus, one sees that in Scenario[$\mathbb{P}$] it suffices to change $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}$(see (12.5.30(p.75))) into $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} = \mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{R}}\mathcal{P}$ above.

■ Moreover, from (III) and (IV) of Table 6.5.1(p.39) it can be easily seen that

$$\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}]. \tag{14.5.4}$$

From all the above discussions it follows that for quite the same reason as that for which Lemma 12.5.1(p.75) was derived we can immediately obtain Lemma 14.5.1(p.104) below.

Lemma 14.5.1 *Let* $A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\texttt{Tom}}\rangle$. *Then* $A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\texttt{Tom}}\rangle$ *where*

$$A_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[A_{\texttt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}]. \quad \square \tag{14.5.5}$$

Finally, also for almost the same reason as that for which Theorem 12.5.1(p.78) is derived from Lemma 12.5.1(p.75) we have Theorem 14.5.1(p.104) below.

Theorem 14.5.1 (symmetry theorem ($\mathbb{P}\to\tilde{\mathbb{P}}$)) *Let* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}]. \quad \square \tag{14.5.6}$$

In addition, we have (see (12.5.53(p.78)))

$$\begin{aligned} \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\}) &\overset{\text{def}}{=} \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\})] && (14.5.7)\\ &= (b^\star, b, a,\ x_{\tilde{L}},\ s_{\tilde{\mathcal{L}}},\ x_{\tilde{K}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t). && (14.5.8)\end{aligned}$$

14.5.2 $\tilde{\text{S}}$cenario[$\mathbb{P}$]

This section provides the inverse of Scenario[$\mathbb{R}$], i.e., the scenario deriving $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}$ (selling model with $\mathbb{P}$-mechanism) from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\}$ (buying model with $\mathbb{P}$-mechanism), denoted by $\tilde{\text{S}}$cenario[$\mathbb{P}$].

■ Before moving on, here let us carry out a review of $\tilde{\text{S}}$cenario[$\mathbb{R}$]. For convenience of reference, below let us copy the transformation process (see (12.8.20(p.84))) of the attribute vectors in Scenario[$\mathbb{R}$].

$$\begin{array}{llrl} \text{Step } 1[\tilde{\mathbb{R}}]: & & & \boldsymbol{\theta}(\,b,\ \mu,\ a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t\,) = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}) \\ & & & \quad\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ \text{Step } 2[\tilde{\mathbb{R}}]: & & \mathcal{R}\to & \boldsymbol{\theta}(\,\hat{b}, \hat{\mu}, \hat{a},\ \hat{\tilde{x}}_L, \hat{\tilde{x}}_K,\ s_{\tilde{\mathcal{L}}}, \hat{\tilde{\kappa}}, \hat{\tilde{T}}, \hat{\tilde{L}}, \hat{\tilde{K}}, \hat{\tilde{\mathcal{L}}}, \hat{V}_t\,) \\ & & & \quad\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ \text{Step } 3[\tilde{\mathbb{R}}]: & \text{Lemma 14.4.1(p.100)} & \check{\mathcal{C}}_{\mathbb{R}}\to & \boldsymbol{\theta}(\,\check{a}, \check{\mu}, \check{b},\ x_{\check{L}}, x_{\check{K}},\ s_{\check{\mathcal{L}}}, \check{\kappa}, \check{T}, \check{L}, \check{K}, \check{\mathcal{L}}, \hat{V}_t\,) \\ & & & \quad\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ \text{Step } 4[\tilde{\mathbb{R}}]: & \text{Lemma 14.4.3(p.102)} & \tilde{\mathcal{I}}_{\mathbb{R}}\to & \boldsymbol{\theta}(\,a, \mu, b,\ x_L, x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t\,) = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}) \end{array} \tag{14.5.9}$$

■ From the above we see that $\tilde{\text{S}}$cenario[$\mathbb{P}$] is the same as $\tilde{\text{S}}$cenario[$\mathbb{R}$] only except that

- b and μ in $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\})$ is replaced $b^\star$ and b in $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\}$ and
- Lemmas 14.4.1(p.100) and 14.4.3(p.102) used there are changed into Lemmas 14.4.2(p.101) and 14.4.4(p.102) respectively.

Therefore the above flow of attribute vectors can be rewritten as follows.

$$\begin{array}{llrl} \text{Step } 1[\tilde{\mathbb{R}}]: & & & \boldsymbol{\theta}(\,b,\ \mu,\ b,\ x_L, x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t\,) \\ \hline & & & \quad\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ \text{Step } 1[\tilde{\mathbb{P}}] & \overset{\downarrow}{\tilde{\text{S}}\text{cenario}}[\mathbb{P}] & & \boldsymbol{\theta}(\,b^\star,\ b,\ a,\ x_{\tilde{L}}, x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t\,) = \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\} \\ & & & \quad\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ \text{Step } 2[\tilde{\mathbb{P}}] & & \mathcal{R}\to & \boldsymbol{\theta}(\,\hat{b}^\star, \hat{b}, \hat{a},\ \hat{\tilde{x}}_L, \hat{\tilde{x}}_K,\ s_{\tilde{\mathcal{L}}}, \hat{\tilde{\kappa}}, \hat{\tilde{T}}, \hat{\tilde{L}}, \hat{\tilde{K}}, \hat{\tilde{\mathcal{L}}}, \hat{V}_t\,) \\ & & & \quad\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ \text{Step } 3[\tilde{\mathbb{P}}] & \text{Lemma 14.4.2(p.101)} & \check{\mathcal{C}}_{\mathbb{P}}\to & \boldsymbol{\theta}(\,\check{a}^\star, \check{a}, \check{b},\ x_{\check{L}}, x_{\check{K}},\ s_{\check{\mathcal{L}}}, \check{\kappa}, \check{T}, \check{L}, \check{K}, \check{\mathcal{L}}, \hat{V}_t\,) \\ & & & \quad\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ \text{Step } 4[\tilde{\mathbb{P}}] & \text{Lemma 14.4.4(p.102)} & \tilde{\mathcal{I}}_{\mathbb{P}}\to & \boldsymbol{\theta}(\,a^\star, a, b,\ x_L, x_K,\ s_{\mathcal{L}}, \kappa, T, L, K, \mathcal{L}, V_t\,) = \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\} \end{array} \tag{14.5.10}$$

Accordingly it follows that the operation which transforms $\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\})$ into $\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\})$ can be eventually reduced to the operation below:

$$\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \overset{\text{def}}{=} \tilde{\mathcal{I}}_{\mathbb{P}}\check{\mathcal{C}}_{\mathbb{P}}\mathcal{R} = \left\{ \begin{array}{cccccccccccc} b^\star, & b, & a, & x_{\tilde{L}_{\mathbb{P}}}, & x_{\tilde{K}_{\mathbb{P}}}, & s_{\tilde{\mathcal{L}}_{\mathbb{P}}}, & \tilde{\kappa}_{\mathbb{P}}, & \tilde{T}_{\mathbb{P}}, & \tilde{L}_{\mathbb{P}}, & \tilde{K}_{\mathbb{P}}, & \tilde{\mathcal{L}}_{\mathbb{P}}, & V_t \\ \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr & \wr \\ a^\star, & a, & b, & x_{L_{\mathbb{P}}}, & x_{K_{\mathbb{P}}}, & s_{\mathcal{L}_{\mathbb{P}}}, & \kappa_{\mathbb{P}}, & T_{\mathbb{P}}, & L_{\mathbb{P}}, & K_{\mathbb{P}}, & \mathcal{L}_{\mathbb{P}}, & V_t \end{array} \right\}. \tag{14.5.11}$$

■ Thus, one sees that in $\tilde{\text{S}}$cenario[$\mathbb{P}$] it suffices to change $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} = \mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}$(see (14.5.3(p.103))) into $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{P}$ above.

■ Moreover, from (III) and (IV) of Table 6.5.1(p.39) it can be easily seen that

$$\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\texttt{A}]\}]. \tag{14.5.12}$$

From all the above discussions it follows that for quite the same reason as that for which Lemma 12.8.1(p.85) was derived we can immediately obtain Lemma 14.5.2(p.105) below.

Lemma 14.5.2 *Let* $A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$. *Then* $A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{C}\langle A_{\mathtt{Tom}}\rangle$ *where*

$$A_{\mathtt{Tom}}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[A_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{14.5.13}$$

Finally, for the same reason as the one for which Theorem 12.8.1(p.85) is derived from Lemma 12.8.1(p.85) we have Theorem 14.5.2(p.105) below.

Theorem 14.5.2 (symmetry theorem ($\tilde{\mathbb{P}} \to \mathbb{P}$)) *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{14.5.14}$$

From (12.8.32(p.85)) we have

$$\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) \overset{\text{def}}{=} \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{14.5.15}$$

$$= (a^\star, a, b, x_L, s_{\mathcal{L}}, x_K, \kappa, T, L, K, \mathcal{L}, V_t). \tag{14.5.16}$$

14.6 Derivation of $\mathscr{A}\{\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}\}$

For the same reason as in Section 12.6(p.78) we see that applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to $\mathscr{A}\{T_{\mathbb{P}}, L_{\mathbb{P}}, K_{\mathbb{P}}, \mathcal{L}_{\mathbb{P}}, \kappa_{\mathbb{P}}\}$ given by Lemmas 13.2.1(p.91)–13.2.6(p.95) yields $\mathscr{A}\{\tilde{T}_{\mathbb{P}}, \tilde{L}_{\mathbb{P}}, \tilde{K}_{\mathbb{P}}, \tilde{\mathcal{L}}_{\mathbb{P}}, \tilde{\kappa}_{\mathbb{P}}\}$.

Lemma 14.6.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$) *For any* $F \in \mathscr{F}$ *we have*:

(a) $\tilde{T}(x)$ *is continuous on* $(-\infty,\infty)$.
(b) $\tilde{T}(x)$ *is nonincreasing on* $(-\infty,\infty)$.
(c) $\tilde{T}(x)$ *is strictly decreasing on* $[a,-\infty)$.
(d) $\tilde{T}(x)+x$ *is nondecreasing on* $(-\infty,\infty)$.
(e) $\tilde{T}(x)+x$ *is strictly increasing on* $(-\infty,b^\star]$.
(f) $\tilde{T}(x)=b-x$ *on* $[b^\star,\infty)$ *and* $\tilde{T}(x)<b-x$ *on* $(-\infty,b^\star)$.
(g) $\tilde{T}(x)<0$ *on* (a,∞) *and* $T(x)=0$ *on* $(-\infty,a]$.
(h) $\tilde{T}(x)\le\min\{0,b-x\}$ *on* $(-\infty,\infty)$.
(i) $\tilde{T}(0)=b$ *if* $b^\star\le 0$ *and* $\tilde{T}(0)=0$ *if* $a>0$.
(j) $\beta\tilde{T}(x)+x$ *is nondecreasing on* $(-\infty,\infty)$ *if* $\beta=1$.
(k) $\beta\tilde{T}(x)+x$ *is strictly increasing on* $(-\infty,\infty)$ *if* $\beta<1$.
(l) *If* $x>y$ *and* $b^\star>y$, *then* $\tilde{T}(x)+x>\tilde{T}(y)+y$.
(m) $\lambda\beta\tilde{T}(\lambda\beta b+s)+s$ *is nondecreasing in* s *and is strictly increasing in* s *if* $\lambda\beta<1$.
(n) $b^\star>b$.

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.1(p.91). ∎
- *Direct proof* See the proof of Lemma A 3.7(p.247). ∎

Applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to (13.2.8(p.93)) - (13.2.13(p.94)), we obtain the relations below:

$$\tilde{L}(x)\begin{cases} = \lambda\beta b+s-\lambda\beta x \text{ on } [b^\star,-\infty) & \cdots(1),\\ < \lambda\beta b+s-\lambda\beta x \text{ on } (-\infty,b^\star) & \cdots(2),\end{cases} \tag{14.6.1}$$

$$\tilde{K}(x)\begin{cases} = \lambda\beta b+s-\delta x \text{ on } [b^\star,\infty) & \cdots(1),\\ < \lambda\beta b+s-\delta x \text{ on } (-\infty,b^\star) & \cdots(2).\end{cases} \tag{14.6.2}$$

$$\tilde{K}(x)\begin{cases} < -(1-\beta)x+s \text{ on } (a,\infty) & \cdots(1),\\ = -(1-\beta)x+s \text{ on } (-\infty,a] & \cdots(2),\end{cases} \tag{14.6.3}$$

$$\tilde{K}(x)+x\le\beta x+s \quad \text{on} \quad (-\infty,\infty). \tag{14.6.4}$$

$$\tilde{K}(x)+x=\begin{cases} \lambda\beta b+s+(1-\lambda)\beta x \text{ on } [b^\star,\infty) & \cdots(1),\\ \beta x+s \text{ on } (-\infty,a] & \cdots(2).\end{cases} \tag{14.6.5}$$

$$\tilde{K}(x_{\tilde{L}})=-(1-\beta)\,x_{\tilde{L}}\cdots(1), \quad \tilde{L}(x_{\tilde{K}})=(1-\beta)\,x_{\tilde{K}}\cdots(2). \tag{14.6.6}$$

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to (13.2.8(p.93))-(13.2.13(p.94)). ∎
- *Direct proof* See the proof of (A 3.1(p.247)) - (A 3.6(p.247)). ∎

Lemma 14.6.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$)

(a) $\tilde{L}(x)$ *is continuous on* $(-\infty,\infty)$.
(b) $\tilde{L}(x)$ *is nonincreasing on* $(-\infty,\infty)$.
(c) $\tilde{L}(x)$ *is strictly decreasing on* $[a,\infty)$.
(d) *Let* $s=0$. *Then* $x_{\tilde{L}}=a$ *where* $x_{\tilde{L}}<(\ge)\,x \Leftrightarrow \tilde{L}(x)<(=)\,0 \Rightarrow \tilde{L}(x)<(\ge)\,0$.

(e) *Let* $s > 0$.

1. $x_{\tilde{L}}$ *uniquely exists with* $x_{\tilde{L}} > a$ *where* $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.
2. $(\lambda\beta b + s)/\lambda\beta \geq (<)\ b^\star \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta b + s)/\lambda\beta < (\geq)\ b^\star$. □

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.2(p.94). ∎
- *Direct proof* See the proof of Lemma A 3.8(p.247) . ∎

Corollary 14.6.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$)

(a) $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (\geq)\ 0$.
(b) $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. □

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Corollary 13.2.1(p.94). ∎
- *Direct proof* See the proof of Corollary A 3.2(p.248) . ∎

Lemma 14.6.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$)

(a) $\tilde{K}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{K}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{K}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{K}(x)$ *is strictly decreasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(e) $\tilde{K}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(f) $\tilde{K}(x) + x$ *is strictly increasing on* $(-\infty, b^\star]$.
(g) $\tilde{K}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\lambda < 1$.
(h) *If* $x > y$ *and* $b^\star > y$, *then* $\tilde{K}(x) + x > \tilde{K}(y) + y$.
(i) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_{\tilde{K}} = a$ *where* $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.
(j) *Let* $\beta < 1$ *or* $s > 0$.

1. *There uniquely exists* $x_{\tilde{K}}$ *where* $x_{\tilde{K}} < (= (>))\ x \Leftrightarrow \tilde{K}(x) < (=>))\ 0$.
2. $(\lambda\beta b + s)/\delta \geq (<)\ b^\star \Leftrightarrow x_{\tilde{K}} = (<)\ (\lambda\beta b + s)/\delta$.
3. *Let* $\tilde{\kappa} < (= (>))\ 0$. *Then* $x_{\tilde{K}} < (= (>))\ 0$. □

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.3(p.94). ∎
- *Direct proof* See the proof of Lemma A 3.9(p.248) . ∎

Corollary 14.6.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0$.
(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. □

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Corollary 13.2.2(p.95). ∎
- *Direct proof* See the proof of Corollary A 3.3(p.249) . ∎

Lemma 14.6.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{P}}/\tilde{K}_{\mathbb{P}}\}$)

(a) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}} = a$.
(b) *Let* $\beta = 1$ *and* $s > 0$. *Then* $x_{\tilde{L}} = x_{\tilde{K}}$.
(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $a < (= (>))\ 0 \Leftrightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}} < (= (=))\ 0$.
(d) *Let* $\beta < 1$ *and* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} \Rightarrow x_{\tilde{K}} < (= (>))\ 0$. □

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.4(p.95). ∎
- *Direct proof* See the proof of Lemma A 3.10(p.249) . ∎

Lemma 14.6.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{P}}\}$)

(a) $\tilde{\mathcal{L}}(s)$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta < 1$.
(b) *Let* $\lambda\beta b \leq a$.

1. $x_{\tilde{L}} \geq \lambda\beta b + s$.
2. *Let* $s > 0$ *and* $\lambda\beta < 1$. *Then* $x_{\tilde{L}} > \lambda\beta b + s$.

(c) *Let* $\lambda\beta b > a$. *Then there exists a* $s_{\tilde{\mathcal{L}}} > 0$ *such that if* $s_{\tilde{\mathcal{L}}} > (\leq)\ s$, *then* $x_{\tilde{L}} < (\geq)\ \lambda\beta b + s$.

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.5(p.95). ∎
- *Direct proof* See the proof of Lemma A 3.11(p.249) . ∎

Lemma 14.6.6 ($\tilde{\kappa}_{\mathbb{P}}$) *We have*:

(a) $\tilde{\kappa} = \lambda\beta b + s$ *if* $b^\star < 0$ *and* $\tilde{\kappa} = s$ *if* $a > 0$.
(b) *Let* $\beta < 1$ *or* $s > 0$. *Then* $\tilde{\kappa} < (= (>))\ 0$. *Then* $x_{\tilde{K}} < (= (>))\ 0$. □

- *Proof by analogy* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 13.2.6(p.95). ∎
- *Direct proof* See the proof of Lemma A 3.12(p.249) . ∎

14.7 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

☐ `Tom` **14.7.1** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) $Ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\,\blacktriangle}$. ☐

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 13.4.1(p.96). ▮

● *Direct proof* See the proof of `Tom` A 4.5(p.254). ▮

☐ `Tom` **14.7.2** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 0$ or $s > 0$. Then, for a given starting time $\tau > 1$:*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \leq a$. Then* $\text{ⓓ}_{\tau}\langle 1\rangle_{\|}$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $b + s \geq b^\star$. Then* $\text{ⓓ}_{\tau>1}\langle 1\rangle_{\|}$.
   - ii. *Let $b + s < b^\star$. Then* $Ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\,\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ⟅$s > 0$⟆.*
   - i. *Let $a < 0$ ⟅$\tilde{\kappa} < 0$⟆. Then* $Ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\,\blacktriangle}$.
   - ii. *Let $a = 0$ ⟅$\tilde{\kappa} = 0$⟆.*
     1. *Let $\beta b + s \geq b^\star$. Then* $\text{ⓓ}_{\tau>1}\langle 1\rangle_{\|}$.
     2. *Let $\beta b + s < b^\star$. Then* $Ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\geq t>1\,\blacktriangle}$.
   - iii. *Let $a > 0$ ⟅$\tilde{\kappa} > 0$⟆.*
     1. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* $\text{ⓓ}_{\tau>1}\langle 1\rangle_{\|}$.
     2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_1$(p.60) $\boxed{Ⓢ\blacktriangle \mid ⓞ\,\|}$ *is true.*

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 13.4.2(p.96). ▮

● *Direct proof* See the proof of `Tom` A 4.6(p.255). ▮

14.8 Optimal Price to Propose

Lemma 14.8.1 ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *The optimal price to propose z_t is nonincreasing in $t > 0$.* ☐

● *Proof* Obvious from `Tom`'s 14.7.1(p.107) (a) and 14.7.2(p.107) (a) and from (6.2.50(p.30)) and Lemma A 3.3(p.244). ▮

Chapter 15

Fifth Step: Analogy Theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$)

In Chaps. 11(p.59) - 14(p.99) we obtained the four assertion systems $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$, $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$, $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$, and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ which are necessary for constructing the integrated theory. In this chapter we clarify the interrelationship between $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$.

15.1 Connection of $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$ and $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$

15.1.1 Assertion System $\mathscr{A}$

First, note the three following relations:

$$\circ\ \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (12.5.52\text{(p.78)})), \tag{15.1.1}$$

$$\bullet\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (13.3.1\text{(p.96)})), \tag{15.1.2}$$

$$\bullet\ \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (14.5.6\text{(p.104)})). \tag{15.1.3}$$

Next, the inverses of the above relations are:

$$\bullet\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \quad (\leftarrow (12.8.31\text{(p.85)})), \tag{15.1.4}$$

$$\circ\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (13.3.6\text{(p.96)})), \tag{15.1.5}$$

$$\circ\ \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \quad (\leftarrow (14.5.14\text{(p.105)})). \tag{15.1.6}$$

Then, from • (15.1.3(p.109)), • (15.1.2(p.109)), and • (15.1.4(p.109)) we obtain the following relation:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \ = \ \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \tag{15.1.7}$$

Finally, from ∘ (15.1.1(p.109)), ∘ (15.1.5(p.109)), and ∘ (15.1.6(p.109)) we obtain the following relation:

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \ = \ \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.8}$$

15.1.2 System of Optimality Equations (SOE)

First, note the following three relations:

$$\circ\ \mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ (\leftarrow (12.5.34\text{(p.75)})), \tag{15.1.9}$$

$$\bullet\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}] \ (\leftarrow (13.3.2\text{(p.96)})), \tag{15.1.10}$$

$$\bullet\ \mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \ (\leftarrow (14.5.4\text{(p.104)})), \tag{15.1.11}$$

Next, the inverses of the above relations are:

$$\bullet\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}] \ (\leftarrow (12.8.25\text{(p.84)})), \tag{15.1.12}$$

$$\circ\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}] \ (\leftarrow (13.3.7\text{(p.96)})), \tag{15.1.13}$$

$$\circ\ \mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}] \ (\leftarrow (14.5.12\text{(p.104)})), \tag{15.1.14}$$

Then, from • (15.1.11(p.109)), • (15.1.10(p.109)), and • (15.1.12(p.109)) we obtain the following relation:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \ = \ \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}], \tag{15.1.15}$$

Finally, from ∘ (15.1.9(p.109)), ∘ (15.1.13(p.109)), and ∘ (15.1.14(p.109)) we obtain the following relation:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \ = \ \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.16}$$

15.1.3 Attribute Vector $\boldsymbol{\theta}$

First, note the following three relations:

$$\circ\ \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (12.5.53(p.78))), \tag{15.1.17}$$
$$\bullet\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (13.3.3(p.96))), \tag{15.1.18}$$
$$\bullet\ \boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (14.5.7(p.104))), \tag{15.1.19}$$

Next, then the inverses of the above relations are:

$$\bullet\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \quad (\leftarrow (12.8.32(p.85))), \tag{15.1.20}$$
$$\circ\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{A}_{\mathbb{P}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (13.3.8(p.96))), \tag{15.1.21}$$
$$\circ\ \boldsymbol{\theta}(\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \quad (\leftarrow (14.5.15(p.105))), \tag{15.1.22}$$

Then, from • (15.1.19(p.110)), • (15.1.18(p.110)), and • (15.1.20(p.110)) we obtain the following relation:

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\})] \tag{15.1.23}$$
$$= (b^\star, b, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \quad (\leftarrow (14.5.8(p.104))). \tag{15.1.24}$$

Finally, from ∘ (15.1.17(p.110)), ∘ (15.1.21(p.110)), and ∘ (15.1.22(p.110)) we obtain the following relation:

$$\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}) = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}[\boldsymbol{\theta}(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\})] \tag{15.1.25}$$
$$= (b, \mu, a,\ x_{\tilde{L}},\ x_{\tilde{K}},\ s_{\tilde{\mathcal{L}}}, \tilde{\kappa}, \tilde{T}, \tilde{L}, \tilde{K}, \tilde{\mathcal{L}}, V_t) \quad (\leftarrow (12.5.54(p.78))). \tag{15.1.26}$$

15.1.4 Symmetry Theorem ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$)

Here let us define

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} \overset{\text{def}}{=} \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}, \tag{15.1.27}$$
$$\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} \overset{\text{def}}{=} \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}. \tag{15.1.28}$$

Then (15.1.7(p.109)) and (15.1.8(p.109)) can be expresses as below.

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}], \tag{15.1.29}$$
$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.30}$$

(15.1.29(p.110)) implies that the following theorem holds.

Theorem 15.1.1 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} \overset{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{15.1.31}$$

Similarly (15.1.30(p.110)) implies that the following theorem (inverse of the above theorem) holds.

Theorem 15.1.2 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P}\times\mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \overset{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{15.1.32}$$

Then (15.1.15(p.109)) and (15.1.16(p.109)) can be expresses as below.

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}], \tag{15.1.33}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}]. \tag{15.1.34}$$

Similarly (15.1.23(p.110)) and (15.1.25(p.110)) can be expresses as below.

$$\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]) = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}])], \tag{15.1.35}$$
$$\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]) = \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\boldsymbol{\theta}(\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}])]. \tag{15.1.36}$$

15.1.5 The Structure of $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$ and $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$

The operation $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}$ given by (15.1.27(p.110)) means that the three operations are applied in the order of $\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \to \mathcal{A}_{\mathbb{R}\to\mathbb{P}} \to \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$. Then, putting this flow in vertically, we have

$$\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \end{array}\right\} \begin{array}{l}\cdots(1)\\ \\ \cdots(2)\end{array} \quad (\leftarrow\ (12.8.21(\text{p.84})))$$

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}} \overset{\text{def}}{=} \left\{\begin{array}{cc} a, & \mu \\ \downarrow & \downarrow \\ a^\star, & a \end{array}\right\} \begin{array}{l}\cdots(3)\\ \\ \cdots(4)\end{array} \quad (\leftarrow\ (13.2.1(\text{p.91})))$$

$$\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} a^\star, & a, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\} \begin{array}{l}\cdots(5)\\ \\ \cdots(6)\end{array} \quad (\leftarrow\ (14.5.3(\text{p.103})))$$

The above flow can be interpreted as follows:

- First, let us focus attention on elements *outside* the dashbox ⬚. Then, we see that first (1)-row changes into (2)-row, next (2)-row is identical to (5)-row, and finally (5)-row changes into (6)-row, which is identical to the original (1)-row. In other words, (1)-row remains unchanged *outside* the dash-box even if these operations are applied.
- Next, let us focus attention on elements *inside* the dashbox ⬚. Then, we see that first (1)-row changes into (2)-row, next (2)-row identical to (5)-row, and finally (5)-row changes into (6)-row. In other words, b and μ in (1)-row change into respectively $b^\star$ and b in (6)-row through the applications of these operations.

From the above we see that the above triple operations can be eventually reduced to the single operation

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} \overset{\text{def}}{=} \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} = \left\{\begin{array}{cc|cccccccccc} b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\}, \tag{15.1.37}$$

called the *analogy replacement operation.* Removing the unchanged elements from the above $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$, eventually we obtain

$$\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} = \{b \to b^\star,\ \mu \to b\}. \tag{15.1.38}$$

Similarly, the operation $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}$ given by (15.1.28(p.110)) means that the three operations are applied in the order of $\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \to \mathcal{A}_{\mathbb{P}\to\mathbb{R}} \to \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$. Then, putting this flow in vertically, we have

$$\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a^\star, & a, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \end{array}\right\} \begin{array}{l}\cdots(1)\\ \\ \cdots(2)\end{array} \quad (\leftarrow\ (14.5.11(\text{p.104})))$$

$$\mathcal{A}_{\mathbb{P}\to\mathbb{R}} \overset{\text{def}}{=} \left\{\begin{array}{cc} a^\star, & a \\ \downarrow & \downarrow \\ a, & \mu \end{array}\right\} \begin{array}{l}\cdots(3)\\ \\ \cdots(4)\end{array} \quad (\leftarrow\ (13.3.5(\text{p.96})))$$

$$\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} \overset{\text{def}}{=} \left\{\begin{array}{cc|cccccccccc} a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{array}\right\} \begin{array}{l}\cdots(5)\\ \\ \cdots(5)\end{array} \quad (\leftarrow\ (12.5.29(\text{p.75})))$$

The above flow can be eventually reduced to as follows.

$$\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} = \{b^\star \to b,\ b \to \mu\}. \tag{15.1.39}$$

From the comparison of `Tom`'s 12.7.2(p.81) and 14.7.2(p.107) we can easily reconfirm that Theorem 15.1.1(p.110) holds in fact.

15.2 Relationship between $\tilde{\mathsf{M}}$:1[$\mathbb{P}$][E] and $\tilde{\mathsf{M}}$:1[$\mathbb{R}$][E]

It can be easily confirmed that the same as in Section 15.1(p.109) holds also for $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]\}$ and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{E}]\}$. Then we have

Theorem 15.2.1 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{E}]\} \overset{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{E}]\}].\ \square \tag{15.2.1}$$

Theorem 15.2.2 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\} \overset{\text{def}}{=} \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}].\ \square \tag{15.2.2}$$

It can be easily confirmed that $\mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}$ and $\mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}}$ are the same as (15.1.38(p.111)) and (15.1.39(p.111)) respectively.

Chapter 16

Flow of Discussions

Let us here again recall Motive 2(p.4) "Does a general theory integrating quadruple-asset-trading-problems exist ?", and this motivation was put an end with a successful construction of the integrated theory.

16.1 Flow of Discussions

The integrated theory is summarized as below.

⟨1⟩ $\mathscr{A}\{T_{\mathbb{R}}\}$ is ***proven*** (see Lemma 10.1.1(p.53)).

⟨2⟩ $\mathscr{A}\{L_{\mathbb{R}}, K_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \kappa_{\mathbb{R}}\}$ is ***proven*** (see Lemmas 10.2.1(p.55) - 10.3.1(p.57)).

⟨3⟩ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ is ***proven*** (see `Tom`'s 11.2.1(p.59) and 11.2.2(p.60)).

⟨4⟩ $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is ***derived*** (see `Tom`'s 12.7.1(p.81) and 12.7.2(p.81)).

⟨5⟩ $\mathscr{A}\{T_{\mathbb{P}}\}$ is ***proven*** (see Lemma 13.2.1(p.91)).

⟨6⟩ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ is ***derived*** (see `Tom`'s 13.4.1(p.96) and 13.4.2(p.96)).

⟨7⟩ $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is ***derived*** (see `Tom`'s 14.7.1(p.107) and 14.7.2(p.107)).

⟨8⟩ The analogous relation between $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ is shown (see Theorems 15.1.1(p.110) and 15.1.2(p.110)).

16.2 Structure of Integrated Theory

The above flow, ⟨1⟩ – ⟨8⟩, can be schematized as in Figure 16.2.1(p.113) below where the three shadow boxes are *directly proven* and the remaining four frame boxes are all *indirectly derived* by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$, $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$, and $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to .

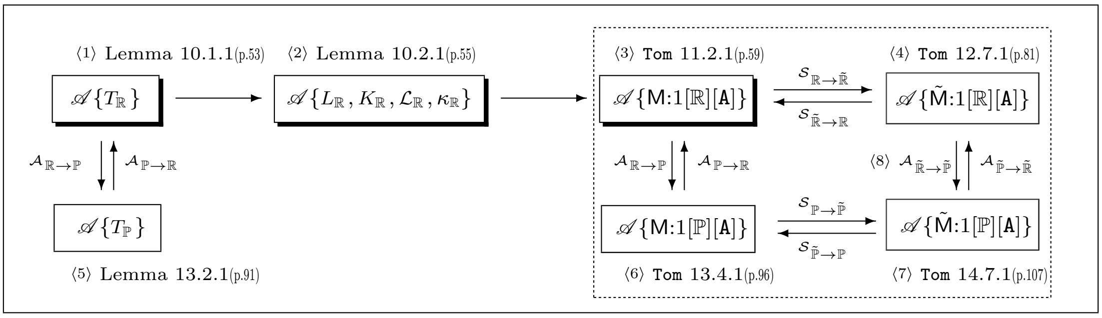


Figure 16.2.1: The whole flow of constructing the integrated theory

16.3 Implications

The interrelationship among the quadruple assertion systems within the dashbox of Figure 16.2.1(p.113) implies the following. First, an assertion system of $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ is *defined* as a *core* within the quadruple-asset-trading-models $\mathcal{Q}\langle\mathsf{M}{:}1[\mathtt{A}]\rangle$ and then *proven* (see Chap. 11(p.59)). Next, the assertion system for each of the remaining three models is *derived* by sequentially applying the operations $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ and $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to the above *core* assertion system (see Chaps. 12(p.67) and 13(p.87)). Finally, $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$ is derived so as to become *symmetrical* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see Chap. 14(p.99)). Since it is proven that any of the above four operations are reversible, even if any other assertion system within $\mathcal{Q}\langle\mathsf{M}{:}1[\mathtt{A}]\rangle$ is selected as a *core*, the same flow as the above can be depicted. Let us refer to the whole structure consisting of the quadruple assertion systems in such a fashion as stated above as the *integrated theory.* In the conventional approach, each of the quadruple assertion systems must be defined *separately* and proven *one by one.* On the other hand, in our approach based on the integrated theory, the number of assertion systems which must be defined and proven is *only one* as a core. In Part 3 that follows we try to apply the integrated theory to all of the remaining five quadruple-asset-trading-models in Table 3.2.1(p.17) except for $\mathcal{Q}\langle\mathsf{M}{:}1[\mathtt{A}]\rangle$ the analyses of which was already ended. From all the above, it will be realized that the integrated theory provides a strong tool for the treatment of asset trading problems.

16.3.1 Limitation of Integration Theory

Here note that the successful construction of the integrated theory is based on the following two premises: one is that price ξ is defined on the total market $(-\infty, \infty)$, the other is that the symmetrical relation between $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ and $\texttt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\texttt{A}]\}$ and the analogy relation between $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\texttt{A}]\}$ and $\texttt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\texttt{A}]\}$ must be satisfied (see ⋆Sections 12.11(p.85) and 13.8(p.97)). However, as seen from ⋆Tables 6.5.3(p.39) - 6.5.6(p.39) , although the symmetrical relation always holds between $\texttt{SOE}\{\mathsf{M}{:}1/2/3[\mathbb{R}/\mathbb{P}][\texttt{A/E}]\}$ and $\texttt{SOE}\{\tilde{\mathsf{M}}{:}1/2/3[\mathbb{R}/\mathbb{P}][\texttt{A/E}]\}$ (compare (I) and (II)), the analogical relation between $\texttt{SOE}\{\mathsf{M}/\tilde{\mathsf{M}}{:}2/3[\mathbb{R}][\texttt{A/E}]\}$ and $\texttt{SOE}\{\mathsf{M}/\tilde{\mathsf{M}}{:}2/3[\mathbb{P}][\texttt{A/E}]\}$ does not hold (compare (I) and (III)). In other words, for Models 1/2/3 the symmetry theorems can be always applied; however, the analogy theorems cannot be applied. Accordingly, it is *only* for discussions related to symmetry that the integrated theory are applicable. For the treatment of the case where the analogy theorem cannot be applied, see Lemma 20.1.1(p.151) and Section 20.1.5(p.164) .

Chapter 17

Market Restriction

17.1 Preliminary

As seen from the whole discussions over Chaps. 10(p.53) –15(p.109) , the integrated theory is constructed under the premise that prices ξ, whether reservation price or posted price, is defined on the Total-dF-Space (see (2.2.5(p.13))), i.e.,

$$\mathscr{F} = \{F \mid -\infty < a < \mu < b < \infty\}, \tag{17.1.1}$$

called the *total market*. However, since the prices ξ in a usual market of the real world are positive, i.e., $\xi \in (0, \infty)$, the above premise, permitting a negative price $\xi \in (-\infty, 0)$, must be said to be unrealistic. This chapter proposes a methodology working through this problem.

17.2 Market Restriction

Let us refer to the restriction of the *total market* $\mathscr{F}$ to a given subset

$$\mathscr{F}' \subseteq \mathscr{F} \tag{17.2.1}$$

as the *market restriction* of $\mathscr{F}$ to $\mathscr{F}'$ and to the $\mathscr{F}'$ as the *restricted market*. Throughout this paper let us consider the following three kinds of restricted markets:

$$\mathscr{F}^+ \stackrel{\text{def}}{=} \{F \mid 0 < a < b\} \quad (\textit{positive market}), \tag{17.2.2}$$

$$\mathscr{F}^\pm \stackrel{\text{def}}{=} \{F \mid a \le 0 \le b\} \quad (\textit{mixed market}), \tag{17.2.3}$$

$$\mathscr{F}^- \stackrel{\text{def}}{=} \{F \mid a < b < 0\} \quad (\textit{negative market}) \tag{17.2.4}$$

where clearly

$$\mathscr{F} = \mathscr{F}^+ \cup \mathscr{F}^\pm \cup \mathscr{F}^-. \tag{17.2.5}$$

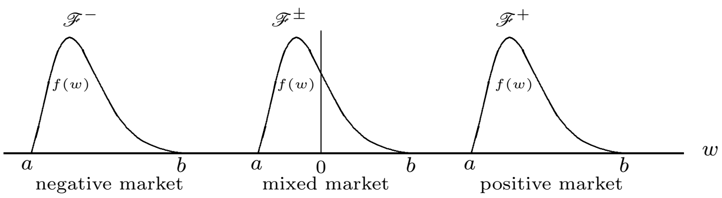


Figure 17.2.1: Three kinds of markets

Definition 17.2.1 In the present paper, let us represent the restriction of $\mathscr{F}$ to the above three restricted markets by the same symbols $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ above, called the *positive market restriction* $\mathscr{F}^+$, the *mixed market restriction* $\mathscr{F}^\pm$, and the *negative market restriction* $\mathscr{F}^-$ respectively. See Section A 7.6(p.264) for an economic implication brought about by the three market restrictions. □

17.3 Market Restricted Models

Throughout the rest of this paper, let us denote the models defined on the restricted markets $\mathscr{F}^+$, $\mathscr{F}^\pm$, and $\mathscr{F}^-$ by Model^+, $\mathsf{Model}^\pm$, and Model^- respectively, called the *market restricted models*. For $x = 1, 2, 3$ and $\mathtt{X} = \mathtt{A}, \mathtt{E}$ let us define the quadruple-asset-trading-models:

$$\mathcal{Q}\langle \mathsf{M}{:}x[\mathtt{X}]^+\rangle \stackrel{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}]^+, \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}]^+, \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}]^+, \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]^+\}, \tag{17.3.1}$$

$$\mathcal{Q}\langle \mathsf{M}{:}x[\mathtt{X}]^\pm\rangle \stackrel{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}]^\pm, \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}]^\pm, \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}]^\pm, \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]^\pm\}, \tag{17.3.2}$$

$$\mathcal{Q}\langle \mathsf{M}{:}x[\mathtt{X}]^-\rangle \stackrel{\text{def}}{=} \{\mathsf{M}{:}x[\mathbb{R}][\mathtt{X}]^-, \tilde{\mathsf{M}}{:}x[\mathbb{R}][\mathtt{X}]^-, \mathsf{M}{:}x[\mathbb{P}][\mathtt{X}]^-, \tilde{\mathsf{M}}{:}x[\mathbb{P}][\mathtt{X}]^-\}. \tag{17.3.3}$$

17.4 Inequalities Caused by Market Restriction

The lemma below will be used to examine what occurs when the market restriction is applied to results derived by using the integrated theory constructed on the total market $\mathscr{F}$.

Lemma 17.4.1 (positive market $\mathscr{F}^+$) *Suppose $0 < a$. Then we have:*

[1] [ref.08078] $0 < a < \mu < b$. Proof: Evident from (2.2.2(p.13)).

[2] [ref.09343] $\beta b \leq b$ *for* $0 < \beta \leq 1$. Proof: Immediate from $0 < \beta b \leq b$ with $\beta = 1$.

[3] [ref.07865] $\beta\mu < b$ *for* $0 < \beta \leq 1$. Proof: Immediate from $0 < \beta\mu < b$ with $\beta = 1$.

[4] [ref.08369] $\beta a < b$ *for* $0 < \beta \leq 1$. Proof: Immediate from $0 < \beta a < b$ with $\beta = 1$.

[5] [ref.09483] $a < \beta\mu$ *and* $\beta\mu \leq a$ *are both possible.* Proof: Since $0 < a < \beta\mu$ with $\beta = 1$, the former is possible for a β sufficiently close to $\beta = 1$ and the latter is possible for any sufficiently small $\beta > 0$.

[6] [ref.06867] $a < \beta b$ *and* $\beta b \leq a$ *are both possible.* Proof: Since $0 < a < \beta b$ with $\beta = 1$, the former is possible for a β sufficiently close to $\beta = 1$ and the latter is possible for any sufficiently small $\beta > 0$.

[7] [ref.06296] $\beta b < b^\star$ *for* $0 < \beta \leq 1$. Proof: Immediate from $0 < \beta b < b^\star$ with $\beta = 1$ due to Lemma 14.6.1(p.105) (n). □

Lemma 17.4.2 (mixed market $\mathscr{F}^\pm$) *Suppose $a \leq 0 \leq b$. Then we have:*

[8] [ref.08062] $a < \beta\mu < b$ *for* $0 < \beta \leq 1$. Proof: Let $\mu = 0$. Then, since $a < \mu < b$ from (2.2.2(p.13)), we have $a < 0 < b$, hence always $a < \beta \times 0 < b$, so $a < \beta\mu < b$. Let $\mu \neq 0$. If $a < \mu < 0$, then $a < \beta\mu < 0 \leq b$ with $\beta = 1$, hence $a < \beta\mu < 0 \leq b$ for $0 < \beta \leq 1$ and if $0 < \mu < b$, then $a \leq 0 < \beta\mu < b$ with $\beta = 1$, hence $a \leq 0 < \beta\mu < b$ for $0 < \beta \leq 1$. Accordingly, whether $a < \mu < 0$ or $0 < \mu < b$, we have $a < \beta\mu < b$ for $0 < \beta \leq 1$. Thus, whether $\mu = 0$ or $\mu \neq 0$, it follows that $a < \beta\mu < b$ for $0 < \beta \leq 1$.

[9] [ref.06907] $\beta a < b$ *for* $0 < \beta \leq 1$. Proof: Let $\beta = 1$. Then $\beta a = a < b$. Let $\beta < 1$. If $a = 0$, then $\beta a = a = 0 < b$ and if $a < 0$, then $\beta a < 0 \leq b$, hence $\beta a < b$ whether $a = 0$ or $a < 0$. Thus, whether $\beta = 1$ or $\beta < 1$ (i.e., $0 < \beta \leq 1$) it follows that we have $\beta a < b$.

[10] [ref.06892] $a < \beta b$ *for* $0 < \beta \leq 1$. Proof: If $b > 0$, then $a \leq 0 < b = \beta b$ with $\beta = 1$, hence $a \leq 0 < \beta b$ for $0 < \beta \leq 1$. If $b = 0$, then $a < b = \beta b = 0$ for $0 < \beta \leq 1$. Therefore, whether $b > 0$ or $b = 0$, we have $a < \beta b$ for $0 < \beta \leq 1$.

[11] [ref.06896] $a^\star < \beta a$ *for* $0 < \beta \leq 1$. Proof: Immediate from $a^\star < \beta a \leq 0$ with $\beta = 1$ due to Lemma 13.2.1(p.91) (n).

[12] [ref.06298] $\beta b < b^\star$ *for* $0 < \beta \leq 1$. Proof: Immediate from $0 \leq \beta b < b^\star$ with $\beta = 1$ due to Lemma 14.6.1(p.105) (n). □

Lemma 17.4.3 (negative market $\mathscr{F}^-$) *Suppose $b < 0$. Then we have:*

[13] [ref.07486] $a < \mu < b < 0$. Proof: Evident from (2.2.2(p.13)).

[14] [ref.06118] $a \leq \beta a$ *for* $0 < \beta \leq 1$. Proof: Immediate from $a \leq \beta a < 0$ with $\beta = 1$.

[15] [ref.08068] $a < \beta\mu$ *for* $0 < \beta \leq 1$. Proof: Immediate from $a < \beta\mu < 0$ with $\beta = 1$.

[16] [ref.07482] $a < \beta b$ *for* $0 < \beta \leq 1$. Proof: Immediate from $a < \beta b < 0$ with $\beta = 1$.

[17] [ref.07478] $\beta\mu < b$ *and* $b \leq \beta\mu$ *are both possible.* Proof: Since $\beta\mu < b < 0$ with $\beta = 1$, the former is true for a β sufficiently close to $\beta = 1$ and the latter is true for a sufficiently small $\beta > 0$.

[18] [ref.08296] $\beta a < b$ *and* $b \leq \beta a$ *are both possible.* Proof: Since $\beta a < b < 0$ with $\beta = 1$, the former is possible for a β sufficiently close to $\beta = 1$ and the latter is possible for a sufficiently small $\beta > 0$.

[19] [ref.06919] $a^\star < \beta a$ *for* $0 < \beta \leq 1$. Proof: Immediate from $a^\star < \beta a < 0$ with $\beta = 1$ due to Lemma 13.2.1(p.91) (n). □

Definition 17.4.1 (market-restriction-free-assertion) When no change occurs even if a market restriction is applied to a given assertion, the assertion is said to be *free from* the market restriction, called the *market-restriction-free assertion.* □

Lemma 17.4.4 *Even if a market restriction is applied to a market-restriction-free assertion, no change occurs.* □

• *Proof* Evident. ∎

17.5 Market Restriction

17.5.1 $\mathscr{A}\{\text{M:1}[\mathbb{R}][\mathtt{A}]\}$

17.5.1.1 Positive Restriction

□ `Pom` **17.5.1** ($\mathscr{A}\{\text{M:1}[\mathbb{R}][\mathtt{A}^+]\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) V_t *is nondecreasing in* $t > 0$.

(b) $Ⓢ_{\tau > 1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{CONDUCT}_{\tau \geq t > 1\blacktriangle}$. □

• *Proof* The same as `Tom` 11.2.1(p.59) due to Lemma 17.4.4(p.116). ∎

□ Pom **17.5.2** ($\mathscr{A}${M:1[$\mathbb{R}$][A]$^+$}) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
(b) *Let* $\beta\mu \geq b$ (impossible). ———
(c) *Let* $\beta\mu < b$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $\mu - s \leq a$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
   ii. *Let* $\mu - s > a$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$,
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $\beta\mu > s$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$ (see *Numerical Example* 1(p.124)).
   ii. *Let* $s \geq \beta\mu$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$ (see *Numerical Example* 2(p.124)).

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Let $\beta < 1$ or $s > 0$. Then $\kappa = \beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.57) (a) with $\lambda = 1$.

(a) The same as (a) of Tom 11.2.2(p.60).

(b,c) Always $\beta\mu < b$ due to [3(p.116)], hence $\beta\mu \geq b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as (c1i,c1ii) of Tom 11.2.2(p.60).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 11.2.2(p.60).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu > s$. Then, since $\kappa > 0$ due to (2), it suffices to consider only (c2i) of Tom 11.2.2(p.60).

(c3ii) Let $\beta\mu \leq s$. Then, since $\kappa \leq 0$ due to (2) and since $\beta\mu - s \leq 0 < a$, it suffices to consider only (c2ii1,c2iii1) of Tom 11.2.2(p.60). ∎

17.5.1.2 Mixed Restriction

□ Mim **17.5.1** ($\mathscr{A}${M:1[$\mathbb{R}$][A]$^\pm$}) *Suppose* $a \leq 0 \leq b$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$. □

● *Proof* The same as Tom 11.2.1(p.59) due to Lemma 17.4.4(p.116). ∎

□ Mim **17.5.2** ($\mathscr{A}${M:1[$\mathbb{R}$][A]$^\pm$}) *Suppose* $a \leq 0 \leq b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
(b) *Let* $\beta\mu \geq b$ (impossible). ———
(c) *Let* $\beta\mu < b$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $\mu - s \leq a$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
   ii. *Let* $\mu - s > a$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < \beta T(0)$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
   ii. *Let* $s = \beta T(0)$.
      1. *Let* $\beta\mu - s \leq a$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
      2. *Let* $\beta\mu - s > a$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
   iii. *Let* $s > \beta T(0)$.
      1. *Let* $\beta\mu - s \leq a$ *or* $s_{\mathcal{L}} \leq s$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
      2. *Let* $\beta\mu - s > a$ *and* $s_{\mathcal{L}} > s$. *Then* S$_1$(p.60) [ⓢ▲ | ⓞ ‖] *is true.*

● *Proof* Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.

(a) The same as Tom 11.2.2(p.60) (a).

(b,c) Always $\beta\mu < b$ due to [8(p.116)], hence $\beta\mu \geq b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as (c1i,c1ii) of Tom 11.2.2(p.60).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, then it suffices to consider only (c2i) of Tom 11.2.2(p.60) and if $b = 0$, then since always $\beta\mu - s = \beta\mu > a$ due to [8], it suffices to consider only (c2ii2) of Tom 11.2.2(p.60). Therefore, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions are immediate from (c2i-c2iii2) of Tom 11.2.2(p.60) with $\kappa = \beta T(0) - s$ from (5.1.7(p.23)) with $\lambda = 1$. ∎

17.5.1.3 Negative Restriction

☐ Nem **17.5.1** ($\mathscr{A}$\{M:1[ℝ][A]$^-$\}) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*
(a) *V_t is nondecreasing in $t > 0$.*
(b) *We have* $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as Tom 11.2.1(p.59) due to Lemma 17.4.4(p.116). ∎

☐ Nem **17.5.2** ($\mathscr{A}$\{M:1[ℝ][A]$^-$\}) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*
(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
(b) *Let $\beta\mu \geq b$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
(c) *Let $\beta\mu < b$.*
  1. *Let $\beta = 1$.*
    i. *Let $\mu - s \leq a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
    ii. *Let $\mu - s > a$. Then* $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$. Then* S$_1$(p.60) [Ⓢ▲ | ◎ ‖] *is true.*
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
    ii. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* S$_1$(p.60) [Ⓢ▲ | ◎ ‖] *is true.*

● *Proof* Suppose $b < 0 \cdots$ **(1)**. Let $\beta < 1$ or $s > 0$. Then, we have $\kappa = -s \cdots$ **(2)** from Lemma 10.3.1(p.57) (a). Moreover, in this case, both $\beta\mu \geq b$ and $\beta\mu < b$ are possible due to [17(p.116)].

(a,b) The same as Tom 11.2.2(p.60) (a,b).

(c) Let $\beta\mu < b$. Then $s_{\mathcal{L}} > 0 \cdots$ **(3)** from Lemma 10.2.4(p.57) (c).

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption $\beta < 1$ or $s > 0$.

(c1i,c1ii) The same as (c1i,c1ii) of Tom 11.2.2(p.60).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii1,c2iii2) of Tom 11.2.2(p.60). Since $\beta\mu - s = \beta\mu > a$ due to [15(p.116)] and since $s = 0 < s_L$ due to (3), we have (c2iii2) of Tom 11.2.2(p.60).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\kappa < 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of Tom 11.2.2(p.60). ∎

17.5.2 $\mathscr{A}$\{$\tilde{\mathsf{M}}$:1[ℝ][A]\}

17.5.2.1 Positive Restriction

☐ Pom **17.5.3** ($\mathscr{A}$\{$\tilde{\mathsf{M}}$:1[ℝ][A]$^+$\}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*
(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as Tom 12.7.1(p.81) due to Lemma 17.4.4(p.116). ∎

☐ Pom **17.5.4** ($\mathscr{A}$\{$\tilde{\mathsf{M}}$:1[ℝ][A]$^+$\}) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*
(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \leq a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
(c) *Let $\beta\mu > a$.*
  1. *Let $\beta = 1$.*
    i. *Let $\mu + s \geq b$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
    ii. *Let $\mu + s < b$. Then* $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* CONDUCT$_{\tau\geq t>1\blacktriangle}$.
  2. *Let $\beta < 1$ and $s = 0$. Then* S$_1$(p.60) [Ⓢ▲ | ◎ ‖] *is true.*
  3. *Let $\beta < 1$ and $s > 0$.*†
    i. *Let $\beta\mu + s \geq b$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
    ii. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* S$_1$(p.60) [Ⓢ▲ | ◎ ‖] *is true* (see Numerical Example 3(p.125)).

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 12.6.6(p.81) (a). Here note that $\mu\beta \leq a$ and $\mu\beta > a$ are both possible due to [5(p.116)].

(a,b) The same as (a,b) of Tom 12.7.2(p.81).

(c) Let $\beta\mu > a$. Then $s_{\tilde{\mathcal{L}}} > 0 \cdots$ **(3)** due to Lemma 12.6.5(p.81) (c) with $\lambda = 1$.

(c1-c1ii) Let $\beta = 1$, hence $s > 0$ due to the assumptions $\beta < 1$ and $s > 0$. Thus, we have (c1i,c1ii) of Tom 12.7.2(p.81).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $\beta\mu + s = \beta\mu < b$ due to [3(p.116)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from (3), due to (1) it suffices to consider only (c2iii2) of Tom 12.7.2(p.81).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of Tom 12.7.2(p.81). ∎

17.5.2.2 Mixed Restriction

□ **Mim 17.5.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{\pm}\}$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) *We have* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as **Tom** 12.7.1(p.81) due to Lemma 17.4.4(p.116). ∎

□ **Mim 17.5.4** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{\pm}\}$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $\beta\mu \leq a$ (impossible). ______

(c) *Let* $\beta\mu > a$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $\mu + s \geq b$. *Then* $\text{❶}_{\tau>1}\langle 1\rangle_{\|}$.
   ii. *Let* $\mu + s < b$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < -\beta\tilde{T}(0)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let* $s = -\beta\tilde{T}(0)$.
      1. *Let* $\beta\mu + s \geq b$. *Then* $\text{❶}_{\tau>1}\langle 1\rangle_{\|}$.
      2. *Let* $\beta\mu + s < b$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let* $s > -\beta\tilde{T}(0)$.
      1. *Let* $\beta\mu + s \geq b$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* $\text{❶}_{\tau>1}\langle 1\rangle_{\|}$.
      2. *Let* $\beta\mu + s < b$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* S_1(p.60) [ⓢ▲ | ◎ ‖] *is true.*

● *Proof* Suppose $a \leq 0 \leq b$.

(a) The same as **Tom** 12.7.2(p.81) (a).

(b,c) Always $\beta\mu > a$ due to [8(p.116)], hence $\beta\mu \leq a$ is impossible. Hence $s_{\tilde{\mathcal{L}}} > 0 \cdots (1)$ due to Lemma 12.6.5(p.81) (c).

(c1-c1ii) The same as (c1-c1ii) of **Tom** 12.7.2(p.81).

(c2) Let $\beta < 1$ and $s = 0$. Let $a < 0$. Then it suffices to consider only (c2i) of **Tom** 12.7.2(p.81). Let $a = 0$. Then $\beta\mu + s = \beta\mu < b$ due to [8(p.116)], hence it suffices to consider only (c2ii2) of **Tom** 12.7.2(p.81). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions become true from (c2i-c2iii2) of **Tom** 12.7.2(p.81) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.16(p.23)). ∎

17.5.2.3 Negative Restriction

□ **Nem 17.5.3** ($\mathscr{A}_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{-}\}$) *Suppose* $b < 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as **Tom** 12.7.1(p.81) due to Lemma 17.4.4(p.116). ∎

□ **Nem 17.5.4** ($\mathscr{A}_{\texttt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{A}]^{-}\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \leq x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $\beta\mu \leq a$ (impossible). ______

(c) *Let* $\beta\mu > a$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $\mu + s \geq b$. *Then* $\text{❶}_{\tau>1}\langle 1\rangle_{\|}$.
   ii. *Let* $\mu + s < b$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $\beta\mu < -s$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let* $\beta\mu \geq -s$. *Then* $\text{❶}_{\tau>1}\langle 1\rangle_{\|}$.

● *Proof* Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$. Then $\tilde{\kappa} = \beta\mu + s \cdots (3)$ due to Lemma 12.6.6(p.81) (a).

(a) The same as **Tom** 12.7.2(p.81) (a).

(b,c) Always $a < \beta\mu$ due to [15(p.116)], hence $\beta\mu \leq a$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of **Tom** 12.7.2(p.81).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c2i) of **Tom** 12.7.2(p.81).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu < -s$, hence $\beta\mu + s < 0$. Then, since $\tilde{\kappa} < 0$ due to (3), it suffices to consider only (c2i) of Tom 12.7.2(p.81).

(c3ii) Let $\beta\mu \geq -s$, hence $\beta\mu + s \geq 0$. Let $\beta\mu + s = 0$. Then, since $\tilde{\kappa} = 0$ due to (3) and $\beta\mu + s > b$ due to (2), it suffices to consider only (c2iii1) of Tom 12.7.2(p.81). Let $\beta\mu + s > 0$. Then, since $\tilde{\kappa} > 0$ due to (3), it suffices to consider only (c2iii) of Tom 12.7.2(p.81). In this case, since $\beta\mu + s > 0 > b$ due to (1), it suffices to consider only (c2ii1) of Tom 12.7.2(p.81). Accordingly, whether $\beta\mu + s = 0$ or $\beta\mu + s > 0$, we have the same result. ∎

17.5.3 $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$

17.5.3.1 Positive Restriction

□ **Pom 17.5.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*

● *Proof* The same as Tom 13.4.1(p.96) due to Lemma 17.4.4(p.116). ∎

□ **Pom 17.5.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta a \geq b$* (impossible). ———

(c) *Let $\beta a < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $a - s \leq a^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
   ii. *Let $a - s > a^\star$. Then $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*
2. *Let $\beta < 1$ and $s = 0$. Then $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*
   ii. *Let $s = \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
      2. *Let $\beta a - s > a^\star$. Then $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*
   iii. *Let $s > \beta T(0)$.*
      1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
      2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_1$(p.60) [Ⓢ▲ | ◎ ‖].

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots$ (1).

(a) The same as Tom 13.4.2(p.96) (a).

(b,c) Always $\beta a < b$ due to [4(p.116)], hence $\beta a \geq b$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of Tom 13.4.2(p.96).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 13.4.2(p.96).

(c3-c3iii2) Immediate from (c2-c2iii2) of Tom 13.4.2(p.96) with $\kappa = \beta T(0) - s$ from (5.1.23(p.24)) with $\lambda = 1$. ∎

17.5.3.2 Mixed Restriction

□ **Mim 17.5.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^\pm\}$) *Suppose $a \leq 0 \leq b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*

● *Proof* The same as Tom 13.4.1(p.96) due to Lemma 17.4.4(p.116). ∎

□ **Mim 17.5.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^\pm\}$) *Suppose $a \leq 0 \leq b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $\beta a \geq b$* (impossible). ———

(c) *Let $\beta a < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $a - s \leq a^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
   ii. *Let $a - s > a^\star$. Then $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*
2. *Let $\beta < 1$ and $s = 0$. Then $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s < \beta T(0)$. Then $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.*

ii. *Let* $s = \beta T(0)$.
   1. *Let* $\beta a - s \le a^\star$. *Then* $\textcircled{d}_{\tau>1}\langle 1\rangle_{\|}$.
   2. *Let* $\beta a - s > a^\star$. *Then* $\textcircled{s}_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.

iii. *Let* $s > \beta T(0)$.
   1. *Let* $\beta a - s \le a^\star$ *or* $s_{\mathcal{L}} \le s$. *Then* $\textcircled{d}_{\tau>1}\langle 1\rangle_{\|}$.
   2. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\mathtt{S}_1$(p.60) $\boxed{\textcircled{s}_{\blacktriangle} \mid \circledcirc_{\|}}$.

● *Proof* Suppose $a \le 0 \le b$.

(a) The same as Tom 13.4.2(p.96) (a).

(b,c) Always $\beta a < b$ due to [9(p.116)], hence $\beta a \ge b$ is impossible. .

(c1-c1ii) The same as (c1-c1ii) of Tom 13.4.2(p.96).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, the assertion is true from Tom 13.4.2(p.96) (c2i) and if $b = 0$, then $\beta a - s = \beta a > a^\star$ from [11(p.116)], hence the assertion become true from Tom 13.4.2(p.96) (c2ii2). Accordingly, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) The same as (c2i-c2iii2) of Tom 13.4.2(p.96) with $\kappa = \beta T(0) - s$ from (5.1.23(p.24))) with $\lambda = 1$. ∎

17.5.3.3 Negative Restriction

□ **Nem 17.5.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose* $b < 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) *We have* $\textcircled{s}_{\tau}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.

● *Proof* Immediate from Tom 13.4.1(p.96) due to Lemma 17.4.4(p.116). ∎

□ **Nem 17.5.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]^-\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $\ge x_K$ *as* $t \to \infty$.

(b) *Let* $\beta a \ge b$. *Then* $\textcircled{d}_{\tau>1}\langle 1\rangle_{\|}$.

(c) *Let* $\beta a < b$.
   1. *Let* $\beta = 1$.
      i. *Let* $a - s \le a^\star$. *Then* $\textcircled{d}_{\tau>1}\langle 1\rangle_{\|}$.
      ii. *Let* $a - s > a^\star$. *Then* $\textcircled{s}_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.
   2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_1$(p.60) $\boxed{\textcircled{s}_{\blacktriangle} \mid \circledcirc_{\|}}$.
   3. *Let* $\beta < 1$ *and* $s > 0$.
      i. *Let* $\beta a - s \le a^\star$ *or* $s_{\mathcal{L}} \le s$. *Then* $\textcircled{d}_{\tau>1}\langle 1\rangle_{\|}$.
      ii. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\mathtt{S}_1$(p.60) $\boxed{\textcircled{s}_{\blacktriangle} \mid \circledcirc_{\|}}$.

● *Proof* Suppose $b < 0 \cdots (1)$, hence $\kappa = -s \cdots (2)$ from Lemma 13.2.6(p.95) (a). Then, both $\beta a \ge b$ and $\beta a < b$ are possible due to [18(p.116)]. If $\beta a < b$, then $s_{\mathcal{L}} > 0 \cdots (3)$ due to Lemma 13.2.5(p.95) (c) with $\lambda = 1$.

(a) The same as Tom 13.4.2(p.96) (a).

(b) Let $\beta a \ge b$. Then, this is the same as (b) of Tom 13.4.2(p.96).

(c) Let $\beta a < b$.

(c1-c1ii) The same as (c1-c1ii) of Tom 13.4.2(p.96).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii) of Tom 13.4.2(p.96). In addition, since $\beta a - s = \beta a > a^\star$ due to [19(p.116)] and since $s_{\mathcal{L}} > 0 = s$ due to (3), it suffices to consider only (c2iii2) of Tom 13.4.2(p.96).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\kappa < 0$ from (2), it suffices to consider only (c2iii - c2iii2) of Tom 13.4.2(p.96). ∎

17.5.4 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

17.5.4.1 Positive Restriction

□ **Pom 17.5.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) *We have* $\textcircled{s}_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{CONDUCT}_{\tau\ge t>1\blacktriangle}$.

● *Proof* The same as Tom 14.7.1(p.107) due to Lemma 17.4.4(p.116). ∎

□ **Pom 17.5.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \ge x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $\beta b \le a$. *Then* $\textcircled{d}_{\tau>1}\langle 1\rangle_{\|}$.

(c) *Let* $\beta b > a$.

1. *Let* $\beta = 1$.
   i. *Let* $b + s \geq b^\star$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$ $\rightarrow$
   ii. *Let* $b + s < b^\star$. *Then* ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\texttt{S}_1$(p.60) [ⓢ▲ | ⊚ ∥].
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* ⓓ$_{\tau}\langle 1\rangle_{\parallel}$.
   ii. *Let* $\beta b + s < b^\star$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\texttt{S}_1$(p.60) [ⓢ▲ | ⊚ ∥].

● *Proof* Suppose $a > 0 \cdots (1)$. Then, $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.106) (a). In this case, $\beta b \leq a$ and $\beta b > a$ are both possible due to [6(p.116)], and if $\beta b > a$, then $s_{\tilde{\mathcal{L}}} > 0 \cdots (3)$ due to Lemma 14.6.5(p.106) (c) with $\lambda = 1$. In addition, we have

(a,b) The same as (a,b) of Tom 14.7.2(p.107).

(c) Let $\beta b > a$.

(c1-c1ii) The same as (c1-c1ii) of Tom 14.7.2(p.107).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii) of Tom 14.7.2(p.107). In this case, since $\beta b + s = \beta b < b^\star$ due to [7(p.116)] and since $s_{\mathcal{L}} > 0 = s$ due to (3), it suffices to consider only (c2iii2) of Tom 14.7.2(p.107).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii-c2iii2) of Tom 14.7.2(p.107). ∎

17.5.4.2 Mixed Restriction

□ **Mim 17.5.7** ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]$^{\pm}\}$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as Tom 14.7.1(p.107) due to Lemma 17.4.4(p.116). ∎

□ **Mim 17.5.8** ($\mathscr{A}\{\tilde{\mathsf{M}}$:1[$\mathbb{P}$][A]$^{\pm}\}$) *Suppose* $a \leq 0 \leq b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nonincreasing in* $t > 0$ *and converges to a finite* $V \geq x_{\tilde{K}}$ *as* $t \rightarrow \infty$.

(b) *Let* $\beta b \leq a$ (impossible). ———

(c) *Let* $\beta b > a$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $b + s \geq b^\star$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
   ii. *Let* $b + s < b^\star$. *Then* ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < -\beta\tilde{T}(0)$. *Then* ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let* $s = -\beta\tilde{T}(0)$.
      1. *Let* $\beta b + s \geq b^\star$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
      2. *Let* $\beta b + s < b^\star$. *Then* ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let* $s > -\beta\tilde{T}(0)$.
      1. *Let* $\beta b + s \geq b^\star$ *or* $s_{\tilde{\mathcal{L}}} \leq s$. *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
      2. *Let* $\beta b + s < b^\star$ *and* $s_{\tilde{\mathcal{L}}} > s$. *Then* $\texttt{S}_1$(p.60) [ⓢ▲ | ⊚ ∥].

● *Proof* Let $b \geq 0 \geq a \cdots (1)$.

(a) The same as Tom 14.7.2(p.107) (a).

(b,c) Always $\beta b > a$ due to [10(p.116)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of Tom 14.7.2(p.107).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2i-c2ii2) of Tom 14.7.2(p.107). Let $a < 0$. Then, the assertion is true from (c2i) of Tom 14.7.2(p.107). Let $a = 0$. Then, since $\beta b + s = \beta b < b^\star$ due to [12(p.116)], it suffices to consider only (c2ii2) of Tom 14.7.2(p.107). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from (c2i-c2iii2) of Tom 14.7.2(p.107) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.25)) with $\lambda = 1$. ∎

17.5.4.3 Negative Restriction

□ **Nem 17.5.7** ($\mathscr{A}\{\tilde{\text{M}}{:}1[\mathbb{P}][\text{A}]^{-}\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.

● *Proof* The same as Tom 14.7.1(p.107) due to Lemma 17.4.4(p.116). ∎

□ **Nem 17.5.8** ($\mathscr{A}\{\tilde{\text{M}}{:}1[\mathbb{P}][\text{A}]^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \geq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \leq a$* (impossible). ———

(c) *Let $\beta b > a$* (always holds).

1. *Let $\beta = 1$.*
   - i. *Let $b + s \geq b^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $b + s < b^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $s < -\beta\tilde{T}(0)$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - ii. *Let $s = -\beta\tilde{T}(0)$.*
     1. *Let $\beta b + s \geq b^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let $\beta b + s < b^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{CONDUCT}_{\tau\geq t>1\blacktriangle}$.
   - iii. *Let $-\beta\tilde{T}(0) < s$.*
     1. *Let $\beta b + s \geq b^\star$ or $s_{\tilde{\mathcal{L}}} \leq s$. Then* ⓓ$_{\tau}\langle 1\rangle_{\parallel}$.
     2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\texttt{S}_1$(p.60) | Ⓢ▲ | ⓞ ∥ |.

● *Proof* Let $b < 0$, hence $a < b < 0 \cdots (1)$.

(a) The same as Tom 14.7.2(p.107) (a).

(b,c) Always $\beta b > a$ due to [16(p.116)], hence $\beta b \leq a$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of Tom 14.7.2(p.107).

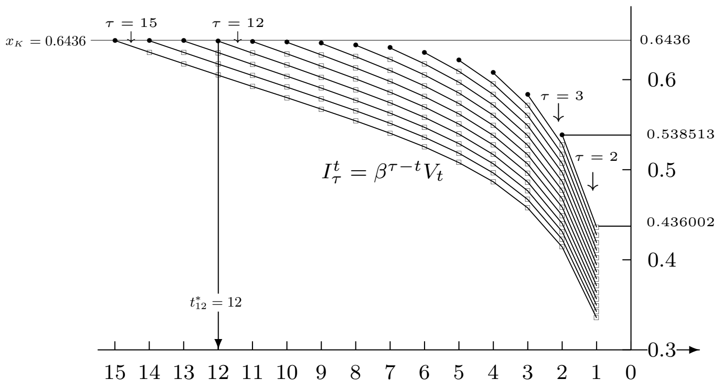
(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 14.7.2(p.107).

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from (c2-c2iii2) of Tom 14.7.2(p.107) with $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.25)) with $\lambda = 1$. ∎

17.6 Numerical Example

Numerical Example **1** **($\mathscr{A}\{$M:1[ℝ][A]$\}^+$ (selling model)**

This is the example for $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ in Pom 17.5.2(p.117) (c3i) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$.† Then, we have $x_K = 0.6436$ (see Section A 6(p.259)). Figure 17.6.1(p.124) below is the graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $\tau = 2, 3, \cdots, 15$ and $t = 1, 2, \cdots, \tau$ (see (7.2.4(p.43))). For example, the two points on the line of $\tau = 2$ are given by $V_2 = 0.538513\,(\bullet)$ and $\beta V_1 = 0.98 \times 0.444900 = 0.436002\,(\circ)$, hence $V_2 > \beta V_1$. Similarly, the three points on the polygonal curve of $\tau = 3$ are given by $V_3 = 0.583152\ (\bullet)$, $\beta V_2 = 0.98 \times 0.538513 = 0.52774274\ (\circ)$, and $\beta^2 V_1 = 0.98^2 \times 0.4449 = 0.42728196\ (\circ)$, hence $V_3 > \beta V_2 > \beta^2 V_1$. Then, the value of t on the horizontal line corresponding to the bullet • provides the optimal initiating time t^*_τ for each of $\tau = 2, 3, \cdots, 15$, i.e., $\texttt{OIT}_\tau\langle t^*_\tau\rangle$, so we have $t^*_2 = 2$, $t^*_3 = 3$, $\cdots$, $t^*_{15} = 15$ (see t^*_τ-column of the table below). This result means $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ for $\tau = 2, 3, \cdots, 15$. Since $V_t - \beta V_t > 0$ for $t = 2, 3, \cdots, 15$ (see values of $V_t - \beta V_t$-column in the table below), we have $L(V_{t-1}) > 0$ from (11.1.1(p.59)), meaning $\texttt{Conduct}_{15\geq t>1\blacktriangle}$ from (11.1.5(p.59)), i.e., it is strictly optimal to conduct the search on $15 \geq t > 1$.



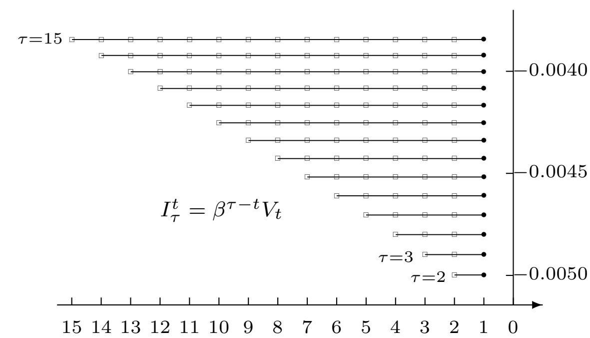
$V_t - \beta V_{t-1}$, t^*_τ (OIT)

t	V_τ	$V_t - \beta V_{t-1}$	t^*_τ	decision
0				
1	0.444900			
2	0.538513	+0.102511	2	$\texttt{Conduct}_\blacktriangle$
3	0.583152	+0.055409	3	$\texttt{Conduct}_\blacktriangle$
4	0.607492	+0.036003	4	$\texttt{Conduct}_\blacktriangle$
5	0.621595	+0.026252	5	$\texttt{Conduct}_\blacktriangle$
6	0.630035	+0.020871	6	$\texttt{Conduct}_\blacktriangle$
7	0.635180	+0.017745	7	$\texttt{Conduct}_\blacktriangle$
8	0.638351	+0.015874	8	$\texttt{Conduct}_\blacktriangle$
9	0.640318	+0.014734	9	$\texttt{Conduct}_\blacktriangle$
10	0.641544	+0.014032	10	$\texttt{Conduct}_\blacktriangle$
11	0.642309	+0.013596	11	$\texttt{Conduct}_\blacktriangle$
12	0.642788	+0.013325	12	$\texttt{Conduct}_\blacktriangle$
13	0.643088	+0.013155	13	$\texttt{Conduct}_\blacktriangle$
14	0.643276	+0.013049	14	$\texttt{Conduct}_\blacktriangle$
15	0.643393	+0.012983	15	$\texttt{Conduct}_\blacktriangle$

Figure 17.6.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \geq \tau \geq 2, \tau \geq t \geq 1)$ where • represents OIT

Numerical Example **2** **($\mathscr{A}\{$M:1[ℝ][A]$\}^+$ (selling model)**

This is the example for $ⓓ_{\tau>1}\langle 1\rangle_{\|}$ in Pom 17.5.2(p.117) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.50$† The bullet • in each of the 14 horizontal straight lines in Figure 17.6.2(p.124) below shows that the optimal initiating time t^*_τ degenerates to time 1 (i.e., $t^*_\tau = 1$ for $\tau = 2, 3, \cdots, 15$) under Preference Rule 7.2.1(p.43), i.e., $ⓓ_{\tau=2,3,\cdots,15}\langle 1\rangle_{\|}$. The result comes from the fact of $V_t - \beta V_t = 0$ for $t = 2, 3, \cdots, 15$ with $t = 2, 3, \cdots, 15$ (see $V_t - \beta V_{t-1}$-column in the table below), leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$ for $\tau = 2, 3, \cdots, 15$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$ for $\tau = 2, 3, \cdots, 15$.



$V_t - \beta V_{t-1}$

t	V_t	$V_t - \beta V_{t-1}$	t^*_τ	$\mathbb{S}_t$
0				
1	−0.005100			
2	−0.004998	0.000000	1	−0.00010200
3	−0.004898	0.000000	1	−0.00021960
4	−0.004800	0.000000	1	−0.00029996
5	−0.004704	0.000000	1	−0.00039600
6	−0.004610	0.000000	1	−0.00049008
7	−0.004517	0.000000	1	−0.00058220
8	−0.004427	0.000000	1	−0.00067334
9	−0.004338	0.000000	1	−0.00076154
10	−0.004252	0.000000	1	−0.00084876
11	−0.004167	0.000000	1	−0.00093304
12	−0.004083	0.000000	1	−0.00101634
13	−0.004002	0.000000	1	−0.00109866
14	−0.003922	0.000000	1	−0.00117804
15	−0.003843	0.000000	1	−0.00125644

Figure 17.6.2: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ $(15 \geq \tau \geq 2, \tau \geq t \geq 1)$ where • represents OIT

Note here that numbers in V_t-column are all negative, meaning that tackling the asset selling problem makes no profits (red ink). Accordingly, if this is of **tE-case** (see H1(p.8) (a)), you must resign to the red ink and if it is of **tA-case** (see H1(p.8) (b)), it suffices to pass over the problem without tackling the selling problem itself. Since $0.5 \times (a + b) = 0.505$ and since $V_t < 0 < 0.01 = a$ for $t = 1, 2, \cdots, 15$ (see V_t-column of the above table), from (A 7.2 (1) (p.261)) we have $T(V_t) = 0.505 - V_t$ for $t = 1, 2, \cdots, 15$, hence we have:

†Note that $a = 0.01 > 0$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$. Then, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949 > 0.05 = s$. Thus, the condition of this assertion is satisfied.

†Note that $a = 0.01 > 0$, $\beta = 0.98 < 1$, and $s = 0.50 > 0$. In addition, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949 < 0.50 = s$. Thus, the condition of the assertion is satisfied.

$$
\begin{aligned}
&T(V_1) = 0.505 - (-0.005100) = 0.510100, && T(V_6) = 0.505 - (-0.004610) = 0.509610, && T(V_{11}) = 0.505 - (-0.004167) = 0.509167,\\
&T(V_2) = 0.505 - (-0.004998) = 0.509998, && T(V_7) = 0.505 - (-0.004517) = 0.509517, && T(V_{12}) = 0.505 - (-0.004083) = 0.509083,\\
&T(V_3) = 0.505 - (-0.004898) = 0.509898, && T(V_8) = 0.505 - (-0.004427) = 0.509427, && T(V_{13}) = 0.505 - (-0.004002) = 0.509002,\\
&T(V_4) = 0.505 - (-0.004800) = 0.509800, && T(V_9) = 0.505 - (-0.004338) = 0.509338, && T(V_{14}) = 0.505 - (-0.003922) = 0.508922,\\
&T(V_5) = 0.505 - (-0.004704) = 0.509704, && T(V_{10}) = 0.505 - (-0.004252) = 0.509252, && T(V_{15}) = 0.505 - (-0.003843) = 0.508843.
\end{aligned}
$$

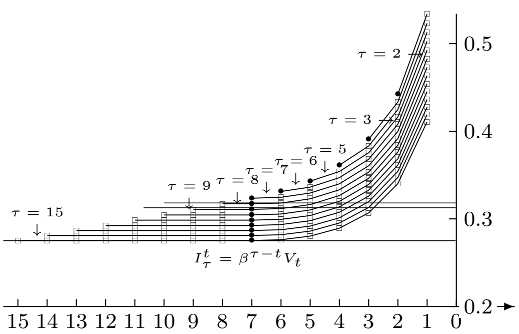
Since $\mathbb{S}_t = 0.98 \times T(V_{t-1}) - 0.5$ from (6.2.13(p.28)), we get

$$
\begin{aligned}
&\mathbb{S}_2 = 0.98 \times 0.510100 - 0.5 = -0.00010200, && \mathbb{S}_7 = 0.98 \times 0.509610 - 0.5 = -0.00058220, && \mathbb{S}_{12} = 0.98 \times 0.509167 - 0.5 = -0.00101634,\\
&\mathbb{S}_3 = 0.98 \times 0.509998 - 0.5 = -0.00021960, && \mathbb{S}_8 = 0.98 \times 0.509517 - 0.5 = -0.00067334, && \mathbb{S}_{13} = 0.98 \times 0.509083 - 0.5 = -0.00109866,\\
&\mathbb{S}_4 = 0.98 \times 0.509898 - 0.5 = -0.00029996, && \mathbb{S}_9 = 0.98 \times 0.509427 - 0.5 = -0.00076154, && \mathbb{S}_{14} = 0.98 \times 0.509002 - 0.5 = -0.00117804,\\
&\mathbb{S}_5 = 0.98 \times 0.509800 - 0.5 = -0.00039600, && \mathbb{S}_{10} = 0.98 \times 0.509338 - 0.5 = -0.00084876, && \mathbb{S}_{15} = 0.98 \times 0.508922 - 0.5 = -0.00125644,\\
&\mathbb{S}_6 = 0.98 \times 0.509704 - 0.5 = -0.00049008, && \mathbb{S}_{11} = 0.98 \times 0.509252 - 0.5 = -0.00093304.
\end{aligned}
$$

From the results of the above numerical calculation we have $\mathbb{S}_t < 0$ for $15 \geq t > 1$, hence it is *strictly optimal* to skip the search over $15 \geq t > 1$ due to (6.2.9(p.28)), i.e., $\texttt{Skip}_\blacktriangle$. However, since $V_t - \beta V_{t-1} = 0$ for $15 \geq t > 1$ (see $(V_t - \beta V_{t-1})$-column in the above table), we have $V_{15} = \beta V_{14} = \cdots = \beta^{14} V_1$, i.e., the profit attained are indifferent over $15 \geq t > 0$. This is not a contradiction, which is a false feeling caused by confusion from the jumble of intuition and theory (see Alice 2(p.42)).

Numerical Example **3** **($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]^+\}$) (buying model)**

This is the numerical example for $⊚_{\tau > t_\tau^\bullet}\langle t_\tau^\bullet \rangle_{\|}$ in $\mathtt{S}_1$(p.60) $\boxed{Ⓢ\blacktriangle \mid ⊚\ \|}$ of **Pom** 17.5.4(p.118) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$.† Then, we have $s_{\tilde{\mathcal{L}}} = 0.323274$ (see Section A 6(p.259)). Hence, the optimal initiating time t_τ^* is given by t attaining $\min_{\tau \geq t > 0} I_\tau^t$ (see (7.2.5(p.43))).‡ The bullet • in Figure 17.6.3(p.125) below shows the optimal initiating time for each of $\tau = 2, 3, \cdots, 15$ (see t_τ^*-column in the table below). From the figure and table we see that $t_\tau^* = \tau$ for $\tau = 2, 3, \cdots, 7$, i.e., $Ⓢ_{7 \geq \tau > 1}\langle \tau \rangle_\blacktriangle$ (see $\mathtt{S}_1$(p.60) (1)) and that $t_\tau^* = 7$ for $\tau = 8, 9, \cdots, 15$, i.e., $⊚_{\tau > 7}\langle 7 \rangle_{\|}$ (see $\mathtt{S}_1$(p.60) (2)). In the numerical example, note the fact that $\tilde{\mathbb{S}} = \tilde{L}(V_{\tau-1})$ are all negative (< 0 $(-)$, i.e., $\texttt{Skip}_\blacktriangle$) for $t = 2, 3, \cdots, 7$ and positive (> 0 $(+)$, i.e., $\texttt{Conduct}_\blacktriangle$) for $t = 8, 9, \cdots, 15$. Moreover, note that we have $V_t - \beta V_{t-1} = 0$ or equivalently $V_t = \beta V_{t-1}$ for $t = 8, 9, \cdots, 15$ and $V_t - \beta V_{t-1} < 0$ or equivalently $V_t < \beta V_{t-1}$ for $t = 2, 3, \cdots, 7$ (see $V_t - \beta V_{t-1}$-column), hence $V_{15} = \beta V_{14} = \beta^2 V_{13} = \cdots = \beta^8 V_7 < \beta^9 V_6 < \beta^{10} V_5 < \cdots < \beta^{14} V_1$ (see $\beta^{15-t} V_t$-column), so we have $⊚_{\tau > 7}\langle 7 \rangle_{\|}$.



t	β^{15-t}	V_t	$V_t - \beta V_{t-1}$	$\beta^{15-t} V_t$	t_τ^*	$\tilde{\mathbb{S}}_t = \tilde{L}(V_{t-1})$	decision
0							
1	0.753641	0.544900		0.410658			
2	0.769022	0.442388	−0.091614	0.340206	2	−0.091614	$\texttt{Conduct}_\blacktriangle$
3	0.784716	0.391004	−0.042535	0.306827	3	−0.042535	$\texttt{Conduct}_\blacktriangle$
4	0.800731	0.361335	−0.021849	0.289332	4	−0.021849	$\texttt{Conduct}_\blacktriangle$
5	0.817072	0.343013	−0.011094	0.280266	5	−0.011094	$\texttt{Conduct}_\blacktriangle$
6	0.833747	0.331264	−0.004889	0.276190	6	−0.004889	$\texttt{Conduct}_\blacktriangle$
7	0.850763	0.323555	−0.001084	0.275268	7	−0.001084	$\texttt{Conduct}_\blacktriangle$
8	0.868125	0.317084	0.000000	0.275268	7	+0.001338	$\texttt{Skip}_\blacktriangle$
9	0.885842	0.310742	0.000000	0.275268	7	+0.003326	$\texttt{Skip}_\blacktriangle$
10	0.903920	0.304527	0.000000	0.275268	7	+0.005233	$\texttt{Skip}_\blacktriangle$
11	0.922368	0.298437	0.000000	0.275268	7	+0.007064	$\texttt{Skip}_\blacktriangle$
12	0.941192	0.292468	0.000000	0.275268	7	+0.008822	$\texttt{Skip}_\blacktriangle$
13	0.960400	0.286618	0.000000	0.275267	7	+0.010508	$\texttt{Skip}_\blacktriangle$
14	0.980000	0.280886	0.000000	0.275268	7	+0.012127	$\texttt{Skip}_\blacktriangle$
15	1.000000	0.275268	0.000000	0.275268	7	+0.013680	$\texttt{Skip}_\blacktriangle$

Figure 17.6.3: Graphs of $I_\tau^t = \beta^{\tau-t} V_t$ $(15 \geq \tau \geq 2, \tau \geq t \geq 1)$

†Note that $a = 0.01 > 0$, $b = 1.00$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$. Then, since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\beta\mu = 0.98 \times 0.505 = 0.4949$, hence $\beta\mu + s = 0.4949 + 0.05 = 0.5449 < 1.00 = b$. In addition, $s_{\tilde{\mathcal{L}}} = 0.323274 > 0.05 = s$. Thus, the conditions for the assertions are satisfied.

‡Note that this is a selling model with cost minimization.

Chapter 18

Conclusions of Part 2 (Integrated Theory)

Below is the summary of the essential points of the whole discussions over Chaps. 10(p.53) - 17(p.115).

C1. **Two preliminary steps**

a. **Proofs of the properties of the underlying functions** (see Chap. 10(p.53))
The first preliminary step in constructing the integrated theory is to clarify the properties of underlying functions $T_{\mathbb{R}}$, $L_{\mathbb{R}}$, $K_{\mathbb{R}}$, and $\mathcal{L}_{\mathbb{R}}$.

b. **Proofs of the properties of the assertion system** $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (see Chap. 11(p.59))
The second preliminary step is to clarify the properties of the assertion system $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ by using the above properties of underlying functions.

C2. **Symmetry and analogy theorems**

a. **Symmetry theorem** ($\mathbb{R} \leftrightarrow \tilde{\mathbb{R}}$) (see Chap. 12(p.67))
The concept of symmetry between a selling problem and a buying problem was first vaguely inspired from the pattern of the *yin-yang principle* in an ancient Chinese philosophy. This rather superstitious and shaky concept was first formalized by the introduction of the *reverse operation* $\mathcal{R}$ (see Section 12.1.1(p.67)). After that, through more than twenty years of trial-and-errors, this concept led us to the *correspondence replacement operation* $\mathcal{C}_{\mathbb{R}}$ (see Lemma 12.3.1(p.69)) and then to *identity replacement operation* $\mathcal{I}_{\mathbb{R}}$ (see Lemma 12.3.3(p.71)). Finally, the above three operations were compiled into a single operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}$ (see (12.5.30(p.75))), called the *symmetry transformation operation*, yielding Theorem 12.5.1(p.78), which derives $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ in `Tom`'s 11.2.1(p.59) and 11.2.2(p.60). In addition, we obtained Theorem 12.8.1(p.85) (the inverse of Theorem 12.5.1(p.78)), which derives $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ from $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$.

b. **Analogy theorem** ($\mathbb{R} \leftrightarrow \mathbb{P}$) (see Chap. 13(p.87))
In the earlier stage of this study, we did not anticipate at all that there would exist a relation between an asset trading problem with $\mathbb{R}$-mechanism and an asset trading problem with $\mathbb{P}$-mechanism (see Section 1.1(p.3)). However, as moving on analyses of both problems, we gradually noticed similarities between the two procedures of treating both problems. This realization led us, as if solving the *jigsaw puzzle*, to the existence of an analogous relation between the above two problems. This little recognition was, before long, materialized by the proof of Lemmas 10.1.1(p.53) and 13.2.1(p.91), which finally leads to the *analogy replacement operation* $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (13.2.1(p.91))). This finding produced Theorems 13.3.1(p.96) and 13.3.2(p.96), which combines $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$.

c. **Symmetry theorem** ($\mathbb{P} \leftrightarrow \tilde{\mathbb{P}}$) (see Chap. 14(p.99))
By applying the way of thinking in C2b(p.127) to the process of deriving the symmetry theorems in C2a(p.127), Theorems 14.5.1(p.104) and 14.5.2(p.105), which combine $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$, we could relatively easily obtain the symmetry theorems, Theorems 14.5.1(p.104) and 14.5.2(p.105), which combine $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$.

d. **Analogy theorem** ($\tilde{\mathbb{R}} \leftrightarrow \tilde{\mathbb{P}}$) (see Chap. 15(p.109))
From the multi-layered relationship among six theorems derived in the above C2a(p.127), C2b(p.127), and C2c(p.127), we can comparatively easily obtain the analogy theorems, Theorems 15.1.1(p.110) and 15.1.2(p.110), which combines $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$.

C3. **Integrated theory**
The most distinguishing results in the present paper is the successful construction of the integrated theory (see Motive 2(p.4) and Chap. 16(p.113)), by use of which all models included in a given structured-unit-of-problems (see Section 3.3(p.18)) can be systematically analyzed. The theory consists of the four symmetry theorems,

Theorems 12.5.1(p.78), 12.8.1(p.85), 14.5.1(p.104), 14.5.2(p.105),

and the four analogy theorems,

Theorems 13.3.1(p.96), 13.3.2(p.96), 15.1.1(p.110), 15.1.2(p.110).

The former four theorems combines an asset selling problem and an asset buying problem and the latter four theorems combines an asset trading problem with $\mathbb{R}$-mechanism and an asset trading problem with $\mathbb{P}$-mechanism. The integrated theory plays a decisively important role in the analyses of not only all models in the present paper but also all variations of the problems which will be dealt with in the future (see Chap. 24(p.235)). However, the integrated theory is not always versatile, which has the following two weak points.

a. **Market restriction**
Let us note here again that the integrated theory can be constructed under the premise that the price ξ is defined on the total market $\mathscr{F}$ (see (17.1.1(p.115))), i.e., $\xi \in (-\infty, \infty)$. Under the integrated theory we clarified that $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ (buying model with $\mathbb{R}$-mechanism) can be derived so as to become *symmetrical* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling model with $\mathbb{R}$-mechanism) and that $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ (selling problem with $\mathbb{P}$-mechanism) can be derived so as to become *analogous* to $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ (selling problem $\mathbb{R}$-mechanism). However, since an asset trading problem on the normal market in the real world is usually conducted on the positive market $\mathscr{F}^+ = (0, \infty)$ (see (17.2.2(p.115))), it is an open question whether symmetry and analogy on $\mathscr{F}$ are inherited to $\mathscr{F}^+$. To approach this problem, in this paper, we employ the methodology of restricting results obtained on $\mathscr{F}$ to $\mathscr{F}^+$ by using Lemmas 17.4.1(p.116) - 17.4.3(p.116). Through this methodology, we will show in C2C(p.132), C3C(p.132), C2C(p.146), and C3C(p.146) that the symmetrical relation and the analogous relation can collapse on $\mathscr{F}^+$.

b. **Collapse of symmetry and analogy among SOE's**
The integrated theory has the following imitation (see Section 16.3.1(p.114)). In Model 1, the successful construction of the integrated theory is based on the fact that the symmetrical relation between $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$ and the analogy relation between $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$ are satisfied (see Sections 12.11(p.85) and 13.8(p.97)). However, from Tables 6.5.1(p.39) - 6.5.6(p.39) we see that although the symmetrical relation always holds between $\mathtt{SOE}\{\mathsf{M}{:}1/2/3[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1/2/3[\mathbb{R}][\mathtt{A}]\}$ (compare (I) and (II)), the analogical relation between $\mathtt{SOE}\{\mathsf{M}{:}2/3[\mathbb{R}][\mathtt{A}]\}$ and $\mathtt{SOE}\{\mathsf{M}{:}2/3[\mathbb{P}][\mathtt{A}]\}$ (compare (I) and (III)) does not always hold. From the above we see that under the integrated theory, although the symmetry theorems can be always applied for Models 1/2/3, the analogy theorems cannot be applied for Models 2/3.

C4. **Summary of operations**
For convenience of reference, let us summarize all operations depicted in Figure 16.2.1(p.113) below.

$$(12.5.29(p.75)) \to \quad \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}} = \left\{\begin{matrix} a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{matrix}\right\}. \tag{18.0.1}$$

$$(12.8.21(p.84)) \to \quad \mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}} = \left\{\begin{matrix} b, & \mu, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & s_{\tilde{\mathcal{L}}}, & \tilde{\kappa}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a, & \mu, & b, & x_L, & x_K, & s_{\mathcal{L}}, & \kappa, & T, & L, & K, & \mathcal{L}, & V_t \end{matrix}\right\}. \tag{18.0.2}$$

$$(14.5.3(p.103)) \to \quad \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}} = \left\{\begin{matrix} a^\star, & a, & b, & x_L, & x_K, & \kappa, & s_{\mathcal{L}}, & T, & L, & K, & \mathcal{L}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & \tilde{\kappa}, & s_{\tilde{\mathcal{L}}}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \end{matrix}\right\}. \tag{18.0.3}$$

$$(14.5.11(p.104)) \to \quad \mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}} = \left\{\begin{matrix} b^\star, & b, & a, & x_{\tilde{L}}, & x_{\tilde{K}}, & \tilde{\kappa}, & s_{\tilde{\mathcal{L}}}, & \tilde{T}, & \tilde{L}, & \tilde{K}, & \tilde{\mathcal{L}}, & V_t \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a^\star, & a, & b, & x_L, & x_K, & \kappa, & s_{\mathcal{L}}, & T, & L, & K, & \mathcal{L}, & V_t \end{matrix}\right\}. \tag{18.0.4}$$

$$(13.2.1(p.91)) \to \quad \mathcal{A}_{\mathbb{R}\to\mathbb{P}} = \{a \to a^\star,\ \mu \to a\}. \tag{18.0.5}$$

$$(13.3.5(p.96)) \to \quad \mathcal{A}_{\mathbb{P}\to\mathbb{R}} = \{a^\star \to a,\ a \to \mu\}. \tag{18.0.6}$$

$$(15.1.38(p.111)) \to \quad \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}} = \{b \to b^\star,\ \mu \to b\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}\,\mathcal{A}_{\mathbb{R}\to\mathbb{P}}\,\mathcal{S}_{\tilde{\mathbb{R}}\to\mathbb{R}}. \tag{18.0.7}$$

$$(15.1.39(p.111)) \to \quad \mathcal{A}_{\tilde{\mathbb{P}}\to\tilde{\mathbb{R}}} = \{b^\star \to b,\ b \to \mu\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}\,\mathcal{A}_{\mathbb{P}\to\mathbb{R}}\,\mathcal{S}_{\tilde{\mathbb{P}}\to\mathbb{P}}. \tag{18.0.8}$$

Part 3

Analyses

In Chaps. 19(p.131), 20(p.151), and 21(p.221) we try to analyze all of the $72 = 3 \times 24$ models include in Model 1, Model 2 and, Model 3 define in this paper by using the integrated theory. In Chap. 22(p.229) the properties of these models are summarized.

Chapter 19

Analysis of Model 1

19.1 Search-Allowed-Model 1: $\mathscr{Q}\{\mathsf{M}{:}1[\mathtt{A}]\} = \{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}1[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

All analyses of the search-Allowed-model 1 already completed in Part 2(p.49). Below, let us summarize the whole conclusions obtained there.

19.1.1 Conclusion 1 (Search-Allowed-Model 1)

C1. Mental Conflict

On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$ we have (see (7.3.1(p.45)) and (7.3.2(p.45)) for the definitions of opt-$\mathbb{R}$-price and opt-$\mathbb{P}$-price below):

a. The opt-$\mathbb{R}$-price V_t in $\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]$ (selling model) is nondecreasing in t as seen in Tom's 11.2.1(p.59) (a) and 11.2.2(p.60) (a) (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

b. The opt-$\mathbb{P}$-price z_t in $\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]$ (selling model) is nondecreasing in t as seen in Lemma 13.7.1(p.97) (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

c. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]$ (buying model) is nonincreasing in t as seen in Tom's 12.7.1(p.81) (a) and 12.7.2(p.81) (a) (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

d. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]$ (buying model) is nonincreasing in t as seen in Lemma 14.8.1(p.107) (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

The above results can be summarized as below.

A. On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict.

C2. Symmetry

a. On $\mathscr{F}^+$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Pom 17.5.3(p.118) $\sim$ Pom 17.5.1(p.116) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^+)$,

Pom 17.5.7(p.121) $\sim$ Pom 17.5.5(p.120) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^+)$.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Pom 17.5.4(p.118) $\not\sim$ Pom 17.5.2(p.117) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^+) \cdots (s^1)$,

Pom 17.5.8(p.121) $\not\sim$ Pom 17.5.6(p.120) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^+) \cdots (s^2)$.

b. On $\mathscr{F}^\pm$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Mim 17.5.3(p.119) $\sim$ Mim 17.5.1(p.117) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm)$,

Mim 17.5.7(p.122) $\sim$ Mim 17.5.5(p.120) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm)$.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Mim 17.5.4(p.119) $\sim$ Mim 17.5.2(p.117) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm)$,

Mim 17.5.8(p.122) $\sim$ Mim 17.5.6(p.120) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm)$.

c. On $\mathscr{F}^-$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Nem}\ 17.5.3(\text{p.119}) \sim \texttt{Nem}\ 17.5.1(\text{p.118}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^- \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^-),$$
$$\texttt{Nem}\ 17.5.7(\text{p.123}) \sim \texttt{Nem}\ 17.5.5(\text{p.121}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^- \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^-).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Nem}\ 17.5.4(\text{p.119}) \not\sim \texttt{Nem}\ 17.5.2(\text{p.118}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^- \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^-)\cdots(s^3),$$
$$\texttt{Nem}\ 17.5.8(\text{p.123}) \not\sim \texttt{Nem}\ 17.5.6(\text{p.121}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^- \not\sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^-)\ \cdots(s^4).$$

The above results can be summarized as below.

A. On $\mathscr{F}^\pm$, for any $\beta \le 1$ and $s \ge 0$, the symmetry is inherited (see C2b(p.131)).

B. On $\mathscr{F}^+$ and $\mathscr{F}^-$, if $\beta = 1$ and $s = 0$, the symmetry is inherited (see C2a1(p.131) and C2c1(p.132)).

C. On $\mathscr{F}^+$ and $\mathscr{F}^-$, if $\beta < 1$ or $s > 0$, the symmetry collapses (see (s^1), (s^2), (s^3), and (s^4)).

C3. Analogy

a. On $\mathscr{F}^+$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Pom}\ 17.5.5(\text{p.120}) \bowtie \texttt{Pom}\ 17.5.1(\text{p.116}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^+),$$
$$\texttt{Pom}\ 17.5.7(\text{p.121}) \bowtie \texttt{Pom}\ 17.5.3(\text{p.118}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^+).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Pom}\ 17.5.6(\text{p.120}) \not\bowtie \texttt{Pom}\ 17.5.2(\text{p.117}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^+)\cdots(a^1),$$
$$\texttt{Pom}\ 17.5.8(\text{p.121}) \bowtie \texttt{Pom}\ 17.5.4(\text{p.118}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^+).$$

b. On $\mathscr{F}^\pm$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Mim}\ 17.5.5(\text{p.120}) \bowtie \texttt{Mim}\ 17.5.1(\text{p.117}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm),$$
$$\texttt{Mim}\ 17.5.7(\text{p.122}) \bowtie \texttt{Mim}\ 17.5.3(\text{p.119}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Mim}\ 17.5.6(\text{p.120}) \bowtie \texttt{Mim}\ 17.5.2(\text{p.117}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm),$$
$$\texttt{Mim}\ 17.5.8(\text{p.122}) \bowtie \texttt{Mim}\ 17.5.4(\text{p.119}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^\pm \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^\pm).$$

c. On $\mathscr{F}^-$, we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

$$\texttt{Nem}\ 17.5.5(\text{p.121}) \bowtie \texttt{Nem}\ 17.5.1(\text{p.118}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^- \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^-),$$
$$\texttt{Nem}\ 17.5.7(\text{p.123}) \bowtie \texttt{Nem}\ 17.5.3(\text{p.119}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^- \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^-).$$

2. Let $\beta < 1$ or $s > 0$. Then we have:

$$\texttt{Nem}\ 17.5.6(\text{p.121}) \bowtie \texttt{Nem}\ 17.5.2(\text{p.118}) \quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}^- \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathtt{A}]\}^-),$$
$$\texttt{Nem}\ 17.5.8(\text{p.123}) \not\bowtie \texttt{Nem}\ 17.5.4(\text{p.119}) \quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}^- \not\bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}^-)\cdots(a^2).$$

The above results can be summarized as below.

A. On $\mathscr{F}^\pm$, for any $\beta \le 1$ and $s \ge 0$, the analogy are inherited (see C3b(p.132)).

B. On $\mathscr{F}^+$ and $\mathscr{F}^-$, if $\beta = 1$ and $s = 0$, the analogy is inherited (see C3a1(p.132) and C3c1(p.132)).

C. On $\mathscr{F}^+$ and $\mathscr{F}^-$, if $\beta < 1$ or $s > 0$, the analogy collapses (see (a^1) and (a^2)).

C4. Optimal Initiation Time (`OIT`)

a. Let $\beta = 1$ and $s = 0$. Then, from

`Pom` 17.5.1(p.116), `Mim` 17.5.1(p.117), `Nem` 17.5.1(p.118),
`Pom` 17.5.3(p.118), `Mim` 17.5.3(p.119), `Nem` 17.5.3(p.119),
`Pom` 17.5.5(p.120), `Mim` 17.5.5(p.120), `Nem` 17.5.5(p.121),
`Pom` 17.5.7(p.121), `Mim` 17.5.7(p.122), `Nem` 17.5.7(p.123)

we obtain Table 19.1.1(p.133) below (the symbol "○" in the table represents "`possible`"):

Table 19.1.1: Possible `OIT` ($\beta = 1$ and $s = 0$)

	$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
Ⓢ$_\tau\langle\tau\rangle_{\parallel}$			
Ⓢ$_\tau\langle\tau\rangle_{\vartriangle}$			
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○
◎$_\tau\langle t_\tau^\bullet\rangle_{\parallel}$			
◎$_\tau\langle t_\tau^\bullet\rangle_{\vartriangle}$			
◎$_\tau\langle t_\tau^\bullet\rangle_{\blacktriangle}$			
🅓$_\tau\langle 0\rangle_{\parallel}$			
🅓$_\tau\langle 0\rangle_{\vartriangle}$			
🅓$_\tau\langle 0\rangle_{\blacktriangle}$			

b. Let $\beta < 1$ or $s > 0$. Then, from

`Pom` 17.5.2(p.117), `Mim` 17.5.2(p.117), `Nem` 17.5.2(p.118),
`Pom` 17.5.4(p.118), `Mim` 17.5.4(p.119), `Nem` 17.5.4(p.119),
`Pom` 17.5.6(p.120), `Mim` 17.5.6(p.120), `Nem` 17.5.6(p.121),
`Pom` 17.5.8(p.121), `Mim` 17.5.8(p.122), `Nem` 17.5.8(p.123)

we obtain Table 19.1.2(p.133) below:

Table 19.1.2: Possible `OIT` ($\beta < 1$ or $s > 0$)

	$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
Ⓢ$_\tau\langle\tau\rangle_{\parallel}$			
Ⓢ$_\tau\langle\tau\rangle_{\vartriangle}$			
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○
◎$_\tau\langle t_\tau^\bullet\rangle_{\parallel}$	○	○	○
◎$_\tau\langle t_\tau^\bullet\rangle_{\vartriangle}$			
◎$_\tau\langle t_\tau^\bullet\rangle_{\blacktriangle}$			
🅓$_\tau\langle 0\rangle_{\parallel}$	○	○	○
🅓$_\tau\langle 0\rangle_{\vartriangle}$			
🅓$_\tau\langle 0\rangle_{\blacktriangle}$			

c. The table below is the list of the occurrence rates of Ⓢ, ◎, and 🅓 (Def. 11.2.4(p.61)) on $\mathscr{F}$ appearing in the primitive `Tom` 11.2.1(p.59) (▣) and `Tom` 11.2.2(p.60) (▣) (see Def. 11.2.2(p.59)).

Table 19.1.3: Occurence rates of Ⓢ, ◎, and 🅓 on $\mathscr{F}$

Ⓢ			◎			🅓		
50.0 % / 5			10.0 % / 1			40.0 % / 4		
Ⓢ$_{\parallel}$	Ⓢ$_{\vartriangle}$	Ⓢ$_{\blacktriangle}$	◎$_{\parallel}$	◎$_{\vartriangle}$	◎$_{\blacktriangle}$	🅓$_{\parallel}$	🅓$_{\vartriangle}$	🅓$_{\blacktriangle}$
—	×	`possible`	`possible`	×	×	`possible`	×	×
– %/ –	0.0 %/ 0	50.0 %/ 5	10.0 %/ 1	0.0 %/ 0	0.0 %/ 0	40.0 %/ 4	0.0 %/ 0	0.0 %/ 0

C5. Null-Time-Zone and Deadline-Engulfing

From Table 19.1.3(p.133) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.43) for the implication of the symbol "▲" representing the strict optimality of the initiating time t_τ^*.

b. As a whole, Ⓢ, ◎, and 🅓 are possible at 50.0%, 10.0%, and 40.0% respectively where

1. Ⓢ$_{\parallel}$ cannot be defined (see Remark **??**(p.??)).
2. ◎$_{\parallel}$ is possible (10.0 %).
3. 🅓$_{\parallel}$ is possible (40.0 %).
4. Ⓢ$_{\vartriangle}$ never occur (0.0 %).
5. ◎$_{\vartriangle}$ never occur (0.0 %).
6. 🅓$_{\vartriangle}$ never occur (0.0 %).
7. Ⓢ$_{\blacktriangle}$ is possible (50.0 %).

8. ◎$_\blacktriangle$ never occur (0.0 %).
9. ⓓ$_\blacktriangle$ never occur (0.0 %).

From the above results we see that on $\mathscr{F}$:

A. ◎$_{\parallel}$ and ⓓ$_{\parallel}$ causing the `null-time-zone` are possible at 50.0% (= 10.0% + 40.0%).

B. ◎$_\blacktriangle$ *strictly* causing the `null-time-zone` is impossible (0.0%).

C. ⓓ$_\blacktriangle$ *strictly* causing the `deadline-engulfing` is impossible (0.0%).

19.2 Search-Enforced-Model 1: $\mathcal{Q}\{\mathsf{M}{:}1[\mathrm{E}]\} = \{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}], \tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}], \mathsf{M}{:}1[\mathbb{P}][\mathrm{E}], \tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$

19.2.1 Preliminary

As ones corresponding to Theorems 12.5.1(p.78), 13.3.1(p.96), and 14.5.1(p.104), let us consider the following three theorems:

Theorem 19.2.1 (symmetry[$\mathbb{R} \to \mathbb{R}$]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}]. \quad \square \tag{19.2.1}$$

Theorem 19.2.2 (analogy[$\mathbb{R} \to \mathbb{P}$)]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}]. \quad \square \tag{19.2.2}$$

Theorem 19.2.3 (symmetry[$\mathbb{P} \to \mathbb{P}$]) *Let $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}]. \quad \square 9039 \tag{19.2.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}], \tag{19.2.4}$$

$$\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}], \tag{19.2.5}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}], \tag{19.2.6}$$

corresponding to (12.5.34(p.75)), (13.2.4(p.91)), and (14.5.4(p.104)). Then, for the same reason as in Chap. 15(p.109) it can be shown that the equality

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}] \tag{19.2.7}$$

holds (corresponding to (15.1.33(p.110))) and that we have the following theorem, corresponding to Theorem 15.1.1(p.110)

Theorem 19.2.4 (analogy[$\mathbb{R} \to \mathbb{P}$]) *Let $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}]. \quad \square \tag{19.2.8}$$

In fact, from the comparisons of (I) and (II), of (I) and (III), of (III) and (IV), and of (II) and (IV) in Table 6.5.2(p.39) we can easily show that (19.2.4(p.134)) - (19.2.7(p.134)) hold.

19.2.2 Proof of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$

19.2.2.1 Analysis

To begin with, let us note that

$$\lambda = 1 \tag{19.2.9}$$

is assumed in the model (see $A2$(p.19)), hence from (10.2.1(p.54)) we have

$$\delta = 1 \tag{19.2.10}$$

□ **`Tom` 19.2.1 (▣ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]\}$)** *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ $\to$ $\to$ Ⓢ$_\blacktriangle$

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.4(p.23)) we have $K(x) = T(x) \geq 0 \cdots$ **(1)** for any x due to Lemma 10.1.1(p.53) (g).

(a) From (6.5.10(p.39)) with $t = 2$ we have $V_2 = K(V_1) + V_1 \geq V_1$ due to (1). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 10.2.2(p.55) (e) we have $V_t \leq K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) From (6.5.9(p.39)) we have $V_1 = \mu < b \cdots$ **(2)**. Suppose $V_{t-1} < b$. Then, from (6.5.10(p.39)) and Lemma 10.2.2(p.55) (h) we have $V_t < K(b) + b = T(b) + b = b$ due to (1) and Lemma 10.1.1(p.53) (g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 10.2.1(p.55) (d), thus $L(V_{t-1}) > 0$ for $\tau \geq t > 1$. Then, from (6.5.10(p.39)) and from (5.1.8(p.23)) we have $V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}) > 0$ for $\tau \geq t > 1$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$. Hence, since $V_\tau > \beta V_{\tau-1}$, $V_{\tau-1} > \beta V_{\tau-2}$, $\cdots$, $V_2 > \beta V_1$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1$, thus $t^*_\tau = \tau$ for $\tau > 1$, i.e., ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$. ▮

For explanatory simplicity, let us define the statement below:

S_2 [ⓢ▲ | ◎‖ | ◎△ | ◎▲] = { For any $\tau > 1$ there exists $t^\bullet_\tau > 1$ such that
(1) ⓢ$_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_\blacktriangle$,
(2) ◎$_{t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle_\triangle$,
(3) ◎$_{\tau > t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle_\parallel$ ⟪ ◎$_{\tau > t^\bullet_\tau+1}\langle t^\bullet_\tau\rangle_\blacktriangle$ ⟫.[†] }

☐ **Tom 19.2.2** (▣$\mathscr{A}${M:1[ℝ][E]}) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \geq b$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\triangle$ → → ⓓ$_\triangle$

(c) *Let $\beta\mu < b$.*

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \leq a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\parallel$ → → ⓓ$_\parallel$
   ii. *Let $\mu - s > a$. Then* ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ → → ⓢ$_\blacktriangle$
2. *Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.*
   i. *Let $b > 0$ ⟪$\kappa > 0$⟫. Then* ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ → → ⓢ$_\blacktriangle$
   ii. *Let $b = 0$ ⟪$\kappa = 0$⟫.*
      1. *Let $\beta\mu - s \leq a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\parallel$ → → ⓓ$_\parallel$
      2. *Let $\beta\mu - s > a$. Then* ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ → → ⓢ$_\blacktriangle$
   iii. *Let $b < 0$ ⟪$\kappa < 0$⟫.*
      1. *Let $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_\triangle$ → → ⓓ$_\triangle$
      2. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* S_2 [ⓢ▲ | ◎‖ | ◎△ | ◎▲] *is true* → → ⓢ$_\blacktriangle$/◎$_\parallel$/◎$_\triangle$/◎$_\blacktriangle$

● *Proof* Let $\beta < 1$ or $s > 0$. From (6.5.10(p.39)) and (5.1.8(p.23)), we have $V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}) \cdots$ **(1)** for $t > 1$. From (6.5.10(p.39)) with $t = 2$ we have $V_2 - V_1 = K(V_1) \cdots$ **(2)**.

(a) Note that $V_1 = \beta\mu - s$ from (6.5.9(p.39)). Then, from Lemma 10.2.2(p.55) (j2) we have $x_K \geq \beta\mu - s$ due to (19.2.9(p.134)) and (19.2.10(p.134)), hence $x_K \geq V_1 \cdots$ **(3)**. Accordingly, since $K(V_1) \geq 0$ due to Lemma 10.2.2(p.55) (j1), we have $V_1 \leq V_2$ from (2). Suppose $V_{t-1} \leq V_t$. Then, from (6.5.10(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \leq K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Note (3). Suppose $V_{t-1} \leq x_K$. Then, from (6.5.10(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \leq K(x_K) + x_K = x_K$. Hence, by induction $V_t \leq x_K$ for $t > 0$, i.e., V_t is upper bounded in t, thus V_t converges to a finite V as $t \to \infty$. Accordingly, from (6.5.10(p.39)) we have $V = K(V) + V$, hence $K(V) = 0$, thus $V = x_K$ due to Lemma 10.2.2(p.55) (j1).

(b) Let $\beta\mu \geq b \cdots$ **(4)**. Then $x_L \leq \beta\mu - s$ from Lemma 10.2.4(p.57) (b1), hence $x_L \leq V_1$ from (6.5.9(p.39)), so $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 10.2.1(p.55) (a), hence $L(V_{t-1}) \leq 0 \cdots$ **(5)** for $\tau \geq t > 1$. Then, since $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 1$ from (1) or equivalently $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau \leq \beta V_{\tau-1}$, $V_{\tau-1} \leq \beta V_{\tau-2}$, $\cdots$, $V_2 \leq \beta V_1$, so $V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1} V_1$, hence it follows that $t^*_\tau = 1$ for $\tau > 1$, i.e., ⓓ$_{\tau>1}\langle 1\rangle_\triangle$.

(c) Let $\beta\mu < b$.

(c1) Let $\beta = 1 \cdots$ **(6)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K \cdots$ **(7)** due to Lemma 10.2.3(p.56) (b), hence $K(x_L) = K(x_K) = 0 \cdots$ **(8)**.

(c1i) Let $\mu - s \leq a$. Then, noting (6), (19.2.9(p.134)), and (19.2.10(p.134)), we have $x_K = \mu - s \cdots$ **(9)** from Lemma 10.2.2(p.55) (j2), hence $x_K = V_1$ from (6.5.9(p.39)). Let $V_{t-1} = x_K$. Then, from (6.5.10(p.39)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ from (7). Then $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, thus $L(V_{t-1}) = 0$ for $\tau \geq t > 1$. Then, since $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (1) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, we have

[†] The outer side of ⟪ ⟫ is for $s = 0$ and the inner side is for $s > 0$.

$V_\tau = \beta V_{\tau-1}$, $V_{\tau-1} = \beta V_{\tau-2}$, $\cdots$, $V_2 = \beta V_1$, so $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1} V_1$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$ (see Preference-Rule 7.2.1(p.43)).

(c1ii) Let $\mu - s > a$. Then, since $V_1 > a$ from (6.5.9(p.39)), we have $V_{t-1} > a$ for $t > 1$ from (a). From (7) and Lemma 10.2.2(p.55) (j2) we have $x_L = x_K > \mu - s = V_1$ from (6.5.9(p.39)). Let $V_{t-1} < x_L$. Then, from (6.5.10(p.39)) and Lemma 10.2.2(p.55) (g) we have $V_t < K(x_L) + x_L = x_L$ due to (8), hence by induction $V_{t-1} < x_L$ for $t > 1$. Thus, since $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma 10.2.1(p.55) (e1), for the same reason as in the proof of Tom 19.2.1(p.134) (b) we obtain ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(c2) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(c2i) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K \cdots$ (10) from Lemma 10.2.3(p.56) (c ⟪d⟫). Now, since $x_K \geq \beta\mu - s$ due to Lemma 10.2.2(p.55) (j2), we have $x_K \geq V_1$ from (6.5.9(p.39)). Suppose $x_K \geq V_{t-1}$. Then, from (6.5.10(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \leq K(x_K) + x_K = x_K$. Thus, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (10). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.55) (a), for the same reason as in the proof of Tom 19.2.1(p.134) (b) we obtain ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(c2ii) Let $b = 0$ ⟪$\kappa = 0$⟫. Then $x_L = x_K \cdots$ (11) from Lemma 10.2.3(p.56) (c ⟪d⟫), hence $K(x_L) = K(x_K) = 0 \cdots$ (12).

(c2ii1) Let $\beta\mu - s \leq a$. Then, since $x_K = \beta\mu - s \cdots$ (13) from Lemma 10.2.2(p.55) (j2), we have $x_K = V_1$ from (6.5.9(p.39)). Let $V_{t-1} = x_K$. Then, from (6.5.10(p.39)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (11). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, for the same reason as in the proof of (c1i) we have ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.

(c2ii2) Let $\beta\mu - s > a$. Then, since $V_1 > a$ from (6.5.9(p.39)), we have $V_{t-1} > a$ for $t > 1$ from (a). From (11) and Lemma 10.2.2(p.55) (j2) we have $x_L = x_K > \beta\mu - s = V_1$. Let $V_{t-1} < x_L$. Then, from (6.5.10(p.39)) and Lemma 10.2.2(p.55) (g) we have $V_t < K(x_L) + x_L = x_L$ due to (12), hence, by induction $V_{t-1} < x_L$ for $t > 1$. Consequently, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary 10.2.1(p.55) (a), for the same reason as in the proof of Tom 19.2.1(p.134) (b) we obtain ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(c2iii) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ (14) from Lemma 10.2.3(p.56) (c ⟪d⟫).

(c2iii1) Let $\beta\mu - s \leq a$, then $x_L < x_K = \beta\mu - s = V_1$ from (14), Lemma 10.2.2(p.55) (j2) and (6.5.9(p.39)), so $x_L \leq V_1$. Let $s_{\mathcal{L}} \leq s$, then $x_L \leq \beta\mu - s$ due to Lemma 10.2.4(p.57) (c), hence $x_L \leq V_1$. Therefore, whether $\beta\mu - s \leq a$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, hence $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of (b) we obtain ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.

(c2iii2) Suppose $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Hence, since $V_1 > a$ from (6.5.9(p.39)), we have $V_{t-1} > a$ for $t > 0$ from (a). Then, since $x_K > x_L > \beta\mu - s = V_1 \cdots$ (15) from (14), Lemma 10.2.4(p.57) (c), and (6.5.9(p.39)), we have $K(V_1) > 0$ from Lemma 10.2.2(p.55) (j1), hence $V_2 > V_1$ from (2). Suppose $V_{t-1} < V_t$. Then, from (6.5.10(p.39)) and Lemma 10.2.2(p.55) (g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Accordingly, by induction we have $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (15). Assume that $V_{t-1} < x_L$ for *all* $t > 1$, hence $V \leq x_L \cdots$ (16) from (a). Then, since $V = x_K$ due to (a), we have the contradiction of $V = x_K > x_L \geq V$ due to (14) and (16). Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 < V_2 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots \quad \cdots (17),$$

from which we have

$$V_{t-1} < x_L, \quad t^\bullet_\tau \geq t > 1, \qquad x_L \leq V_{t^\bullet_\tau}, \qquad x_L < V_{t-1}, \quad t > t^\bullet_\tau + 1. \tag{19.2.11}$$

Hence, we have

$$\begin{aligned}
&L(V_{t-1}) > 0 && \cdots (18)\ t^\bullet_\tau \geq t > 1 && (\leftarrow \text{Corollary 10.2.1(p.55) (a)})\\
&L(V_{t^\bullet_\tau}) \leq 0 && \cdots (19) && (\leftarrow \text{Corollary 10.2.1(p.55) (a)})\\
&L(V_{t-1}) = ⟪< 0⟫^\dagger && \cdots (20)\ t > t^\bullet_\tau + 1 && (\leftarrow \text{Lemma 10.2.1(p.55) (d⟪e1⟫)})
\end{aligned}$$

- Let $t^\bullet_\tau \geq \tau > 1$. Then $L(V_{t-1}) > 0 \cdots$ (21) for $\tau \geq t > 1$ from (18). Since $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (1) and (21), we have $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, hence $V_\tau > \beta V_{\tau-1}$, $V_{\tau-1} > \beta V_{\tau-2}$, $\cdots$, $V_2 > \beta V_1$. Therefore, since $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1$, we obtain $t^*_\tau = \tau$ for $t^\bullet_\tau \geq \tau > 1$, i.e., ⓢ$_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_{\blacktriangle}$, thus $\mathtt{S}_2(1)$ is true. Let us note here that when $\tau = t^\bullet_\tau$, we have $V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots$ (22).
- Let $\tau = t^\bullet_\tau + 1$. From (1) with $t = t^\bullet_\tau + 1$ and (19) we have $V_{t^\bullet_\tau + 1} - \beta V_{t^\bullet_\tau} \leq 0$, hence $V_{t^\bullet_\tau + 1} \leq \beta V_{t^\bullet_\tau}$. Accordingly, from (22) we have

$$V_{t^\bullet_\tau + 1} \leq \beta V_{t^\bullet_\tau} > \beta^2 V_{t^\bullet_\tau - 1} > \beta^3 V_{t^\bullet_\tau - 2} > \cdots > \beta^{t^\bullet_\tau} V_1 \cdots (23),$$

thus $t^*_{t^\bullet_\tau + 1} = t^\bullet_\tau$, i.e., ◎$_{t^\bullet_\tau + 1}\langle t^\bullet_\tau\rangle_{\vartriangle}$, thus $\mathtt{S}_2(2)$ is true.
- Let $\tau > t^\bullet_\tau + 1$. Since $L(V_{t^\bullet_\tau + 1}) =$ ⟪<⟫ 0 from (20) with $t = t^\bullet_\tau + 2$, we have $V_{t^\bullet_\tau + 2} =$ ⟪<⟫ $\beta V_{t^\bullet_\tau + 1}$ from (1), hence from (23) we have

$$V_{t^\bullet_\tau + 2} = ⟪<⟫\ \beta V_{t^\bullet_\tau + 1} \leq \beta^2 V_{t^\bullet_\tau} > \beta^3 V_{t^\bullet_\tau - 1} > \beta^4 V_{t^\bullet_\tau - 2} > \cdots > \beta^{t^\bullet_\tau + 1} V_1$$

† If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

Similarly we have

$$V_{t^\bullet_\tau+3} = (\!\!<\!\!)\ \beta V_{t^\bullet_\tau+2} = (\!\!<\!\!)\ \beta^2 V_{t^\bullet_\tau+1} \le \beta^3 V_{t^\bullet_\tau} > \beta^4 V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau+2} V_1.$$

By repeating the same procedure, for $\tau = t^\bullet_\tau + 2, t^\bullet_\tau + 3, \cdots$ we obtain

$$V_\tau = (\!\!<\!\!)\ \beta V_{\tau-1} = (\!\!<\!\!)\ \cdots = (\!\!<\!\!)\ \beta^{\tau-t^\bullet_\tau-2} V_{t^\bullet_\tau+2} = (\!\!<\!\!)\ \beta^{\tau-t^\bullet_\tau-1} V_{t^\bullet_\tau+1} \le \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{\tau-1} V_1. \cdots (24)$$

- Let $s = 0$. Then (24) can be written as

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-t^\bullet_\tau-2} V_{t^\bullet_\tau+2} = \beta^{\tau-t^\bullet_\tau-1} V_{t^\bullet_\tau+1} \le \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{\tau-1} V_1,$$

  hence we have $t^*_\tau = t^\bullet_\tau$, i.e., $\circledcirc_{\tau > t^\bullet_\tau+1} \langle t^\bullet_\tau \rangle_{\parallel}$ (see Preference Rule 7.2.1(p.43)), hence $\mathtt{S}_2(3)$ is true.
- Let $s > 0$. Then (24) can be written as

$$V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-t^\bullet_\tau-2} V_{t^\bullet_\tau+2} < \beta^{\tau-t^\bullet_\tau-1} V_{t^\bullet_\tau+1} \le \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \cdots > \beta^{\tau-1} V_1, \quad (19.2.12)$$

  hence we have $t^*_\tau = t^\bullet_\tau$, i.e., $\circledcirc_{\tau > t^\bullet_\tau+1} \langle t^\bullet_\tau \rangle_{\blacktriangle}$, hence $\mathtt{S}_2(3)$ is true. ∎

19.2.2.2 Market Restriction

19.2.2.2.1 Positive Restriction

☐ **Pom 19.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have $\circledS_{\tau>1} \langle \tau \rangle_{\blacktriangle}$.*

● *Proof* The same as Tom 19.2.1(p.134) due to Lemma 17.4.4(p.116). ∎

☐ **Pom 19.2.2** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \ge b$* (impossible).

(c) *Let $\beta\mu < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \le a$. Then* $\text{ⓓ}_{\tau>1} \langle 1 \rangle_{\parallel}$.
   ii. *Let $\mu - s > a$. Then* $\circledS_{\tau>1} \langle \tau \rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\circledS_{\tau>1} \langle \tau \rangle_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $\beta\mu > s$. Then* $\circledS_{\tau>1} \langle \tau \rangle_{\blacktriangle}$.
   ii. *Let $\beta\mu \le s$. Then* $\text{ⓓ}_{\tau>1} \langle 1 \rangle_{\vartriangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.57) (a) with $\lambda = 1$.

(a) The same as Tom 19.2.2(p.135) (a).

(b,c) Always $\beta\mu < b$ from [3(p.116)], hence $\beta\mu \ge b$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of Tom 19.2.2(p.135).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 19.2.2(p.135).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu > s$, hence $\kappa > 0$ due to (2). Hence it suffices to consider only (c2i) of Tom 19.2.2(p.135).

(c3ii) Let $\beta\mu \le s$, hence $\kappa \le 0$ due to (2). Then, since $\beta\mu - s \le 0 < a$, it suffices to consider only (c2iii1) of Tom 19.2.2(p.135). ∎

19.2.2.2.2 Mixed Restriction

☐ **Mim 19.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^\pm\}$) *Suppose $a \le 0 \le 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have $\circledS_{\tau>1} \langle \tau \rangle_{\blacktriangle}$.*

● *Proof* The same as Tom 19.2.1(p.134) due to Lemma 17.4.4(p.116). ∎

☐ **Mim 19.2.2** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathrm{E}]^\pm\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \ge b$* (impossible).

(c) *Let $\beta\mu < b$* (always holds).

1. *Let $\beta = 1$.*
   i. *Let $\mu - s \le a$. Then* $\text{ⓓ}_{\tau>1} \langle 1 \rangle_{\parallel}$.
   ii. *Let $\mu - s > a$. Then* $\circledS_{\tau>1} \langle \tau \rangle_{\blacktriangle}$.

2. *Let $\beta < 1$ and $s = 0$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $s < \beta T(0)$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - ii. *Let $s = \beta T(0)$.*
     1. *Let $\beta\mu - s \le a$. Then* 🅓$_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let $\beta\mu - s > a$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - iii. *Let $s > \beta T(0)$.*
     1. *Let $\beta\mu - s \le a$ or $s_{\mathcal{L}} \le s$. Then* 🅓$_{\tau>1}\langle 1\rangle_{\triangle}$.
     2. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_2$ $\boxed{\circledS_{\blacktriangle} \mid \circledcirc_{\parallel} \mid \circledcirc_{\triangle} \mid \circledcirc_{\blacktriangle}}$ *is true.*

● *Proof* Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.

(a) The same as `Tom` 19.2.2(p.135) (a).

(b,c) Always $\beta\mu < b$ due to [8(p.116)], hence $\beta\mu \ge b$ is impossible.

(c1) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$".

(c1i,c1ii) The same as (c1i,c1ii) of `Tom` 19.2.2(p.135).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, then it suffices to consider only (c2i) of `Tom` 19.2.2(p.135) and if $b = 0$, then $\beta\mu - s = \beta\mu > a$ due to [8(p.116)], hence it suffices to consider only (c2ii2) of `Tom` 19.2.2(p.135). Therefore, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions are immediate from (c2i-c2iii2) of `Tom` 19.2.2(p.135) with $\kappa = \beta T(0) - s$ from (5.1.6(p.23)) with $\lambda = 1$. ∎

19.2.2.2.3 Negative Restriction

☐ **`Nem` 19.2.1** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^-\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.1(p.134) due to Lemma 17.4.4(p.116). ∎

☐ **`Nem` 19.2.2** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]^-\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta\mu \ge b$. Then* 🅓$_{\tau>1}\langle 1\rangle_{\triangle}$.

(c) *Let $\beta\mu < b$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu - s \le a$. Then* 🅓$_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $\mu - s > a$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_2$ $\boxed{\circledS_{\blacktriangle} \mid \circledcirc_{\parallel} \mid \circledcirc_{\triangle} \mid \circledcirc_{\blacktriangle}}$ *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $\beta\mu - s \le a$ or $s_{\mathcal{L}} \le s$. Then* 🅓$_{\tau>1}\langle 1\rangle_{\triangle}$.
   - ii. *Let $\beta\mu - s > a$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_2$ $\boxed{\circledS_{\blacktriangle} \mid \circledcirc_{\parallel} \mid \circledcirc_{\triangle} \mid \circledcirc_{\blacktriangle}}$ *is true.*

● *Proof* Suppose $b < 0$, hence $a < \mu < b < 0 \cdots (1)$. Hence $\kappa = -s \cdots (2)$ from Lemma 10.3.1(p.57) (a) with $\lambda = 1$. In addition, $\beta\mu \ge b$ and $\beta\mu < b$ are both possible due to [17(p.116)].

(a,b) The same as (a,b) of `Tom` 19.2.2(p.135).

(c) Let $\beta\mu < b$.

(c1-c1ii) The same as (c1-c1ii) of `Tom` 19.2.2(p.135).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $b < 0$ due to (1), it suffices to consider only (c2iii) of `Tom` 19.2.2(p.135). In this case, since $\beta\mu - s = \beta\mu > \beta a > a$ due to (1) and since $s_{\mathcal{L}} > 0 = s$ due to Lemma 10.2.4(p.57) (c), it suffices to consider only (c2iii2) of `Tom` 19.2.2(p.135).

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\kappa < 0$ due to (2). Thus, it suffices to consider only (c2iii1,c2iii2) of `Tom` 19.2.2(p.135). ∎

19.2.3 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}$

19.2.3.1 Analysis

☐ **`Tom` 19.2.3** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128))) to Tom 19.2.1(p.134). ▮

□ **Tom 19.2.4** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \le a$. Then* $❶_{\tau>1}\langle 1\rangle_{\vartriangle}$.

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* $❶_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* $Ⓢ_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
   - i. *Let $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. Then* $Ⓢ_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
   - ii. *Let $a = 0\,(\!(\tilde{\kappa} = 0)\!)$.*†
     1. *Let $\beta\mu + s \ge b$. Then* $❶_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let $\beta\mu + s < b$. Then* $Ⓢ_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
   - iii. *Let $a > 0\,(\!(\tilde{\kappa} > 0)\!)$.*
     1. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $❶_{\tau>1}\langle 1\rangle_{\vartriangle}$.
     2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_2$ [ⓢ▲ | ◎∥ | ◎△ | ◎▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128))) to Tom 19.2.2(p.135). ▮

19.2.3.2 Market Restriction

19.2.3.2.1 Positive Restriction

□ **Pom 19.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* $Ⓢ_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.

● *Proof* The same as Tom 19.2.3(p.138) due to Lemma 17.4.4(p.116). ▮

□ **Pom 19.2.4** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta\mu \le a$. Then* $❶_{\tau>1}\langle 1\rangle_{\vartriangle}$.

(c) *Let $\beta\mu > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* $❶_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* $Ⓢ_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_2$ [ⓢ▲ | ◎∥ | ◎△ | ◎▲] *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* $❶_{\tau>1}\langle 1\rangle_{\vartriangle}$.
   - ii. *Let $\beta\mu + s < b$ and $s < s_{\tilde{\mathcal{L}}}$. Then* $\mathtt{S}_2$ [ⓢ▲ | ◎∥ | ◎△ | ◎▲] *is true* (see *Numerical Example* 4(p.145)).

● *Proof* Suppose $a > 0 \cdots (1)$, hence $\tilde{\kappa} = s \cdots (2)$ from Lemma 12.6.6(p.81) (a). Then $\mu\beta \le a$ and $\mu\beta > a$ are both possible due to [5(p.116)].

(a,b) The same as (a,b) of Tom 19.2.4(p.139).

(c) Let $\beta\mu > a$. Then $s_{\tilde{\mathcal{L}}} > 0 \cdots (3)$ due to Lemma 12.6.5(p.81) (c) with $\lambda = 1$.

(c1-c1ii) Let $\beta = 1$, hence $s > 0$ due to the assumptions $\beta < 1$ and $s > 0$. Thus, we have (c1i,c1ii) of Tom 19.2.4(p.139).

(c2) Let $\beta < 1$ and $s = 0$. Then, since $\beta\mu + s = \beta\mu < b$ due to [3(p.116)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from (3), due to (1) it suffices to consider only (c2iii2) of Tom 19.2.4(p.139).

(c3-c3ii) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to (2), it suffices to consider only (c2iii1,c2iii2) of Tom 19.2.4(p.139). ▮

19.2.3.2.2 Mixed Restriction

☐ `Mim` **19.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]\}^{\pm}$) *Suppose $a \le 0 \le b$. Let $\beta = 1$ and $s = 0$.*

(a) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.3(p.138) due to Lemma 17.4.4(p.116). ∎

☐ `Mim` **19.2.4** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \le a$* (impossible). ———
(c) *Let $\beta\mu > a$* (always holds).

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* ❶$_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $s < -\beta\tilde{T}(0)$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - ii. *Let $s = -\beta\tilde{T}(0)$.*
     1. *Let $\beta\mu + s \ge b$. Then* ❶$_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let $\beta\mu + s < b$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - iii. *Let $s > -\beta\tilde{T}(0)$.*
     1. *Let $\beta\mu + s \ge b$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* ❶$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
     2. *Let $\beta\mu + s < b$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_2$ [Ⓢ▲ | ◎∥ | ◎△ | ◎▲] *is true.*

● *Proof* Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.

(a) The same as `Tom` 19.2.4(p.139) (a).

(b,c) Always $\beta\mu > a$ due to [8(p.116)], hence $\beta\mu \le a$ is impossible. Then $s_{\tilde{\mathcal{L}}} > 0$ due to Lemma 12.6.5(p.81) (c).

(c1-c1ii) The same as (c-c1ii) of `Tom` 19.2.4(p.139).

(c2) Let $\beta < 1$ and $s = 0$. Let $a < 0$. Then it suffices to consider only (c2i) of `Tom` 19.2.4(p.139). Let $a = 0$. Now, since $\beta\mu + s = \beta\mu < b$ due to [8(p.116)], it suffices to consider only (c2ii2) of `Tom` 19.2.4(p.139). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.16(p.23)), the assertions become true from (c2i-c2iii2) of `Tom` 19.2.4(p.139). ∎

19.2.3.2.3 Negative Restriction

☐ `Nem` **19.2.3** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{-}\}$) *Suppose $b < 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*
(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.4(p.139) due to Lemma 17.4.4(p.116). ∎

☐ `Nem` **19.2.4** ($\mathscr{A}_{\mathtt{Tom}}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathrm{E}]^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta\mu \le a$* (impossible). ———
(c) *Let $\beta\mu > a$* (always holds).

1. *Let $\beta = 1$.*
   - i. *Let $\mu + s \ge b$. Then* ❶$_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $\mu + s < b$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $\beta\mu < -s$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - ii. *Let $\beta\mu \ge -s$. Then* ❶$_{\tau>1}\langle 1\rangle_{\vartriangle}$.

● *Proof* Suppose $b < 0 \cdots (1)$, hence $a < b < 0 \cdots (2)$. Then $\tilde{\kappa} = \beta\mu + s \cdots (3)$ due to Lemma 12.6.6(p.81) (a).

(a) The same as `Tom` 19.2.4(p.139) (a).

(b,c) Always $a < \beta\mu$ due to [15(p.116)], hence $\beta\mu \le a$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of `Tom` 19.2.4(p.139).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c2i) of `Tom` 19.2.4(p.139).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i) Let $\beta\mu < -s$, hence $\beta\mu + s < 0$. Then, since $\tilde{\kappa} < 0$ due to (3), it suffices to consider only (c2i) of `Tom` 19.2.4(p.139).

(c3ii) Let $\beta\mu \ge -s$, hence $\beta\mu + s \ge 0$. Let $\beta\mu + s = 0$. Then, since $\tilde{\kappa} = 0$ due to (3) and since $\beta\mu + s = 0 > b$ due to (2), it suffices to consider only (c2ii1) of `Tom` 19.2.4(p.139). Let $\beta\mu + s > 0$. Then, since $\tilde{\kappa} > 0$ due to (3), it suffices to consider only (c2iii) of `Tom` 19.2.4(p.139). Then, since $\beta\mu + s > 0 > b$ due to (1), it suffices to consider only (c2iii1) of `Tom` 19.2.4(p.139). Accordingly, whether $\beta\mu + s = 0$ or $\beta\mu + s > 0$, we have the same result. ∎

19.2.4 Derivation of $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$

19.2.4.1 Analysis

☐ **Tom 19.2.5** (⊡ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof by analogy* The same as Tom 19.2.1(p.134) due to Lemma 13.6.1(p.97). ∎

☐ **Tom 19.2.6** (⊡ $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta a \geq b$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.

(c) *Let $\beta a < b$.*

1. *Let $\beta = 1$.*
   - i. *Let $a - s \leq a^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $a - s > a^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $b > 0$ ($\kappa > 0$). Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - ii. *Let $b = 0$ ($\kappa = 0$).*
     1. *Let $\beta a - s \leq a^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let $\beta a - s > a^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - iii. *Let $b < 0$ ($\kappa < 0$).*
     1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
     2. *Let $\beta a - s > a^\star$ and $s_{\mathcal{L}} > s$. Then* $\mathtt{S}_2$ [Ⓢ▲ | ◎‖ | ◎△ | ◎▲] *is true.* □

● *Proof by analogy* Immediate from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.128))) to Tom 19.2.2(p.135). ∎

Corollary 19.2.1 (optimal price to propose) *The optimal price to propose z_t is nondecreasing in $t > 0$.* □

● *Proof* Immediate from Tom's 19.2.5(p.141) (a) and 19.2.6(p.141) (a) and from (6.2.34(p.29)) and Lemma 13.1.3(p.87). ∎

19.2.4.2 Market Restriction

19.2.4.2.1 Positive Restriction

☐ **Pom 19.2.5** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t > 0$.*

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof* The same as Tom 19.2.5(p.141) due to Lemma 17.4.4(p.116). ∎

☐ **Pom 19.2.6** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $\beta a \geq b$* (impossible).

(c) *Let $\beta a < b$* (always holds).

1. *Let $\beta = 1$.*
   - i. *Let $a - s \leq a^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
   - ii. *Let $a - s > a^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $s < \beta T(0)$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - ii. *Let $s = \beta T(0)$.*
     1. *Let $\beta a - s \leq a^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
     2. *Let $\beta a - s > a^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - iii. *Let $s > \beta T(0)$.*
     1. *Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
     2. *Let $\beta a - s > a^\star$ and $s < s_{\mathcal{L}}$. Then* $\mathtt{S}_2$ [Ⓢ▲ | ◎‖ | ◎△ | ◎▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$.

(a) The same as Tom 19.2.6(p.141) (a).

(b,c) Always $\beta a < b$ from [4(p.116)], hence $\beta a \geq b$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of Tom 19.2.6(p.141).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 19.2.6(p.141).

(c3) Let $\beta < 1$ and $s > 0$.

(c3i-c3iii2) Immediate from Tom 19.2.6(p.141) (c2i-c2iii2) since $\kappa = \beta T(0) - s \cdots (2)$ from (5.1.23(p.24)). ∎

19.2.4.2.2 Mixed Restriction

☐ `Mim` **19.2.5** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^\pm\}$) *Suppose* $a \le 0 \le b$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.

● *Proof* The same as `Tom` 19.2.5(p.141) due to Lemma 17.4.4(p.116). ∎

☐ `Mim` **19.2.6** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^\pm\}$) *Suppose* $a \le 0 \le b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.

(b) *Let* $\beta a \ge b$ (impossible). ———

(c) *Let* $\beta a < b$ (always holds).

1. *Let* $\beta = 1$.
   i. *Let* $a - s \le a^\star$. *Then* 🅓$_{\tau>1}\langle 1\rangle_{\parallel}$.
   ii. *Let* $a - s > a^\star$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $s < \beta T(0)$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
   ii. *Let* $s = \beta T(0)$.
      1. *Let* $\beta a - s \le a^\star$. *Then* 🅓$_{\tau>1}\langle 1\rangle_{\parallel}$.
      2. *Let* $\beta a - s > a^\star$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
   iii. *Let* $s > \beta T(0)$.
      1. *Let* $\beta a - s \le a^\star$ *or* $s_{\mathcal{L}} \le s$. *Then* 🅓$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
      2. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$. *Then* $\mathtt{S}_2$ [Ⓢ▲ | ⊚‖ | ⊚△ | ⊚▲] *is true* ☐

● *Proof* Suppose $a \le 0 \le b$.

(a) The same as `Tom` 19.2.6(p.141) (a).

(b,c) Always $\beta a < b$ due to [9(p.116)], hence $\beta a \ge b$ is impossible. .

(c1-c1ii) The same as (c1-c1ii) of `Tom` 19.2.6(p.141).

(c2) Let $\beta < 1$ and $s = 0$. If $b > 0$, the assertion is true from (c2i) of `Tom` 19.2.6(p.141) and if $b = 0$, then $\beta a - s = \beta a > a^\star$ from [11(p.116)], hence the assertion become true from (c2ii2) of `Tom` 19.2.6(p.141). Accordingly, whether $b > 0$ or $b = 0$, we have the same result.

(c3-c3iii2) The same as `Tom` 19.2.6(p.141) (c2i-c2iii2) since $\kappa = \beta T(0) - s$ from (5.1.23(p.24))) with $\lambda = 1$. ∎

19.2.4.2.3 Negative Restriction

☐ `Nem` **19.2.5** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^-\}$) *Suppose* $b < 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t > 0$.

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.

● *Proof* The same as `Tom` 19.2.5(p.141) due to Lemma 17.4.4(p.116). ∎

☐ `Nem` **19.2.6** ($\mathscr{A}\{$M:1[$\mathbb{P}$][E]$^-\}$) *Suppose* $b < 0$. *Let* $\beta < 1$ *or* $s > 0$.

(a) V_t *is nondecreasing in* $t > 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.

(b) *Let* $\beta a \ge b$. *Then* 🅓$_{\tau>1}\langle 1\rangle_{\vartriangle}$.

(c) *Let* $\beta a < b$.

1. *Let* $\beta = 1$.
   i. *Let* $a - s \le a^\star$. *Then* 🅓$_{\tau>1}\langle 1\rangle_{\parallel}$.
   ii. *Let* $a - s > a^\star$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_2$ [Ⓢ▲ | ⊚‖ | ⊚△ | ⊚▲] *is true.*
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $\beta a - s \le a^\star$ *or* $s_{\mathcal{L}} \le s$. *Then* 🅓$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
   ii. *Let* $\beta a - s > a^\star$ *and* $s < s_{\mathcal{L}}$. *Then* $\mathtt{S}_2$ [Ⓢ▲ | ⊚‖ | ⊚△ | ⊚▲] *is true.*

● *Proof* Suppose $b < 0$. Then, $\kappa = -s \cdots (1)$ from Lemma 13.2.6(p.95) (a). In addition, $\beta a \ge b$ and $\beta a < b$ are both possible due to [18(p.116)].

(a,b) The same as (a,b) of `Tom` 19.2.6(p.141).

(c) Let $\beta a < b$.

(c1-c1ii) The same as (c1-c1ii) of `Tom` 19.2.6(p.141).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2iii-c2iii2) of `Tom` 19.2.6(p.141). In this case, since $\beta a - s = \beta a > a^\star$ due to [19(p.116)] and since $s_{\mathcal{L}} > 0 = s$ due to Lemma 13.2.5(p.95) (c), it suffices to consider only (c2iii2) of `Tom` 19.2.6(p.141).

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\kappa < 0$ due to (1). Hence, it suffices to consider only (c2iii1,c2iii2) of `Tom` 19.2.6(p.141). ∎

19.2.5 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$

19.2.5.1 Analysis

☐ `Tom` **19.2.7** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.128))) to `Tom` 19.2.5(p.141). ▮

☐ `Tom` **19.2.8** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \le a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $b + s \ge b^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
   - ii. *Let $b + s < b^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ⟅$s > 0$⟆ .*
   - i. *Let $a < 0$ ⟅$\tilde{\kappa} < 0$⟆ . Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - ii. *Let $a = 0$ ⟅$\tilde{\kappa} = 0$⟆ .*
     1. *Let $\beta b + s \ge b^\star$.*[†] *Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
     2. *Let $\beta b + s < b^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   - iii. *Let $a > 0$ ⟅$\tilde{\kappa} > 0$⟆ .*
     1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
     2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_2$ [Ⓢ▲ | ⊙‖ | ⊙△ | ⊙▲] *is true.* ▯

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.128))) to `Tom` 19.2.6(p.141). ▮

Corollary 19.2.2 (optimal price to propose) *The optimal price to propose z_t is nonincreasing in $t > 0$.* ▯

● *Proof* Immediate from `Tom`'s 19.2.7(p.143) (a) and 19.2.8(p.143) (a) and from (6.2.50(p.30)) and Lemma A 3.3(p.244)). ▮

19.2.5.2 Market Restriction

19.2.5.2.1 Positive Restriction

☐ `Pom` **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

● *Proof* The same as `Tom` 19.2.7(p.143) due to Lemma 17.4.4(p.116). ▮

☐ `Pom` **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{+}\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \le a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $b + s \ge b^\star$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
   - ii. *Let $b + s < b^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_2$ [Ⓢ▲ | ⊙‖ | ⊙△ | ⊙▲] *is true.*
3. *Let $\beta < 1$ and $s > 0$.*
   - i. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
   - ii. *Let $\beta b + s < b^\star$ and $s < s_{\tilde{\mathcal{L}}}$. Then* $\mathtt{S}_2$ [Ⓢ▲ | ⊙‖ | ⊙△ | ⊙▲] *is true.*

● *Proof* Suppose $a > 0 \cdots$ **(1)**. Then, $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 14.6.6(p.106) (a). In addition, $\beta b \le a$ and $\beta b > a$ are both possible due to [6(p.116)].

(a,b) The same as (a,b) of `Tom` 19.2.8(p.143).

(c) Let $\beta b > a$.

(c1-c1ii) The same as (c1-c1ii) of `Tom` 19.2.8(p.143).

(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2iii-c2iii2) of `Tom` 19.2.8(p.143). In this case, since $\beta b + s = \beta b < b^\star$ due to [7(p.116)] and since $s_{\tilde{\mathcal{L}}} > 0 = s$ from Lemma 14.6.5(p.106) (c) with $\lambda = 1$, it suffices to consider only (c2iii2) of `Tom` 19.2.8(p.143).

(c3-c3ii) Let $\beta < 1$ and $s > 0$, hence $\tilde{\kappa} > 0$ due to (2). Hence, it suffices to consider only (c2iii1,c2iii2) of `Tom` 19.2.8(p.143). ▮

19.2.5.2.2 Mixed Restriction

☐ `Mim` **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]\}^{\pm}$) *Suppose $a \le 0 \le b$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have $ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*

● *Proof* The same as `Tom` 19.2.7(p.143) due to Lemma 17.4.4(p.116). ▮

☐ `Mim` **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{\pm}\}$) *Suppose $a \le 0 \le b$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \ge x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta b \le a$* (impossible). ———
(c) *Let $\beta b > a$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $b + s \ge b^\star$. Then ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.*
    ii. *Let $b + s < b^\star$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
  2. *Let $\beta < 1$ and $s = 0$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < -\beta\tilde{T}(0)$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
    ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \ge b^\star$. Then ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.*
      2. *Let $\beta b + s < b^\star$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
    iii. *Let $s > -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. Then ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.*
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_2$ [ⓢ▲ | ◎‖ | ◎△ | ◎▲] *is true.*

● *Proof* Let $b \ge 0 \ge a \cdots$ **(1)**.

(a) The same as `Tom` 19.2.8(p.143) (a).

(b,c) Always $\beta b > a$ due to [10(p.116)], hence $\beta b \le a$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of `Tom` 19.2.8(p.143).

(c2) Let $\beta < 1$ and $s = 0$. Then, it suffices to consider only (c2i-c2ii2) of `Tom` 19.2.8(p.143). Let $a < 0$. Then, the assertion is true from (c2i) of `Tom` 19.2.8(p.143). Let $a = 0$. Then, since $\beta b + s = \beta b < b^\star$ due to [12(p.116)], it suffices to consider only (c2ii2) of `Tom` 19.2.8(p.143). Accordingly, whether $a < 0$ or $a = 0$, we have the same result.

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from (c2i-c2iii2) of `Tom` 19.2.8(p.143) since $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.25)) with $\lambda = 1$. ▮

19.2.5.2.3 Negative Restriction

☐ `Nem` **19.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*

● *Proof* The same as `Tom` 19.2.7(p.143) due to Lemma 17.4.4(p.116). ▮

☐ `Nem` **19.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathrm{E}]^{-}\}$) *Suppose $b < 0$. Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \ge x_{\tilde{K}}$ as $t \to \infty$.*
(b) *Let $\beta b \le a$* (impossible). ———
(c) *Let $\beta b > a$* (always holds).
  1. *Let $\beta = 1$.*
    i. *Let $b + s \ge b^\star$. Then ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.*
    ii. *Let $b + s < b^\star$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
  2. *Let $\beta < 1$ and $s = 0$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
  3. *Let $\beta < 1$ and $s > 0$.*
    i. *Let $s < -\beta\tilde{T}(0)$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
    ii. *Let $s = -\beta\tilde{T}(0)$.*
      1. *Let $\beta b + s \ge b^\star$. Then ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.*
      2. *Let $\beta b + s < b^\star$. Then ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*
    iii. *Let $-\beta\tilde{T}(0) < s$.*
      1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{\mathcal{L}}} \le s$. Then ⓓ$_{\tau>1}\langle 1\rangle_{\vartriangle}$.*
      2. *Let $\beta b + s < b^\star$ and $s_{\tilde{\mathcal{L}}} > s$. Then* $\mathtt{S}_2$ [ⓢ▲ | ◎‖ | ◎△ | ◎▲] *is true.*
      is true.

• *Proof* Let $b < 0$, hence $a < b < 0 \cdots (1)$.

(a) The same as Tom 19.2.8(p.143) (a).

(b,c) Always $\beta b > a$ due to [16(p.116)], hence $\beta b \le a$ is impossible.

(c1-c1ii) The same as (c1-c1ii) of Tom 19.2.8(p.143).

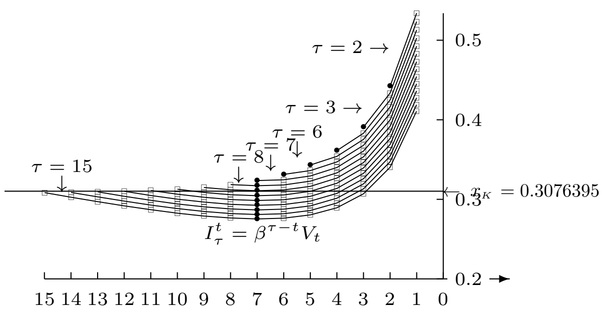
(c2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i) of Tom 19.2.8(p.143).

(c3-c3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions hold from (c2-c2iii2) of Tom 19.2.8(p.143) since $\tilde{\kappa} = \beta\tilde{T}(0) + s$ from (5.1.36(p.25)) with $\lambda = 1$. ▮

19.2.6 Numerical Calculation

Numerical Example **4** **($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]^+\}$) (buying model)**

This is the example for [Ⓢ▲] and [◎▲] of $\mathtt{S}_2$(p.135) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] in Pom 19.2.4(p.139) (c3ii) with $a = 0.01$, $b = 1.00$, $\beta = 0.98$, and $s = 0.05$ where $x_{\tilde{K}} = 0.3076395$ and $s_{\tilde{\mathcal{L}}} = 0.3232736$.† Note that the example is for the model of a buying problem with the cost minimization. The figure below is the graph of $I^t_\tau = \beta^{\tau-t}V_t$ where the symbol • shows the optimal initiating time (OIT) for each $\tau = 2, 3, \cdots, 15$ (see t^*-column in the table below). In addition, note that each of polygonal curves for $\tau = 2, 3, \cdots, 7$ is strictly decreasing in $t = 1, 2, \cdots, 7$ and that each of polygonal curves for $\tau = 8, 9, \cdots, 15$ is *strictly* decreasing in $t = 1, 2, \cdots, 7$ and *strictly* increasing in $t = 7, 8, \cdots, 15$. The fact implies that the optimal initiating time t^*_τ degenerates to the starting time $\tau = 2, 3, \cdots, 7$, i.e., $Ⓢ_\tau\langle\tau\rangle_\blacktriangle$ and that it is given by $t^*_\tau = 7$ (non-degenerate) for each of $\tau = 8, 9, \cdots, 15$, i.e., $◎_\tau\langle 7\rangle_\blacktriangle$ (see t^*-column in the table below). Finally, note here that the leftmost point V_t in each curves converges to $x_{\tilde{K}} = 0.3076395$ as $\tau \to \infty$ (see Pom 19.2.4(p.139) (a)).



t	V_t	$\Delta_\beta V_t$	t^*_τ
0			
1	0.544900		
2	0.442388	−0.091614	2
3	0.391004	−0.042535	3
4	0.361335	−0.021849	4
5	0.343013	−0.011094	5
6	0.331264	−0.004889	6
7	0.323555	−0.001084	7
8	0.318422	+0.001338	7
9	0.314972	+0.002918	7
10	0.312638	+0.003965	7
11	0.311053	+0.004667	7
12	0.309973	+0.005141	7
13	0.309236	+0.005462	7
14	0.308732	+0.005681	7
15	0.308388	+0.005830	7

$\Delta_\beta V_t = V_t - \beta V_{t-1}$

Figure 19.2.1: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $\tau = 2, 3, \cdots, 15$ and $t = 1, 2, \cdots, \tau$

19.2.7 Conclusion 2 (Search-Enforced-Model 1)

C1. Mental Conflict

On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$ we have (see (7.3.1(p.45)) and (7.3.2(p.45)) for the definitions of opt-$\mathbb{R}$-price and opt-$\mathbb{P}$-price below):

a. The opt-$\mathbb{R}$-price V_t in $\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]$ (selling model) is nondecreasing in t as seen in Tom's 19.2.1(p.134) (a) and 19.2.2(p.135) (a) (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

b. The opt-$\mathbb{P}$-price z_t in $\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]$ is nondecreasing (selling model) in t as seen in Corollary 19.2.1(p.141)) (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

c. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]$ (buying model) is nonincreasing in t as seen in Tom's 19.2.3(p.138) (a) and 19.2.4(p.139) (a) (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

d. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]$ (buying model) is nonincreasing in t as seen in Corollary 19.2.2(p.143) (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

The above results can be summarized as below.

A. On $\mathscr{F}$, for any $\beta \le 1$ and $s \ge 0$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict.

C2. Symmetry

a. On $\mathscr{F}^+$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Pom 19.2.3(p.139) ∾ Pom 19.2.1(p.137) ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^+$),

Pom 19.2.7(p.143) ∾ Pom 19.2.5(p.141) ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^+$).

†Since $a = 0.01 > 0$, $b = 1.00$, $\beta = 0.98 < 1$, and $s = 0.05 > 0$, we have $\mu = (0.01 + 1.00)/2 = 0.525$, $\beta\mu = 0.98 \times 0.525 = 0.5145 > 0.01 = a$, $\beta\mu + s = 0.5145 + 0.05 = 0.5645 < 1.00 = b$, and $s = 0.05 < 0.3232736 = s_{\tilde{\mathcal{L}}}$. Thus, the condition of this assertion is satisfied.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Pom 19.2.4(p.139) $\nsim$ Pom 19.2.2(p.137) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{+} \nsim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{+})\cdots(s^1)$,

Pom 19.2.8(p.143) $\nsim$ Pom 19.2.6(p.141) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{+} \nsim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{+})\cdots(s^2)$.

b. On $\mathscr{F}^{\pm}$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Mim 19.2.3(p.140) $\sim$ Mim 19.2.1(p.137) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm})$,

Mim 19.2.7(p.144) $\sim$ Mim 19.2.5(p.142) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm})$.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Mim 19.2.4(p.140) $\sim$ Mim 19.2.2(p.137) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm})$,

Mim 19.2.8(p.144) $\sim$ Mim 19.2.6(p.142) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm})$.

c. On $\mathscr{F}^{-}$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Nem 19.2.3(p.140) $\sim$ Nem 19.2.1(p.138) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{-} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{-})$,

Nem 19.2.7(p.144) $\sim$ Nem 19.2.5(p.142) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{-} \sim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{-})$.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Nem 19.2.4(p.140) $\nsim$ Nem 19.2.2(p.138) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{-} \nsim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{-})\cdots(s^3)$,

Nem 19.2.8(p.144) $\nsim$ Nem 19.2.6(p.142) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{-} \nsim \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{-})\cdots(s^4)$.

The above results can be summarized as below.

A. On $\mathscr{F}^{\pm}$, for any $\beta \le 1$ and $s \ge 0$, the symmetry is inherited (see C2b(p.146)).

B. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, the symmetry is inherited (see C2a1(p.145) and C2c1(p.146)).

C. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, the symmetry collapses (see (s^1), (s^2), (s^3), and (s^4)).

C3. Analogy

a. On $\mathscr{F}^{+}$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Pom 19.2.5(p.141) $\bowtie$ Pom 19.2.1(p.137) $\quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{+} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{+})$,

Pom 19.2.7(p.143) $\bowtie$ Pom 19.2.3(p.139) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{+} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{+})$.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Pom 19.2.6(p.141) $\not\bowtie$ Pom 19.2.2(p.137) $\quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{+} \not\bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{+})\cdots(a^1)$,

Pom 19.2.8(p.143) $\bowtie$ Pom 19.2.4(p.139) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{+} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{+})$.

b. On $\mathscr{F}^{\pm}$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Mim 19.2.5(p.142) $\bowtie$ Mim 19.2.1(p.137) $\quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm})$,

Mim 19.2.7(p.144) $\bowtie$ Mim 19.2.3(p.140) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm})$.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Mim 19.2.6(p.142) $\bowtie$ Mim 19.2.2(p.137) $\quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm})$,

Mim 19.2.8(p.144) $\bowtie$ Mim 19.2.4(p.140) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{\pm} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{\pm})$.

c. On $\mathscr{F}^{-}$ we have:

1. Let $\beta = 1$ and $s = 0$. Then we have:

Nem 19.2.5(p.142) $\bowtie$ Nem 19.2.1(p.138) $\quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{-} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{-})$,

Nem 19.2.7(p.144) $\bowtie$ Nem 19.2.3(p.140) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{-} \bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{-})$.

2. Let $\beta < 1$ or $s > 0$. Then we have:

Nem 19.2.6(p.142) $\bowtie$ Nem 19.2.2(p.138) $\quad (\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathsf{E}]\}^{-} \bowtie \mathscr{A}\{\mathsf{M}{:}1[\mathbb{R}][\mathsf{E}]\}^{-})$,

Nem 19.2.8(p.144) $\not\bowtie$ Nem 19.2.4(p.140) $\quad (\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathsf{E}]\}^{-} \not\bowtie \mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathsf{E}]\}^{-})\cdots(a^2)$.

The above results can be summarized as below.

A. On $\mathscr{F}^{\pm}$, for any $\beta \le 1$ and $s \ge 0$, the analogy is inherited (see C3b(p.146)).

B. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, then the analogy is inherited (see C3a1(p.146) and C3c1(p.146)).

C. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, then the analogy collapses (see (a^1) and (a^2)).

C4. Optimal initiating time (OIT)

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 19.2.1(p.137), Mim 19.2.1(p.137), Nem 19.2.1(p.138),
Pom 19.2.3(p.139), Mim 19.2.3(p.140), Nem 19.2.3(p.140),
Pom 19.2.5(p.141), Mim 19.2.5(p.142), Nem 19.2.5(p.142),
Pom 19.2.7(p.143), Mim 19.2.7(p.144), Nem 19.2.7(p.144)

we obtain the following table (the symbol "○" in the table below represents "possible"):

Table 19.2.1: Possible OIT ($\beta = 1$ and $s = 0$)

	$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
Ⓢ$_\tau\langle\tau\rangle_{\|}$			
Ⓢ$_\tau\langle\tau\rangle_{\vartriangle}$			
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○
◎$_\tau\langle t^\bullet_\tau\rangle_{\|}$			
◎$_\tau\langle t^\bullet_\tau\rangle_{\vartriangle}$			
◎$_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$			
ⓓ$_\tau\langle 0\rangle_{\|}$			
ⓓ$_\tau\langle 0\rangle_{\vartriangle}$			
ⓓ$_\tau\langle 0\rangle_{\blacktriangle}$			

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 19.2.2(p.137), Mim 19.2.2(p.137), Nem 19.2.2(p.138),
Pom 19.2.4(p.139), Mim 19.2.4(p.140), Nem 19.2.4(p.140),
Pom 19.2.6(p.141), Mim 19.2.6(p.142), Nem 19.2.6(p.142),
Pom 19.2.8(p.143), Mim 19.2.8(p.144), Nem 19.2.8(p.144)

we obtain the following table:

Table 19.2.2: Possible OIT ($\beta < 1$ or $s > 0$)

	$\mathscr{F}^+$	$\mathscr{F}^\pm$	$\mathscr{F}^-$
Ⓢ$_\tau\langle\tau\rangle_{\|}$			
Ⓢ$_\tau\langle\tau\rangle_{\vartriangle}$			
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○
◎$_\tau\langle t^\bullet_\tau\rangle_{\|}$	○	○	○
◎$_\tau\langle t^\bullet_\tau\rangle_{\vartriangle}$	○	○	○
◎$_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$	○	○	○
ⓓ$_\tau\langle 0\rangle_{\|}$	○	○	○
ⓓ$_\tau\langle 0\rangle_{\vartriangle}$	○	○	○
ⓓ$_\tau\langle 0\rangle_{\blacktriangle}$			

c. The table below is the list of the occurrence rates of Ⓢ, ◎, and ⓓ (Def. 11.2.4(p.61)) on $\mathscr{F}$ appearing in the primitive Tom 19.2.1(p.134) (▣) and Tom 19.2.2(p.135) (▣) (see Def. 11.2.2(p.59)).

Table 19.2.3: Occurence rates of Ⓢ, ◎, and ⓓ on $\mathscr{F}$

Ⓢ			◎			ⓓ		
41.7 % / 5			25.0 % / 3			33.3 % / 4		
Ⓢ$_{\|}$	Ⓢ$_{\vartriangle}$	Ⓢ$_{\blacktriangle}$	◎$_{\|}$	◎$_{\vartriangle}$	◎$_{\blacktriangle}$	ⓓ$_{\|}$	ⓓ$_{\vartriangle}$	ⓓ$_{\blacktriangle}$
—	×	possible	possible	possible	possible	possible	possible	×
– % / –	0.0 % / 0	41.7 % / 5	8.3 % / 1	8.3 % / 1	8.3 % / 1	16.7 % / 2	16.7 % / 2	0.0 % / 0

C5. Null-time-zone and deadline-engulfing

From Table 19.2.3(p.147) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.43) for the implication of the symbol "$\blacktriangle$" representing the strict optimality of the initiating time t^*_τ.

b. As a whole, Ⓢ, ◎, and ⓓ are possible at 41.7%, 25.0%, and 33.3% respectively where

1. Ⓢ$_{\parallel}$ cannot be defined (see Remark ??(p.??)).
2. ◎$_{\parallel}$ is possible (8.3 %).
3. 🅓$_{\parallel}$ is possible (16.7 %).

4. Ⓢ$_{\vartriangle}$ never occur (0.0 %).
5. ◎$_{\vartriangle}$ is possible (8.3 %).
6. 🅓$_{\vartriangle}$ is possible (16.7 %).

7. Ⓢ$_{\blacktriangle}$ is possible (41.7 %).
8. ◎$_{\blacktriangle}$ is possible (8.3 %).
9. 🅓$_{\blacktriangle}$ never occur (0.0 %).

From the above results we see that on $\mathscr{F}$:

A. ◎$_{\parallel}$ and 🅓$_{\parallel}$ causing the `null-time-zone` are possible at 25.0% (= 8.3% + 16.7%).

B. ◎$_{\blacktriangle}$ *strictly* causing the `null-time-zone` is possible at 8.3%.

C. 🅓$_{\blacktriangle}$ *strictly* causing the `deadline-engulfing` is impossible (0.0%).

19.3 Conclusions of Model 1

Conclusions 1 (p.131) and 2 (p.145) can be summarized as below.

C1. Mental Conflict

On $\mathscr{F}$, from C1A(p.131) and C1A(p.145), for any $\beta \leq 1$ and $s \geq 0$, whether selling problem or buying problem, whether $\mathbb{R}$-model or $\mathbb{P}$-model, and whether search-**A**llowed-model or search-**E**nforced-model, we have the normal mental conflict in *Examples* 1.4.1(p.6) - 1.4.4(p.6).

C2. Symmetry

a. On $\mathscr{F}^{\pm}$, for any $\beta \leq 1$ and $s \geq 0$, the symmetry is inherited (see C2A(p.132) and C2A(p.146)).

b. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, the symmetry is inherited (see C2B(p.132) and C2B(p.146)).

c. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, the symmetry *may* collapse on $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$ (see C2C(p.132) and C2C(p.146)).

C3. Analogy

a. On $\mathscr{F}^{\pm}$, for any $\beta \leq 1$ and $s \geq 0$, the analogy are inherited (see C3A(p.132) and C3A(p.146)).

b. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta = 1$ and $s = 0$, the analogy are inherited (see C3B(p.132) and C3B(p.146)).

c. On $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$, if $\beta < 1$ or $s > 0$, the analogy collapse on $\mathscr{F}^{+}$ and $\mathscr{F}^{-}$ (see C3C(p.132) and C3C(p.146)).

C4. Optimal initiating time (`OIT`)

On $\mathscr{F}^{+}$, $\mathscr{F}^{\pm}$, and $\mathscr{F}^{-}$, we have:

a. Let $\beta = 1$ and $s = 0$. Then only Ⓢ$_{\blacktriangle}$ is possible (see Tables 19.1.1(p.133) and 19.2.1(p.147)).

b. Let $\beta < 1$ or $s > 0$. Then:

   1. For search-**A**llowed-model we have only Ⓢ$_{\blacktriangle}$, ◎$_{\parallel}$, and 🅓$_{\parallel}$ (see Table 19.1.2(p.133)).
   2. For search-**E**nforced-model we have Ⓢ$_{\blacktriangle}$, ◎$_{\parallel}$, ◎$_{\vartriangle}$, ◎$_{\blacktriangle}$, 🅓$_{\parallel}$, and 🅓$_{\vartriangle}$ (see Table 19.2.2(p.147)).

c. Joining Tables 19.1.3(p.133) and 19.2.3(p.147) produces the following table:

Table 19.3.1: Occurance rates of Ⓢ, ◎, and 🅓 on $\mathscr{F}$

Ⓢ			◎			🅓		
45.5 % / 10			18.2 % / 4			36.3 % / 8		
Ⓢ$_{\parallel}$	Ⓢ$_{\vartriangle}$	Ⓢ$_{\blacktriangle}$	◎$_{\parallel}$	◎$_{\vartriangle}$	◎$_{\blacktriangle}$	🅓$_{\parallel}$	🅓$_{\vartriangle}$	🅓$_{\blacktriangle}$
−	×	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`	×
− % / −	0.0 % / 0	45.5 % / 10	9.0 % / 2	4.6 % / 1	4.6 % / 1	27.3 % / 6	9.0 % / 2	0.0 % / 0

C5. Null-time-zone and deadline-engulfing

From Table 19.3.1(p.148) above we see that on $\mathscr{F}$

a. See Remark 7.2.2(p.43) for the implication of the symbol "$\blacktriangle$" representing the strict optimality of the initiating time t^*_τ.

b. As a whole we have Ⓢ, ◎, and 🅓 at 45.5%, 18.2%, and 36.3% respectively where

   1. Ⓢ$_{\parallel}$ cannot be defined (see Remark ??(p.??)).
   2. ◎$_{\parallel}$ is possible (9.0 %).

3. ⓓ$_{\parallel}$ is possible (27.3 %).
4. Ⓢ$_{\triangle}$ never occur (0.0 %).
5. ◎$_{\triangle}$ is possible (4.6 %).
6. ⓓ$_{\triangle}$ is possible (9.0 %).
7. Ⓢ$_{\blacktriangle}$ is possible (45.5%),
8. ◎$_{\blacktriangle}$ is possible(4.6%).
9. ⓓ$_{\blacktriangle}$ never occur (0.0%).

From the above results we see that:

A. ◎$_{\parallel}$ and ⓓ$_{\parallel}$ causing the `null-time-zone` are possible at 36.3% (= 9.0% + 27.3%).

B. ◎$_{\blacktriangle}$ *strictly* causing the `null-time-zone` is possible at 4.6%.

C. ⓓ$_{\blacktriangle}$ *strictly* causing the `deadline-engulfing` is impossible (0.0%).

Chapter 20

Analysis of Model 2

20.1 Search-Allowed-Model 2: $\mathcal{Q}\{\mathsf{M}{:}2[\mathtt{A}]\} = \{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$

20.1.1 Preliminary

As ones corresponding to Theorems 12.5.1(p.78), 13.3.1(p.96), and 14.5.1(p.104), let us consider the following three theorems:

Theorem 20.1.1 (symmetry$[\mathbb{R} \to \tilde{\mathbb{R}}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.1}$$

Theorem 20.1.2 (analogy$[\mathbb{R} \to \mathbb{P}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.2}$$

Theorem 20.1.3 (symmetry$[\mathbb{P} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{20.1.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}], \tag{20.1.4}$$

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}], \tag{20.1.5}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}], \tag{20.1.6}$$

corresponding to (12.5.34(p.75)), (13.2.4(p.91)), and (14.5.4(p.104)). Then, for the same reason as in Chap. 15(p.109) it can be shown that the equality

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}] \tag{20.1.7}$$

holds (corresponding to (15.1.33(p.110))) and that we have the following theorem, corresponding to Theorem 15.1.1(p.110)

Theorem 20.1.4 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{20.1.8}$$

In fact, from the comparisons of (I) and (II), of (I) and (III), of (III) and (IV), and of (II) and (IV) in Table 6.5.3(p.39) we can easily show that (20.1.4(p.151)) - (20.1.7(p.151)) hold.

20.1.2 A Lemma

The following lemma provides the conditions which determine if each of Theorems 20.1.1(p.151), 20.1.2(p.151), and 20.1.3(p.151) holds by testing whether or not each of (20.1.4(p.151)), (20.1.5(p.151)), and (20.1.6(p.151)) is true.

Lemma 20.1.1 ($\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$)

(a) Theorem 20.1.1(p.151) *holds.*
(b) Theorem 20.1.3(p.151) *holds.*
(c) *If* $\rho \le a^\star$ *or* $b \le \rho$, *then* Theorem 20.1.2(p.151) *holds.*
(d) *If* $a^\star < \rho < b$, *then* Theorem 20.1.2(p.151) *does not always hold.* $\square$

• *Proof* (a) From Table 6.5.3(p.39) (I) we have, for *any* $\rho \in (-\infty, \infty)$,

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\} = \{V_0 = \rho,\ V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 0\} \tag{20.1.9}$$

First, applying the operation $\mathcal{R}$ (see Step 2 (p.73)) to this leads to

$$\begin{aligned}
\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] &= \{-\hat{V}_0 = \rho,\ -\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\}, \quad t > 0\} \\
&= \{-\hat{V}_0 = \rho,\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\} \\
&= \{\hat{V}_0 = -\rho,\ \hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\} \\
&= \{\hat{V}_0 = \hat{\rho},\ -\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}
\end{aligned} \tag{20.1.10}$$

Then, applying $\mathcal{C}_{\mathbb{R}}$ (see Step 3 (p.73)) to this yields

$$\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_0 = \hat{\rho},\ \hat{V}_t = \min\{\check{\hat{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}. \tag{20.1.11}$$

Finally, applying $\mathcal{I}_{\mathbb{R}}$ (see Step 4 (p.74)) to this produces

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] = \{\hat{V}_0 = \hat{\rho},\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}. \tag{20.1.12}$$

Since this holds for any $\rho \in (-\infty, \infty)$, it holds also for $\hat{\rho} \in (-\infty, \infty)$, hence holds also for the $\hat{\hat{\rho}}$, i.e.,

$$\begin{aligned}
\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] &= \{\hat{V}_0 = \hat{\hat{\rho}},\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\} \\
&= \{\hat{V}_0 = \rho,\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 0\}
\end{aligned} \tag{20.1.13}$$

due to $\rho = \hat{\hat{\rho}}$. Now, we have $\hat{V}_0 = \rho = V_0$ from (6.5.17(p.39)). Suppose $\hat{V}_{t-1} = V_{t-1}$. Then, the second term in the r.h.s. of (20.1.13(p.152)) can be rewritten as $\hat{V}_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$. Thus, by induction $\hat{V}_t = V_t$ for $t \geq 0$. Accordingly (20.1.13(p.152)) can be rewritten as

$$\mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] = \{V_0 = \rho,\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 0, \tag{20.1.14}$$

which is identical to $\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$ (see Table 6.5.3(p.39) (II)), i.e.,

$$\begin{aligned}
\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\} &= \mathcal{I}_{\mathbb{R}}\mathcal{C}_{\mathbb{R}}\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] \\
&= \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}] \quad \text{(see (12.5.30(p.75)))}.
\end{aligned} \tag{20.1.15}$$

Hence, since (20.1.4(p.151)) holds, it follows that Theorem 20.1.1(p.151) holds.

(b) From Table 6.5.3(p.39) (III) we have, for *any* $\rho \in (-\infty, \infty)$,

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\} = \left\{\begin{array}{l}
V_0 = \rho, \\
V_1 = \max\{\lambda\beta\max\{0, a - \rho\} + \beta\rho - s, \beta\rho\}, \\
V_t = \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 1
\end{array}\right\}$$

Applying the operation $\mathcal{R}$ to this leads to

$$\begin{aligned}
\mathcal{R}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}] &= \left\{\begin{array}{l}
-\hat{V}_0 = \rho, \\
-\hat{V}_1 = \max\{\lambda\beta\max\{0, -\hat{a} - \rho\} + \beta\rho - s, \beta\rho\}, \\
-\hat{V}_t = \max\{-\hat{K}(V_{t-1}) - \hat{V}_{t-1}, -\beta\hat{V}_{t-1}\}, \quad t > 1
\end{array}\right\} \\
&= \left\{\begin{array}{l}
-\hat{V}_0 = \rho, \\
-\hat{V}_1 = \max\{-\lambda\beta\min\{0, \hat{a} + \rho\} + \beta\rho - s, \beta\rho\}, \\
-\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1
\end{array}\right\} \\
&= \left\{\begin{array}{l}
-\hat{V}_0 = \rho, \\
-\hat{V}_1 = -\min\{\lambda\beta\min\{0, \hat{a} + \rho\} - \beta\rho + s, -\beta\rho\}, \\
-\hat{V}_t = -\min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1
\end{array}\right\} \\
&= \left\{\begin{array}{l}
\hat{V}_0 = -\rho, \\
\hat{V}_1 = \min\{\lambda\beta\min\{0, \hat{a} + \rho\} + \beta\rho - s, \beta\rho\}, \\
\hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1
\end{array}\right\} \\
&= \left\{\begin{array}{l}
\hat{V}_0 = \hat{\rho}, \\
\hat{V}_1 = \min\{\lambda\beta\min\{0, \hat{a} - \hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\
\hat{V}_t = \min\{\hat{K}(V_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1
\end{array}\right\}.
\end{aligned}$$

Applying $\mathcal{C}_{\mathbb{P}}$ to this yields

$$\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, \check{b} - \hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\check{\tilde{K}}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\}.$$

Applying $\mathcal{I}_{\mathbb{P}}$ to this produces

$$\mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} \hat{V}_0 = \hat{\rho}, \\ \hat{V}_1 = \min\{\lambda\beta\min\{0, b - \hat{\rho}\} + \beta\hat{\rho} + s, \beta\hat{\rho}\}, \\ \hat{V}_t = \min\{\tilde{K}(\hat{V}_{t-1}) + \hat{V}_{t-1}, \beta\hat{V}_{t-1}\}, \quad t > 1 \end{array}\right\}.$$

For the same reason as in the proof of (a), we can replace $\hat{\rho}$ by ρ, hence we obtain.

$$\mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] = \left\{\begin{array}{l} V_0 = \rho, \\ V_1 = \min\{\lambda\beta\min\{0, b - \rho\} + \beta\rho + s, \beta\rho\}, \\ V_t = \min\{\tilde{K}(V_{t-1}) + V_{t-1}, \beta V_{t-1}\}, \quad t > 1 \end{array}\right\},$$

which is the same as $\texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\texttt{A}]\}$ given by Table 6.5.3(p.39) (IV), hence we have

$$\begin{aligned} \texttt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\texttt{A}]\} &= \mathcal{I}_{\mathbb{P}}\mathcal{C}_{\mathbb{P}}\mathcal{R}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}] \qquad (20.1.16) \\ &= \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\}] \quad (\text{see } (14.5.3(\text{p.103}))). \qquad (20.1.17) \end{aligned}$$

Hence, since (20.1.6(p.151)) holds, it follows that Theorem 20.1.3(p.151) holds.

(c) Let $\rho \le a^\star$ or $b \le \rho$.

1. Let $\rho \le a^\star$. Then, since $\rho \le a^\star < a$ due to Lemma 13.2.1(p.91) (n), we have $\max\{0, a-\rho\} = a - \rho \cdots (1)$. In addition, since $T_{\mathbb{R}}(\rho) = \mu - \rho$ from Lemma 10.1.1(p.53) (f) and since $T_{\mathbb{P}}(\rho) = a - \rho$ from Lemma 13.2.1(p.91) (f), we have
$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mu - \rho] = a - \rho = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\} \cdots (2) \quad (\text{due to } (1))$$
2. Let $b \le \rho$. Then, since $a < b < \rho$, we have $\max\{0, a-\rho\} = 0 \cdots (3)$. In addition, since $T_{\mathbb{R}}(\rho) = 0$ from Lemma 10.1.1(p.53) (g) and since $T_{\mathbb{P}}(\rho) = 0$ from Lemma 13.2.1(p.91) (g), we have
$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = 0 = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\} \cdots (4) \quad (\text{due to } (3)).$$

From (2) and (4), whether $\rho \le a^\star$ or $b \le \rho$, we have

$$\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] = T_{\mathbb{P}}(\rho) = \max\{0, a-\rho\}, \qquad (20.1.18)$$

hence from (5.1.4(p.23)) we have

$$\begin{aligned} \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[K_{\mathbb{R}}(\rho)] &= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\lambda\beta T_{\mathbb{R}}(\rho) - (1-\beta)\rho - s] \\ &= \lambda\beta\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[T_{\mathbb{R}}(\rho)] - (1-\beta)\rho - s \\ &= \lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s. \qquad (20.1.19) \end{aligned}$$

Accordingly, we have

$$\begin{aligned} &\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(6.5.18(\text{p.39})) \text{ with } t = 1] \\ &= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{V_1 = \max\{K_{\mathbb{R}}(V_0) + V_0, \beta V_0\}\}] \\ &= \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\{V_1 = \max\{K_{\mathbb{R}}(\rho) + \rho, \beta\rho\}\}] \\ &= \{V_1 = \max\{\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[K_{\mathbb{R}}(\rho)] + \rho, \beta\rho\}\} \\ &= \{V_1 = \max\{\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s + \rho, \beta\rho\}\} \quad (\text{due to } (20.1.19(\text{p.153}))) \\ &= \{V_1 = \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\}\} \\ &= \{(6.5.22(\text{p.39}))\}. \end{aligned}$$

The above result means that $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}[(6.5.18(\text{p.39}))$ with "$t > 0$" is separated into the two cases, (6.5.22(p.39)) with "$t = 1$" and (6.5.23(p.39)) "with "$t > 1$". This fact implies that

$$\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\texttt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\texttt{A}]\}]. \qquad (20.1.20)$$

Accordingly, since (20.1.5(p.151)) holds whether $\rho \le a^\star$ or $b \le \rho$, it follows that Theorem 20.1.2(p.151) holds.

(d) Let $a^\star < \rho < b$. Then, since the same reasoning as in the proof of (c) does not always hold, it follows that Theorem 20.1.2(p.151) does not always hold. ∎

Remark 20.1.1 (pseudo-reversible element ρ) Let us recall here that $\mathcal{R}$ is an operation applied *only* to attribute elements which depend on the distribution function F (see Section 12.1.1(p.67)). Accordingly, the operation cannot be applied to the constant ρ which is not related to F, implying that the $\hat{\rho}$ in the proofs of (a,b) is one resulting from *merely rearranging* the expression $-\hat{V}_1 = \rho$ as $\hat{V}_1 = -\rho \to \hat{V}_1 = \hat{\rho}$. However, superficially this transformation $\rho \to \hat{\rho}$ seems to be the application of the reversible operation $\mathcal{R}$ defined in Section 12.1.1(p.67). For this reason, regarding this ρ, which is *originally* a non-reversible element, as a *reversible element* of a sort (see Def. 12.3.3(p.71)), let us call it the *pseudo-reversible element*. □

20.1.3 Proof of $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$

20.1.3.1 Preliminary

From (6.5.18(p.39)) and (5.1.8(p.23)) we have

$$\begin{aligned} V_t &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 0, \end{aligned} \tag{20.1.21}$$

hence

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\}, \quad t > 0. \tag{20.1.22}$$

Then, for $t > 0$ we have

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1} \quad \text{if } L(V_{t-1}) \geq 0 \quad (\text{see } (5.1.9(\text{p.23}))), \tag{20.1.23}$$

$$V_t = \beta V_{t-1} \quad \text{if } L(V_{t-1}) \leq 0. \tag{20.1.24}$$

Finally, from (6.2.75(p.31)) and from (6.2.71(p.31)) and (6.2.73(p.31)) we have

$$\mathbb{S}_t = L(V_{t-1}) \geq (\leq)\ 0 \Rightarrow \mathtt{Conduct}_{t\vartriangle}\ (\mathtt{Skip}_{t\vartriangle}), \quad t > 0, \tag{20.1.25}$$

$$\mathbb{S}_t = L(V_{t-1}) > (<)\ 0 \Rightarrow \mathtt{Conduct}_{t\blacktriangle}\ (\mathtt{Skip}_{t\blacktriangle}), \quad t > 0. \tag{20.1.26}$$

20.1.3.2 Analysis

20.1.3.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 20.1.1** (▣$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \geq b$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$ $\rightarrow$ → ⓓ$_{\parallel}$

(c) *Let $\rho < b$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → Ⓢ$_{\blacktriangle}$

● *Proof* Let $\beta = 1$ and $s = 0$, hence $x_L = x_K = b \cdots$ **(1)** from Lemmas 10.2.3(p.56) (a). Then, since $K(x) = \lambda T(x) \cdots$ **(2)** for any x from (5.1.4(p.23)), due to Lemma 10.1.1(p.53) (g) we have $K(x) \geq 0 \cdots$ **(3)** for any x and $K(b) = 0 \cdots$ **(4)**.

(a) From (6.5.18(p.39)) we have $V_t \geq K(V_{t-1}) + V_{t-1}$ for $t > 0$, hence $V_t \geq V_{t-1}$ for $t > 0$ due to (3). Thus V_t is nondecreasing in $t \geq 0$.

(b) Let $\rho \geq b$, hence $\rho \geq x_L$ due to (1). Then, since $V_0 \geq x_L$ from (6.5.17(p.39)), we have $V_{t-1} \geq x_L$ for $t > 0$ from (a). Hence, since $L(V_{t-1}) = 0$ for $t > 0$ from Lemma 10.2.1(p.55) (d), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.154)), thus $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 0$, i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$. Hence, since $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0$, we have $t^*_\tau = 0$ for $\tau > 0$ due to Preference Rule 7.2.1(p.43), i.e., ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) Let $\rho < b$. Then $V_0 < b$ from (6.5.17(p.39)). Suppose $V_{t-1} < b$. Then, from Lemma 10.2.2(p.55) (h) and (6.5.18(p.39)) with $\beta = 1$ we have $V_t < \max\{K(b) + b, b\} = \max\{b, b\}$ due to (4), hence $V_t < b$. Accordingly, by induction $V_{t-1} < b \cdots$ **(5)** for $t > 0$, so $V_{t-1} < x_L$ for $t > 0$ due to (1). Thus, since $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.55) (d), we have $L(V_{t-1}) > 0 \cdots$ **(6)** for $\tau \geq t > 0$. Accordingly, from (20.1.22(p.154)) we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 0$, i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 0$, hence $V_\tau > \beta V_{\tau-1} > \cdots > \beta^\tau V_0$. Accordingly, we have $t^*_\tau = \tau$ for $\tau > 0$, i.e., Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$. Then $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 0$ due to (6) and (20.1.26(p.154)). ∎

20.1.3.2.2 Case of $\beta <$ or $s > 0$

For explanatory simplicity, let us define

$$\mathtt{S}_3\boxed{\text{Ⓢ}\blacktriangle\ \big|\ \text{◎}\ \parallel} = \left\{ \begin{array}{l} \text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 0 \text{ such that} \\ (1)\ \text{Ⓢ}_{t^\bullet_\tau \geq \tau > 0}\langle\tau\rangle_{\blacktriangle} \text{ where } \mathtt{Conduct}_{\tau\geq t>0\blacktriangle}, \\ (2)\ \text{◎}_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_{\parallel} \text{ where } \mathtt{Conduct}_{t^\bullet_\tau\geq t>0\blacktriangle}. \end{array} \right\}.$$

□ **Tom 20.1.2** (▣$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V \geq x_K$ as $t \rightarrow \infty$.*

(b) *Let $x_L \leq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$ $\rightarrow$ → ⓓ$_{\parallel}$

(c) *Let $\rho < x_L$.*

1. Ⓢ$_1\langle 1\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$* $\rightarrow$ → Ⓢ$_{\blacktriangle}$
2. *Let $\beta = 1$.*
   i. *Let $a < \rho$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → Ⓢ$_{\blacktriangle}$
   ii. *Let $\rho \leq a$.*
      1. *Let $(\lambda\mu - s)/\lambda \leq a$.*
         i. *Let $\lambda = 1$. Then* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ $\rightarrow$ → ◎$_{\parallel}$
         ii. *Let $\lambda < 1$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$ $\rightarrow$ → Ⓢ$_{\blacktriangle}$

2. *Let* $(\lambda\mu - s)/\lambda > a$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\rightarrow$ $\rightarrow \circledS_\blacktriangle$

3. *Let* $\beta < 1$ *and* $s = 0$ $(\!(s > 0)\!)$.

   i. *Let* $a < \rho$.

   1. *Let* $b \ge 0$ $(\!(\kappa \ge 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\rightarrow$ $\rightarrow \circledS_\blacktriangle$

   2. *Let* $b < 0$ $(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_3$(p.154) $\boxed{\circledS_\blacktriangle \mid \circledcirc_\parallel}$ *is true* $\rightarrow$ $\rightarrow \circledS_\blacktriangle/\circledcirc_\parallel$

   ii. *Let* $\rho \le a$.

   1. *Let* $(\lambda\beta\mu - s)/\delta \le a$.

      i. *Let* $\lambda = 1$.

      2. *Let* $b > 0$ $(\!(\kappa > 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\rightarrow$ $\rightarrow \circledS_\blacktriangle$

      3. *Let* $b \le 0$ $(\!(\kappa \le 0)\!)$. *Then* $\circledcirc_{\tau>1}\langle 1\rangle_\parallel$ *where* $\mathtt{Conduct}_{1\blacktriangle}$ $\rightarrow$ $\rightarrow \circledcirc_\parallel$

      ii. *Let* $\lambda < 1$.

      4. *Let* $b \ge 0$ $(\!(\kappa \ge 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\rightarrow$ $\rightarrow \circledS_\blacktriangle$

      5. *Let* $b < 0$ $(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_3$(p.154) $\boxed{\circledS_\blacktriangle \mid \circledcirc_\parallel}$ *is true* $\rightarrow$ $\rightarrow \circledS_\blacktriangle/\circledcirc_\parallel$

   2. *Let* $(\lambda\beta\mu - s)/\delta > a$.

      i. *Let* $b \ge 0$ $(\!(\kappa \ge 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\rightarrow$ $\rightarrow \circledS_\blacktriangle$

      ii. *Let* $b < 0$ $(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_3$(p.154) $\boxed{\circledS_\blacktriangle \mid \circledcirc_\parallel}$ *is true* $\rightarrow$ $\rightarrow \circledS_\blacktriangle/\circledcirc_\parallel$

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho < x_K \cdots$ **(1)**. Then $V_0 < x_K \cdots$ **(2)** from (6.5.17(p.39)) and $K(\rho) > 0 \cdots$ **(3)** due to Lemma 10.2.2(p.55) (j1). Accordingly, from (6.5.18(p.39)) with $t = 1$ we have $V_1 - V_0 = V_1 - \rho = \max\{K(\rho), \beta\rho - \rho\} \ge K(\rho) > 0$ due to (3), hence $V_1 > V_0 \cdots$ **(4)**.

(a) Note (4), hence $V_0 \le V_1$. Suppose $V_{t-1} \le V_t$. Then, from (6.5.18(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \le \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_t \ge V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \ge 0$. Again note (4). Suppose $V_{t-1} < V_t$. If $\lambda < 1$, from Lemma 10.2.2(p.55) (f) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, and if $a < \rho$, then $a < V_0$ from (6.5.17(p.39)), hence $a < V_t$ for $t \ge 0$ due to (a), thus from Lemma 10.2.2(p.55) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Therefore, whether $\lambda < 1$ or $a < \rho$, by induction $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \ge 0$. Consider a sufficiently large $M > 0$ with $\rho \le M$ and $b \le M$, hence $V_0 \le M$ from (6.5.17(p.39)). Suppose $V_{t-1} \le M$. Then, from Lemma 10.2.2(p.55) (e) and (6.5.18(p.39)) we have $V_t \le \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\}$ due to (10.2.7 (2) (p.55)), hence $V_t \le \max\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Thus, by induction $V_t \le M$ for $t \ge 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta V\}$ from (6.5.18(p.39)), we have $0 = \max\{K(V), -(1-\beta)V\}$, hence $K(V) \le 0$, so $V \ge x_K$ due to Lemma 10.2.2(p.55) (j1).

(b) Let $x_L \le \rho$. Then, since $x_L \le V_0$ from (6.5.17(p.39)), we have $x_L \le V_{t-1}$ for $t > 0$ due to (a), hence $L(V_{t-1}) \le 0$ for $t > 0$ due to Corollary 10.2.1(p.55) (a). Accordingly, since $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.154)), we have $V_t - \beta V_{t-1} = 0$ for $\tau \ge t > 0$ or equivalently $V_t = \beta V_{t-1}$ for $\tau \ge t > 0$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0$, implying that $t^*_\tau = 0$ for $\tau > 0$, i.e., 🅓$_{\tau>0}\langle 0\rangle_\parallel$.

(c) Let $\rho < x_L \cdots$ **(5)**, hence $V_0 < x_L \cdots$ **(6)** from (6.5.17(p.39)).

(c1) Since $L(V_0) = L(\rho) > 0 \cdots$ **(7)** due to (5) and Corollary 10.2.1(p.55) (a), we have $V_1 = L(V_0) + \beta V_0 \cdots$ **(8)** due to (20.1.23(p.154)) with $t = 1$, hence $V_1 > \beta V_0 \cdots$ **(9)**. Accordingly, we have $t^*_1 = 1$, i.e., $\circledS_1\langle 1\rangle_\blacktriangle \cdots$ **(10)** and $\mathtt{Conduct}_{1\blacktriangle} \cdots$ **(11)** due to (7) and (20.1.26(p.154)) with $t = 1$. Below let $\tau > 1$.

(c2) Let $\beta = 1$, hence $s > 0 \cdots$ **(12)** due to the assumption "$\beta < 1$ or $s > 0$". Then $\delta = \lambda \cdots$ **(13)** from (10.2.1(p.54)) and $x_L = x_K \cdots$ **(14)** from Lemma 10.2.3(p.56) (b), hence $K(x_L) = K(x_K) = 0 \cdots$ **(15)**. Then, from (5) and (14) we have $\rho < x_K \cdots$ **(16)**.

(c2i) Let $a < \rho$. Then $a < V_0$ from (6.5.17(p.39)), hence $a < V_{t-1}$ for $t > 0$ due to (a). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.5.18(p.39)) with $\beta = 1$ and Lemma 10.2.2(p.55) (g) we have $V_t < \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, thus $V_{t-1} < x_L$ for $t > 0$ due to (14). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.55) (e1), for almost the same reason as in the proof of $\mathtt{Tom}$ 20.1.1(p.154) (c) we have $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\mathtt{CONDUCT}_{\tau\ge t>0\blacktriangle}$.

(c2ii) Let $\rho \le a \cdots$ **(17)**. Then $V_0 \le a \cdots$ **(18)** from (6.5.17(p.39)). Here note that (8) can be rewritten as $V_1 = K(V_0) + V_0 = K(\rho) + \rho \cdots$ **(19)** due to (5.1.9(p.23)). Then, from (17) and (10.2.7 (1) (p.55)) with $\beta = 1$ we have $V_1 = \lambda\mu - s + (1-\lambda)\rho \cdots$ **(20)**

(c2ii1) Let $(\lambda\mu - s)/\lambda \le a$. Then $x_K = (\lambda\mu - s)/\lambda \le a \cdots$ **(21)** from Lemma 10.2.2(p.55) (j2) and (13). Hence $K(a) \le 0 \cdots$ **(22)** from Lemma 10.2.2(p.55) (j1). Note (18). Suppose $V_{t-1} \le a$. Then, from Lemma 10.2.2(p.55) (e) and (6.5.18(p.39)) with $\beta = 1$ we have $V_t \le \max\{K(a) + a, a\} = a$ due to (22), hence by induction $V_{t-1} \le a$ for $t > 0$. Accordingly, from (6.5.18(p.39)) with $\beta = 1$ and (10.2.7 (1) (p.55)) we have $V_t = \max\{\lambda\mu - s + (1-\lambda)V_{t-1}, V_{t-1}\} \cdots$ **(23)** for $t > 0$.

(c2ii1i) Let $\lambda = 1$. Then, since $x_K = \mu - s$ from (21), we have $V_1 = \mu - s = x_K \cdots$ **(24)** from (20). In addition, from (23) we have $V_t = \max\{\mu - s, V_{t-1}\} = \max\{x_K, V_{t-1}\}$ for $t > 0$. Note (24). Suppose $V_{t-1} = x_K$. Then $V_t = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, thus $V_{t-1} = x_L$ for $t > 1$ due to (14). Hence $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, so $L(V_{t-1}) = 0 \cdots$ **(25)** for $\tau \ge t > 1$. Then, from (20.1.22(p.154)) we have $V_t - \beta V_{t-1} = 0$ for $\tau \ge t > 1$, i.e., $V_t = \beta V_{t-1}$ for

$\tau \geq t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$. From this and (9) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$. Then, from (7) and (20.1.26(p.154)) with $t = 1$ we have $\texttt{Conduct}_{1\blacktriangle}$.

(c2ii1ii) Let $\lambda < 1$. Note (6). Suppose $V_{t-1} < x_L$. Then, since $L(V_{t-1}) > 0$ due to Lemma 10.2.1(p.55) (e1), from (20.1.23(p.154)) and Lemma 10.2.2(p.55) (f) we have $V_t = K(V_{t-1}) + V_{t-1} < K(x_L) + x_L = x_L$ due to (15). Accordingly, by induction $V_{t-1} < x_L$ for $t > 0$, so $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.55) (e1). Hence, for almost the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c2ii2) Let $(\lambda\mu - s)/\lambda > a$. Then $x_K > (\lambda\mu - s)/\lambda > a \cdots$ **(26)** from Lemma 10.2.2(p.55) (j2). Note (6). Suppose $V_{t-1} < x_L$. Then $L(V_{t-1}) > 0$ from Lemma 10.2.1(p.55) (e1), hence $V_t = K(V_{t-1}) + V_{t-1}$ from (20.1.23(p.154)). Now, since $a < x_K = x_L$ due to (26) and (14), from Lemma 10.2.2(p.55) (h) we have $V_t < K(x_L) + x_L = x_L$ due to (15). Accordingly, by induction $V_{t-1} < x_L \cdots$ **(27)** for $t > 0$, thus $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.55) (e1). Hence, for almost the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(c3i) Let $a < \rho \cdots$ **(28)**. Then, we have $a < V_0$ from (6.5.17(p.39)), hence $a < V_{t-1} \cdots$ **(29)** for $t > 0$ from (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 10.2.2(p.55) (g) and (6.5.18(p.39)) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, hence by induction $V_{t-1} < V_t$ for $t > 0$. Accordingly, it follows that V_{t-1} is *strictly increasing* in $t > 0 \cdots$ **(30)**.

(c3i1) Let $b \geq 0$ ⟪$\kappa \geq 0$⟫. Then, $x_L \geq x_K \geq 0 \cdots$ **(31)** from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (2). Suppose $V_{t-1} < x_K$. Then, from (29), Lemma 10.2.2(p.55) (g), and (6.5.18(p.39)) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$. Then, since $V_{t-1} < x_L$ for $t > 0$ due to (31), we have $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a). Consequently, for almost the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ † and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3i2) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ **(32)** from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (6), hence $V_0 \leq x_L$. Assume that $V_{t-1} \leq x_L$ for *all* $t > 0$, hence $V \leq x_L$ due to (a). Then, since $x_K \leq V \cdots$ **(33)** due to (a), we have the contradiction of $V \leq x_L < x_K \leq V$ from (32). Accordingly, it is impossible that $V_{t-1} \leq x_L$ for *all* $t > 0$. Therefore, from (6) and (30) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots.$$

Hence, for almost the same reason as in the proof of $\texttt{Tom}$ 11.2.2(p.60) (c2iii2) we immediately see that $\texttt{S}_3$ is true.‡

(c3ii) Let $\rho \leq a \cdots$ **(34)**, hence $V_0 \leq a$ from (6.5.17(p.39)). Then, from (19) and (10.2.7 (1) (p.55)) we have $V_1 = \lambda\beta\mu - s + (1-\lambda)\beta\rho$.

(c3ii1) Let $(\lambda\beta\mu - s)/\delta \leq a$. Then, since $x_K = (\lambda\beta\mu - s)/\delta \leq a \cdots$ **(35)** from Lemma 10.2.2(p.55) (j2), we have $\delta x_K = \lambda\beta\mu - s$, hence $V_1 = \delta x_K + (1-\lambda)\beta\rho \cdots$ **(36)**.

(c3ii1i) Let $\lambda = 1$. Then, since $\delta = 1$ from (10.2.1(p.54)), we have $x_K = \beta\mu - s \leq a$ from (35) and $V_1 = x_K \leq a \cdots$ **(37)** from (36).

(c3ii1i1) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K > 0 \cdots$ **(38)** due to Lemma 10.2.3(p.56) (c ⟪d⟫). Note (37). Suppose $V_{t-1} = x_K$. Then, from (6.5.18(p.39)) we have $V_t = \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Hence, by induction $V_{t-1} = x_K$ for $t > 1$, thus $V_{t-1} < x_L$ for $t > 1$ due to (38). Accordingly $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 10.2.1(p.55) (a), hence $L(V_{t-1}) > 0$ for $t > 0$ due to (7). Therefore, for almost the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii1i2) Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then, since $x_L \leq x_K$ due to Lemma 10.2.3(p.56) (c ⟪d⟫), from (37) we have $V_1 \geq x_L$, hence $V_{t-1} \geq x_L$ for $t > 1$ from (a), so $V_{t-1} \geq x_L$ for $\tau \geq t > 1$. Accordingly, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$ from Corollary 10.2.1(p.55) (a), we obtain $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (20.1.22(p.154)) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$. From this and (9) we obtain $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$. Then, we have $\texttt{Conduct}_{1\blacktriangle}$ from (7) and (20.1.26(p.154)) with $t = 1$.

(c3ii1ii) Let $\lambda < 1$. Note (4). Suppose $V_{t-1} < V_t$. Then, from (6.5.18(p.39)) and Lemma 10.2.2(p.55) (f) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$, hence by induction $V_{t-1} < V_t$ for $t > 0$. Accordingly, it follows that V_t is *strictly increasing* in $t \geq 0 \cdots$ **(39)**.

(c3ii1ii1) Let $b \geq 0$ ⟪$\kappa \geq 0$⟫. Then $x_L \geq x_K \geq 0 \cdots$ **(40)** from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.5.18(p.39)) and Lemma 10.2.2(p.55) (f) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, thus $V_{t-1} < x_L$ for $t > 0$ due to (40). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a), for almost the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(c3ii1ii2) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K$ from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (6), hence $V_0 \leq x_L$. Assume that $V_{t-1} \leq x_L$ for *all* $t > 0$, hence $V \leq x_L$. Then, since $x_K \leq V$ from (a), we have the contradiction of $V \leq x_L < x_K \leq V$. Accordingly, it is impossible that $V_{t-1} \leq x_L$ for all $t > 0$. Therefore, from (6) and (39) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots,$$

---

†Note that we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ instead of ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ due to (c1).

‡Note the fine difference between $\texttt{S}_3$ and $\texttt{S}_1$(p.60).

hence for almost the same reason as in the proof of `Tom` 11.2.2(p.60) (c2iii2) we have S_3[‡] is true.

(c3ii2) Let $(\lambda\beta\mu - s)/\lambda > a \cdots$ (41). Then $x_K > (\lambda\beta\mu - s)/\delta > a \cdots$ (42) from Lemma 10.2.2(p.55) (j2). Let us note here that:

1. Let $\lambda < 1$. Then V_t is *strictly increasing* in $t \geq 0$ for the same reason as in the proof of (c3ii1ii).
2. Let $\lambda = 1$. Then $\beta\mu - s > a \cdots$ (43) from (41). Now, since $K(\rho) + \rho = \beta\mu - s$ from (10.2.7 (1) (p.55)) and (34), we have $V_1 = \beta\mu - s$ from (19), hence $V_1 > a$ from (43), so $V_{t-1} > a$ for $t > 1$ from (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from (6.5.18(p.39)) and Lemma 10.2.2(p.55) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly by induction $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t > 0$.

Consequently, whether $\lambda < 1$ or $\lambda = 1$, it follows that V_t is *strictly increasing* in $t > 0 \cdots$ (44).

(c3ii2i) Let $b \geq 0$ ⟪$\kappa \geq 0$⟫. Then $x_L \geq x_K \geq 0 \cdots$ (45) from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.5.18(p.39)) and from (42) and Lemma 10.2.2(p.55) (h) we have $V_t < \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K \geq 0$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ from (45), so $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a). Hence, for almost the same reason as in the proof of `Tom` 20.1.1(p.154) (c) we have ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and `Conduct`$_{\tau\geq t>0\blacktriangle}$.

(c3ii2ii) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ (46) from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (6). Assume that $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L \cdots$ (47). Now, since $x_K \leq V$ from (a), we have the contradiction of $V \leq x_L < x_K \leq V$. Accordingly, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. Therefore, from (44) and (6) we see that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < \cdots,$$

hence for almost the same reason as in the proof of `Tom` 11.2.2(p.60) (c2iii2) we have S_3 is true. ∎

□ `Tom` **20.1.3** (▣ $\mathscr{A}$ {M:2[ℝ][A]}) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\beta = 1$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$ ↦ → ⓓ$_{\|}$

(c) *Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.*

1. *Let $b > 0$ ⟪$\kappa > 0$⟫. Then* ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$ ↦ → ⓢ$_\blacktriangle$
2. *Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$ ↦ → ⓓ$_{\|}$

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Then $V_0 = x_K \cdots$ (1) from (6.5.17(p.39)), hence $K(V_0) = K(x_K) = 0 \cdots$ (2).

(a) We obtain $V_1 \geq K(V_0) + V_0 = V_0 \cdots$ (3) from (6.5.18(p.39)) with $t = 1$ and (2). Suppose $V_{t-1} \leq V_t$. Then, from Lemma 10.2.2(p.55) (e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(b) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K$ from Lemma 10.2.3(p.56) (b). Note (1). Suppose $V_{t-1} = x_K$. Then, from (6.5.18(p.39)) we have $V_t = \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 0$, hence $V_{t-1} = x_L$ for $t > 0$, so $L(V_{t-1}) = L(x_L) = 0$ for $t > 0$. Accordingly, for the same reason as in the proof of `Tom` 20.1.1(p.154) (b) we obtain ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(c) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.

(c1) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K > 0 \cdots$ (4) from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (1). Suppose $V_{t-1} = x_K$. Then, from (6.5.18(p.39)) we have $V_t = \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ due to (4), so $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 10.2.1(p.55) (a). Therefore, for the same reason as in the proof of `Tom` 20.1.1(p.154) (c) we have ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ and `Conduct`$_{\tau\geq t>0\blacktriangle}$.

(c2) Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then, since $x_L \leq x_K$ from Lemma 10.2.3(p.56) (c ⟪d⟫), we have $x_L \leq V_0$ from (1), hence $x_L \leq V_{t-1}$ for $t > 0$ from (a), so $L(V_{t-1}) \leq 0$ for $t > 0$ due to Corollary 10.2.1(p.55) (a). Then, since $V_t - \beta V_{t-1} = 0$ for $t > 0$ from(20.1.22(p.154)), for the same reason as in the proof of `Tom` 20.1.1(p.154) (b) we obtain ⓓ$_{\tau>0}\langle 0\rangle_{\|}$. ∎

S_4 [ⓢ▲ | ●‖ | C⇝SΔ | C⇝S▲] = { There exist $t^\bullet_\tau$ and t°_τ ($t^\bullet_\tau > t^\circ_\tau \geq 0$) such that
(1) ⓓ$_{t^\bullet_\tau\geq\tau>0}\langle 0\rangle_{\|}$,
(2) ⓢ$_{\tau>t^\bullet_\tau}\langle\tau\rangle_\blacktriangle$ where `Conduct`$_{\tau\geq t>t^\bullet_\tau\blacktriangle}$ $\cdots$ (1*) and
where C⇝S$_{t^\bullet_\tau\geq t>t^\circ_\tau\triangle}$ $\cdots$ (2*) and
C⇝S$_{t^\circ_\tau\geq t>0\triangle}$ ⟪C⇝S$_{t^\circ_\tau\geq t>0\blacktriangle}$⟫ $\cdots$ (3*).[†] }

□ `Tom` **20.1.4** (▣ $\mathscr{A}$ {M:2[ℝ][A]}) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*[‡]

1. *$V_t = \rho$ for $t \geq 0$.*

[‡]Note the fine difference between S_3 and S_1(p.60).

[†]See Def. 2.2.1(p.12) for the definition of the symbol C⇝S.

2. *Let* $x_L \le \rho$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$ $\mapsto$ $\rightarrow$ ⓓ$_{\|}$
3. *Let* $x_L > \rho$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\mapsto$ $\rightarrow$ Ⓢ$_{\blacktriangle}$

(b) *Let* $\beta<1$ *and* $\rho>0$ *and let* $s=0$ $(\!(s>0)\!)$.

1. V_t *is nonincreasing in* $t\ge 0$ *and converges to a finite* $V\ge x_K$ *as* $t\rightarrow\infty$.
2. *Let* $b\le 0$ $(\!(\kappa\le 0)\!)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$ $\mapsto$ $\rightarrow$ ⓓ$_{\|}$
3. *Let* $b>0$ $(\!(\kappa>0)\!)$.
   i. *Let* $\rho< x_L$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\mapsto$ $\rightarrow$ Ⓢ$_{\blacktriangle}$
   ii. *Let* $\rho= x_L$. *Then* ⓓ$_1\langle 0\rangle_{\|}$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\mapsto$ $\rightarrow$ ⓓ$_{\|}$/Ⓢ$_{\blacktriangle}$
   iii. *Let* $x_L<\rho$. *Then* $\texttt{S}_4$ [Ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true* $\mapsto$ $\rightarrow$ Ⓢ$_{\blacktriangle}$/ⓓ$_{\|}$/(c⇝s)$_{\triangle}$/(c⇝s)$_{\blacktriangle}$

(c) *Let* $\beta<1$ *and* $\rho<0$ *and let* $s=0$ $(\!(s>0)\!)$.

1. V_t *is nondecreasing in* $t\ge 0$.
2. *Let* $b\le 0$ $(\!(\kappa\le 0)\!)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$ $\mapsto$ $\rightarrow$ ⓓ$_{\|}$
3. *Let* $b>0$ $(\!(\kappa>0)\!)$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ $\mapsto$ $\rightarrow$ Ⓢ$_{\blacktriangle}$

● *Proof* Let $\beta<1$ or $s>0$ and let $\rho> x_K\cdots$**(1)**. Hence $V_0> x_K\cdots$**(2)** from (6.5.17(p.39)) and $K(\rho)<0\cdots$**(3)** due to Lemma 10.2.2(p.55) (j1). Note that $V_0\ge x_K$. Suppose $V_{t-1}\ge x_K$. Then, from (6.5.18(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t\ge K(V_{t-1})+V_{t-1}\ge K(x_K)+x_K = x_K$. Hence, by induction $V_{t-1}\ge x_K\cdots$**(4)** for $t>0$. From (6.5.18(p.39)) with $t=1$ we have

$$V_1-V_0=V_1-\rho=\max\{K(V_0)+V_0,\beta V_0\}-\rho=\max\{K(\rho)+\rho,\beta\rho\}-\rho=\max\{K(\rho),-(1-\beta)\rho\}\cdots\textbf{(5)}.$$

(a) Let $\beta=1$ or $\rho=0$.

(a1) Then, since $-(1-\beta)\rho=0$, due to (3) we have $V_1-V_0=0$ from (5), i.e., $V_0=V_1$. Suppose $V_{t-1}=V_t$. Then, from (6.5.18(p.39)) we have $V_t=\max\{K(V_t)+V_t,\beta V_t\}=V_{t+1}$. Thus, by induction $V_{t-1}=V_t$ for $t>0$, i.e., $V_0=V_1=V_2=\cdots$, hence $V_t=V_0=\rho$ for $t\ge 0$.

(a2) Let $x_L\le\rho$. Then, since $x_L\le V_{t-1}$ for $t>0$ from (a1), we have $L(V_{t-1})\le 0$ for $t>0$ due to Corollary 10.2.1(p.55) (a), hence $V_t-\beta V_{t-1}=0$ for $t>0$ from (20.1.22(p.154)). Accordingly, for the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (b) we obtain ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(a3) Let $x_L>\rho$. Then, since $x_L>V_{t-1}$ for $t>0$ from (a1), we have $L(V_{t-1})>0$ for $t>0$ due to Corollary 10.2.1(p.55) (a), hence for the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we obtain Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ and $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

(b) Let $\beta<1\cdots$**(6)** and $\rho>0\cdots$**(7)** and let $s=0$ $(\!(s>0)\!)$. Then, since $-(1-\beta)\rho<0\cdots$**(8)**, from (5) and (3) we have $V_1-V_0<0$, so $V_1>V_0$, hence $\rho=V_0>V_1\cdots$**(9)** from (6.5.17(p.39)).

(b1) We have $V_0\ge V_1$ from (9). Suppose $V_{t-1}\ge V_t$. Then, from (6.5.18(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t\ge\max\{K(V_t)+V_t,\beta V_t\}=V_{t+1}$. Hence, by induction $V_{t-1}\ge V_t$ for $t>0$, i.e., V_t is nonincreasing in $t\ge 0$. In addition, since V_t is lower bounded in t due to (4), it follows that V_t converges to a finite V as $t\rightarrow\infty$. Accordingly, from (4) we have $V\ge x_K$.

(b2) Let $b\le 0$ $(\!(\kappa\le 0)\!)$. Then, since $x_L\le x_K$ due to Lemma 10.2.3(p.56) (c (d)), from (4) we have $V_{t-1}\ge x_L$ for $t>0$. Accordingly, since $L(V_{t-1})\le 0$ for $t>0$ from Corollary 10.2.1(p.55) (a), we have $V_t-\beta V_{t-1}=0$ for $t>0$ from (20.1.22(p.154)), hence for the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (b) we obtain ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(b3) Let $b>0$ $(\!(\kappa>0)\!)$. Then $x_L> x_K>0\cdots$**(10)** from Lemma 10.2.3(p.56) (c (d)).

(b3i) Let $\rho< x_L$. Then, since $V_0< x_L$ from (6.5.17(p.39)), we have $V_{t-1}< x_L$ for $t>0$ due to (b1). Therefore, since $L(V_{t-1})>0$ for $t>0$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we have Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ and $\texttt{CONDUCT}_{\tau\ge t>0\blacktriangle}$.

(b3ii) Let $\rho= x_L\cdots$**(11)**. Then, since $V_0= x_L$ from (6.5.17(p.39)), we have $L(V_0)=L(x_L)=0\cdots$**(12)**, hence from (20.1.24(p.154)) with $t=1$ we have $V_1=\beta V_0\cdots$**(13)**, so $t_1^*=0$, i.e., ⓓ$_1\langle 0\rangle_{\|}$. Below let $\tau>1$. From (9) and (11) we have $V_1< V_0= x_L$. Accordingly, since $V_{t-1}< x_L$ for $t>1$ from (b1), we have $L(V_{t-1})>0\cdots$**(14)** for $t>1$ from Corollary 10.2.1(p.55) (a), hence $L(V_{t-1})>0\cdots$**(15)** for $\tau\ge t>1$. Therefore, $V_t-\beta V_{t-1}>0$ for $\tau\ge t>1$ from (20.1.22(p.154)), hence $V_t>\beta V_{t-1}$ for $\tau\ge t>1$, so that $V_\tau>\beta V_{\tau-1}>\cdots>\beta^{\tau-1}V_1$. From this and (13) we obtain $V_\tau>\beta V_{\tau-1}>\cdots>\beta^{\tau-1}V_1=\beta^\tau V_0$ for $\tau>1$, hence $t_\tau^*=\tau$ for $\tau>1$, i.e., Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau\ge t>1$ due to (15) and (20.1.26(p.154)).

(b3iii) Let $x_L<\rho$, hence $x_L< V_0\cdots$**(16)** from (6.5.17(p.39)), so $x_L\le V_0$. Suppose $x_L\le V_{t-1}\cdots$**(17)** for *all* $t>0$. Then, since $L(V_{t-1})\le 0$ for $t>0$ from Corollary 10.2.1(p.55) (a), we have $V_t=\beta V_{t-1}$ for $t>0$ from (20.1.24(p.154)), hence $V_t=\beta^t V_0=\beta^t\rho>0$ for $t\ge 0$ due to (7). Then, since $\lim_{t\rightarrow\infty}V_t=0$ due to (6), from (10) we have $x_L> x_K>V_t>0$ for a sufficiently large t, which contradicts (17). Hence, it is impossible that $x_L\le V_{t-1}$ for *all* $t>0$. Accordingly, from (16) and (b1) we see that there exist t_τ° and $t_\tau^\bullet$ $(t_\tau^\circ<t_\tau^\bullet)$ such that

$$V_0\ge V_1\ge\cdots\ge V_{t_\tau^\circ-1}>V_{t_\tau^\circ}=V_{t_\tau^\circ+1}=\cdots=V_{t_\tau^\bullet-1}= x_L>V_{t_\tau^\bullet}\ge V_{t_\tau^\bullet+1}\ge\cdots\cdots\textbf{(18)}$$

Hence, we have

[†]The inverse of the condition "$\beta=1$ or $\rho=0$" is "$\beta<1$ and $\rho\ne 0$", which is classified into the two cases of "$\beta<1$ and $\rho>0$" and "$\beta<1$ and $\rho<0$", leading to the conditions in (b) and (c) that follows.

$$x_L > V_{t^\bullet_\tau},\ \ x_L > V_{t^\bullet_\tau+1},\ \cdots,$$
$$V_{t^\circ_\tau} = x_L,\ \ V_{t^\circ_\tau+1} = x_L,\ \cdots,\ V_{t^\bullet_\tau-1} = x_L,$$
$$V_0 > x_L,\ V_1 > x_L,\ \cdots,\ V_{t^\circ_\tau-1} > x_L,$$

or equivalently

$$x_L > V_{t-1} \cdots (19), \quad t > t^\bullet_\tau,$$
$$V_{t-1} = x_L \cdots (20), \quad t^\bullet_\tau \geq t > t^\circ_\tau,$$
$$V_{t-1} > x_L \cdots (21), \quad t^\circ_\tau \geq t > 0.$$

Accordingly, we have:

1. Let $t^\bullet_\tau \geq \tau > 0$. Then, since $V_{t-1} \geq x_L$ for $\tau \geq t > 0$ from (20) and (21), we have $L(V_{t-1}) \leq 0 \cdots (22)$ for $\tau \geq t > 0$ from Corollary 10.2.1(p.55) (a), hence $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 0$ from (20.1.22(p.154)), i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 0$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^\tau V_0 \cdots (23)$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 0$, i.e., ⓓ$_{t^\bullet_\tau \geq \tau > 0}\langle 0\rangle_{\|}$. Accordingly, $S_4(1)$ is true. Then, from (23) with $\tau = t^\bullet_\tau$ we have $V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau-1} = \cdots = \beta^{t^\bullet_\tau} V_0 \cdots (24)$,
2. Let $\tau > t^\bullet_\tau$. Then, since $x_L > V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (19), we have $L(V_{t-1}) > 0 \cdots (25)$ for $\tau \geq t > t^\bullet_\tau$ from Corollary 10.2.1(p.55) (a), hence $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > t^\bullet_\tau$ from (20.1.22(p.154)), i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (26)$. From this and (24) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} = \cdots = \beta^\tau V_0,$$

hence $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_\blacktriangle$, so the former half of $S_4(2)$ is true.

(i) We have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > t^\bullet_\tau \cdots (27)$ form (25) and (20.1.26(p.154)). Hence the latter half (1*) of $S_4(2)$ is true.

Below let us show the latter half (2*) and (3*) of $S_4(2)$.

(ii) If $t^\bullet_\tau \geq t > t^\circ_\tau$, then $L(V_{t-1}) = L(x_L) = 0$ from (20), hence we have $\texttt{Skip}_{t\vartriangle}$ from (20.1.25(p.154)), implying $\texttt{C}{\rightsquigarrow}\texttt{S}_{t\vartriangle}$ (see Figure 7.2.1(p.42) (II)) or equivalently $\texttt{C}{\rightsquigarrow}\texttt{S}_{t^\bullet_\tau \geq t > t^\circ_\tau\vartriangle}$. Hence the latter half (2*) of $S_4(2)$ is true.

(iii) If $t^\circ_\tau \geq t > 0$, then $L(V_{t-1}) =$ ⟪<⟫ 0 [‡] from (21) and Lemma 10.2.1(p.55) (d ⟪e1⟫), hence we have $\texttt{Skip}_{t\vartriangle}$ ⟪$\texttt{Skip}_{t\blacktriangle}$⟫ from (20.1.25(p.154)) ⟪(20.1.26(p.154))⟫, implying $\texttt{C}{\rightsquigarrow}\texttt{S}_{t\vartriangle}$ ⟪$\texttt{C}{\rightsquigarrow}\texttt{S}_{t\blacktriangle}$⟫ or equivalently $\texttt{C}{\rightsquigarrow}\texttt{S}_{t^\circ_\tau \geq t > 0\vartriangle}$ ⟪$\texttt{C}{\rightsquigarrow}\texttt{S}_{t^\circ_\tau \geq t > 0\blacktriangle}$⟫. Hence the latter half (3*) of $S_4(2)$ is true..

(c) Let $\beta < 1$ and $\rho < 0 \cdots (28)$ and let $s = 0$ ⟪$s > 0$⟫.

(c1) Since $-(1-\beta)\rho > 0$, from (5) we have $V_1 - V_0 > 0$, i.e., $V_0 < V_1$, hence $V_0 \leq V_1$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.18(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \leq \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 0$, i.e., V_t is nondecreasing in $t \geq 0$.

(c2) Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then $x_L \leq x_K$ due to Lemma 10.2.3(p.56) (c ⟪d⟫), hence from (4) we have $V_{t-1} \geq x_L$ for $t > 0$. Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > 0$ from (20.1.22(p.154)), hence for the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (b) we obtain ⓓ$_{\tau > 0}\langle 0\rangle_{\|}$.

(c3) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K > 0 \cdots (29)$ from Lemma 10.2.3(p.56) (c ⟪d⟫). Then, since $\rho < 0 < x_K$ from (28) and (29), we have $V_0 < x_K$ from (6.5.17(p.39)), hence $V_0 \leq x_K$. Suppose $V_{t-1} \leq x_K$, hence $V_{t-1} < x_L$ form (29), thus $L(V_{t-1}) > 0$ from Corollary 10.2.1(p.55) (a). Accordingly, from (20.1.23(p.154)) and Lemma 10.2.2(p.55) (e) we have $V_t = K(V_{t-1}) + V_{t-1} \leq K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \leq x_K$ for $t > 0$, so $V_{t-1} < x_L$ for $t > 0$ from (29). Therefore, since $L(V_{t-1}) > 0 \cdots (30)$ for $t > 0$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of $\texttt{Tom}$ 20.1.1(p.154) (c) we have ⓢ$_{\tau > 0}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau \geq t > 0\blacktriangle}$. ∎

20.1.3.3 Market Restriction

20.1.3.3.1 Positive Restriction

□ **Pom 20.1.1** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \geq b$. Then* ⓓ$_{\tau > 0}\langle 0\rangle_{\|}$.

(c) *Let $\rho < b$. Then* ⓢ$_{\tau > 0}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau \geq t > 0\blacktriangle}$.

• *Proof* The same as $\texttt{Tom}$ 20.1.1(p.154) due to Lemma 17.4.4(p.116). ∎

□ **Pom 20.1.2** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$) *Suppose $a > 0$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a \leq \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* ⓓ$_{\tau > 0}\langle 0\rangle_{\|}$.

(c) *Let $\rho < x_L$.*

---

‡ If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

1. $\circledS_1\langle 1\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
2. *Let* $\beta = 1$, *hence* $s > 0$.
    - i. *Let* $a \le \rho$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
    - ii. *Let* $\rho < a$.
        1. *Let* $(\lambda\mu - s)/\lambda \le a$.
            - i. *Let* $\lambda = 1$. *Then* $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
            - ii. *Let* $\lambda < 1$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
        2. *Let* $(\lambda\mu - s)/\lambda > a$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
4. *Let* $\beta < 1$ *and* $s > 0$.
    - i. *Let* $a < \rho$.
        1. *Let* $\lambda\beta\mu \ge s$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
        2. *Let* $\lambda\beta\mu < s$. *Then* $\texttt{S}_3$(p.154) $\boxed{\circledS^{\blacktriangle} \mid \circledcirc\ \|}$ *is true.*
    - ii. *Let* $\rho \le a$.
        1. *Let* $(\lambda\beta\mu - s)/\delta \le a$.
            - i. *Let* $\lambda = 1$.
                1. *Let* $\beta\mu > s$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
                2. *Let* $\beta\mu \le s$. *Then* $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
            - ii. *Let* $\lambda < 1$.
                1. *Let* $\lambda\beta\mu \ge s$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
                2. *Let* $\lambda\beta\mu < s$. *Then* $\texttt{S}_3$(p.154) $\boxed{\circledS^{\blacktriangle} \mid \circledcirc\ \|}$ *is true.*
        2. *Let* $(\lambda\beta\mu - s)/\delta > a$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

● *Proof* Suppose $a > 0 \cdots (1)$, hence $b > a > 0 \cdots (2)$. Then, we have $\kappa = \lambda\beta\mu - s \cdots (3)$ from Lemma 10.3.1(p.57) (a).

(a-c2ii2) The same as (a-c2ii2) of Tom 20.1.2(p.154).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (2) it suffices to consider only (c3i1,c3ii1i1, c3ii1ii1,c3ii2i) of Tom 20.1.2(p.154).

(c4) Let $\beta < 1$ and $s > 0$.

(c4i-c4ii1ii2) Immediate from (3) and (c3i-c3ii1ii2) of Tom 20.1.2(p.154) with κ.

(c4ii2) Let $(\lambda\beta\mu - s)/\delta > a$. Then, since$(\lambda\beta\mu - s)/\delta > a > 0$ due to (1), we have $\lambda\beta\mu - s > 0$, so that $\kappa > 0$ due to (3). Hence, it suffices to consider only (c3ii2i) of Tom 20.1.2(p.154). ∎

□ **Pom 20.1.3** ($\mathscr{A}\{\text{M:2}[\mathbb{R}][\text{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) V_t *is nondecreasing in* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

(d) *Let* $\beta < 1$ *and* $s > 0$.

1. *Let* $\lambda\beta\mu > s$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.
2. *Let* $\lambda\beta\mu \le s$. *Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then, we have $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.57) (a).

(a,b) The same as (a,b) of Tom 20.1.3(p.157).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 20.1.3(p.157).

(d) Let $\beta < 1$ and $s > 0$.

(d1,d2) Immediate from (2) and (c1,c2) of Tom 20.1.3(p.157) with κ. ∎

□ **Pom 20.1.4** ($\mathscr{A}\{\text{M:2}[\mathbb{R}][\text{A}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.

1. $V_t = \rho$ *for* $t \ge 0$.
2. *Let* $x_L \le \rho$. *Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let* $x_L > \rho$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$.

1. V_t *is nonincreasing in* $t \ge 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
2. *Let* $\rho < x_L$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.

3. *Let* $\rho = x_L$. *Then* $ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
4. *Let* $x_L < \rho$. *Then* $\texttt{S}_4$ | ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲ | *is true.*

(c) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s > 0$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
2. *Let* $\lambda\beta\mu \leq s$. *Then* $❶_{\tau>0}\langle 0\rangle_{\parallel}$.
3. *Let* $\lambda\beta\mu > s$.
   i. *Let* $\rho < x_L$. *Then* $ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $\rho = x_L$. *Then* $ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle\ \vartriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let* $x_L < \rho$. *Then* $\texttt{S}_4$ | ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲ | *is true* (see *Numerical Example* 5(p.184)).

(d) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$.

1. V_t *is nondecreasing in* $t \geq 0$.
2. $ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(e) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s > 0$.

1. V_t *is nondecreasing in* $t \geq 0$.
2. *Let* $\lambda\beta\mu \leq s$. *Then* $❶_{\tau>0}\langle 0\rangle_{\parallel}$.
3. *Let* $\lambda\beta\mu > s$. *Then* $ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* Suppose $a > 0$, hence $b > \mu > a > 0 \cdots$ **(1)**. Then $\kappa = \lambda\beta\mu - s \cdots$ **(2)** from Lemma 10.3.1(p.57) (a).

(a-a3) The same as (a-a3) of `Tom` 20.1.4(p.157).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. First, (b1) is the same as (b1) of `Tom` 20.1.4(p.157). Next, due to (1) it suffices to consider only (b3i-b3iii) of `Tom` 20.1.4(p.157).

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. First, (c1) is the same as (b1) of `Tom` 20.1.4(p.157). Next, due to (1) and (2) it suffices to consider only (b3i-b3iii) of `Tom` 20.1.4(p.157).

(d-d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. First, (d1) is the same as (c1) of `Tom` 20.1.4(p.157). Next, since $\kappa = \lambda\beta\mu > 0$ due to (2) and (1), it suffices to consider only (c3) of `Tom` 20.1.4(p.157).

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. First, (e1) is the same as (c1) of `Tom` 20.1.4(p.157). Next, (e2,e3) are the same as (c2,c3) of `Tom` 20.1.4(p.157) due to (2). ∎

20.1.3.3.2 Mixed Restriction

Omitted.

20.1.3.3.3 Negative Restriction

Omitted.

20.1.4 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$

20.1.4.1 Preliminary

Due to Lemma 20.1.1(p.151) (a), we see that the following `Tom`'s 20.1.5(p.161) − 20.1.8(p.162) can be obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128))) to `Tom`'s 20.1.1(p.154) − 20.1.4(p.157) (see Theorem 20.1.1(p.151)).

20.1.4.2 Analysis

20.1.4.2.1 Case of $\beta = 1$ and $s = 0$

□ `Tom` **20.1.5** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t > 0$.

(b) *Let* $\rho \leq a$. *Then* $❶_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let* $\rho > a$. *Then* $ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 20.1.1(p.154). ∎

20.1.4.2.2 Case of $\beta < 1$ or $s > 0$

☐ `Tom` **20.1.6** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t \ge 0$ if $\lambda < 1$ or $b > \rho$, and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \ge \rho$. Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $\circledS_1\langle 1\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   - i. *Let $b > \rho$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
   - ii. *Let $\rho \ge b$.*
     1. *Let $(\lambda\mu + s)/\lambda \ge b$.*
        - i. *Let $\lambda = 1$. Then* $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
     2. *Let $(\lambda\mu + s)/\lambda < b$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.*
   - i. *Let $b > \rho$.*
     1. *Let $a \le 0$ $(\!(\tilde\kappa \le 0)\!)$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
     2. *Let $a > 0$ $(\!(\tilde\kappa > 0)\!)$. Then* $\mathtt{S}_3$(p.154) $\boxed{\circledS\blacktriangle \mid \circledcirc\,\|}$ *is true.*
   - ii. *Let $\rho \ge b$.*
     1. *Let $(\lambda\beta\mu + s)/\delta \ge b$.*
        - i. *Let $\lambda = 1$.*
          1. *Let $a < 0$ $(\!(\tilde\kappa < 0)\!)$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
          2. *Let $a \ge 0$ $(\!(\tilde\kappa \ge 0)\!)$. Then* $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
        - ii. *Let $\lambda < 1$.*
          1. *Let $a \le 0$ $(\!(\tilde\kappa \le 0)\!)$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
          2. *Let $a > 0$ $(\!(\tilde\kappa > 0)\!)$. Then* $\mathtt{S}_3$(p.154) $\boxed{\circledS\blacktriangle \mid \circledcirc\,\|}$ *is true.*
     2. *Let $(\lambda\beta\mu + s)/\delta < b$.*
        - i. *Let $a \le 0$ $(\!(\tilde\kappa \le 0)\!)$. Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
        - ii. *Let $a > 0$ $(\!(\tilde\kappa > 0)\!)$. Then* $\mathtt{S}_3$(p.154) $\boxed{\circledS\blacktriangle \mid \circledcirc\,\|}$ *is true.* ▯

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 20.1.2(p.154). ∎

☐ `Tom` **20.1.7** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$.*

(b) *Let $\beta = 1$. Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.*

1. *Let $a < 0$ $(\!(\tilde\kappa < 0)\!)$. Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
2. *Let $a \ge 0$ $(\!(\tilde\kappa \ge 0)\!)$. Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\|}$. ▯

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 20.1.3(p.157). ∎

☐ `Tom` **20.1.8** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \ge 0$.*
2. *Let $x_{\tilde{L}} \ge \rho$. Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ $(\!(s > 0)\!)$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \ge 0$ $(\!(\tilde\kappa \ge 0)\!)$. Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $a < 0$ $(\!(\tilde\kappa < 0)\!)$.*
   - i. *Let $\rho > x_{\tilde{L}}$. Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
   - ii. *Let $\rho = x_{\tilde{L}}$. Then* $\text{ⓓ}_1\langle 0\rangle_{\|}$ *and* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$.
   - iii. *Let $x_{\tilde{L}} > \rho$. Then* $\mathtt{S}_4$ $\boxed{\circledS\blacktriangle \mid \bullet\,\| \mid \mathrm{C}\rightsquigarrow\mathrm{S}\,\triangle \mid \mathrm{C}\rightsquigarrow\mathrm{S}\,\blacktriangle}$ *is true.*

(c) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ $(\!(s > 0)\!)$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \ge 0$ $(\!(\tilde\kappa \ge 0)\!)$. Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $a < 0$ $(\!(\tilde\kappa < 0)\!)$. Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\ge t>0\blacktriangle}$. ▯

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see to `Tom` 20.1.4(p.157). ∎

20.1.4.3 Market Restriction

20.1.4.3.1 Positive Restriction

□ Pom **20.1.5** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{R}][\mathtt{A}]^{+}\}$) *Suppose $a>0$. Let $\beta=1$ and $s=0$.*

(a) *V_t is nonincreasing in $t\geq 0$.*

(b) *Let $\rho\leq a$. Then* ❶$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\rho>a$. Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* The same as Tom 20.1.5(p.161) due to Lemma 17.4.4(p.116). ∎

□ Pom **20.1.6** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{R}][\mathtt{A}]^{+}\}$) *Suppose $a>0$. Let $\beta<1$ or $s>0$ and let $\rho>x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t\geq 0$, is strictly decreasing in $t\geq 0$ if $\lambda<1$ or $b\geq\rho$, and converges to a finite $V\leq x_{\tilde{K}}$ as $t\to\infty$.*

(b) *Let $x_{\tilde{L}}\geq\rho$. Then* ❶$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\rho>x_{\tilde{L}}$.*

1. ⓢ$_1\langle 1\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let $\tau>1$.*
2. *Let $\beta=1$.*
   i. *Let $b>\rho$. Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $\rho\geq b$.*
      1. *Let $(\lambda\mu+s)/\lambda\geq b$.*
         i. *Let $\lambda=1$. Then* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
         ii. *Let $\lambda<1$. Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
      2. *Let $(\lambda\mu+s)/\lambda<b$. Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta<1$ and $s=0$. Then we have* $\mathtt{S}_3$(p.154) [ⓢ▲ | ◎ ‖].
4. *Let $\beta<1$ and $s>0$.*
   i. *Let $b>\rho$. Then* $\mathtt{S}_3$(p.154) [ⓢ▲ | ◎ ‖] *is true.*
   ii. *Let $\rho\geq b$.*
      1. *Let $(\lambda\beta\mu+s)/\delta\geq b$.*
         i. *Let $\lambda=1$. Then* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
         ii. *Let $\lambda<1$. Then* $\mathtt{S}_3$(p.154) [ⓢ▲ | ◎ ‖] *is true.*
      2. *Let $(\lambda\beta\mu+s)/\delta<b$. Then* $\mathtt{S}_3$(p.154) [ⓢ▲ | ◎ ‖] *is true.* □

● *Proof* Suppose $a>0\cdots$**(1)**, hence $b>a>0\cdots$**(2)** and $\tilde{\kappa}=s\cdots$**(3)** from Lemma 12.6.6(p.81) (a).

(a-c2ii2) The same as (a-c2ii2) of Tom 20.1.6(p.162).

(c3) Let $\beta<1$ and $s=0$. Assume $(\lambda\beta\mu+s)/\delta\geq b$. Then, since $\lambda\beta\mu/\delta\geq b$, we have $\lambda\beta\mu\geq\delta b$, hence $\lambda\beta\mu\geq\delta b\geq\lambda b$ due to (2) and (10.2.2 (1) (p.54)), so $\beta\mu\geq b$, which contradicts [3(p.116)]. Thus, it must be that $(\lambda\beta\mu+s)/\delta<b$. From this and (1) it suffices to consider only (c3ii2ii) of Tom 20.1.6(p.162).

(c4-c4ii2) If $\beta<1$ and $s>0$, then $\kappa>0$ due to (3), hence it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) of Tom 20.1.6(p.162). ∎

□ Pom **20.1.7** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{R}][\mathtt{A}]^{+}\}$) *Suppose $a>0$. Let $\beta<1$ or $s>0$ and let $\rho=x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t\geq 0$.*

(b) *We have* ❶$_{\tau>0}\langle 0\rangle_{\parallel}$. □

● *Proof* Suppose $a>0\cdots$**(1)**. Then $\tilde{\kappa}=s\cdots$**(2)** from Lemma 12.6.6(p.81) (a).

(a) The same as (a) of Tom 20.1.7(p.162).

(b) Let $\beta=1$. Then it suffices to consider only (b) of Tom 20.1.7(p.162). Let $\beta<1$. If $s=0$, due to (1) it suffices to consider only (c2) of Tom 20.1.7(p.162) and if $s>0$, then $\tilde{\kappa}>0$ due to (2), hence it suffices to consider only (c2) of Tom 20.1.7(p.162), thus, whether $s=0$ or $s>0$ we have the same result. Accordingly, whether $\beta=1$ or $\beta<1$, it follows that we have the same result. ∎

□ Pom **20.1.8** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}2[\mathbb{R}][\mathtt{A}]^{+}\}$) *Suppose $a>0$. Let $\beta<1$ or $s>0$ and let $\rho<x_{\tilde{K}}$.*

(a) *Let $\beta=1$ or $\rho=0$.*

1. *$V_t=\rho$ for $t\geq 0$.*
2. *Let $x_{\tilde{L}}\geq\rho$. Then* ❶$_{\tau>0}\langle 0\rangle_{\parallel}$.
3. *Let $x_{\tilde{L}}<\rho$. Then* ⓢ$_{\tau}\langle\tau\rangle_{\blacktriangle}$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta<1$ and $\rho<0$.*

1. *V_t is nondecreasing in $t\geq 0$ and converges to a finite $V\leq x_{\tilde{K}}$ as $t\to\infty$.*
2. ❶$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\beta<1$ and $\rho>0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{\kappa}}$ as $t \to \infty$.*
2. ⓓ$_{\tau>0}\langle 0 \rangle_{\|}$.

● *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 12.6.6(p.81) (a).

(a-a3) The same as (a-a3) of Tom 20.1.8(p.162).

(b) Let $\beta < 1$ and $\rho < 0$.

(b1) The same as (b1) of Tom 20.1.8(p.162).

(b2) If $s = 0$, then due to (1) it suffices to consider only (b2) of Tom 20.1.8(p.162) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b2) of Tom 20.1.8(p.162). Accordingly, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $\beta < 1$ and $\rho > 0$.

(c1) The same as Tom 20.1.8(p.162) (c1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 20.1.8(p.162) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of Tom 20.1.8(p.162). Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.1.4.3.2 Mixed Restriction

Omitted.

20.1.4.3.3 Negative Restriction

Unnecessary.

20.1.5 Proof and Derivation of $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$

20.1.5.1 Preliminary

From (6.5.23(p.39)) and from (5.1.21(p.24)) and (5.1.20(p.24)) we have

$$\begin{aligned} V_t &= \max\{K(V_{t-1}) + (1-\beta)V_{t-1}, 0\} + \beta V_{t-1} \\ &= \max\{L(V_{t-1}), 0\} + \beta V_{t-1}, \quad t > 1, \end{aligned} \tag{20.1.27}$$

hence

$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\}, \quad t > 1. \tag{20.1.28}$$

Then, for $t > 1$ we have

$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1} \quad \text{if } L(V_{t-1}) \geq 0 \tag{20.1.29}$$

$$V_t = \beta V_{t-1} \quad \text{if } L(V_{t-1}) \leq 0. \tag{20.1.30}$$

Now, from (6.2.107(p.33)) and from (6.2.103(p.33)) and (6.2.105(p.33)) we have, for $t > 1$,

$$\mathbb{S}_t = L(V_{t-1}) \geq (\leq)\ 0 \Rightarrow \mathtt{Conduct}_{t\vartriangle}(\mathtt{Skip}_{t\vartriangle}), \tag{20.1.31}$$

$$\mathbb{S}_t = L(V_{t-1}) > (<)\ 0 \Rightarrow \mathtt{Conduct}_{t\blacktriangle}(\mathtt{Skip}_{t\blacktriangle}). \tag{20.1.32}$$

From (6.5.22(p.39)) we have

$$V_1 = \max\{\lambda\beta \max\{0, a-\rho\} - s, 0\} + \beta\rho, \tag{20.1.33}$$

hence

$$V_1 - \beta V_0 = V_1 - \beta\rho = \max\{\lambda\beta \max\{0, a-\rho\} - s, 0\} \geq 0. \tag{20.1.34}$$

From the comparison of the two terms within { } in the r.h.s. of (20.1.33(p.164)) it can be seen that

$$\mathbb{S}_1 = \lambda\beta \max\{0, a-\rho\} \geq (\leq)\ s \Rightarrow \mathtt{Conduct}_{1\vartriangle}(\mathtt{Skip}_{1\vartriangle}), \tag{20.1.35}$$

$$\mathbb{S}_1 = \lambda\beta \max\{0, a-\rho\} > (<)\ s \Rightarrow \mathtt{Conduct}_{1\blacktriangle}(\mathtt{Skip}_{1\blacktriangle}). \tag{20.1.36}$$

20.1.5.2 Analysis

20.1.5.2.1 Case of $\beta = 1$ and $s = 0$

20.1.5.2.1.1 Preliminary

Let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.24)), (5.1.20(p.24)), and Lemma 13.2.1(p.91) (g) we have

$$K(x) = L(x) = \lambda T(x) \geq 0 \quad \text{for any } x. \tag{20.1.37}$$

In addition, from (20.1.28(p.164)) we have

$$V_t - \beta V_{t-1} = \max\{\lambda T(V_{t-1}), 0\} = \lambda T(V_{t-1}) \geq 0, \quad t > 1. \tag{20.1.38}$$

Finally, from (20.1.33(p.164)) we have

$$V_1 = \max\{\lambda \max\{0, a-\rho\}, 0\} + \rho \tag{20.1.39}$$

$$= \lambda \max\{0, a-\rho\} + \rho \quad (\text{due to } \lambda \max\{0, a-\rho\} \geq 0) \tag{20.1.40}$$

$$= \max\{\rho, \lambda a + (1-\lambda)\rho\}. \tag{20.1.41}$$

20.1.5.2.1.2 Case of $\rho \leq a^\star$

In this case, due to Lemma 20.1.1(p.151) (c), we can obtain Tom 20.1.1(p.165) below by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.128))) to Tom 20.1.1(p.154) with the condition $\rho \leq a^\star$ (see Theorem 20.1.2(p.151)).

Proposition 20.1.1 ($\rho \leq a^\star$) Assume $\rho \leq a^\star$ and let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* Assume $\rho \leq a^\star$ and let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.1.1(p.154) (a).

(b) Due to the assumption $\rho \leq a^\star$ we have $\rho \leq a^\star < a < b$ from Lemma 13.2.1(p.91) (n). Hence it suffices to consider only (c) of Tom 20.1.1(p.154) . ∎

20.1.5.2.1.3 Case of $b \leq \rho$

In this case, due to Lemma 20.1.1(p.151) (c), we can obtain Tom 20.1.2(p.165) below by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.128))) to Tom 20.1.1(p.154) with the condition $b \leq \rho$ (see Theorem 20.1.2(p.151)).

Proposition 20.1.2 ($b \leq \rho$) Assume $b \leq \rho$ and let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$. □

● *Proof* Assume $b \leq \rho \cdots$ **(1)** and let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.1.1(p.154) (a).

(b) Due to (1) it suffices to consider only (b) of Tom 20.1.1(p.154) . ∎

20.1.5.2.1.4 Case of $a^\star < \rho < b$

In this case, Theorem 20.1.2(p.151) does not always hold due to Lemma 20.1.1(p.151) (d), hence $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}$ must be directly found.

Proposition 20.1.3 ($a^\star < \rho < b$) Assume $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \geq 0$.

(b) Let $a \leq \rho$. Then ⓓ$_1\langle 0\rangle_{\parallel}$ and Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$ and $\texttt{C}\rightsquigarrow\texttt{S}_{1\vartriangle}$.

(c) Let $\rho < a$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof* Assume $a^\star < \rho < b \cdots$ **(1)** and let $\beta = 1$ and $s = 0$. Then, from (5.1.20(p.24)) and (5.1.21(p.24)) we have $L(x) = K(x) = \lambda T(x) \geq 0 \cdots$ **(2)** for any x from Lemma 13.2.1(p.91) (g). Then, since $\rho < b$ and $a < b$, from (20.1.41(p.164)) we obtain $V_1 < \max\{b, \lambda b + (1-\lambda)b\} = \max\{b, b\} = b$. Suppose $V_{t-1} < b$. Then, since $a^\star < b$ due to (1), from (6.5.23(p.39)) with $\beta = 1$ we have $V_t < \max\{K(b)+b, b\}$ from Lemma 13.2.3(p.94) (h), hence $V_t < \max\{\beta b - s, b\}$ from (13.2.12 (2) (p.94)), so $V_{t-1} < \max\{b, b\} = b$ due to the assumption "$\beta = 1$ and $s = 0$". Accordingly, by induction we have $V_{t-1} < b \cdots$ **(3)** for $t > 1$, hence $T(V_{t-1}) > 0 \cdots$ **(4)** for $t > 1$ from Lemma 13.2.1(p.91) (g). Accordingly, $V_t - \beta V_{t-1} > 0$ for $t > 1$ from (20.1.38(p.164)), i.e., $V_t > \beta V_{t-1}$ for $t > 1$. Then, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 \cdots$ **(5)** for $\tau > 1$. In addition, since $L(V_{t-1}) = \lambda T(V_{t-1}) > 0 \cdots$ **(6)** for $\tau \geq t > 1$ due to (4), we have $\texttt{Conduct}_{\tau\geq t>1\blacktriangle} \cdots$ **(7)** from (20.1.32(p.164)).

(a) From (20.1.40(p.164)) and (6.5.21(p.39)) we have $V_1 - V_0 = V_1 - \rho = \lambda \max\{0, a-\rho\} \geq 0$, hence $V_1 \geq V_0 \cdots$ **(8)**. Since $V_2 \geq K(V_1) + V_1$ from (6.5.23(p.39)) with $t = 2$, we have $V_2 - V_1 \geq K(V_1) \geq 0$ due to (2), hence $V_2 \geq V_1 \cdots$ **(9)**. Suppose $V_t \geq V_{t-1}$. Then from (6.5.23(p.39)) and Lemma 13.2.3(p.94) (e) we have $V_{t+1} = \max\{K(V_t) + V_t, \beta V_t\} \geq \max\{K(V_{t-1}) + V_{t-1}, \beta V_{t-1}\} = V_t$. Hence, by induction $V_t \geq V_{t-1}$ for $t > 1$. From this and (8) we have $V_t \geq V_{t-1}$ for $t > 0$, hence it follows that V_t is nondecreasing in $t \geq 0$.

(b) Let $a \leq \rho \cdots$ **(10)**, hence $V_1 = \rho$ from (20.1.40(p.164)), so $V_1 < b$ due to (1). Then $V_1 - \beta V_0 = V_1 - V_0 = \rho - \rho = 0$ from (6.5.21(p.39)), hence $V_1 = \beta V_0 \cdots$ **(11)**, so $t_1^* = 0$, i.e., ⓓ$_1\langle 0\rangle_{\parallel}$. Below let $\tau > 1$. Then, from (5) and (11) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$. Here note $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ from (7). In addition, since $\lambda \max\{0, a-\rho\} = 0$ due to (10), we have $\lambda \max\{0, a-\rho\} = 0 \leq s$ for any $s \geq 0$, hence $\texttt{Skip}_{1\vartriangle}$ due to (20.1.35(p.164)). Accordingly, it follows that we have $\texttt{C}\rightsquigarrow\texttt{S}_{1\vartriangle}$ (see Remark 7.2.1(p.42)).

(c) Let $\rho < a \cdots$ **(12)**, hence $V_1 = \lambda(a-\rho) + \rho$ due to (20.1.40(p.164)). Then, from (6.5.21(p.39)) we have $V_1 - \beta V_0 = V_1 - V_0 = V_1 - \rho = \lambda(a-\rho) > 0$, i.e., $V_1 > \beta V_0 \cdots$ **(13)**, hence $t_1^* = 1$, i.e., Ⓢ$_1\langle 1\rangle_\blacktriangle \cdots$ **(14)**. Below let $\tau > 1$. Then, from (5) and (13) we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 > \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$. From the result and (14) we have Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$. Since $a - \rho > 0$ due to (12), we have $\lambda \max\{0, a-\rho\} > 0 = s$, implying that we have $\texttt{Conduct}_{1\blacktriangle}$ due to (20.1.36(p.164)). From this and (7) it follows that $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. ∎

20.1.5.2.1.5 Integration of Propositions 20.1.1(p.165) − 20.1.3(p.165)

☐ Tom **20.1.9** (▣$\mathscr{A}${M:2[ℙ][A]}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \leq a^\star$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$ ↦ → Ⓢ$_\blacktriangle$

(c) *Let $b \leq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_\|$ ↦ → ⓓ$_\|$

(d) *Let $a^\star < \rho < b$.*

1. *Let $a \leq \rho$. Then* ⓓ$_1\langle 0\rangle_\|$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>1\blacktriangle}$ *and* C⇝S$_{1\triangle}$ ↦ → ⓓ$_\|$/Ⓢ$_\blacktriangle$/[C⇝S]$_\triangle$
2. *Let $\rho < a$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$ ↦ → Ⓢ$_\blacktriangle$

• *Proof* (a) The same as Tom's 20.1.1(p.165) (a), 20.1.2(p.165) (a), and 20.1.3(p.165) (a).

(b) The same as Tom 20.1.1(p.165) (b).

(c) The same as Tom 20.1.2(p.165) (b).

(d-d2) The same as Tom 20.1.3(p.165) (b,c). ▮

Corollary 20.1.1 *Let $\beta = 1$ and $s = 0$. Then, the optimal price to propose z_t is nondecreasing in t.* ☐

• *Proof* Immediate from Lemma 20.1.9(p.166) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ▮

20.1.5.2.2 Case of $\beta < 1$ or $s > 0$

20.1.5.2.2.1 Case of $\rho \leq a^\star$

In this case, Theorem 20.1.2(p.151) holds due to Lemma 20.1.1(p.151) (c), hence Tom's 20.1.10(p.166) –20.1.12(p.167) below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.128))) to Tom's 20.1.2(p.154) –20.1.4(p.157). In the proofs below, let us represent what results from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to a given Tom by Tom′, i.e.,

$$\text{Tom}' = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\text{Tom}]. \tag{20.1.42}$$

☐ Tom **20.1.10** (☐$\mathscr{A}${M:2[ℙ][A]}) *Assume $\rho \leq a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a < \rho$, and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let $x_L \leq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_\|$.

(c) *Let $\rho < x_L$.*

1. Ⓢ$_1\langle 1\rangle_\blacktriangle$ *where* Conduct$_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
      1. *Let $\lambda = 1$. Then* ◎$_{\tau>1}\langle 1\rangle_\|$ *where* Conduct$_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *and* Conduct$_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
   i. *Let $(\lambda\beta a - s)/\delta \leq a^\star$.*
      1. *Let $\lambda = 1$.*
         i. *Let $b > 0$ ⦅$\kappa > 0$⦆. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.
         ii. *Let $b \leq 0$ ⦅$\kappa \leq 0$⦆. Then* ◎$_{\tau>1}\langle 1\rangle_\|$ *where* Conduct$_{1\blacktriangle}$.
      2. *Let $\lambda < 1$.*
         i. *Let $b \geq 0$ ⦅$\kappa \geq 0$⦆. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.
         ii. *Let $b < 0$ ⦅$\kappa < 0$⦆. Then* S$_3$(p.154) [Ⓢ▲|◎ ‖] *is true.*
   ii. *Let $(\lambda\beta a - s)/\delta > a^\star$.*
      1. *Let $b \geq 0$ ⦅$\kappa \geq 0$⦆. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.
      2. *Let $b < 0$ ⦅$\kappa < 0$⦆. Then* S$_3$(p.154) [Ⓢ▲|◎ ‖] *is true.* ☐

• *Proof by analogy* Consider the Tom′ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 20.1.2(p.154). Then "$a < \rho$" in Tom 20.1.2(p.154) (c2i,c3i) changes into "$a^\star < \rho$" in the Tom′, which contradicts the assumption $\rho \leq a^\star$. Accordingly, removing all assertions with "$a^\star < \rho$" from the Tom′ leads to Tom 20.1.10 above. ▮

Corollary 20.1.2 ($\mathscr{A}${M:2[ℙ][A]}) *Assume $\rho \leq a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho < x_K$. Then, the optimal price to propose z_t is nondecreasing in $t \geq 0$.* ▮

• *Proof* Immediate from Tom 20.1.10(p.166) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ▮

☐ Tom **20.1.11** (☐$\mathscr{A}${M:2[ℙ][A]}) *Assume $\rho \leq a^\star$, let $\beta < 1$ or $s > 0$, and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let* $\beta = 1$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\beta < 1$ *and* $s = 0$ $(\!(s > 0)\!)$.

1. *Let* $b > 0$ $(\!(\kappa > 0)\!)$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\,\blacktriangle}$.
2. *Let* $b \leq 0$ $(\!(\kappa \leq 0)\!)$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$. □

● *Proof by analogy* The same as `Tom` 20.1.3(p.157) due to Lemma 13.6.1(p.97) . ∎

Corollary 20.1.3 ($\mathscr{A}${M:2[$\mathbb{P}$][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho = x_K$. *Then, the optimal price to propose* z_t *is nondecreasing in* $t \geq 0$. ∎

● *Proof* Immediate from `Tom` 20.1.11(p.166) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ∎

□ `Tom` **20.1.12** (⊡$\mathscr{A}${M:2[$\mathbb{P}$][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.

1. $V_t = \rho$ *for* $t \geq 0$.
2. *Let* $x_L \leq \rho$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let* $x_L > \rho$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\,\blacktriangle}$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ $(\!(s > 0)\!)$.

1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. *Let* $b \leq 0$ $(\!(\kappa \leq 0)\!)$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let* $b > 0$ $(\!(\kappa > 0)\!)$.
   i. *Let* $\rho < x_L$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\,\blacktriangle}$.
   ii. *Let* $\rho = x_L$. *Then* $ⓓ_1\langle 0\rangle_{\|}$ *and* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\,\blacktriangle}$.
   iii. *Let* $\rho > x_L$. *Then* $\texttt{S}_4$ $\boxed{\circledS\blacktriangle \mid \bullet\| \mid c\rightsquigarrow s\triangle \mid c\rightsquigarrow s\blacktriangle}$ *is true.*

(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ $(\!(s > 0)\!)$.

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. *Let* $b \leq 0$ $(\!(\kappa \leq 0)\!)$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let* $b > 0$ $(\!(\kappa > 0)\!)$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\,\blacktriangle}$. □

● *Proof by analogy* The same as `Tom` 20.1.4(p.157) (see Lemma 13.6.1(p.97)). ∎

Corollary 20.1.4 ($\mathscr{A}${M:2[$\mathbb{P}$][A]}) *Assume* $\rho \leq a^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$. *Then* $z_t = z(\rho)$ *for* $t \geq 0$, *i.e., constant in* $t \geq 0$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ $(\!(s > 0)\!)$. *Then* z_t *is nonincreasing in* $t \geq 0$.

(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ $(\!(s > 0)\!)$. *Then* z_t *is nondecreasing in* $t \geq 0$. □

● *Proof* Immediate from `Tom` 20.1.12(p.167) (a1,b1,c1) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ∎

20.1.5.2.2.2 Case of $b \leq \rho$

In this case, Theorem 20.1.2(p.151) holds due to Lemma 20.1.1(p.151) (c), hence the following `Tom`'s 20.1.13(p.167) –20.1.15(p.168) can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.128))) to `Tom`'s 20.1.2(p.154) –20.1.4(p.157) :

□ `Tom` **20.1.13** (⊡$\mathscr{A}${M:2[$\mathbb{P}$][A]}) *Assume* $b \leq \rho$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho < x_K$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.

(b) *Let* $x_L \leq \rho$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\rho < x_L$.

1. $\circledS_1\langle 1\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{1\,\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\,\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$ $(\!(s > 0)\!)$.
   i. *Let* $b \geq 0$ $(\!(\kappa \geq 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\,\blacktriangle}$.
   ii. *Let* $b < 0$ $(\!(\kappa < 0)\!)$. *Then* $\texttt{S}_3$(p.154) $\boxed{\circledS\blacktriangle \mid \circledcirc\ \|}$ *is true.* □

● *Proof by analogy* Consider the `Tom`′ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to `Tom` 20.1.2(p.154) . Then "$\rho \leq a$" in `Tom` 20.1.2(p.154) (c2i,c3i) changes into "$\rho \leq a^\star$" in the `Tom`′, hence $\rho \leq a^\star < a < b$ due to Lemma 13.2.1(p.91) (n), which contradicts the assumption $b \leq \rho$. Accordingly, removing all assertions with "$\rho \leq a$" from the `Tom`′ leads to `Tom` 20.1.13 above. ∎

Corollary 20.1.5 *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho <\ x_K$. Then z_t is nondecreasing in $t \ge 0$.* □

● *Proof* Immediate from Tom 20.1.13(p.167) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ▮

□ **Tom 20.1.14** **(⊡ 𝒜{M:2[ℙ][A]})** *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho =\ x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\beta = 1$. Then ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.*

(c) *Let $\beta < 1$ and $s = 0$ ⟅$s > 0$⟆ .*

  1. *Let $b > 0$ ⟅$\kappa > 0$⟆ . Then ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*
  2. *Let $b \le 0$ ⟅$\kappa \le 0$⟆ . Then ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.* □

● *Proof by analogy* The same as Tom 20.1.3(p.157) due to Lemma 13.6.1(p.97) . ▮

Corollary 20.1.6 *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho =\ x_K$. Then z_t is nondecreasing in $t \ge 0$.* □

● *Proof* Immediate from Tom 20.1.14(p.168) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ▮

□ **Tom 20.1.15** **(⊡ 𝒜{M:2[ℙ][A]})** *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho >\ x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

  1. *$V_t = \rho$ for $t \ge 0$.*
  2. *Let $x_L \le \rho$. Then ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.*
  3. *Let $x_L > \rho$. Then ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⟅$s > 0$⟆ .*

  1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge\ x_K$ as $t \to \infty$.*
  2. *Let $b \le 0$ ⟅$\kappa \le 0$⟆ . Then ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.*
  3. *Let $b > 0$ ⟅$\kappa > 0$⟆ .*
     i. *Let $\rho <\ x_L$. Then ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*
     ii. *Let $\rho =\ x_L$. Then ⓓ$_{1}\langle 0\rangle_{\parallel}$ and ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*
     iii. *Let $x_L < \rho$. Then $\texttt{S}_4$ [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] is true.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⟅$s > 0$⟆ .*

  1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge\ x_K$ as $t \to \infty$.*
  2. *Let $b \le 0$ ⟅$\kappa \le 0$⟆ . Then ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.*
  3. *Let $b > 0$ ⟅$\kappa > 0$⟆ . Then ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.* □

● *Proof by analogy* The same as Tom 20.1.4(p.157) due to Lemma 13.6.1(p.97) . ▮

Corollary 20.1.7 *Assume $b \le \rho$, let $\beta < 1$ or $s > 0$, and let $\rho >\ x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$. Then $z_t = z(\rho)$ for $t \ge 0$.*

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⟅$s > 0$⟆ . Then z_t is nonincreasing in $t \ge 0$.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⟅$s > 0$⟆ . Then z_t is nondecreasing in $t \ge 0$.* □

● *Proof* Immediate from Tom 20.1.15(p.168) (a1,b1,c1) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ▮

20.1.5.2.2.3 Case of $a^\star < \rho < b$

In this case, Theorem 20.1.2(p.151) does not always hold due to Lemma 20.1.1(p.151) (d), hence 𝒜{M:2[ℙ][A]} must be directly found. For convenience of reference, below let us copy (20.1.33(p.164))

$$V_1 = \max\{\lambda\beta\max\{0, a-\rho\} - s, 0\} + \beta\rho. \tag{20.1.43}$$

Lemma 20.1.2

(a) *Let $V_1 \le\ x_K$. Then V_t is nondecreasing in $t > 0$.*

(b) *Let $V_1 >\ x_K$.*

  1. *Let $\beta = 1$ or $V_1 = 0$. Then $V_t = V_1$ for $t > 0$.*
  2. *Let $\beta < 1$ and $V_1 > 0$. Then V_t is nonincreasing in $t > 0$.*
  3. *Let $\beta < 1$ and $V_1 < 0$. Then V_t is nondecreasing in $t > 0$.* □

● *Proof* (a) Let $V_1 \le x_K$. Then, $K(V_1) \ge 0$ due to Corollary 13.2.2(p.95) (b), hence from (6.5.23(p.39)) with $t = 2$ we have $V_2 \ge K(V_1) + V_1 \ge V_1$. Suppose $V_{t-1} \le V_t$. Then, from (6.5.23(p.39)) and Lemma 13.2.3(p.94) (e) we have $V_t \le \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \le V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Let $V_1 > x_K$. Then $K(V_1) \le 0 \cdots (1)$ due to Corollary 13.2.2(p.95) (a). Hence, from (6.5.23(p.39)) with $t = 2$, hence $V_2 - V_1 = \max\{K(V_1) + V_1, \beta V_1\} - V_1 = \max\{K(V_1), -(1-\beta)V_1\} \cdots (2)$.

(b1) Let $\beta = 1$ or $V_1 = 0$. Then, since $-(1-\beta)V_1 = 0$, from (2) we have $V_2 - V_1 = \max\{K(V_1), 0\} = 0$ due to (1), hence $V_2 = V_1$. Suppose $V_{t-1} = V_1$. Then from (6.5.23(p.39)) we have $V_t = \max\{K(V_1) + V_1, \beta V_1\} = V_2 = V_1$. Hence, by induction we have $V_t = V_1$ for $t > 0$.

Below note that $\overline{\beta = 1 \text{ or } V_1 = 0}$ (the negation of $\beta = 1$ or $V_1 = 0$) is "$\beta < 1$ and $V_1 \ne 0$", which can be classified into the two cases, "$\beta < 1$ and $V_1 > 0$" and "$\beta < 1$ and $V_1 > 0$".

(b2) Let $\beta < 1$ and $V_1 > 0$. Then, since $-(1-\beta)V_1 < 0$, from (2) we have $V_2 - V_1 \le 0$ due to (1), hence $V_2 \le V_1$. Suppose $V_{t-1} \le V_{t-2}$. Then, from (6.5.23(p.39)) and Lemma 13.2.3(p.94) (e) we have $V_t \le \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = V_{t-1}$. Hence, by induction we have $V_t \le V_{t-1}$ for $t > 1$, thus V_t nonincreasing in $t > 0$.

(b3) Let $\beta < 1$ and $V_1 < 0$. Then, since $-(1-\beta)V_1 > 0$, from (2) we have $V_2 - V_1 > 0$ or equivalently $V_2 > V_1$, so $V_2 \ge V_1$. Suppose $V_{t-1} \ge V_{t-2}$. Then from (6.5.23(p.39)) and Lemma 13.2.3(p.94) (e) we have $V_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = V_{t-1}$. Hence, by induction we have $V_t \ge V_{t-1}$ for $t > 1$, thus V_t nondecreasing in $t > 0$. ∎

Let us define:

$\mathtt{S}_5\boxed{\circledS\blacktriangle \mid \circledcirc\,\|}$ = {
There exists $t^\bullet_\tau > 1$ such that:
(1) $t^\bullet_\tau \ge \tau > 1 \Rightarrow \circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\mathtt{Conduct}_{\tau\ge t>1\,\blacktriangle}$
(2) $\tau > t^\bullet_\tau \Rightarrow \circledcirc_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\|$ where $\mathtt{Conduct}_{t^\bullet_\tau\ge t>1\,\blacktriangle}$.
}

$\mathtt{S}_6\boxed{\circledS\blacktriangle \mid \circledcirc\,\| \mid \bullet\| \mid \mathtt{C}{\rightsquigarrow}\mathtt{S}\triangle \mid \mathtt{C}{\rightsquigarrow}\mathtt{S}\blacktriangle}$ = {
There exists $t^{\bullet\,\dagger}_\tau$ and t°_τ ($t^\bullet_\tau > t^\circ_\tau > 1$) such that:
(1) $t^\bullet_\tau \ge \tau > 1 \Rightarrow$ If $\lambda\beta\max\{0, a-\rho\} \le s$, then Ⓓ$_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle_\|$.
If $\lambda\beta\max\{0, a-\rho\} > s$, then $\circledcirc_{t^\bullet_\tau\ge\tau>1}\langle 1\rangle_\|$ where $\mathtt{Conduct}_{1\,\blacktriangle}$.
(2) $\tau > t^\bullet_\tau \Rightarrow \circledS_{\tau>t^\bullet_\tau}\langle\tau\rangle_\blacktriangle$ where $\mathtt{Conduct}_{\tau\ge t>t^\bullet_\tau\,\blacktriangle}$,
where $\mathtt{pSKIP}_{t^\bullet_\tau\ge\tau>t^\circ_\tau\,\triangle}$ ($\mathtt{C}{\rightsquigarrow}\mathtt{S}_{\,t^\bullet_\tau\ge t>t^\circ_\tau\,\triangle}$), and
where $\mathtt{pSKIP}_{t^\circ_\tau\ge t>1\,\triangle}$ ⟪$\mathtt{pSKIP}_{t^\circ_\tau\ge t>1\,\blacktriangle}$⟫ ($\mathtt{C}{\rightsquigarrow}\mathtt{S}_{\,t^\circ_\tau\ge t>1\,\triangle}$ ⟪$\mathtt{C}{\rightsquigarrow}\mathtt{S}_{\,t^\circ_\tau\ge t>1}$⟫).
}

$\mathtt{S}_7\boxed{\circledS\blacktriangle \mid \circledcirc\,\| \mid \bullet\| \mid \mathtt{C}{\rightsquigarrow}\mathtt{S}\triangle}$ = {
There exists $t^\bullet_\tau > 1$ such that:
(1) $t^\bullet_\tau \ge \tau > 1 \Rightarrow$ If $\lambda\beta\max\{0, a-\rho\} \le s$, then Ⓓ$_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle_\|$.
If $\lambda\beta\max\{0, a-\rho\} > s$, then $\circledcirc_{t^\bullet_\tau\ge\tau>1}\langle 1\rangle_\|$ where $\mathtt{Conduct}_{1\,\blacktriangle}$.
(2) $\tau > t^\bullet_\tau \Rightarrow \circledS_{\tau>t^\bullet_\tau}\langle\tau\rangle_\blacktriangle$ where $\mathtt{Conduct}_{\tau\ge t>t^\bullet_\tau\,\blacktriangle}$ and where $\mathtt{pSKIP}_{t^\bullet_\tau\ge\tau>1\,\triangle}$.
}

Remark 20.1.2 For explanatory convenience, let us represent "$\beta = 1$ or $V_1 = 0$" as $\{\beta = 1 \cup V_1 = 0\}$. Then, its negation $\overline{\{\beta = 1 \cup V_1 = 0\}}$ can be written as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap V_1 \ne 0\} = \{\beta < 1 \cap V_1 > 0\} \cup \{\beta < 1 \cap V_1 < 0\}.$$

Without loss of generality, this can be further expressed as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap s \ge 0 \cap V_1 > 0\} \cup \{\beta < 1 \cap s \ge 0 \cap V_1 < 0\}.$$

Furthermore, since $\{s \ge 0\}$ can be denoted by $\{s = 0 \,(\!(s > 0)\!)\}$, it follows that the above expression can be rewritten as

$$\overline{\{\beta = 1 \cup V_1 = 0\}} = \{\beta < 1 \cap \{s = 0 \,(\!(s > 0)\!)\} \cap \{V_1 > 0\}\} \cup \{\beta < 1 \cap \{s = 0 \,(\!(s > 0)\!)\} \cap \{V_1 < 0\}\}. \;\square$$

☐ **Tom 20.1.16** (▣ $\mathscr{A}$\{M:2[P][A]\}) *Assume $a^\star < \rho < b$ and let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, a-\rho\} \le s$, then* Ⓓ$_1\langle 0\rangle_\|$*, or else* $\circledS_1\langle 1\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{1\,\blacktriangle}$. *Below let* $\tau > 1$ ↦ → Ⓓ$_\|$/$\circledS_\blacktriangle$

(b) *Let $V_1 \le x_K$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $V_1 \ge x_L$. Then, if $\lambda\beta\max\{0, a-\rho\} \le s$, we have* Ⓓ$_{\tau>1}\langle 0\rangle_\|$*, or else* $\circledcirc_{\tau>1}\langle 1\rangle_\|$ *where* $\mathtt{Conduct}_{1\,\blacktriangle}$ → Ⓓ$_\|$/$\circledcirc_\|$
3. *Let $V_1 < x_L$.*
   i. *Let $\beta = 1$. Then $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ where* $\mathtt{Conduct}_{\tau\ge t>1\,\blacktriangle}$ ↦ → $\circledS_\blacktriangle$
   ii. *Let $\beta < 1$ and $s = 0 \,(\!(s > 0)\!)$.*
      1. *Let $b > 0 \,(\!(\kappa > 0)\!)$. Then $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$ where* $\mathtt{Conduct}_{\tau\ge t>1\,\blacktriangle}$ ↦ → $\circledS_\blacktriangle$
      2. *Let $b \le 0 \,(\!(\kappa \le 0)\!)$. Then* $\mathtt{S}_5\boxed{\circledS\blacktriangle \mid \circledcirc\,\|}$ *is true*↦ → $\circledS_\blacktriangle$/$\circledcirc_\|$

(c) *Let $V_1 > x_K$.*

1. *Let $\beta = 1$ or $V_1 = 0$.*
   i. *$V_t = V_1$ for $t > 0$.*

ii. *If* $\lambda \max\{0, a-\rho\} \le s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ $\mapsto$ → ⓓ$_{\|}$/◎$_{\|}$

2. *Let* $\beta < 1$ *and* $s = 0\,(\!(s>0)\!)$ (see Remark 20.1.2(p.169) above)

i. *Let* $V_1 > 0$.

1. V_t *is nonincreasing in* $t \ge 0$ *and converges to* $V \ge x_K$ *as* $t \to \infty$.
2. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then*

i. *Let* $V_1 > x_L$. *Then* $\texttt{S}_6$ [Ⓢ▲ | ◎ ‖ | ●‖ | c⇝s△ | c⇝s▲] *is true* $\mapsto$ → Ⓢ$_\blacktriangle$/◎$_{\|}$/ⓓ$_{\|}$/[c⇝s]$_\triangle$/[c⇝s]$_\blacktriangle$

ii. *Let* $V_1 = x_L$. *Then* $\texttt{S}_7$ [Ⓢ▲ | ◎ ‖ | ●‖ | c⇝s△] *is true* $\mapsto$ → Ⓢ$_\blacktriangle$/◎$_{\|}$/ⓓ$_{\|}$/[c⇝s]$_\triangle$

iii. *Let* $V_1 < x_L$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.

3. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *If* $\lambda\beta \max\{0, a-\rho\} \le s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ → ⓓ$_{\|}$/◎$_{\|}$

ii. *Let* $V_1 < 0$.

1. V_t *is nondecreasing in* $t \ge 0$ *and converges to a finite* $V \ge x_K$ *as* $t \to \infty$.
2. *Let* $b > 0\,(\!(\kappa > 0)\!)$.

i. *Let* $V_1 \ge x_L$. *If* $\lambda\beta \max\{0, a-\rho\} \le s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ → ⓓ$_{\|}$/◎$_{\|}$

ii. *Let* $V_1 < x_L$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.

3. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *If* $\lambda\beta \max\{0, a-\rho\} \le s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$ → ⓓ$_{\|}$/◎$_{\|}$

• *Proof* Assume $a^\star < \rho < b \cdots (1)$ and let $\beta < 1$ or $s > 0$.

(a)

i. Let $\lambda\beta \max\{0, a-\rho\} \le s$. Then, since $\lambda\beta \max\{0, a-\rho\} - s \le 0$, we have $V_1 - \beta V_0 = 0$ from (20.1.34(p.164)), i.e., $V_1 = \beta V_0 \cdots (2)$, hence $t_1^* = 0$, i.e., ⓓ$_1\langle 0\rangle_{\|}$.

ii. Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, since $\lambda\beta \max\{0, a-\rho\} - s > 0$, we have $V_1 - \beta V_0 > 0$ from (20.1.34(p.164)), i.e., $V_1 > \beta V_0 \cdots (3)$, hence $t_1^* = 1$, i.e., Ⓢ$_1\langle 1\rangle_\blacktriangle$. Then, since $\lambda\beta \max\{0, a-\rho\} - s > 0$, from the comparison of the two terms within { } in the r.h.s. of (20.1.33(p.164)) it follows that conducting the search is *strictly* optimal at time $t = 1$, i.e., $\texttt{Conduct}_{1\blacktriangle} \cdots (4)$.

Below let $\tau > 1$.

(b) Let $V_1 \le x_K \cdots (5)$.

(b1) V_t is nondecreasing in $t > 0$ due to Lemma 20.1.2(p.168) (a). Consider a sufficiently large $M > 0$ with $b \le M$ and $V_1 \le M$. Suppose $V_{t-1} \le M$. Then, from (6.5.23(p.39)) and Lemma 13.2.3(p.94) (e) we have $V_t \le \max\{K(M)+M, \beta M\} = \max\{\beta M - s, \beta M\}$ due to (13.2.12 (2) (p.94)), hence $V_t \le \max\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Accordingly, by induction $V_t \le M$ for $t > 0$, i.e., V_t is upper bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta M\} \cdots (6)$ from (6.5.23(p.39)), we have $0 = \max\{K(V), -(1-\beta)V\} \cdots (7)$, hence $K(V) \le 0$, so $V \ge x_K$ due to Lemma 13.2.3(p.94) (j1).

(b2) Let $V_1 \ge x_L$. Then, since $V_{t-1} \ge x_L$ for $t > 1$ due to (b1), we have $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a), hence $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (20.1.28(p.164)), i.e., $V_t = \beta V_{t-1}$ for $t > 1$. Then, since $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$, we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots (8)$.

i. Let $\lambda\beta \max\{0, a-\rho\} \le s$. Then, from (8) and (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t_\tau^* = 0$ for $\tau > 1$, i.e., ⓓ$_{\tau>1}\langle 0\rangle_{\|}$.

ii. Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, from (8) and (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t_\tau^* = 1$ for $\tau > 1$, i.e., ◎$_{\tau>1}\langle 1\rangle_{\|}$. In addition, we have $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(b3) Let $V_1 < x_L \cdots (9)$.

(b3i) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$", thus $x_L = x_K \cdots (10)$ from Lemma 13.2.4(p.95) (b). Now, since $V_1 \ge \beta\rho$ from (6.5.22(p.39)), we have $V_1 \ge \rho$ due to the assumption $\beta = 1$, hence $a^\star < V_1$ due to (1). Accordingly, it follows that $a^\star \le V_{t-1}$ for $t > 1$ due to (b1). Note $V_1 < x_K$ from (9) and (10). Suppose $V_{t-1} < x_K$. Then, from Lemma 13.2.3(p.94) (f) and (6.5.23(p.39)) with $\beta = 1$ we have $V_t < \max\{K(x_K) + x_K, x_K\} = \max\{x_K, x_K\} = x_K$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ due to (10), so $L(V_{t-1}) > 0$ for $t > 1$ from Lemma 13.2.2(p.94) (e1). Then, since $L(V_{t-1}) > 0 \cdots (11)$ for $\tau \ge t > 1$, we have $V_t - \beta V_{t-1} > 0$ for $\tau \ge t > 1$ from (20.1.28(p.164)), i.e., $V_t > \beta V_{t-1}$ for $\tau \ge t > 1$, hence $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$. In addition, since $V_1 \ge \beta V_0$ from (20.1.34(p.164)), we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \ge \beta^\tau V_0$, hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$. Then, we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$ from (11) and (20.1.32(p.164)).

(b3ii) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(b3ii1) Let $b > 0\,(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots (12)$ from Lemma 13.2.4(p.95) (c (d)). Here note (9) and (b1). Then suppose there exists a t' such that $V_{t-1} \ge x_L$ for $t \ge t'$. Then $L(V_{t-1}) \le 0$ for $t \ge t'$ from Corollary 13.2.1(p.94) (a), hence $V_t = \beta V_{t-1}$ for $t \ge t'$ due to (20.1.30(p.164)). Therefore, we have $V_t = \beta^{t-t'+1} V_{t'-1}$ for $t \ge t'$, leading to $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (12), which contradicts $V \ge x_K$ in (b1). Accordingly, it follows that $V_{t-1} < x_L$ for all $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a). Thus, for the same reason as in the proof of (b3i) we have Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau\ge t>1\blacktriangle}$.

(b3ii2) Let $b \le 0\,(\!(\kappa \le 0)\!)$.

- Let $b = 0\,(\!(\kappa = 0)\!)$. Then $x_L = x_K = 0 \cdots (13)$ from Lemma 13.2.4(p.95) (c (d)), hence $V \geq x_K = x_L = 0$ from (b1). Here assume $V > x_K = 0$. Then, since $-(1-\beta)V < 0$, we have $K(V) = 0$ from (7), leading to the contradiction $V = x_K$ due to Lemma 13.2.3(p.94) (j1). Thus it must be that $V = x_K = 0$. Accordingly, due to (b1) and due to $V_1 < x_L = x_K = V$ from (9) and (13) it follows that there exists a $t^\bullet_\tau > 1$ such that
$$V_1 \leq V_2 \leq \cdots \leq V_{t^\bullet_\tau -1} < x_L = x_K = V_{t^\bullet_\tau} = V_{t^\bullet_\tau +1} = \cdots,^\dagger$$
where $t^\bullet_\tau$ might be infinity (i.e., $t^\bullet_\tau = \infty$). Hence $V_{t-1} < x_L$ for $t^\bullet_\tau \geq t > 1$ and $V_{t-1} = x_L$ for $t > t^\bullet_\tau$. Thus, from Corollary 13.2.1(p.94) (a) we have
$$L(V_{t-1}) > 0 \text{ for } t^\bullet_\tau \geq t > 1 \text{ and } L(V_{t-1}) = 0 \text{ (hence } L(V_{t-1}) \leq 0) \text{ for } t > t^\bullet_\tau \cdots (14).$$
- Let $b < 0\,(\!(\kappa < 0)\!)$. Then $x_L < x_K$ from Lemma 13.2.4(p.95) (c (d)). Since $V_1 < x_L$ from (9) and since $x_L < x_K \leq V$ from (b1), there exists $t^\bullet_\tau$ such that
$$V_1 \leq V_2 \leq \cdots \leq V_{t^\bullet_\tau -1} < x_L \leq V_{t^\bullet_\tau} \leq V_{t^\bullet_\tau +1} \leq \cdots,$$
hence $V_{t-1} < x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \leq V_{t-1}$ for $t > t^\bullet_\tau$. Accordingly, from Corollary 13.2.1(p.94) (a) we have
$$L(V_{t-1}) > 0 \text{ for } t^\bullet_\tau \geq t > 1 \text{ and } L(V_{t-1}) \leq 0 \text{ for } t > t^\bullet_\tau \cdots (15).$$

From (14) and (15) we have, whether $b = 0\,(\!(\kappa = 0)\!)$ or $b < 0\,(\!(\kappa < 0)\!)$ (or equivalently $b \leq 0\,(\!(\kappa \leq 0)\!)$),
$$L(V_{t-1}) > 0 \cdots (16) \text{ for } t^\bullet_\tau \geq t > 1,$$
$$L(V_{t-1}) \leq 0 \cdots (17) \text{ for } t > t^\bullet_\tau.$$

Accordingly, from (20.1.28(p.164)) we have $V_t - \beta V_{t-1} > 0$ for $t^\bullet_\tau \geq t > 1$ due to (16) and $V_t - \beta V_{t-1} = 0$ for $t > t^\bullet_\tau$ due to (17) or equivalently
$$V_t > \beta V_{t-1} \cdots (18), \quad t^\bullet_\tau \geq t > 1, \qquad V_t = \beta V_{t-1} \cdots (19), \quad t > t^\bullet_\tau.$$

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $V_t > \beta V_{t-1} \cdots (20)$ for $\tau \geq t > 1$ due to (18), for the same reason as in the proof of (b3i) we have $ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau \geq t > 1\blacktriangle}$. Hence (1) of $\texttt{S}_5$ holds. Then, since (20) with $\tau = t^\bullet_\tau$ can be rewritten as $V_t > \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, we have
$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau -1} > \cdots > \beta^{t^\bullet_\tau -1} V_1 \cdots (21).$$
2. Let $\tau > t^\bullet_\tau$. Then $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ due to (19), hence
$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (22).$$
Hence, from (22) and (21) and from the fact that $V_1 \geq \beta V_0$ due to (2) and (3) we obtain
$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau +1} V_{t^\bullet_\tau -1} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0,$$
so we have $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$, i.e., $◎_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\parallel$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (16) and (20.1.32(p.164)). From the above we see that (2) of $\texttt{S}_5$ holds.

(c) Let $V_1 > x_K \cdots (23)$.

(c1) Let $\beta = 1$ or $V_1 = 0$.

(c1i) The same as Lemma 20.1.2(p.168) (b1).

(c1ii) Since $V_\tau = V_{\tau-1} = \cdots = V_1$ for $\tau > 0$ from (c1i), we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots (24)$.

i. Let $\lambda \max\{0, a-\rho\} \leq s$. Then, from (2) and (24) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $ⓓ_{\tau>1}\langle 0\rangle_\parallel$.

ii. Let $\lambda \max\{0, a-\rho\} > s$. Then, from (3) and (24) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $◎_{\tau>1}\langle 1\rangle_\parallel$ where $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(c2) Let $\beta < 1 \cdots (25)$ and $s = 0\,(\!(s > 0)\!)$.

(c2i) Let $V_1 > 0$.

(c2i1) The former half is the same as Lemma 20.1.2(p.168) (b2). The latter half can be proven as follows. Note (23), hence $V_1 \geq x_K$. Suppose $V_{t-1} \geq x_K$. Then from (6.5.23(p.39)) we have $V_t \geq K(V_{t-1}) + V_{t-1} \geq K(x_K) + x_K$ due to Lemma 13.2.3(p.94) (e), hence $V_t \geq x_K$ since $K(x_K) = 0$. Accordingly, by induction $V_t \geq x_K$ for $t > 0$, i.e., V_t is lower bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Then, since $V = \max\{K(V) + V, \beta V\}$ from (6.5.23(p.39)), we have $0 = \max\{K(V), -(1-\beta)V\}$, hence $K(V) \leq 0$, so $V \geq x_K$ due to Lemma 13.2.3(p.94) (j1).

---

†Since $V_t \leq V$ for any $t > 0$ due to (b1), if $V \leq V_t$ for a t, then $V = V_t$.

(c2i2) Let $b > 0\,(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots \mathbf{(26)}$ from Lemma 13.2.4(p.95) (c (\!(d)\!)).

(c2i2i) Let $V_1 > x_L \cdots \mathbf{(27)}$, hence $V_1 \ge x_L$. Suppose $V_{t-1} \ge x_L$ for *all* $t > 1$. Then, since $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (20.1.28(p.164)), i.e., $V_t = \beta V_{t-1}$ for all $t > 1$, hence $V_t = \beta^{t-1} V_1$. Accordingly, we have $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (26), which contradicts $V \ge x_K$ in (c2i1). Thus it is impossible that $x_L \le V_{t-1}$ for *all* $t > 0$. Accordingly, due to (27) and (c2i1) it follows that there exist $t^\bullet_\tau$ and t°_τ ($t^\bullet_\tau > t^\circ_\tau > 0$) such that

$$V_1 \ge V_2 \ge \cdots \ge V_{t^\circ_\tau - 1} > x_L = V_{t^\circ_\tau} = V_{t^\circ_\tau + 1} = \cdots = V_{t^\bullet_\tau - 1} > V_{t^\bullet_\tau} \ge V_{t^\bullet_\tau + 1} \ge \cdots .$$

Hence, we have

$$\begin{aligned} & x_L > V_{t^\bullet_\tau},\ x_L > V_{t^\bullet_\tau+1}, \cdots, \\ & V_{t^\circ_\tau} = x_L,\ V_{t^\circ_\tau+1} = x_L, \cdots, V_{t^\bullet_\tau-1} = x_L, \\ & V_1 > x_L,\ V_2 > x_L, \cdots, V_{t^\circ_\tau-1} > x_L, \end{aligned}$$

or equivalently

$$\begin{aligned} x_L > V_{t-1} \cdots \mathbf{(28)}, &\qquad t > t^\bullet_\tau, \\ V_{t-1} = x_L \cdots \mathbf{(29)}, &\qquad t^\bullet_\tau \ge t > t^\circ_\tau, \\ V_{t-1} > x_L \cdots \mathbf{(30)}, &\qquad t^\circ_\tau \ge t > 1. \end{aligned}$$

Accordingly, we have:

1. Let $t^\bullet_\tau \ge \tau > 1$. Then, since $V_{t-1} \ge x_L$ for $\tau \ge t > 1$ from (29) and (30), we have $L(V_{t-1}) \le 0 \cdots \mathbf{(31)}$ for $\tau \ge t > 1$ from Corollary 13.2.1(p.94) (a), hence $V_t - \beta V_{t-1} = 0$ for $\tau \ge t > 1$ from (20.1.28(p.164)), i.e., $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$, leading to $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots \mathbf{(32)}$.

   i. Let $\lambda \max\{0, a-\rho\} \le s$. Then, from (2) and (32) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \ge \tau > 1$, i.e., $\text{ⓓ}_{t^\bullet_\tau \ge \tau > 1}\langle 0\rangle_{\|}$.

   ii. Let $\lambda \max\{0, a-\rho\} > s$. Then, from (3) and (32) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \ge \tau > 1$, i.e., $\circledcirc_{t^\bullet_\tau \ge \tau > 1}\langle 1\rangle_{\|}$ where $\texttt{Conduct}_{1\blacktriangle}$ from (4).

   Accordingly $\texttt{S}_6(1)$ holds. From (32) with $\tau = t^\bullet_\tau$ we have $V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1} V_1 \cdots \mathbf{(33)}$.

2. Let $\tau > t^\bullet_\tau$. Then, since $x_L > V_{t-1}$ for $\tau \ge t > t^\bullet_\tau$ from (28), due to Corollary 13.2.1(p.94) (a) we have $L(V_{t-1}) > 0 \cdots \mathbf{(34)}$ for $\tau \ge t > t^\bullet_\tau$. Accordingly, from (20.1.28(p.164)) we have $V_t - \beta V_{t-1} > 0$ for $\tau \ge t > t^\bullet_\tau$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \ge t > t^\bullet_\tau$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau}$. From this and (33) we have

   $$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1. \cdots \mathbf{(35)}.$$

   Since $V_1 \ge \beta V_0$ due to (2) and (3), from (35) we have

   $$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1 \ge \beta^\tau V_0.$$

   Hence, we have $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., $\text{ⓢ}_{\tau > t^\bullet_\tau}\langle \tau\rangle_\blacktriangle$, thus the former half of $\texttt{S}_6(2)$ holds. The latter half can be proven as follows.

   (i) If $\tau \ge t > t^\bullet_\tau$, then $\texttt{Conduct}_{t\blacktriangle}$ from (34) and (20.1.32(p.164)).

   (ii) If $t^\bullet_\tau \ge t > t^\circ_\tau$, then $V_{t-1} = x_L$ from (29), hence $L(V_{t-1}) = L(x_L) = 0$, so $\texttt{Skip}_{t\vartriangle}$ from (20.1.31(p.164)), implying that we have $\texttt{C}{\rightsquigarrow}\texttt{S}_{\,t^\bullet_\tau \ge t > t^\circ_\tau\,\vartriangle}$ (see Figure 7.2.1(p.42) (II).

   (iii) If $t^\circ_\tau \ge t > 1$, then $V_{t-1} > x_L$ from (30), hence $L(V_{t-1}) = (\!(<)\!)\ 0^\ddagger$ from Lemma 13.2.2(p.94) (d (\!(e1)\!)); i.e., $\texttt{Skip}_{t\vartriangle}$ (\!($\texttt{Skip}_{t\blacktriangle}$)\!) due to (20.1.31(p.164)) (\!((20.1.32(p.164)))\!), implying that we have $\texttt{C}{\rightsquigarrow}\texttt{S}_{\,t^\circ_\tau \ge t > 1\,\vartriangle}$ (\!($\texttt{C}{\rightsquigarrow}\texttt{S}_{\,t^\circ_\tau \ge t > 1}$)\!).

   From the above results we see that the latter half of $\texttt{S}_6(2)$ holds.

(c2i2ii) Let $V_1 = x_L$. Suppose $V_{t-1} = x_L$ for *all* $t > 1$. Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t - \beta V_{t-1} = 0$ for *all* $t > 1$ from (20.1.28(p.164)), i.e., $V_t = \beta V_{t-1}$ for all $t > 1$, hence $V_t = \beta^{t-1} V_1$. Then $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (26), which contradicts $V \ge x_K$ in (c2i1). Hence, since V_{t-1} is not equal to x_L for *all* $t > 1$, due to (c2i1) it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 = V_2 = \cdots = V_{t^\bullet_\tau - 1} = x_L > V_{t^\bullet_\tau} \ge V_{t^\bullet_\tau + 1} \ge \cdots ,$$

or equivalently $V_{t-1} = x_L$ for $t^\bullet_\tau \ge t > 1$ and $x_L > V_{t-1}$ for $t > t^\bullet_\tau$. Thus, due to Corollary 13.2.1(p.94) (a) we have

$$L(V_{t-1}) = L(x_L) = 0 \cdots \mathbf{(36)}, \quad t^\bullet_\tau \ge t > 1, \qquad L(V_{t-1}) > 0 \cdots \mathbf{(37)}, \quad t > t^\bullet_\tau.$$

Accordingly, we have:

---

‡If $s = 0$, then "$= 0$", or else "< 0".

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, from (36) and (20.1.28(p.164)) we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, from which we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1$.
   i. Let $\lambda\beta \max\{0, a-\rho\} \leq s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., ⓓ$_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle_{\parallel}$.
   ii. Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., ◎$_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle_{\parallel}$. In addition, we have $\texttt{Conduct}_{1\blacktriangle}$ from (4).

   Accordingly, it follows that $\texttt{S}_7(1)$ holds.

2. Let $\tau > t^\bullet_\tau$. Then $L(V_{t-1}) > 0 \cdots (38)$ for $\tau \geq t > t^\bullet_\tau$ from (37), hence due to (20.1.28(p.164)) we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > t^\bullet_\tau$ or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$, leading to $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (39)$. In addition, since $V_t - \beta V_{t-1} = 0$ for $t^\bullet_\tau \geq t > 1$ from (36) and (20.1.28(p.164)), we have $V_t = \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, leading to
$$V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1} V_1 \cdots (40).$$

   From (39) and (40) we have
$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1.$$

   In addition, since $V_1 \geq \beta^\tau V_0$ from (2) and (3), we eventually obtain
$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1} V_1 \geq \beta^\tau V_0 \cdots (41).$$

   Thus $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., ⓢ$_{\tau > t^\bullet_\tau}\langle \tau\rangle_\blacktriangle$, hence the former half of $\texttt{S}_7(2)$ holds. Then, we have that $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > t^\bullet_\tau$ due to (38) and (20.1.32(p.164)). Moreover, we have $\texttt{Skip}_{t\vartriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (36) and (20.1.31(p.164)), so it follows that we have $\texttt{pSKIP}_{t\vartriangle}$ for $t^\bullet_\tau \geq t > 1$ (see Figure 7.2.1(p.42))(II)) or equivalently $\texttt{pSKIP}_{t^\bullet_\tau \geq t > 1\vartriangle}$. Hence the latter half of $\texttt{S}_7(2)$ holds.

(c2i2iii) Let $V_1 < x_L$. Then $V_{t-1} < x_L$ for $t > 1$ due to (c2i1), hence $L(V_{t-1}) > 0 \cdots (42)$ for $t > 1$ from Corollary 13.2.1(p.94) (a). Accordingly, since $L(V_{t-1}) > 0 \cdots (43)$ for $\tau \geq t > 1$, we have $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$ from (20.1.28(p.164)) or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, hence
$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1.$$

Since $V_1 \geq \beta V_0$ from (2) and (3), we have
$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0,$$

hence we have $t^*_\tau = \tau$ for $\tau > 1$, i.e., ⓢ$_{\tau > 1}\langle \tau\rangle_\blacktriangle$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (43) and (20.1.32(p.164)).

(c2i3) Let $b \leq 0$ ($\kappa \leq 0$), hence $x_L \leq x_K \cdots (44)$ from Lemma 13.2.4(p.95) (c (d)). Then, from (23) and (c2i1) we have $V_{t-1} \geq x_K$ for all $t > 1$, hence $V_{t-1} \geq x_L$ for all $t > 1$ due to (44), thus $L(V_{t-1}) \leq 0$ for all $t > 1$ from Corollary 13.2.1(p.94) (a). Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$, we have $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (20.1.28(p.164)) or equivalently $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, hence
$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1.$$

i. Let $\lambda\beta \max\{0, a-\rho\} \leq s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., ⓓ$_{\tau > 1}\langle 0\rangle_{\parallel}$.

ii. Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., ◎$_{\tau > 1}\langle 1\rangle_{\parallel}$. Then $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(c2ii) Let $V_1 < 0$.

(c2ii1) The same as the proof of (c2i1).

(c2ii2) Let $b > 0$ ($\kappa > 0$), hence $x_L > x_K > 0 \cdots (45)$ from Lemma 13.2.4(p.95) (c (d)).

(c2ii2i) Let $V_1 \geq x_L$. Then, since $V_{t-1} \geq x_L$ for $t > 1$ due to (c2ii1), we have $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a), hence $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Thus $V_t - \beta V_{t-1} = 0$ for $\tau \geq t > 1$ from (20.1.28(p.164)), i.e., $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$, so
$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1.$$

i. Let $\lambda\beta \max\{0, a-\rho\} \leq s$. Then, from (2) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 1$, i.e., $\texttt{dOIT}_{\tau > 1}\langle 0\rangle_{\parallel}$.

ii. Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, from (3) we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., ◎$_{\tau > 1}\langle 1\rangle_{\parallel}$. Then $\texttt{Conduct}_{1\blacktriangle}$ from (4).

(c2ii2ii) Let $V_1 < x_L$. Suppose that there exists $t' > 1$ such that $x_L \le V_{t-1}$ for $t > t'$. Then, since $L(V_{t-1}) \le 0$ for $t > t'$ from Corollary 13.2.1(p.94) (a), we have $V_t - \beta V_{t-1} = 0$ for $t > t'$ due to (20.1.28(p.164)), hence $V_t = \beta V_{t-1}$ for $t > t'$, so

$$V_t = \beta V_{t-1} = \beta^2 V_{t-2} = \cdots = \beta^{t-t'} V_{t'}.$$

Accordingly $V = \lim_{t\to\infty} V_t = 0 < x_K$ due to (25) and (45), which contradicts $V \ge x_K$ in (c2ii1), hence it must be that $V_{t-1} < x_L$ for $t > 1$. Then, since $V_{t-1} < x_L$ for $\tau \ge t > 1$, we have $L(V_{t-1}) > 0 \cdots$ (46) for $\tau \ge t > 1$ from Corollary 13.2.1(p.94) (a), hence $V_t - \beta V_{t-1} > 0$ for $\tau \ge t > 1$ from (20.1.28(p.164)) or equivalently $V_t > \beta V_{t-1}$ for $\tau \ge t > 1$, thus

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1.$$

Since $V_1 \ge \beta V_0$ from (2) and (3), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \ge \beta^\tau V_0,$$

hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$. From (46) and (20.1.32(p.164)) we have $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2ii3) Let $b \le 0$ ⟪$\kappa \le 0$⟫, hence $x_L \le x_K \cdots$ (47) from Lemma 13.2.4(p.95) (c ⟪d⟫). Then, due to (23) and (c2ii1) we have $V_{t-1} > x_K$ for $t > 1$, hence $V_{t-1} > x_L$ for $t > 1$ from (47), thus $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a). Accordingly, the assertion is true for the same reason as in the proof of (c2ii2i). ∎

Corollary 20.1.8 *Assume $a^\star < \rho < b$ and let $\beta < 1$ or $s > 0$.*

(a) *Let $V_1 \le x_K$. Then z_t is nondecreasing in $t > 0$.*

(b) *Let $V_1 > x_K$.*

1. *Let $\beta = 1$ or $V_1 = 0$. Then $z_t = z(V_1)$ for $t > 0$.*
2. *Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.*
   i. *Let $V_1 > 0$. Then z_t is nonincreasing in $t > 0$.*
   ii. *Let $V_1 < 0$. Then z_t is nondecreasing in $t > 0$.* □

• *Proof* Immediate from Tom 20.1.16(p.169) (b1,c1i,c2i1,c2ii1) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ∎

20.1.5.3 Market Restriction

20.1.5.3.1 Positive Restriction

20.1.5.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 20.1.9** ($\mathscr{A}\{$M:2[ℙ][A]$^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\rho \le a^\star$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*

(c) *Let $b \le \rho$. Then ❶$_{\tau>0}\langle 0\rangle_{\parallel}$.*

(d) *Let $a^\star < \rho < b$.*

1. *Let $a \le \rho$. Then ❶$_1\langle 0\rangle_{\parallel}$ and Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$ and* pSKIP$_1$ (C⇝S)
2. *Let $\rho < a$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*

• *Proof* The same as Lemma 20.1.9(p.166) due to Lemma 17.4.4(p.116). ∎

20.1.5.3.1.2 Case of $\beta < 1$ or $s > 0$

20.1.5.3.1.2.1 Case of $\rho \le a^\star$

□ **Pom 20.1.10** ($\mathscr{A}\{$M:2[ℙ][A]$^+\}$) *Suppose $a > 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $x_L \le \rho$. Then ❶$_{\tau>0}\langle 0\rangle_{\parallel}$.*

(c) *Let $\rho < x_L$.*

1. Ⓢ$_1\langle 1\rangle_\blacktriangle$ *where $\texttt{Conduct}_{1\blacktriangle}$. Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda a - s)/\lambda \le a^\star$.*
      1. *Let $\lambda = 1$. Then ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ where $\texttt{Conduct}_{1\blacktriangle}$.*
      2. *Let $\lambda < 1$. Then Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*
   ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ and $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*
3. *Let $\beta < 1$ and $s = 0$. Then Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\ge t>0\blacktriangle}$.*
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $(\lambda\beta a - s)/\delta \le a^\star$.*

1. *Let* $\lambda = 1$.
   i. *Let* $s < \lambda\beta T(0)$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $s \geq \lambda\beta T(0)$. *Then* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.
2. *Let* $\lambda < 1$.
   i. *Let* $s \leq \lambda\beta T(0)$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   ii. *Let* $s > \lambda\beta T(0)$. *Then* $\texttt{S}_3$(p.154) [ⓢ▲ | ◎ ‖] *is true.*

ii. *Let* $(\lambda\beta a - s)/\delta > a^{\star}$.
   1. *Let* $s \geq \lambda\beta T(0)$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   2. *Let* $s < \lambda\beta T(0)$. *Then* $\texttt{S}_3$(p.154) [ⓢ▲ | ◎ ‖] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.24)).

(a-c2ii) The same as (a-c2ii) of Tom 20.1.10(p.166).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i1i,c3i2i,c3ii1) of Tom 20.1.10(p.166).

(c4-c4ii2) The same as (c3-c3ii2) of Tom 20.1.10(p.166). ∎

□ **Pom 20.1.11** ($\mathscr{A}$\{M:2[ℙ][A]$^+$\}) *Suppose* $a > 0$. *Assume* $\rho \leq a^{\star}$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) V_t *is nondecreasing in* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(d) *Let* $\beta < 1$ *and* $s > 0$.
   1. *Let* $s < \beta\mu T(0)$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   2. *Let* $s \geq \beta\mu T(0)$. *Then* ⓓ$_{\tau}\langle 0\rangle_{\parallel}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.24)).

(a,b) The same as (a,b) of Tom 20.1.11(p.166).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 20.1.11(p.166).

(d-d2) The same as (c1,c2) of Tom 20.1.11(p.166). ∎

□ **Pom 20.1.12** ($\mathscr{A}$\{M:2[ℙ][A]$^+$\}) *Suppose* $a > 0$. *Assume* $\rho \leq a^{\star}$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.
   1. $V_t = \rho$ *for* $t \geq 0$.
   2. *Let* $x_L \leq \rho$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.
   3. *Let* $x_L > \rho$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$.
   1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* V *as* $t \to \infty$.
   2. *Let* $\rho < x_L$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   3. *Let* $\rho = x_L$. *Then* ⓓ$_{1}\langle 0\rangle_{\parallel}$ *and* ⓢ$_{\tau>0}\langle\tau\rangle_{\triangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   4. *Let* $x_L < \rho$. *Then* $\texttt{S}_4$ [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true.*

(c) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s > 0$.
   1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* V *as* $t \to \infty$.
   2. *Let* $s \geq \beta\mu T(0)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.
   3. *Let* $s < \beta\mu T(0)$.
      i. *Let* $\rho < x_L$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
      ii. *Let* $\rho = x_L$. *Then* ⓓ$_{1}\langle 0\rangle_{\parallel}$ *and* ⓢ$_{\tau>0}\langle\tau\rangle_{\triangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.
      iii. *Let* $x_L < \rho$. *Then* $\texttt{S}_4$ [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true.*

(d) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$.
   1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* V *as* $t \to \infty$.
   2. ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(e) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s > 0$.
   1. V_t *is nondecreasing in* t $(\tau \geq t \geq 0)$ *and converges to a finite* V *as* $t \to \infty$.
   2. *Let* $s \geq \beta\mu T(0)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.
   3. *Let* $s < \beta\mu T(0)$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.24)).

(a-a3) The same as (a-a3) of Tom 20.1.12(p.167).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (b1,b3i-b3iii) of Tom 20.1.12(p.167).

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. Then, we have the same as (b1-b3iii) of Tom 20.1.12(p.167).

(d-d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (c1,c3) of Tom 20.1.12(p.167).

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. Then, we have the same as (c1-c3) of Tom 20.1.12(p.167). ▮

20.1.5.3.1.2.2 Case of $b \le \rho$

☐ Pom **20.1.13** ($\mathscr{A}\{$M:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V \ge x_K$ as $t \to \infty$.*

(b) *Let $x_L \le \rho$. Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\rho < x_L$.*

1. $\textcircled{s}_1\langle 1\rangle_\blacktriangle$ *where* Conduct$_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$. Then* $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $s \le \lambda\beta T(0)$. Then* $\textcircled{s}_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
   ii. *Let $s > \lambda\beta T(0)$. Then* S$_3$(p.154) [$\textcircled{s}\blacktriangle$ | $\circledcirc\ \|$] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.24)).

(a-c2) The same as (a-c2) of Tom 20.1.13(p.167).

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i) of Tom 20.1.13(p.167).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then, we have the same as (c3i,c3ii) of Tom 20.1.13(p.167). ▮

☐ Pom **20.1.14** ($\mathscr{A}\{$M:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$.*

(b) *Let $\beta = 1$. Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$. Then* $\textcircled{s}_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.

(d) *Let $\beta < 1$ and $s > 0$.*

1. *Let $s < \lambda\beta T(0)$. Then* $\textcircled{s}_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
2. *Let $s \ge \lambda\beta T(0)$. Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.24)).

(a,b) The same as (a,b) of Tom 20.1.14(p.168).

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of Tom 20.1.14(p.168).

(d-d2) Let $\beta < 1$ and $s > 0$. Then, we have the same as (c1,c2) of Tom 20.1.14(p.168). ▮

☐ Pom **20.1.15** ($\mathscr{A}\{$M:2[$\mathbb{P}$][A]$^+\}$) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \ge 0$.*
2. *Let $x_L \le \rho$. Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $x_L > \rho$. Then* $\textcircled{s}_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $\rho < x_L$. Then* $\textcircled{s}_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
3. *Let $\rho = x_L$. Then* $\textcircled{d}_1\langle 0\rangle_{\|}$ *and* $\textcircled{s}_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
4. *Let $x_L < \rho$. Then* S$_4$ [$\textcircled{s}\blacktriangle$ | $\bullet\|$ | $c{\rightsquigarrow}s\vartriangle$ | $c{\rightsquigarrow}s\blacktriangle$] *is true.*

(c) *Let $\beta < 1$ and $\rho > 0$ and let $s > 0$.*

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \ge x_K$ as $t \to \infty$.*
2. *Let $s \ge \lambda\beta T(0)$. Then* $\textcircled{d}_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $s < \lambda\beta T(0)$.*
   i. *Let $\rho < x_L$. Then* $\textcircled{s}_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.
   ii. *Let $\rho = x_L$. Then* $\textcircled{d}_1\langle 0\rangle_{\|}$ *and* $\textcircled{s}_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\ge t>0\blacktriangle}$.

iii. *Let* $x_L < \rho$. *Then* $\mathtt{S}_4$ [Ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true.*

(d) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$.

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(e) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s > 0$.

1. V_t *is nondecreasing in* t $(\tau \geq t \geq 0)$.
2. *Let* $s \geq \lambda\beta T(0)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_\|$.
3. *Let* $s < \lambda\beta T(0)$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.24)).

(a-a3) The same as (a-a3) of Tom 20.1.15(p.168).

(b-b4) Let $\beta < 1$ and $\rho > 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (b1,b3i-b3iii) of Tom 20.1.15(p.168).

(c-c3iii) Let $\beta < 1$ and $\rho > 0$ and let $s > 0$. Then, we have the same as (b1-b3iii) of Tom 20.1.15(p.168).

(d,d2) Let $\beta < 1$ and $\rho < 0$ and let $s = 0$. Then, due to (1) it suffices to consider only (c1,c3) of Tom 20.1.15(p.168).

(e-e3) Let $\beta < 1$ and $\rho < 0$ and let $s > 0$. Then, we have the same as (c1-c3) of Tom 20.1.15(p.168). ∎

20.1.5.3.1.2.3 Case of $a^\star < \rho < b$

□ Pom **20.1.16** ($\mathscr{A}${M:2[ℙ][A]$^+$}) *Suppose* $a > 0$. *Assume* $a^\star \leq \rho < b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *If* $\lambda\beta\max\{0, a-\rho\} \leq s$, *then* ⓓ$_1\langle 0\rangle_\|$, *or else* Ⓢ$_1\langle 1\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{1\blacktriangle}$. *Below let* $\tau > 1$.

(b) *Let* $V_1 \leq x_K$.

1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
2. *Let* $V_1 \geq x_L$. *Then, if* $\lambda\beta\max\{0, a-\rho\} \leq s$, *we have* ⓓ$_{\tau>1}\langle 0\rangle_\|$, *or else* ◎$_{\tau>1}\langle 1\rangle_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
3. *Let* $V_1 < x_L$.
   - i. *Let* $\beta = 1$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   - ii. *Let* $\beta < 1$ *and* $s = 0$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   - iii. *Let* $\beta < 1$ *and* $s > 0$.
     1. *Let* $s < \lambda\beta T(0)$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
     2. *Let* $s \geq \lambda\beta T(0)$. *Then* $\mathtt{S}_5$ [Ⓢ▲ | ◎‖] *is true.*

(c) *Let* $V_1 > x_K$.

1. *Let* $\beta = 1$ *or* $V_1 = 0$.
   - i. $V_t = V_1$ *for* $t > 0$.
   - ii. *If* $\lambda\max\{0, a-\rho\} \leq s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_\|$, *or else* ◎$_{\tau>1}\langle 1\rangle_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$.
   - i. *Let* $V_1 > 0$.
     1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
     2. *Let* $V_1 > x_L$. *Then* $\mathtt{S}_6$ [Ⓢ▲ | ◎‖ | ●‖ | c⇝s△ | c⇝s▲] *is true.*
     3. *Let* $V_1 = x_L$. *Then* $\mathtt{S}_7$ [Ⓢ▲ | ◎‖ | ●‖ | c⇝s△] *is true.*
     4. *Let* $V_1 < x_L$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let* $V_1 < 0$.
     1. *Then* V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
     2. *Let* $V_1 \geq x_L$. *If* $\lambda\beta\max\{0, a-\rho\} \leq s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_\|$, *or else* ◎$_{\tau>1}\langle 1\rangle_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$.
     3. *Let* $V_1 < x_L$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>1\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s > 0$.
   - i. *Let* $V_1 > 0$.
     1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \geq x_K$ *as* $t \to \infty$.
     2. *Let* $s < \lambda\beta T(0)$.
        - i. *Let* $V_1 > x_L$. *Then* $\mathtt{S}_6$ [Ⓢ▲ | ◎‖ | ●‖ | c⇝s△ | c⇝s▲] *is true.*
        - ii. *Let* $V_1 = x_L$. *Then* $\mathtt{S}_7$ [Ⓢ▲ | ◎‖ | ●‖ | c⇝s△] *is true.*
        - iii. *Let* $V_1 < x_L$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* $\mathtt{Conduct}_{\tau\geq t>0\blacktriangle}$.
     3. *Let* $s \geq \lambda\beta T(0)$. *If* $\lambda\beta\max\{0, a-\rho\} \leq s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_\|$, *or else* ◎$_{\tau>1}\langle 1\rangle_\|$ *where* $\mathtt{Conduct}_{1\blacktriangle}$
   - ii. *Let* $V_1 < 0$.

1. *Then V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*
2. *Let $s < \lambda\beta T(0)$.*
   i. *Let $V_1 \geq x_L$. If $\lambda\beta \max\{0, a-\rho\} \leq s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* `Conduct`$_{1\blacktriangle}$.
   ii. *Let $V_1 < x_L$. Then* ⓢ$_{\tau>}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>1\blacktriangle}$.
3. *Let $s \geq \lambda\beta T(0)$. If $\lambda\beta \max\{0, a-\rho\} \leq s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* `Conduct`$_{1\blacktriangle}$

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Here note $\kappa = \lambda\beta T(0) - s$ from (5.1.23(p.24)).

(a-b3i) The same as `Tom` 20.1.16(p.169) (a-b3i).

(b3ii) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (b3ii1) of `Tom` 20.1.16(p.169).

(b3iii-b3iii2) Let $\beta < 1$ and $s > 0$. Then, the assertions are immediate from `Tom` 20.1.16(p.169) (b3ii1,b3ii2) with κ.

(c-c1ii) The same as `Tom` 20.1.16(p.169) (c-c1ii).

(c2-c2i4) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c2i-c2i1,c2i2i-c2i2iii) of `Tom` 20.1.16(p.169).

(c2ii-c2ii3) Due to (1) it suffices to consider only (c2ii-c2ii2ii) of `Tom` 20.1.16(p.169).

(c3-c3i3) Let $\beta < 1$ and $s > 0$. Then, we have the same as (c2-c2i1,c2i2i-c2i2iii) of `Tom` 20.1.16(p.169) with κ.

(c3ii-c3ii3) We have the same as (c2ii-c2ii2ii) of `Tom` 20.1.16(p.169) with κ. ∎

20.1.5.3.2 Mixed Restriction

Omitted.

20.1.5.3.3 Negative Restriction

Omitted.

20.1.6 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$\}$

20.1.6.1 Preliminary

Since Theorem 20.1.3(p.151) holds due to Lemma 20.1.1(p.151) (b), we can derive $\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$\}$ by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.128))) to $\mathscr{A}\{\mathsf{M}$:2[$\mathbb{P}$][A]$\}$.

20.1.6.2 Analysis

20.1.6.2.1 Case of $\beta = 1$ and $s = 0$

□ `Tom` **20.1.17** (⊡$\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\rho \geq b^\star$. Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$.

(c) *Let $a \geq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(d) *Let $b^\star > \rho > a$.*
   1. *Let $b \geq \rho$. Then* ⓓ$_1\langle 0\rangle_{\|}$ *and* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>1\blacktriangle}$ *and* `pSKIP`$_{1\vartriangle}$.
   2. *Let $\rho > b$. Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 20.1.9(p.166). ∎

Corollary 20.1.9 *Let $\beta = 1$ and $s = 0$. Then z_t is nonincreasing in $t \geq 0$.* □

● *Proof* Immediate from `Tom` 20.1.17(p.178) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ∎

20.1.6.2.2 Case of $\beta < 1$ or $s > 0$

20.1.6.2.2.1 Case of $\rho \geq b^{\star\dagger}$

□ `Tom` **20.1.18** (⊡$\mathscr{A}\{\tilde{\mathsf{M}}$:2[$\mathbb{P}$][A]$\}$) *Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\rho > x_{\tilde{L}}$.*
   1. ⓢ$_1\langle 1\rangle_{\blacktriangle}$ *where* `Conduct`$_{1\blacktriangle}$. *Below let $\tau > 1$.*
   2. *Let $\beta = 1$.*
      i. *Let $(\lambda b + s)/\lambda \geq b^\star$.*
         1. *Let $\lambda = 1$. Then* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* `Conduct`$_{1\blacktriangle}$.

---

†The condition of $\rho \geq b^\star$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition $\rho \leq a^\star$ in Section 20.1.5.2.2.1(p.166).

2. *Let* $\lambda < 1$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

ii. *Let* $(\lambda b + s)/\lambda < b^\star$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

3. *Let* $\beta < 1$ *and* $s = 0$ $(\!(s > 0)\!)$.

 i. *Let* $(\lambda\beta b + s)/\delta \geq b^\star$.

  1. *Let* $\lambda = 1$.

   i. *Let* $a < 0$ $(\!(\tilde\kappa < 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

   ii. *Let* $a \geq 0$ $(\!(\tilde\kappa \geq 0)\!)$. *Then* $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$ *where* $\texttt{Conduct}_{1\blacktriangle}$.

  2. *Let* $\lambda < 1$.

   i. *Let* $a \leq 0$ $(\!(\tilde\kappa \leq 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

   ii. *Let* $a > 0$ $(\!(\tilde\kappa > 0)\!)$. *Then* $\texttt{S}_3$(p.154) [Ⓢ▲ | ◎ ‖] *is true.*

 ii. *Let* $(\lambda\beta b + s)/\delta < b^\star$.

  1. *Let* $a \leq 0$ $(\!(\tilde\kappa \leq 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

  2. *Let* $a > 0$ $(\!(\tilde\kappa > 0)\!)$. *Then* $\texttt{S}_3$(p.154) [Ⓢ▲ | ◎ ‖] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.10(p.166). ▮

Corollary 20.1.10 *Assume* $\rho \geq b^\star$, *let* $\beta < 1$ *or* $s > 0$, *and let* $\rho > x_{\tilde K}$. *Then* z_t *is nonincreasing in* $t \geq 0$. □

● *Proof* Immediate from Tom 20.1.18(p.178) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

□ **Tom 20.1.19** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde K}$. *Then, for a given starting time* $\tau > 0$:

(a) V_t *is nonincreasing in* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\beta < 1$ *and* $s = 0$ $(\!(s > 0)\!)$.

 1. *Let* $a < 0$ $(\!(\tilde\kappa < 0)\!)$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *and* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

 2. *Let* $a \geq 0$ $(\!(\tilde\kappa \geq 0)\!)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$. □

● *Proof by symmetry* Clear from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.11(p.166). ▮

Corollary 20.1.11 *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde K}$. *Then* z_t *is nonincreasing in* $t \geq 0$. □

● *Proof* Immediate from Tom 20.1.19(p.179) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

□ **Tom 20.1.20** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}$) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde K}$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$.

 1. $V_t = \rho$ *for* $t \geq 0$.

 2. *Let* $x_{\tilde L} \geq \rho$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

 3. *Let* $x_{\tilde L} < \rho$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

(b) *Let* $\beta < 1$ *and* $\rho > 0$ *and let* $s = 0$ $(\!(s > 0)\!)$.

 1. V_t *is nondecreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde K}$ *as* $t \to \infty$.

 2. *Let* $a \geq 0$ $(\!(\tilde\kappa \geq 0)\!)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

 3. *Let* $a < 0$ $(\!(\tilde\kappa < 0)\!)$.

  i. *Let* $\rho > x_{\tilde L}$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

  ii. *Let* $\rho = x_{\tilde L}$. *Then* ⓓ$_1\langle 0\rangle_{\|}$ *where* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$.

  iii. *Let* $\rho < x_{\tilde L}$. *Then* $\texttt{S}_4$ [Ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true.*

(c) *Let* $\beta < 1$ *and* $\rho < 0$ *and let* $s = 0$ $(\!(s > 0)\!)$.

 1. V_t *is nonincreasing in* $t \geq 0$ *and converges to a finite* $V \leq x_{\tilde K}$ *as* $t \to \infty$.

 2. *Let* $a \geq 0$ $(\!(\tilde\kappa \geq 0)\!)$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

 3. *Let* $a < 0$ $(\!(\tilde\kappa < 0)\!)$. *Then* $\circledS_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>0\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.1.12(p.167). ▮

Corollary 20.1.12 *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde K}$.

(a) *Let* $\beta = 1$ *or* $\rho = 0$. *Then* z_t *is constant in* t ($z_t = z(\rho)$ *for* $t \geq 0$).

(b) *Let* $\beta < 1$ *and* $\rho > 0$. *Then* z_t *is nondecreasing in* $t \geq 0$ *for any* $s \geq 0$.

(c) *Let* $\beta < 1$ *and* $\rho < 0$. *Then* z_t *is nonincreasing in* $t \geq 0$ *for any* $s \geq 0$. □

● *Proof by symmetry* Evident from Tom 20.1.20(p.179) (a1,b1,c1) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

20.1.6.2.2.2 Case of $a \geq \rho$[†]

☐ `Tom` **20.1.21** (⊡$\mathscr{A}${M̃:2[ℙ][A]}) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. ⓢ$_1\langle 1\rangle_{\blacktriangle}$ *where* `Conduct`$_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.*
   - i. *Let $a \leq 0$ ⟪$\tilde{\kappa} \leq 0$⟫. Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $a > 0$ ⟪$\tilde{\kappa} > 0$⟫. Then* $\mathtt{S}_3$(p.154) [ⓢ▲ | ◎ ‖] *is true.* ☐

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.1.13(p.167). ▮

Corollary 20.1.13 *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* ☐

● *Proof* Evident from `Tom` 20.1.21(p.180) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

☐ `Tom` **20.1.22** (⊡$\mathscr{A}${M̃:2[ℙ][A]}) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\beta = 1$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫.*

1. *Let $a < 0$ ⟪$\tilde{\kappa} < 0$⟫. Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *and* `Conduct`$_{\tau\geq t>0\blacktriangle}$.
2. *Let $a \geq 0$ ⟪$\tilde{\kappa} \geq 0$⟫. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$. ☐

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.1.14(p.168). ▮

Corollary 20.1.14 *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then z_t is nonincreasing in $t \geq 0$.* ☐

● *Proof* Evident from `Tom` 20.1.22(p.180) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

☐ `Tom` **20.1.23** (⊡$\mathscr{A}${M̃:2[ℙ][A]}) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⟪$s > 0$⟫.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ ⟪$\tilde{\kappa} \geq 0$⟫. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $a < 0$ ⟪$\tilde{\kappa} < 0$⟫.*
   - i. *Let $\rho > x_{\tilde{L}}$. Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$.
   - ii. *Let $\rho = x_{\tilde{L}}$. Then* ⓓ$_1\langle 0\rangle_{\|}$ *where* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$.
   - iii. *Let $x_{\tilde{L}} > \rho$. Then* $\mathtt{S}_4$ [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] *is true.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⟪$s > 0$⟫.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $a \geq 0$ ⟪$\tilde{\kappa} \geq 0$⟫. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.
3. *Let $a < 0$ ⟪$\tilde{\kappa} < 0$⟫. Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>0\blacktriangle}$. ☐

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.1.15(p.168). ▮

Corollary 20.1.15 *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$. Then $z_t = z(\rho)$ for $t \geq 0$.*

(b) *Let $\beta < 1$ and $\rho > 0$ and let $s = 0$ ⟪$s > 0$⟫. Then z_t is nondecreasing in $t \geq 0$.*

(c) *Let $\beta < 1$ and $\rho < 0$ and let $s = 0$ ⟪$s > 0$⟫. Then z_t is nonincreasing in $t \geq 0$.* ☐

● *Proof* Evident from `Tom` 20.1.23(p.180) (a1,b1,c1) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

[†] The condition of $a \geq \rho$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition of $b \leq \rho$ in Section 20.1.5.2.2.2(p.167).

20.1.6.2.2.3 Case of $b^\star > \rho > a$[†]

Let us here note that (20.1.43(p.168)) changes as follows.

$$V_1 = \min\{\lambda\beta\min\{0, b-\rho\} + s, 0\} + \beta\rho.^{\dagger} \tag{20.1.44}$$

□ `Tom` **20.1.24** (⊡$\mathscr{A}\{\tilde{\text{M}}$:2[ℙ][A]$\}$) *Assume $b^\star > \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\min\{0, \rho-b\} \ge -s$, then* ⓓ$_1\langle 0\rangle_{\parallel}$*, or else* Ⓢ$_1\langle 1\rangle_{\blacktriangle}$ *where* `Conduct`$_{1\blacktriangle}$*. Below let $\tau > 1$.*

(b) *Let $V_1 \ge x_{\tilde{K}}$.*
  1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*
  2. *Let $V_1 \le x_{\tilde{L}}$. Then, if $\lambda\beta\min\{0, \rho-b\} \ge -s$, we have* ⓓ$_{\tau>1}\langle 0\rangle_{\parallel}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* `Conduct`$_{1\blacktriangle}$*.*
  3. *Let $V_1 > x_{\tilde{L}}$.*
     - i. *Let $\beta = 1$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\ge t>1\blacktriangle}$*.*
     - ii. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
       1. *Let $a < 0$ ⦅$\tilde{\kappa} < 0$⦆. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\ge t>1\blacktriangle}$*.*
       2. *Let $a \ge 0$ ⦅$\tilde{\kappa} \ge 0$⦆. Then* $\texttt{S}_5$ [Ⓢ▲ | ◎ ‖] *is true.*

(c) *Let $V_1 < x_{\tilde{K}}$.*
  1. *Let $\beta = 1$ or $V_1 = 0$.*
     - i. *$V_t = V_1$ for $t > 0$.*
     - ii. *If $\lambda\min\{0, \rho-b\} \ge -s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\parallel}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* `Conduct`$_{1\blacktriangle}$*.*
  2. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*[†]
     - i. *Let $V_1 < 0$.*
       1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $\tau \to \infty$.*
       2. *Let $a < 0$ ⦅$\tilde{\kappa} < 0$⦆. Then*
          - i. *Let $V_1 < x_{\tilde{L}}$. Then* $\texttt{S}_6$ [Ⓢ▲ | ●‖ | ◎ ‖ | c⇝s△ | c⇝s▲] *is true.*
          - ii. *Let $V_1 = x_{\tilde{L}}$. Then* $\texttt{S}_7$ [Ⓢ▲ | ●‖ | ◎ ‖ | c⇝s△] *is true.*
          - iii. *Let $V_1 > x_{\tilde{L}}$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\ge t>0\blacktriangle}$*.*
       3. *Let $a \ge 0$ ⦅$\tilde{\kappa} \ge 0$⦆. If $\lambda\beta\min\{0, \rho-b\} \ge -s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\parallel}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* `Conduct`$_{1\blacktriangle}$*.*
     - ii. *Let $V_1 > 0$.*
       1. *Then V_t is nonincreasing in $t \ge 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $\tau \to \infty$.*
       2. *Let $a < 0$ ⦅$\tilde{\kappa} < 0$⦆. Then*
          - i. *Let $V_1 \le x_{\tilde{L}}$. If $\lambda\beta\min\{0, \rho-b\} \ge -s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\parallel}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* `Conduct`$_{1\blacktriangle}$*.*
          - ii. *Let $V_1 > x_{\tilde{L}}$. Then* Ⓢ$_{\tau>}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\ge t>1\blacktriangle}$*.*
       3. *Let $a \ge 0$ ⦅$\tilde{\kappa} \ge 0$⦆. If $\lambda\beta\min\{0, \rho-b\} \ge -s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\parallel}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* `Conduct`$_{1\blacktriangle}$*.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.1.16(p.169). ▮

Corollary 20.1.16 *Assume $b^\star > \rho > a$. Let $\beta < 1$ or $s > 0$:*

(a) *Let $V_1 \ge x_{\tilde{K}}$. Then z_t is nonincreasing in $t > 0$.*

(b) *Let $V_1 < x_{\tilde{K}}$. Then*
  1. *Let $\beta = 1$ or $V_1 = 0$. Then z_t is constant in $t > 0$ ($z_t = z(V_1)$ for $t > 0$).*
  2. *Let $\beta < 1$.*
     - i. *Let $V_1 < 0$. Then z_t is nondecreasing in $t > 0$ for any $s \ge 0$.*
     - ii. *Let $V_1 > 0$. Then z_t is nonincreasing in $t > 0$ for any $s \ge 0$.* □

● *Proof* Immediate from `Tom` 20.1.24(p.181) (b1,c1i,c2i1,c2ii1) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

[†] The condition of $b^\star > \rho > a$ is what results from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to the condition of $a^\star < \rho < b$ in Section 20.1.5.2.2.3(p.168).

[†]
$$\begin{aligned}
-\hat{V}_1 &= \max\{\lambda\beta\max\{0, -\hat{a}+\hat{\rho}\} - s, 0\} - \beta\hat{\rho} && \text{(apply the reverse to (20.1.43(p.168)))}\\
\hat{V}_1 &= -\max\{\lambda\beta\max\{0, -\hat{a}+\hat{\rho}\} - s, 0\} + \beta\hat{\rho} && \text{(multiply the above by } -1)\\
&= \min\{-\lambda\beta\max\{0, -\hat{a}+\hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(arrangement the above)}\\
&= \min\{\lambda\beta\min\{0, \hat{a}-\hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(arrangement the above)}\\
\hat{V}_1 &= \min\{\lambda\beta\min\{0, \check{b}-\hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(apply } \mathcal{I}_{\mathbb{R}} \text{ to the above)}\\
\hat{V}_1 &= \min\{\lambda\beta\min\{0, b-\hat{\rho}\} + s, 0\} + \beta\hat{\rho} && \text{(apply } \mathcal{C}_{\mathbb{R}} \text{ to the above)}\\
V_1 &= \min\{\lambda\beta\min\{0, b-\rho\} + s, 0\} + \beta\rho && \text{(remove the hat symbol } \hat{\ })
\end{aligned}$$

[†] See Remark 20.1.2(p.169).

20.1.6.3 Market Restriction

20.1.6.3.1 Positive Restriction

20.1.6.3.1.1 Case of $\beta = 1$ and $s = 0$

□ Pom **20.1.17** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{A}]^{+}\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *Let $\rho \geq b^\star$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.

(c) *Let $a \geq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.

(d) *Let $b^\star > \rho > a$.*

1. *Let $b \geq \rho$. Then* ⓓ$_1\langle 0\rangle_{\parallel}$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$ *and* C⇝S$_{1\vartriangle}$.
2. *Let $\rho > b$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$. □

● *Proof* The same as Tom 20.1.17(p.178) due to Lemma 17.4.4(p.116). ∎

20.1.6.3.1.2 Case of $\beta < 1$ or $s > 0$

20.1.6.3.1.2.1 Case of $\rho \geq b^\star$

□ Pom **20.1.18** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{A}]^{+}\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. Ⓢ$_1\langle 1\rangle_\blacktriangle$ *and* Conduct$_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda b + s)/\lambda \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* Conduct$_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.
   ii. *Let $(\lambda b + s)/\lambda < b^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$ and $s > 0$. Then we have* S3(p.154) [Ⓢ▲ | ◎ ‖].
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $(\lambda\beta b + s)/\delta \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$ *where* Conduct$_{1\blacktriangle}$.
      2. *Let $\lambda < 1$. Then* S3(p.154) [Ⓢ▲ | ◎ ‖] *is true.*
   ii. *Let $(\lambda\beta b + s)/\delta < b^\star$. Then* S3(p.154) [Ⓢ▲ | ◎ ‖] *is true.* □

● *Proof* Suppose $a > 0 \cdots (1)$, hence $b > a > 0 \cdots (2)$ and $b^\star > 0 \cdots (3)$ from Lemma 14.6.1(p.105) (n) and (2). Then we have $\tilde{\kappa} = s \cdots (4)$ from Lemma 14.6.6(p.106) (a).

(a-c2ii) The same as (a-c2ii) of Tom 20.1.18(p.178).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta b + s)/\delta \geq b^\star$. Then since $\lambda\beta b/\delta \geq b^\star$, we have $\lambda\beta b \geq \delta b^\star$ from (10.2.2 (1) (p.54)), hence $\lambda\beta b \geq \delta b^\star \geq \lambda b^\star$ due to (10.2.2 (1) (p.54)), so $\beta b \geq b^\star$, which contradicts [7(p.116)]. Thus it must be that $(\lambda\beta b + s)/\delta < b^\star$. From this it suffices to consider only (c3ii2) of Tom 20.1.18(p.178).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\tilde{\kappa} > 0$ due (4), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of Tom 20.1.18(p.178); accordingly, whether $s = 0$ or $s > 0$, we have the same result. ∎

□ Pom **20.1.19** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{A}]^{+}\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *We have* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$. □

● *Proof* Suppose $a > 0$. Then $\tilde{\kappa} = s \cdots (1)$ from Lemma 14.6.6(p.106) (a).

(a) The same as Tom 20.1.19(p.179) (a).

(b) Let $\beta = 1$. Then, we have ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$ from (b) of Tom 20.1.19(p.179). Let $\beta < 1$. Then, if $s = 0$, it suffices to consider only (c2) of Tom 20.1.19(p.179) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (1), hence it suffices to consider only (c2) of Tom 20.1.19(p.179); accordingly, whether $s = 0$ or $s > 0$, we have the same results. Therefore, whether $\beta = 1$ or $\beta < 1$, we have the same result. ∎

□ Pom **20.1.20** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{A}]^{+}\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $❹_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\beta < 1$ and $\rho < 0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $❹_{\tau>0}\langle 0\rangle_{\parallel}$. □

• *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.106) (a).

(a-a3) The same as (a-a3) of Tom 20.1.20(p.179).

(b-b2) Let $\beta < 1$ and $\rho > 0$. First, we have the same as (b1) of Pom 20.1.20. Next, if $s = 0$, then due to (1) it suffices to consider only (b2) of Tom 20.1.20(p.179) and if $s > 0$, then since $\tilde{\kappa} > 0$ from (2), it suffices to consider only (b2) of Tom 20.1.20(p.179). Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c-c2) Let $\beta < 1$ and $\rho < 0$. First, we have the same as (c1) of Pom 20.1.20. Next, if $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 20.1.20(p.179) and if $s > 0$, then since $\tilde{\kappa} > 0$ from (2), it suffices to consider only (c2) of Tom 20.1.20(p.179). Thus, whether $s = 0$ or $s > 0$, we have the same result. ▮

20.1.6.3.1.2.2 Case of $a \geq \rho$

□ **Pom 20.1.21** ($\mathscr{A}\{\tilde{M}{:}2[\mathbb{P}][A]^+\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $❹_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\rho > x_{\tilde{L}}$.*

1. $Ⓢ_1\langle 1\rangle_{\blacktriangle}$ *where* Conduct$_{1\blacktriangle}$. *Below let $\tau > 1$.*
2. *Let $\beta = 1$. Then* $Ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.
3. *Let $\beta < 1$. Then* S$_3$(p.154) [Ⓢ▲ | ◎ ‖] *is true.* □

• *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.106) (a).

(a-c2) The same as (a-c2) of Tom 20.1.21(p.180).

(c3) Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (c3ii) of Tom 20.1.21(p.180) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c3ii) of Tom 20.1.21(p.180). Thus, whether $s = 0$ or $s > 0$, we have the same result. ▮

□ **Pom 20.1.22** ($\mathscr{A}\{\tilde{M}{:}2[\mathbb{P}][A]^+\}$) *Suppose $a > 0$. Assume $a > \rho$. Let $\beta < 1$ or $s > 0$, and let $\rho = x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$.*

(b) *We have* $❹_{\tau>0}\langle 0\rangle_{\parallel}$. □

• *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.106) (a).

(a) The same as Tom 20.1.22(p.180) (a).

(b) Let $\beta = 1$. Then $❹_{\tau>0}\langle 0\rangle_{\parallel}$ from (b) of Tom 20.1.22(p.180). Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (c2) of Tom 20.1.22(p.180) and if $s > 0$, then $\tilde{\kappa} \geq 0$ due to (2), hence it suffices to consider only (c2) of Tom 20.1.22(p.180) with $\tilde{\kappa}$. Accordingly, whether $s = 0$ or $s > 0$, we have $❹_{\tau>0}\langle 0\rangle_{\parallel}$. Thus, whether $\beta = 1$ or $\beta < 1$, we have $❹_{\tau>0}\langle 0\rangle_{\parallel}$. ▮

□ **Pom 20.1.23** ($\mathscr{A}\{\tilde{M}{:}2[\mathbb{P}][A]^+\}$) *Suppose $a > 0$. Assume $a > \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *Let $\beta = 1$ or $\rho = 0$.*

1. *$V_t = \rho$ for $t \geq 0$.*
2. *Let $x_{\tilde{L}} \geq \rho$. Then* $❹_{\tau>0}\langle 0\rangle_{\parallel}$.
3. *Let $x_{\tilde{L}} < \rho$. Then* $Ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ *where* Conduct$_{\tau\geq t>0\blacktriangle}$.

(b) *Let $\beta < 1$ and $\rho > 0$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $❹_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\beta < 1$ and $\rho < 0$.*

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. $❹_{\tau>0}\langle 0\rangle_{\parallel}$. □

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $b > a > 0$. Then $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 14.6.6(p.106) (a).

(a-a3) The same as `Tom` 20.1.23(p.180) (a-a3).

(b) Let $\beta < 1$ and $\rho > 0$.

(b1) The same as (b1) of `Pom` 20.1.23.

(b2) If $s = 0$, then due to (1) it suffices to consider only (b2) of `Tom` 20.1.23(p.180) and if $s > 0$, then $\tilde{\kappa} > 0$ from (2), hence it suffices to consider only (b2) of `Tom` 20.1.23(p.180). Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c1) The same as `Pom` c1(b1).

(c2) If $s = 0$, then due to (1) it suffices to consider only (c2) of `Tom` 20.1.23(p.180) and if $s > 0$, then $\tilde{\kappa} > 0$ from (2), hence it suffices to consider only (c2) of `Tom` 20.1.23(p.180). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.1.6.3.1.2.3 Case of $b^\star > \rho > b$

□ `Pom` **20.1.24** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{A}]^+\}$) *Suppose $a > 0$. Assume $b^\star > \rho > b$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta \min\{0, \rho - b\} \geq -s$, then* ⓓ$_1\langle 0\rangle_{\|}$, *or else* ⓢ$_1\langle 1\rangle_{\blacktriangle}$ *where* `Conduct`$_{1\blacktriangle}$. *Below let $\tau > 1$.*

(b) *Let $V_1 \geq x_{\tilde{K}}$.*
1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $V_1 \leq x_{\tilde{L}}$. Then, if $\lambda\beta \max\{0, \rho - b\} \leq s$, we have* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* `Conduct`$_{1\blacktriangle}$.
3. *Let $V_1 > x_{\tilde{L}}$.*
   i. *Let $\beta = 1$. Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *where* `Conduct`$_{\tau\geq t>1\blacktriangle}$.
   ii. *Let $\beta < 1$. Then* $\mathtt{S}_5$ [ⓢ▲ | ◎ ‖] *is true.*

(c) *Let $V_1 < x_{\tilde{K}}$.*
1. *Let $\beta = 1$ or $V_1 = 0$. Then:*
   i. *$V_t = V_1$ for $t > 0$.*
   ii. *If $\lambda \min\{0, \rho - b\} \geq -s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* `Conduct`$_{1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*[†]
   i. *Let $V_1 < 0$.*
      1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
      2. *If $\lambda\beta \min\{0, \rho - b\} \geq -s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* `Conduct`$_{1\blacktriangle}$.
   ii. *Let $V_1 > 0$.*
      1. *Then V_t is nonincreasing in $t \geq 0$ and converges to a finite $V \leq x_{\tilde{K}}$ as $t \to \infty$.*
      2. *If $\lambda\beta \min\{0, \rho - b\} \geq -s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\|}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\|}$ *where* `Conduct`$_{1\blacktriangle}$.

● *Proof* Suppose $a > 0 \cdots$ **(1)**. Then $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 14.6.6(p.106) (a).

(a-b3i) The same as (a-b3i) of `Tom` 20.1.24(p.181).

(b3ii) Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (b3ii2) of `Tom` 20.1.24(p.181) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b3ii2) of `Tom` 20.1.24(p.181). Accordingly, whether $s = 0$ or $s > 0$, we have the same result.

(c) Let $V_1 < x_{\tilde{K}}$.

(c1-c1ii) The same as (c1-c1ii) of `Tom` 20.1.24(p.181).

(c2) Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.

(c2i,c2i1) The same as (c2i,c2i1) of `Tom` 20.1.24(p.181).

(c2i2) The same as `Tom` 20.1.24(p.181) (c2i3).

(c2ii,c2ii1) The same as (c2ii,c2ii1) of `Tom` 20.1.24(p.181).

(c2ii2) If $s = 0$, then due to (1) it suffices to consider only (c2ii3) of `Tom` 20.1.24(p.181) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence t suffices to consider only (c2ii3) of `Tom` 20.1.24(p.181). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.1.6.3.1.2.4 Mixed Restriction

Omitted.

20.1.6.3.1.2.5 Negative Restriction

Unnecessary.

20.1.7 Numerical Calculation

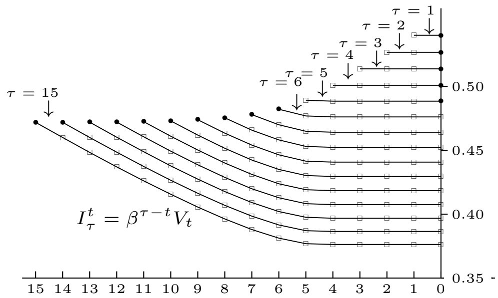
Numerical Example **5** ($\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{A}]^+\}$ **(selling model)**

This is the example for [c⇝s▲] of $\mathtt{S}_4$ [ⓢ▲ | ●‖ | c⇝s△ | c⇝s▲] in `Pom` 20.1.4(p.160) (c3iii) in which $a > 0$, $\rho > x_K$, $\beta < 1$, $\rho > 0$, $s > 0$, and $x_L < \rho$. As an example let $a = 0.01$, $b = 1.00$, $\lambda = 0.7$, $\beta = 0.98$, $s = 0.1$, and $\rho = 0.5$where $x_L = 0.462767$.[†] The graph below is for $I^t_\tau = \beta^{\tau-t}V_t$, $\tau = 1, 2, \cdots, 15$ and $t = 0, 1, \cdots, \tau$, where • represents the optimal initiating time (`OIT`) for each $\tau = 1, 2, \cdots, 15$ (see t^*_τ-column in the table below).

[†]See Remark 20.1.2(p.169).

[†]Note that $a = 0.01 > 0$, $\rho = 0.5 > 0$, $\beta = 0.98 < 1$, and $s = 0.1 > 0$. In addition, since $\mu = (1.00 + 0.01)/2 = 0.505$, we have $\lambda\beta\mu = 0.34643 > 0.1 = s$. Furthermore, we have $x_L = 0.4627674 < 0.5 = \rho$. Thus the condition of the assertion is satisfied.

1. Since $\Delta_\beta V_1 = \Delta_\beta V_2 = \Delta_\beta V_3 = \Delta_\beta V_4 = 0$ (see $\Delta_\beta V_t$-column in the table below), we have $V_4 = \beta V_3$, $V_3 = \beta V_2$, $V_2 = \beta V_1$, and $V_1 = \beta V_0$, implying that it becomes *indifferent* to skip the search up to the deadline $t_d = 0$ on $t = 4, 3, 2, 1$ (see Preference Rule 7.2.1(p.43)), i.e., ⓓ$_{\tau=4,3,2,1}\langle 0\rangle_\blacktriangle$. On the other hand, since $L(V_{t-1}) < 0$ for $1 \le t \le 4$ (see $L(V_{t-1})$-column in the table below), it follows that it is *strictly optimal* to skip the search up to the deadline 0 (see (20.1.26(p.154))) for $1 \le t \le \tau = 4$, i.e., ⓓ$_{\tau=4,3,2,1}\langle 0\rangle_\blacktriangle$. Although the above two results "*indifferent*" and "*strictly optimal*" seem to contradict each other at a glance, it is what is caused by the jumble of intuition and theory (see Alice 2(p.42)).
2. Each of the graphs for $\tau = 6, 7, \cdots, 15$ shows that the optimal initiating time is *strictly*, i.e., Ⓢ$_{6\le\tau\le15}\langle\tau\rangle_\blacktriangle$, meaning that the immediate initiation is strictly optimal and that conducting the search is *strictly optimal* at time $t = 6, 7, \cdots, 15$ (**Conduct**$_\blacktriangle$) and skipping the search becomes *strictly optimal* at time $t = 5, 4, 3, 2, 1$ after that (see $L(V_{t-1})$-column in the table below), implying that we have C⇝S$_\blacktriangle$ (see Remark 7.2.1(p.42)) occurs.



t	V_t	$\Delta_\beta V_t = V_t - \beta V_{t-1}$	t_τ^*	$L(V_{\tau-1})$
0	0.5000000			
1	0.4900000	0.0000000	0	−0.0133838 (SKIP$_\blacktriangle$)
2	0.4802000	0.0000000	0	−0.0098846 (SKIP$_\blacktriangle$)
3	0.4705960	0.0000000	0	−0.0063880 (SKIP$_\blacktriangle$)
4	0.4611841	0.0000000	0	−0.0028969 (SKIP$_\blacktriangle$)
5	0.4525469	+0.0005865	5	+0.0005865 (Conduct$_\blacktriangle$)
6	0.4473331	+0.0038371	6	+0.0038371 (Conduct$_\blacktriangle$)
7	0.4442109	+0.0058244	7	+0.0058244 (Conduct$_\blacktriangle$)
8	0.4423501	+0.0070235	8	+0.0070235 (Conduct$_\blacktriangle$)
9	0.4412444	+0.0077413	9	+0.0077413 (Conduct$_\blacktriangle$)
10	0.4405885	+0.0081690	10	+0.0081690 (Conduct$_\blacktriangle$)
11	0.4401998	+0.0084231	11	+0.0084231 (Conduct$_\blacktriangle$)
12	0.4399696	+0.0085738	12	+0.0085738 (Conduct$_\blacktriangle$)
13	0.4398333	+0.0086631	13	+0.0086631 (Conduct$_\blacktriangle$)
14	0.4397527	+0.0087160	14	+0.0087160 (Conduct$_\blacktriangle$)
15	0.4397049	+0.0087473	15	+0.0087473 (Conduct$_\blacktriangle$)

Figure 20.1.1: Graphs of $I_\tau^t = \beta^{\tau-t} V_t$ $(15 \ge \tau > 1, \tau \ge t > 0)$

20.1.8 Conclusion 3 (Search-Allowed-Model 2)

C1. Mental Conflict

On $\mathscr{F}^+$, we have (see (7.3.1(p.45)) and (7.3.2(p.45)) for the definitions of opt-ℝ-price and opt-ℙ-price below):

a. Let $\beta = 1$ and $s = 0$.

1. The opt-ℝ-price V_t in M:2[ℝ][A] (selling model) is nondecreasing in t $\blacktriangle^{a}$ (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).
2. The opt-ℙ-price z_t in M:2[ℙ][A] (selling model) is nondecreasing in t $\blacktriangle^{b}$ (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).
3. The opt-ℝ-price V_t in M̃:2[ℝ][A] (buying model) is nonincreasing in t $\blacktriangledown^{c}$ (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)).
4. The opt-ℙ-price z_t in M̃:2[ℙ][A] (buying model) is nonincreasing in t $\blacktriangledown^{d}$ (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)).

- $\blacktriangle^{a}$ ← Tom's 20.1.1(p.154) (a).
- $\blacktriangle^{b}$ ← Corollaries 20.1.1(p.166) .
- $\blacktriangledown^{c}$ ← Tom's 20.1.5(p.161) (a).
- $\blacktriangledown^{d}$ ← Corollaries 20.1.9(p.178) .

b. Let $\beta < 1$ or $s > 0$.

1. The opt-ℝ-price V_t in M:2[ℝ][A] (selling model) is nondecreasing $\blacktriangle^{a}$, constant $\|^{a}$, or nonincreasing in t $\blacktriangledown^{a}$ (see Figure 7.3.2(p.45) (I)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).
2. The opt-ℙ-price z_t in M:2[ℙ][A] (selling model) is nondecreasing $\blacktriangle^{b}$, constant $\|^{b}$, or nonincreasing in t $\blacktriangledown^{b}$ (see Figure 7.3.2(p.45) (I)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).
3. The opt-ℝ-price V_t in M̃:2[ℝ][A] (buying model) is nondecreasing $\blacktriangle^{c}$, constant $\|^{c}$, or nonincreasing in t $\blacktriangledown^{c}$ (see Figure 7.3.2(p.45) (II)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).
4. The opt-ℙ-price z_t in M̃:2[ℙ][A] (buying model) is nondecreasing $\blacktriangle^{d}$, constant $\|^{d}$, or nonincreasing in t $\blacktriangledown^{d}$ (see Figure 7.3.2(p.45) (II)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).

- $\blacktriangle^{a}$ ← 20.1.2(p.154) (a), 20.1.3(p.157) (a), 20.1.4(p.157) (c1).
  $\|^{a}$ ← Tom 20.1.4(p.157) (a1)).
  $\blacktriangledown^{a}$ ← Tom 20.1.4(p.157) (b1).
- $\blacktriangle^{b}$ ← 20.1.2(p.166) , 20.1.3(p.167) ,20.1.4(p.167) (c),
  20.1.5(p.168) , 20.1.6(p.168) , 20.1.7(p.168) (c), 20.1.8(p.174) (a,b2ii).
  $\|^{b}$ ← Corollary 20.1.4(p.167) (a), 20.1.7(p.168) (a), 20.1.8(p.174) (b1).
  $\blacktriangledown^{b}$ ← Corollaries 20.1.4(p.167) (b), 20.1.7(p.168) (b), 20.1.8(p.174) (b2i).
- $\blacktriangle^{c}$ ← Tom 20.1.8(p.162) (b1).
  $\|^{c}$ ← Tom 20.1.8(p.162) (a1).
  $\blacktriangledown^{c}$ ← 20.1.6(p.162) (a), 20.1.7(p.162) (a), 20.1.8(p.162) (c1).

- $\blacktriangle^{\mathrm{d}}$ ← Corollaries 20.1.12(p.179) (b), 20.1.15(p.180) (b),20.1.16(p.181) (b2i).
- $\|^{\mathrm{d}}$ ← Corollaries 20.1.15(p.180) (a), 20.1.16(p.181) (b1).
- $\blacktriangledown^{\mathrm{d}}$ ← 20.1.10(p.179) , 20.1.11(p.179) , 20.1.12(p.179) (c),
  20.1.13(p.180) , 20.1.14(p.180) , 20.1.15(p.180) (c), 20.1.16(p.181) (a),b2ii).

The above results can be summarized as below.

A. If $\beta = 1$ and $s = 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict in *Examples* 1.4.1(p.6) - 1.4.4(p.6) .

B. If $\beta < 1$ or $s > 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the abnormal mental conflict.

C2. Symmetry

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$. Then we have:

$$
\begin{aligned}
&\texttt{Pom}\ 20.1.5\text{(p.163)} \sim \texttt{Pom}\ 20.1.1\text{(p.159)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.17\text{(p.182)} \sim \texttt{Pom}\ 20.1.9\text{(p.174)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+).
\end{aligned}
$$

b. Let $\beta < 1$ or $s > 0$. Then we have

$$
\begin{aligned}
&\texttt{Pom}\ 20.1.6\text{(p.163)} \nsim \texttt{Pom}\ 20.1.2\text{(p.159)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.7\text{(p.163)} \nsim \texttt{Pom}\ 20.1.3\text{(p.160)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.8\text{(p.163)} \nsim \texttt{Pom}\ 20.1.4\text{(p.160)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.18\text{(p.182)} \nsim \texttt{Pom}\ 20.1.10\text{(p.174)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.19\text{(p.182)} \nsim \texttt{Pom}\ 20.1.11\text{(p.175)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.20\text{(p.182)} \nsim \texttt{Pom}\ 20.1.12\text{(p.175)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.21\text{(p.183)} \nsim \texttt{Pom}\ 20.1.13\text{(p.176)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.22\text{(p.183)} \nsim \texttt{Pom}\ 20.1.14\text{(p.176)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.23\text{(p.183)} \nsim \texttt{Pom}\ 20.1.15\text{(p.176)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.24\text{(p.184)} \nsim \texttt{Pom}\ 20.1.16\text{(p.177)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \nsim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+).
\end{aligned}
$$

The above results can be summarized as below.

A. Let $\beta = 1$ and $s = 0$. Then the symmetry is inherited.

B. Let $\beta < 1$ or $s > 0$. Then the symmetry collapses.

C3. Analogy

On $\mathscr{F}^+$, for any $\beta \le 1$ and $s \ge 0$ we have:

a. We have:

$$
\begin{aligned}
&\texttt{Pom}\ 20.1.9\text{(p.174)} \bowtie \texttt{Pom}\ 20.1.1\text{(p.159)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.10\text{(p.174)} \bowtie \texttt{Pom}\ 20.1.2\text{(p.159)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.17\text{(p.182)} \bowtie \texttt{Pom}\ 20.1.5\text{(p.163)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]\}^+),\\
&\texttt{Pom}\ 20.1.18\text{(p.182)} \bowtie \texttt{Pom}\ 20.1.6\text{(p.163)} && (\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]\}^+).
\end{aligned}
$$

The above results can be summarized as below.

A. The analogy collapses.

C4. Optimal initiating time (OIT)

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 20.1.1(p.159) , Pom 20.1.5(p.163) , Pom 20.1.9(p.174) , Pom 20.1.17(p.182) ,

we have the following table:

Table 20.1.1: Possible OIT ($\beta = 1$ and $s = 0$)

	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$
Ⓢ$_\tau\langle\tau\rangle_{\|}$				
Ⓢ$_\tau\langle\tau\rangle_{\triangle}$				
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○	○
◎$_\tau\langle t^\bullet_\tau\rangle_{\|}$				
◎$_\tau\langle t^\bullet_\tau\rangle_{\triangle}$				
◎$_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$				
ⓓ$_\tau\langle 0\rangle_{\|}$	○	○	○	○
ⓓ$_\tau\langle 0\rangle_{\triangle}$				
ⓓ$_\tau\langle 0\rangle_{\blacktriangle}$				

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 20.1.2(p.159), Pom 20.1.3(p.160), Pom 20.1.4(p.160), Pom 20.1.5(p.163), Pom 20.1.6(p.163),
Pom 20.1.7(p.163), Pom 20.1.8(p.163), Pom 20.1.10(p.174), Pom 20.1.11(p.175), Pom 20.1.12(p.175),
Pom 20.1.13(p.176), Pom 20.1.14(p.176), Pom 20.1.15(p.176), Pom 20.1.16(p.177), Pom 20.1.19(p.182),
Pom 20.1.20(p.182), Pom 20.1.21(p.183), Pom 20.1.22(p.183), Pom 20.1.23(p.183), Pom 20.1.24(p.184),

we have the following table:

Table 20.1.2: Possible `OIT` ($\beta <$ or $s > 0$)

	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$
Ⓢ$_\tau\langle\tau\rangle_{\|}$				
Ⓢ$_\tau\langle\tau\rangle_{\triangle}$				
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○	○
◎$_\tau\langle t^\bullet_\tau\rangle_{\|}$	○	○	○	○
◎$_\tau\langle t^\bullet_\tau\rangle_{\triangle}$				
◎$_\tau\langle t^\bullet_\tau\rangle_{\blacktriangle}$				
ⓓ$_\tau\langle 0\rangle_{\|}$	○	○	○	○
ⓓ$_\tau\langle 0\rangle_{\triangle}$				
ⓓ$_\tau\langle 0\rangle_{\blacktriangle}$				

c. The table below is the list of the occurrence rates of Ⓢ, ◎, and ⓓ (Def. 11.2.4(p.61)) on $\mathscr{F}$ (See the primitive Tom's 20.1.1(p.154) (▣), 20.1.2(p.154) (▣), 20.1.3(p.157) (▣), 20.1.4(p.157) (▣), 20.1.9(p.166) (▣), and 20.1.16(p.169) (▣).

Table 20.1.3: Occurence rates of Ⓢ, ◎, and ⓓ on $\mathscr{F}$

Ⓢ			◎			ⓓ		
46.6 % / 28			21.6 % / 13			31.6 % / 19		
Ⓢ$_{\|}$	Ⓢ$_{\triangle}$	Ⓢ$_{\blacktriangle}$	◎$_{\|}$	◎$_{\triangle}$	◎$_{\blacktriangle}$	ⓓ$_{\|}$	ⓓ$_{\triangle}$	ⓓ$_{\blacktriangle}$
—	×	possible	possible	×	×	possible	×	×
− % / −	0.0 % / 0	46.6 % / 28	21.6 % / 13	0.0 % / 0	0.0 % / 0	31.6 % / 19	0.0 % / 0	0.0 % / 0

C5. Null-time-zone and deadline-engulfing

From Table 20.1.3(p.187) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.43) for the implication of the symbol "▲" representing the strict optimality of the initiating time t^*_τ.

b. As a whole, Ⓢ, ◎, and ⓓ are possible at 47.5%, 21.3%, and 31.2% respectively where

1. Ⓢ$_{\|}$ cannot be defined (see Remark **??**(p.??)).
2. ◎$_{\|}$ is possible (21.6 %).
3. ⓓ$_{\|}$ is possible (31.6 %).
4. Ⓢ$_{\triangle}$ never occur (0.0 %).
5. ◎$_{\triangle}$ never occur (0.0 %).
6. ⓓ$_{\triangle}$ never occur (0.0 %).

7. ⓢ$_\blacktriangle$ is possible (46.6 %).
8. ◎$_\blacktriangle$ never occurs (0.0 %).
9. ⓓ$_\blacktriangle$ never occurs (0.0 %).

From the above results we see that:

A. ◎$_\parallel$ and ⓓ$_\parallel$ causing the `null-time-zone` are possible at 53.2% (= 21.6% + 31.6%).

B. ◎$_\blacktriangle$ *strictly* causing the `null-time-zone` is impossible (0.0%).

C. ⓓ$_\blacktriangle$ *strictly* causing the `deadline-engulfing` is impossible (0.0%).

C6. `C⇝S` On $\mathscr{F}^+$, we have (see (A5b(p.12))):

Let $\beta < 1$ or $s > 0$. Then, from `Pom`'s 20.1.4(p.160), 20.1.12(p.175), and 20.1.15(p.176) we have the following table:

Table 20.1.4: `C⇝S` ($\beta < 1$ or $s > 0$)

	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]^+\}$	$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]^+\}$
(a) `C⇝S`$_\triangle$	○		○	
(b) `C⇝S`$_\blacktriangle$	○		○	

A. `C⇝S`$_\triangle$ and `C⇝S`$_\blacktriangle$ occur only for $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]^+$ and $\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]^+$ (both are a selling model).

20.2 Search-Enforced-Model 2: $\mathcal{Q}\{\mathsf{M}{:}2[\mathsf{E}]\} = \{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}], \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}], \mathsf{M}{:}2[\mathbb{P}][\mathsf{E}], \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}$

20.2.1 Theorems

As ones corresponding to Theorems 19.2.1(p.134), 19.2.2(p.134), and 19.2.3(p.134), let us consider here the following three theorems:

Theorem 20.2.1 (symmetry$[\mathbb{R} \to \mathbb{R}]$**))** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{20.2.1}$$

Theorem 20.2.2 (analogy$[\mathbb{R} \to \mathbb{P}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{20.2.2}$$

Theorem 20.2.3 (symmetry$[\mathbb{P} \to \mathbb{P}]$**)** *Let* $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}]. \quad \square \tag{20.2.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}], \tag{20.2.4}$$

$$\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}], \tag{20.2.5}$$

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}], \tag{20.2.6}$$

corresponding to (19.2.4(p.134)), (19.2.5(p.134)), and (19.2.6(p.134)). Then, for the same reason as in Chap. 15(p.109) it can be shown that the equality

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]\}] \tag{20.2.7}$$

holds (corresponding to (19.2.7(p.134))) and that we have the following theorem, corresponding to Theorem 19.2.4(p.134).

Theorem 20.2.4 (analogy $[\tilde{\mathbb{R}} \to \tilde{\mathbb{P}}]$**)** *Let* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$*. Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\tilde{\mathbb{R}}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}]. \quad \square \tag{20.2.8}$$

In fact, from the comparison of (I) and (II) and of (III) and (IV) in Table 6.5.4(p.39) it can be easily shown that (20.2.4(p.188)) and (20.2.6(p.188)) hold; however, from the comparison of (I) and (III) in Table 6.5.4(p.39) we can immediately see that (20.2.5(p.188)) does not always hold.

20.2.2 A Lemma

The following lemma provides the conditions on which whether each of Theorems 20.2.1(p.188), 20.2.2(p.188), and 20.2.3(p.188) holds or not.

Lemma 20.2.1

(a) Theorem 20.2.1(p.188) *always hold.*
(b) Theorem 20.2.3(p.188) *always hold.*
(c) *Let* $\rho \le a^\star$ *or* $b \le \rho$. *Then* Theorem 20.2.2(p.188) *holds.*
(d) *Let* $a^\star < \rho < b$. *Then* Theorem 20.2.2(p.188) *does not always hold.* □

● *Proof* (a,b) From the comparisons of (I) and (II) in Table 6.5.4(p.39) and that of (III) and (IV) in Table 6.5.4(p.39) we see that (20.2.4(p.188)) and (20.2.6(p.188)) hold, hence Theorems 20.2.1(p.188) and 20.2.3 hold.

(c,d) From the comparison of (I) and (III) in Table 6.5.4(p.39) we see that (20.2.5(p.188)) does not always hold, hence it follows that Theorem 20.2.2(p.188) does not always hold. The proofs for the two assertions (c,d) are the same as those of Lemma 20.1.1(p.151) (c,d). ∎

20.2.3 Proof of $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}$

20.2.3.1 Preliminary

From (6.5.28(p.39)) and (5.1.8(p.23)) we have

$$V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}), \quad t > 0. \tag{20.2.9}$$

20.2.3.2 Analysis

20.2.3.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 20.2.1** (■$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nondecreasing in* $t \ge 0$.
(b) *Let* $\rho \ge b$. *Then* Ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$ $\mapsto$ → Ⓓ$_{\parallel}$
(c) *Let* $\rho < b$. *Then* Ⓢ$_{\tau>0}\langle \tau\rangle_{\blacktriangle}$ $\mapsto$ → Ⓢ$_{\blacktriangle}$

● *Proof* Let $\beta = 1$ and $s = 0$. Then, since $K(x) = \lambda T(x) \cdots (1)$ from (5.1.4(p.23)), we have $K(x) \ge 0 \cdots (2)$ for any x due to Lemma 10.1.1(p.53) (g).

(a) From (6.5.28(p.39)) and (2) we obtain $V_t \ge V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \ge 0$.

(b) Let $\rho \ge b$. Then, since $b \le V_0$ from (6.5.27(p.39)), we have $b \le V_{t-1}$ for $t > 0$ from (a), hence $L(V_{t-1}) = 0$ for $t > 0$ from Lemma 10.2.1(p.55) (d), thus $V_t = \beta V_{t-1}$ for $t > 0$ from (20.2.9(p.189)). Then, since $V_t = \beta V_{t-1}$ for $\tau \ge t > 0$, we have $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 0$, i.e., Ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$ (see Preference Rule 7.2.1(p.43)).

(c) Let $\rho < b$. Then $V_0 < b \cdots (3)$ from (6.5.27(p.39)). Let $V_{t-1} < b$. Then, since $V_t < K(b) + b$ from (6.5.28(p.39)) and Lemma 10.2.2(p.55) (h), we have $V_t < \beta b - s = b$ from (10.2.7 (2) (p.55)) and the assumptions "$\beta = 1$ and $s = 0$". Hence, by induction $V_{t-1} < b$ for $t > 0$, so $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.55) (d). Accordingly, $V_t - \beta V_{t-1} > 0$ for $t > 0$ from (20.2.9(p.189)) or equivalently $V_t > \beta V_{t-1}$ for $t > 0$. Then, since $V_t > \beta V_{t-1}$ for $\tau \ge t > 0$, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0$, hence $t^*_\tau = \tau$ for $\tau > 0$, i.e., Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$. ∎

20.2.3.2.2 Case of $\beta < 1$ or $s > 0$

Let us define

$$\mathtt{S}_8\boxed{Ⓢ\blacktriangle \mid ◎\parallel \mid ◎\vartriangle \mid ◎\blacktriangle} = \left\{\begin{array}{l} \text{For any } \tau > 0 \text{ there exists } t^\bullet_\tau > 0 \text{ such that} \\ (1)\ Ⓢ_{t^\bullet_\tau \ge \tau > 0}\langle\tau\rangle_{\blacktriangle}, \\ (2)\ ◎_{t^\bullet_\tau + 1}\langle t^\bullet_\tau\rangle_{\vartriangle}, \\ (3)\ ◎_{\tau > t^\bullet_\tau + 1}\langle t^\bullet_\tau\rangle_{\parallel}\ (\!(◎_{\tau > t^\bullet_\tau + 1}\langle t^\bullet_\tau\rangle_{\blacktriangle})\!). \end{array}\right\}.$$

Remark 20.2.1 $\mathtt{S}_8$ is the same as $\mathtt{S}_2$(p.135) except that the inequalities of $\tau > 1$, $t^\bullet_\tau > 1$, and $t^\bullet_\tau \ge \tau > 1$ in $\mathtt{S}_2$ changes into $\tau > 0$, $t > 0$, and $t^\bullet_\tau \ge \tau > 0$ respectively in $\mathtt{S}_8$. □

□ **Tom 20.2.2** (■$\mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$.

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t \ge 0$ *if* $\lambda < 1$ *or* $a < \rho$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
(b) *Let* $x_L \le \rho$. *Then* Ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$ $\mapsto$ → Ⓓ$_{\vartriangle}$
(c) *Let* $\rho < x_L$.
  1. Ⓢ$_1\langle 1\rangle_{\blacktriangle}$. *Below let* $\tau > 1$ $\mapsto$ → Ⓢ$_{\blacktriangle}$
  2. *Let* $\beta = 1$.

i. *Let* $a < \rho$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow \circledS_\blacktriangle$

ii. *Let* $\rho \le a$.

1. *Let* $(\lambda\mu - s)/\lambda \le a$.

i. *Let* $\lambda = 1$. *Then* $\circledcirc_{\tau>1}\langle 1\rangle_\parallel \mapsto$ $\rightarrow \circledcirc_\parallel$

ii. *Let* $\lambda < 1$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow \circledS_\blacktriangle$

2. *Let* $(\lambda\mu - s)/\lambda > a$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow \circledS_\blacktriangle$

3. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.

i. *Let* $a < \rho$.

1. *Let* $b \ge 0\,(\!(\kappa \ge 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow \circledS_\blacktriangle$

2. *Let* $b < 0\,(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_8$ $\boxed{\circledS_\blacktriangle \mid \circledcirc_\parallel \mid \circledcirc_\vartriangle \mid \circledcirc_\blacktriangle}$ *is true* $\mapsto$ $\rightarrow \circledS_\blacktriangle/\circledcirc_\parallel/\circledcirc_\vartriangle/\circledcirc_\blacktriangle$

ii. *Let* $\rho \le a$.

1. *Let* $(\lambda\beta\mu - s)/\delta \le a$.

i. *Let* $\lambda = 1$.

3. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow \circledS_\blacktriangle$

4. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\circledcirc_{\tau>1}\langle 1\rangle_\vartriangle \mapsto$ $\rightarrow \circledcirc_\vartriangle$

ii. *Let* $\lambda < 1$.

5. *Let* $b \ge 0\,(\!(\kappa \ge 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow \circledS_\blacktriangle$

6. *Let* $b < 0\,(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_8$ $\boxed{\circledS_\blacktriangle \mid \circledcirc_\parallel \mid \circledcirc_\vartriangle \mid \circledcirc_\blacktriangle}$ *is true* $\mapsto$ $\rightarrow \circledS_\blacktriangle/\circledcirc_\parallel/\circledcirc_\vartriangle/\circledcirc_\blacktriangle$

2. *Let* $(\lambda\beta\mu - s)/\delta > a$.

i. *Let* $b \ge 0\,(\!(\kappa \ge 0)\!)$. *Then* $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow \circledS_\blacktriangle$

ii. *Let* $b < 0\,(\!(\kappa < 0)\!)$. *Then* $\mathtt{S}_8$ $\boxed{\circledS_\blacktriangle \mid \circledcirc_\parallel \mid \circledcirc_\vartriangle \mid \circledcirc_\blacktriangle}$ *is true* $\mapsto$ $\rightarrow \circledS_\blacktriangle/\circledcirc_\parallel/\circledcirc_\vartriangle/\circledcirc_\blacktriangle$

• *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho < x_K \cdots$ **(1)**. Then $V_0 < x_K \cdots$ **(2)** from (6.5.27(p.39)) and $K(\rho) > 0$ due to Lemma 10.2.2(p.55) (j1). Since $V_1 = K(\rho) + \rho \cdots$ **(3)** from (6.5.28(p.39)) with $t = 1$, we have $V_1 - V_0 = V_1 - \rho = K(\rho) > 0$, hence $V_1 > V_0 \cdots$ **(4)**.

(a) Note (4), hence $V_0 \le V_1$. Suppose $V_{t-1} \le V_t$. Then, due to Lemma 10.2.2(p.55) (e) we have $V_t \le K(V_t) + V_t = V_{t+1}$ from (6.5.28(p.39)). Hence, by induction $V_t \ge V_{t-1}$ for $t > 0$, i.e., V_t is nondecreasing in $t \ge 0$. Note again (4). Suppose $V_{t-1} < V_t$. If $\lambda < 1$, from Lemma 10.2.2(p.55) (f) we have $V_t < K(V_t) + V_t = V_{t+1}$. If $a < \rho$, then $a < V_0$ from (6.5.27(p.39)), hence $a < V_{t-1}$ for $t > 0$ due to the nondecreasing of V_t, so from Lemma 10.2.2(p.55) (g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Therefore, whether $\lambda < 1$ or $a < \rho$, by induction we have $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \ge 0$. Consider a sufficiently large $M > 0$ with $\rho \le M$ and $b \le M$, hence from (6.5.27(p.39)) we have $V_0 \le M$. Suppose $V_{t-1} \le M$. Then, from Lemma 10.2.2(p.55) (e) we have $V_t \le K(M) + M = \beta M - s$ due to (10.2.7 (2) (p.55)), hence $V_t \le M$ due to the assumptions "$\beta \le 1$ and $s \ge 0$". Accordingly, by induction $V_t \le M$ for $t \ge 0$, i.e., V_t is upper bounded in t. Hence V_t converges to a finite V as $t \to \infty$. Thus $V = K(V) + V$ from (6.5.28(p.39)), hence $K(V) = 0$, so $V = x_K$ due to Lemma 10.2.2(p.55) (j1).

(b) Let $x_L \le \rho$. Then, since $x_L \le V_0$ from (6.5.27(p.39)), we have $x_L \le V_{t-1}$ for $t > 0$ due to (a), hence $L(V_{t-1}) \le 0$ for $t > 0$ due to Corollary 10.2.1(p.55) (a), thus $V_t - \beta V_{t-1} \le 0$ for $t > 0$ from (20.2.9(p.189)) or equivalently $V_t \le \beta V_{t-1}$ for $t > 0$. Accordingly, since $V_t \le \beta V_{t-1}$ for $\tau \ge t > 0$, we have $V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^\tau V_0$, hence $t^*_\tau = 0$ for $\tau > 0$, i.e., ⓓ$_{\tau>0}\langle 0\rangle_\vartriangle$.

(c) Let $\rho < x_L \cdots$ **(5)**. Then $V_0 < x_L \cdots$ **(6)** from (6.5.27(p.39)), hence $L(V_0) > 0 \cdots$ **(7)** due to Corollary 10.2.1(p.55) (a).

(c1) Since $V_1 - \beta V_0 = L(V_0) > 0$ from (20.2.9(p.189)) with $t = 1$ and (7), we have $V_1 > \beta V_0$, hence $t^*_1 = 1$, i.e., $\circledS_1\langle 1\rangle_\blacktriangle \cdots$ **(8)**.

Below let $\tau > 1 \cdots$ **(9)**.

(c2) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $\delta = \lambda$ from (10.2.1(p.54)) and $x_L = x_K \cdots$ **(10)** from Lemma 10.2.3(p.56) (b), hence $K(x_L) = K(x_K) = 0 \cdots$ **(11)**.

(c2i) Let $a < \rho$. Then $a < V_0$ from (6.5.27(p.39)), hence $a < V_{t-1}$ for $t > 0$ due to (a). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.5.28(p.39)) and Lemma 10.2.2(p.55) (g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$. Then, since $V_{t-1} < x_L$ for $t > 0$ due to (10), we have $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.55) (e1), hence for the same reason as in the proof of `Tom` 20.2.1(p.189) (c) we have $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(c2ii) Let $\rho \le a$, hence $V_0 \le a \cdots$ **(12)** from (6.5.27(p.39)). Then, from (3) and (10.2.7 (1) (p.55)) we have $V_1 = \lambda\mu - s + (1-\lambda)\rho$.

(c2ii1) Let $(\lambda\mu - s)/\lambda \le a$. Then $x_K = (\lambda\mu - s)/\lambda \le a \cdots$ **(13)** from Lemma 10.2.2(p.55) (j2). Hence $K(a) \le 0$ from Lemma 10.2.2(p.55) (j1). Note (12). Suppose $V_{t-1} \le a$. Then, from (6.5.28(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \le K(a) + a \le a$, hence by induction $V_{t-1} \le a$ for $t > 0$. Accordingly, from (6.5.28(p.39)) and (10.2.7 (1) (p.55)) we have $V_t = \lambda\mu - s + (1 - \lambda)V_{t-1} \cdots$ **(14)** for $t > 0$.

(c2ii1i) Let $\lambda = 1$. Then, we have $x_K = \mu - s$ from (13) and $V_t = \mu - s$ for $t > 0$ from (14), hence $V_t = x_K$ for $t > 0$, so $V_{t-1} = x_K$ for $t > 1$. Accordingly, $V_{t-1} = x_L$ for $t > 1$ due to (10). Then $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, hence $V_t - \beta V_{t-1} = 0$ for $t > 1$ from (20.2.9(p.189)) or equivalently $V_t = \beta V_{t-1}$ for $t > 1$. Then, since $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$, we have

$V_\tau = \beta V_{\tau-1} \cdots = \beta^{\tau-1} V_1$ for $\tau > 1$. From this and (4) we have $V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-1} V_1 > \beta^\tau V_0$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\circledcirc_{\tau>1}\langle 1 \rangle_{\|}$.

(c2ii1ii) Let $\lambda < 1$. Note (6). Suppose $V_{t-1} < x_L$. Then, we have $V_t < K(x_L) + x_L = x_L$ from Lemma 10.2.2(p.55) (f) and (11). Accordingly, by induction $V_{t-1} < x_L$ for $t > 0$, hence $L(V_{t-1}) > 0$ for $t > 0$ from Lemma 10.2.1(p.55) (e1). Thus, for the same reason as in the proof of Tom 20.2.1(p.189) (c) we have $\circledS_{\tau>1}\langle \tau \rangle_\blacktriangle$.

(c2ii2) Let $(\lambda\mu - s)/\lambda > a$. Then $x_K > (\lambda\mu - s)/\lambda > a$ from Lemma 10.2.2(p.55) (j2), hence $x_L > a$ from (10). Note (6). Suppose $V_{t-1} < x_L$. Then, we have $V_t < K(x_L) + x_L = x_L$ from
Lemma 10.2.2(p.55) (h) and (11). Accordingly, by induction $V_{t-1} < x_L \cdots$(15) for $t > 0$, hence $L(V_{t-1}) > 0$ for $t > 0$ due to Lemma 10.2.1(p.55) (e1). Consequently, for the same reason as in the proof of Tom 20.2.1(p.189) (c) we obtain $\circledS_{\tau>1}\langle \tau \rangle_\blacktriangle$.

(c3) Let $\beta < 1$ and $s = 0$ ($s > 0$).

(c3i) Let $a < \rho \cdots$(16). Then, since $a < V_0$ from (6.5.27(p.39)), we have $a < V_{t-1}$ for $t > 0$ due to (a).

(c3i1) Let $b \geq 0$ ($\kappa \geq 0$). Then $x_L \geq x_K \cdots$(17) from Lemma 10.2.3(p.56) (c (d)). Note (2). Suppose $V_{t-1} < x_K$. Then, from (6.5.28(p.39)) and Lemma 10.2.2(p.55) (g) we have $V_t < K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} < x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ due to (17). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of Tom 20.2.1(p.189) (c) we obtain $\circledS_{\tau>1}\langle \tau \rangle_\blacktriangle$.

(c3i2) Let $b < 0$ ($\kappa < 0$). Then $x_L < x_K \cdots$(18) from Lemma 10.2.3(p.56) (c (d)). Note (6). Suppose $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \leq x_L$. Now, since $V = x_K$ due to (a), we have $x_L < V$ due to (18), which is a contradiction. Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. In addition, from (6) and the *strict increasingness* of V_t due to (a), it follows that there exists $t^\bullet_\tau > 0$ such that

$$V_0 < V_1 < \cdots < V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} < V_{t^\bullet_\tau + 1} < V_{t^\bullet_\tau + 2} < \cdots.$$

from which we have

$$V_{t-1} < x_L,\ t^\bullet_\tau \geq t > 0, \qquad x_L \leq V_{t^\bullet_\tau}, \qquad x_L < V_{t-1},\ t > t^\bullet_\tau + 1. \tag{20.2.10}$$

Hence, we have

$$L(V_{t-1}) > 0 \quad \cdots(19), \quad t^\bullet_\tau \geq t > 0 \quad \text{(due to Corollary 10.2.1(p.55) (a))}$$

$$L(V_{t^\bullet_\tau}) \leq 0 \quad \cdots(20), \quad \text{(due to Corollary 10.2.1(p.55) (a))}$$

$$L(V_{t-1}) = (< 0)^\dagger \cdots(21), \quad t > t^\bullet_\tau + 1 \quad \text{(due to Lemma 10.2.1(p.55) (d(e1)))}$$

- Let $t^\bullet_\tau \geq \tau > 0$. Then $L(V_{t-1}) > 0 \cdots$(22) for $\tau \geq t > 0$ from (19). Hence, for the same reason as in Tom 20.2.1(p.189) (c) we obtain $\circledS_\tau\langle \tau \rangle_\blacktriangle$ for $t^\bullet_\tau \geq \tau > 0$. Accordingly, $\mathtt{S}_8(1)$ is true. Now, since $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 0$ from (20.2.9(p.189)) and (22), we have $V_t > \beta V_{t-1}$ for $\tau \geq t > 0$, hence
$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0.$$
Accordingly, when $\tau = t^\bullet_\tau$, we have
$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \cdots > \beta^{t^\bullet_\tau} V_0 \cdots (23)$$
- Let $\tau = t^\bullet_\tau + 1$. From (20.2.9(p.189)) with $t = t^\bullet_\tau + 1$ and (20) we have $V_{t^\bullet_\tau + 1} - \beta V_{t^\bullet_\tau} = L(t^\bullet_\tau) \leq 0$, hence $V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau}$. Accordingly, from (23) we have
$$V_{t^\bullet_\tau+1} \leq \beta V_{t^\bullet_\tau} > \beta^2 V_{t^\bullet_\tau - 1} > \beta^3 V_{t^\bullet_\tau - 2} > \cdots > \beta^{t^\bullet_\tau + 1} V_0 \cdots (24),$$
thus $t^*_{t^\bullet_\tau+1} = t^\bullet_\tau$, i.e., $\circledcirc_{t^\bullet_\tau+1}\langle t^\bullet_\tau \rangle_\vartriangle$, so that $\mathtt{S}_8(2)$ is true.
- Let $\tau > t^\bullet_\tau + 1$. Since $L(V_{t^\bullet_\tau+1}) = (<)\ 0$ from (21) with $t = t^\bullet_\tau + 2$, we have $V_{t^\bullet_\tau+2} = (<)\ \beta V_{t^\bullet_\tau+1}$ from (20.2.9(p.189)), hence from (24) we have
$$V_{t^\bullet_\tau+2} = (<)\ \beta V_{t^\bullet_\tau+1} \leq \beta^2 V_{t^\bullet_\tau} > \beta^3 V_{t^\bullet_\tau-1} > \beta^4 V_{t^\bullet_\tau-2} > \cdots > \beta^{t^\bullet_\tau+2} V_0$$
Similarly we have
$$V_{t^\bullet_\tau+3} = (<)\ \beta V_{t^\bullet_\tau+2} = (<)\ \beta^2 V_{t^\bullet_\tau+1} \leq \beta^3 V_{t^\bullet_\tau} > \beta^4 V_{t^\bullet_\tau-1} > \cdots > \beta^{t^\bullet_\tau+3} V_0.$$
By repeating the same procedure, for $\tau = t^\bullet_\tau + 2, t^\bullet_\tau + 3, \cdots$ we obtain
$$V_\tau = (<)\ \beta V_{\tau-1} = (<)\ \cdots = (<)\ \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = (<)$$
$$\beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^\tau V_0. \cdots (25)$$
  - Let $s = 0$. Then (25) can be written as
$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau - t^\bullet_\tau - 2} V_{t^\bullet_\tau+2} = \beta^{\tau - t^\bullet_\tau - 1} V_{t^\bullet_\tau+1} \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \cdots > \beta^\tau V_0,$$
  hence $t^*_\tau = t^\bullet_\tau$, i.e., $\circledcirc_{\tau > t^\bullet_\tau+1}\langle t^\bullet_\tau \rangle_{\|}$ (see Preference Rule 7.2.1(p.43)), hence $\mathtt{S}_8(3)$ is true.

---

† If $s = 0$, then $L(V_{t-1}) = 0$, or else $L(V_{t-1}) < 0$.

∘ Let $s > 0$. Then (25) can be written as

$$V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-t_\tau^\bullet-2} V_{t_\tau^\bullet+2} < \beta^{\tau-t_\tau^\bullet-1} V_{t_\tau^\bullet+1} \le \beta^{\tau-t_\tau^\bullet} V_{t_\tau^\bullet} > \beta^{\tau-t_\tau^\bullet+1} V_{t_\tau^\bullet-1} > \cdots > \beta^\tau V_0, \tag{20.2.11}$$

hence $t_\tau^* = t_\tau^\bullet$, i.e., $◎_{\tau > t_\tau^\bullet+1}\langle t^\circ\rangle_\blacktriangle$, hence $\mathtt{S}_8(3)$ is true.

(c3ii) Let $\rho \le a$, hence $V_0 \le a$ from (6.5.27(p.39)). Then, from (3) and (10.2.7 (1) (p.55)) we have $V_1 = \lambda\beta\mu - s + (1-\lambda)\beta\rho$.

(c3ii1) Let $(\lambda\beta\mu - s)/\delta \le a$. Then $x_K = (\lambda\beta\mu - s)/\delta \le a \cdots$ **(26)** from Lemma 10.2.2(p.55) (j2(p.56)). Hence $V_1 = \delta x_K + (1-\lambda)\beta\rho \cdots$ **(27)**.

(c3ii1i) Let $\lambda = 1$, hence $\delta = 1$ from (10.2.1(p.54)). Thus, from (26) and (27) we have $x_K = \beta\mu - s \le a$ and $V_1 = x_K \le a \cdots$ **(28)**.

(c3ii1i1) Let $b > 0$ ⟪$\kappa > 0$⟫. Then $x_L > x_K \cdots$ **(29)** due to Lemma 10.2.3(p.56) (c ⟪d⟫). Note (28). Suppose $V_{t-1} = x_K$. Then, from (6.5.28(p.39)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ due to (29), thus $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 10.2.1(p.55) (a). Hence, from (7) we obtain $L(V_{t-1}) > 0$ for $t > 0$. Accordingly, for almost the same reason as in the proof of `Tom` 20.2.1(p.189) (c) we obtain $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(c3ii1i2) Let $b \le 0$ ⟪$\kappa \le 0$⟫. Then, since $x_L \le x_K$ from Lemma 10.2.3(p.56) (c ⟪d⟫), we have $V_1 \ge x_L$ from (28), hence $V_{t-1} \ge x_L$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 10.2.1(p.55) (a), we have $L(V_{t-1}) \le 0$ for $\tau \ge t > 1$, thus $V_t - \beta V_{t-1} \le 0$ for $\tau \ge t > 1$ from (20.2.9(p.189)), i.e., $V_t \le \beta V_{t-1}$ for $\tau \ge t > 1$. Hence $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 \cdots$ **(30)**. Now, from (6.5.27(p.39)), (4), (28), and (29) we have $\rho = V_0 < V_1 = x_K < x_L$, hence $L(\rho) > 0$ from Corollary 10.2.1(p.55) (a). In addition, from (3) and (6.5.27(p.39)) we have $V_1 - \beta V_0 = V_1 - \beta\rho = K(\rho) + \rho - \beta\rho = K(\rho) + (1-\beta)\rho = L(\rho) > 0$ from (5.1.8(p.23)), hence $V_1 > \beta V_0$. Accordingly, from (30) we have $V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^{\tau-1} V_1 > \beta^\tau V_0$ for $\tau > 1$, hence $t_\tau^* = 1$ for $\tau > 1$, i.e., $◎_{\tau>1}\langle 1\rangle_\vartriangle$.

(c3ii1ii) Let $\lambda < 1$.

(c3ii1ii1) Let $b \ge 0$ ⟪$\kappa \ge 0$⟫. Then $x_L \ge x_K \cdots$ **(31)** from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (2). Suppose $V_{t-1} < x_K$. Then, from Lemma 10.2.2(p.55) (f) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} < x_K$ for $t > 0$, so $V_{t-1} < x_L$ for $t > 0$ due to (31). Accordingly, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of `Tom` 20.2.1(p.189) (c) we obtain $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(c3ii1ii2) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ **(32)** from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (6). Assume that $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \le x_L$ due to (a). Now, since $V = x_K$ from (a), we have the contradiction $x_L < V$ from (32). Hence, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. From this and the strict increasingness of V_t due to (a), it follows that there exists $t_\tau^\bullet > 0$ such that

$$V_0 < V_1 < \cdots < V_{t_\tau^\bullet-1} < x_L \le V_{t_\tau^\bullet} < V_{t_\tau^\bullet+1} < V_{t_\tau^\bullet+2} < \cdots \to x_K.$$

Accordingly, for the same reason as in the proof of (c3i2) we have $\mathtt{S}_8$ [Ⓢ▲ | ◎∥ | ◎△ | ◎▲].

(c3ii2) Let $(\lambda\beta\mu - s)/\delta > a \cdots$ **(33)**. Then $x_K > (\lambda\beta\mu - s)/\delta > a$ from Lemma 10.2.2(p.55) (j2).

1. Let $\lambda < 1$. Then V_t is *strictly increasing* in $t \ge 0$ due to (a).
2. Let $\lambda = 1$, hence $\delta = 1$ from (10.2.1(p.54)), so $\beta\mu - s > a$ from (33). Now $K(x) \ge \beta\mu - s - x$ for any x from (10.2.4(p.55)) or equivalently $K(x) + x \ge \beta\mu - s$ for any x, so $V_1 \ge \beta\mu - s > a$ from (3). Accordingly $V_{t-1} > a$ for $t > 1$ due to (a). Note (4). Suppose $V_{t-1} < V_t$. Then, from Lemma 10.2.2(p.55) (g) we have $V_t < K(V_t) + V_t = V_{t+1}$. Accordingly, by induction we have $V_{t-1} < V_t$ for $t > 0$, i.e., V_t is *strictly increasing* in $t \ge 0$.

From the above, whether $\lambda < 1$ or $\lambda = 1$, we see that V_t is *strictly increasing* in $t > 0$.

(c3ii2i) Let $b \ge 0$ ⟪$\kappa \ge 0$⟫. Then $x_L \ge x_K \cdots$ **(34)** from Lemma 10.2.2(p.55) (c ⟪d⟫). From the above strict increasingness of V_t in $t \ge 0$ and (a) we have $V_{t-1} < V = x_K$ for $t > 0$, hence $V_{t-1} < x_L$ for $t > 0$ from (34). Thus, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of `Tom` 20.2.1(p.189) (c) we obtain $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(c3ii2ii) Let $b < 0$ ⟪$\kappa < 0$⟫. Then $x_L < x_K \cdots$ **(35)** from Lemma 10.2.3(p.56) (c ⟪d⟫). Note (6). Suppose $V_{t-1} < x_L$ for *all* $t > 0$, hence $V \le x_L$. Now, since $V = x_K$ from (a), we have $x_L < V$ from (35), which is a contradiction. Accordingly, it is impossible that $V_{t-1} < x_L$ for *all* $t > 0$. From this, (6), and the above strict increasingness of V_t in $t \ge 0$ it follows that there exists $t_\tau^\bullet > 0$ such that

$$V_0 < V_1 < \cdots < V_{t_\tau^\bullet-1} < x_L \le V_{t_\tau^\bullet} < V_{t_\tau^\bullet+1} < V_{t_\tau^\bullet+2} < \cdots \to x_K.$$

Accordingly, for the same reason as in the proof of (c3i2) we can immediately see that the assertion holds true. ∎

□ `Tom` **20.2.3** (▣ $\mathscr{A}$ {M:2[ℝ][E]}) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $❻_{\tau>0}\langle 0\rangle_{\|} \mapsto$ $\to ❻_{\|}$

(c) *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.

1. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\text{Ⓢ}_{\tau>0}\langle\tau\rangle_{\blacktriangle} \mapsto$ $\rightarrow \text{Ⓢ}_{\blacktriangle}$
2. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\vartriangle} \mapsto$ $\rightarrow \text{ⓓ}_{\vartriangle}$

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Hence $V_0 = \rho = x_K \cdots$ **(1)** from (6.5.27(p.39)).

(a) Note (1). Suppose $V_{t-1} = x_K$. Then, from (6.5.28(p.39)) we have $V_t = K(x_K) + x_K = x_K$. Hence, by induction $V_t = x_K = \rho$ for $t \ge 0$.

(b) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K$ from Lemma 10.2.3(p.56) (b). Accordingly, since $V_{t-1} = x_L$ for $t > 0$ from (a), we have $L(V_{t-1}) = L(x_L) = 0$ for $t > 0$, hence for the same reason as in the proof of `Tom` 20.2.1(p.189) (b) we obtain $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.

(c1) Let $b > 0\,(\!(\kappa > 0)\!)$. Then, since $x_L > x_K$ from Lemma 10.2.3(p.56) (c (\!(d)\!), we have $x_L > x_K = V_{t-1}$ for $t > 0$ from (a), hence $L(V_{t-1}) > 0$ for $t > 0$ due to Corollary 10.2.1(p.55) (a), thus for the same reason as in the proof of `Tom` 20.2.1(p.189) (c) we obtain $\text{Ⓢ}_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c2) Let $b \le 0\,(\!(\kappa \le 0)\!)$. Then $x_L \le x_K$ from Lemma 10.2.3(p.56) (c (\!(d)\!). Hence, since $x_L \le x_K = V_{t-1}$ for $t > 0$ from (a), we have $L(V_{t-1}) \le 0$ for $t > 0$ due to Corollary 10.2.1(p.55) (a), hence $V_t - \beta V_{t-1} \le 0$ for $t > 0$ from (20.2.9(p.189)) or equivalently $V_t \le \beta V_{t-1}$ for $t > 0$. Accordingly, since $V_t \le \beta V_{t-1}$ for $\tau \ge t > 0$, we have $V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^\tau V_0$, thus $t^*_\tau = 0$ for $\tau > 0$, i.e., $\texttt{dOIT}_{\tau>0}\langle 0\rangle_{\vartriangle}$. ▮

$$\texttt{S}_9\boxed{\text{Ⓢ}\vartriangle}\boxed{\bullet\vartriangle}\boxed{\bullet\blacktriangle} = \left\{\begin{array}{l}\text{For any } \tau > 0 \text{ there exists } t^\bullet > 0 \text{ such that} \\ (1)\ \ \text{ⓓ}_{\tau=1}\langle 0\rangle_{\parallel}\ (\text{ⓓ}_{\tau=1}\langle 0\rangle_{\blacktriangle}), \\ (2)\ \ \text{Ⓢ}_{\tau>t^\bullet}\langle\tau\rangle_{\vartriangle} \text{ or } \text{ⓓ}_{\tau>t^\bullet}\langle 0\rangle_{\vartriangle}, \\ (3)\ \ \text{ⓓ}_{t^\bullet\ge\tau>1}\langle 0\rangle_{\vartriangle}\ (\!(\text{ⓓ}_{t^\bullet\ge\tau>1}\langle 0\rangle_{\blacktriangle})\!). \end{array}\right\}$$

□ **`Tom` 20.2.4** (■$\mathscr{A}${M:2[ℝ][E]}) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) V_t *is nonincreasing in* $t \ge 0$*, is strictly decreasing in* $t > 0$ *if* $\lambda < 1$*, and converges to a finite* $V = x_K$ *as to* $\rightarrow \infty$.

(b) *Let* $\rho < x_L$. *Then* $\text{Ⓢ}_{\tau>0}\langle\tau\rangle_{\blacktriangle} \mapsto$ $\rightarrow \text{Ⓢ}_{\blacktriangle}$

(c) *Let* $\rho = x_L$. *Then* $\text{ⓓ}_1\langle 0\rangle_{\parallel}$ *and* $\text{Ⓢ}_{\tau>1}\langle\tau\rangle_{\blacktriangle} \mapsto$ $\rightarrow \text{ⓓ}_{\parallel}/\text{Ⓢ}_{\blacktriangle}$

(d) *Let* $\rho > x_L$.

1. *Let* $\beta = 1$. *Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\vartriangle} \mapsto$ $\rightarrow \text{ⓓ}_{\vartriangle}$
2. *Let* $\beta < 1$ *and* $s = 0\,(\!(s > 0)\!)$.
   - i. *Let* $b \le 0\,(\!(\kappa \le 0)\!)$. *Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\vartriangle}\ (\!(\text{ⓓ}_{\tau>0}\langle 0\rangle_{\blacktriangle})\!) \mapsto$ $\rightarrow \text{ⓓ}_{\vartriangle}/\text{ⓓ}_{\blacktriangle}$
   - ii. *Let* $b > 0\,(\!(\kappa > 0)\!)$. *Then* $\texttt{S}_9\boxed{\text{Ⓢ}\vartriangle}\boxed{\bullet\vartriangle}\boxed{\bullet\blacktriangle}$ *is true* $\mapsto$ $\rightarrow \text{Ⓢ}_{\vartriangle}/\text{ⓓ}_{\vartriangle}/\text{ⓓ}_{\blacktriangle}$

● *Proof* Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. Then $V_0 > x_K \cdots$ **(1)** from (6.5.27(p.39)) and $K(\rho) < 0 \cdots$ **(2)** from Lemma 10.2.2(p.55) (j1). From (6.5.28(p.39)) with $t = 1$ and from (6.5.27(p.39)) we have $V_1 - V_0 = K(V_0) = K(\rho) < 0$, hence $V_1 < V_0 \cdots$ **(3)**. In addition, from (20.2.9(p.189)) with $t = 1$ we have $V_1 - \beta V_0 = L(V_0) = L(\rho) \cdots$ **(4)** from (6.5.27(p.39)).

(a) Note (3), hence $V_0 \ge V_1$. Suppose $V_{t-1} \ge V_t$. Then, from (6.5.28(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \ge K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 0$, i.e., V_t is nonincreasing in $t \ge 0$. Let $\lambda < 1$. Note again (3). Suppose $V_{t-1} > V_t$. Then, from Lemma 10.2.2(p.55) (f) we have $V_t > K(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} > V_t$ for $t > 0$, i.e., V_t is *strictly decreasing* in $t \ge 0$. Note (1), hence $V_0 \ge x_K$. Suppose $V_{t-1} \ge x_K$. Then, from (6.5.28(p.39)) and Lemma 10.2.2(p.55) (e) we have $V_t \ge K(x_K) + x_K = x_K$. Hence, by induction $V_{t-1} \ge x_K \cdots$ **(5)** for $t > 0$, i.e., V_t is lower bounded in t. Thus, it follows that V_t converges to a finite V as $t \rightarrow \infty$. Hence, since $V = K(V) + V$ from (6.5.28(p.39)), we have $K(V) = 0$, thus $V = x_K$ due to Lemma 10.2.2(p.55) (j1).

(b) Let $\rho < x_L$. Then, since $V_0 < x_L$ from (6.5.27(p.39)), we have $V_{t-1} < x_L$ for $t > 0$ due to (a). Therefore, since $L(V_{t-1}) > 0$ for $t > 0$ from Corollary 10.2.1(p.55) (a), for the same reason as in the proof of `Tom` 20.2.1(p.189) (c) we obtain $\text{Ⓢ}_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c) Let $\rho = x_L \cdots$ **(6)**. Then, since $L(\rho) = L(x_L) = 0$, we have $V_1 - \beta V_0 = 0$ from (4) or equivalently $V_1 = \beta V_0 \cdots$ **(7)**, hence $\text{ⓓ}_1\langle 0\rangle_{\parallel}$. Below, let $\tau > 1$. Now, since $V_1 = K(\rho) + \rho < \rho$ from (6.5.28(p.39)) with $t = 1$ and (2), we have $V_{t-1} < \rho$ for $t > 1$ from (a), hence $V_{t-1} < x_L$ for $t > 1$ due to (6), so $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 10.2.1(p.55) (a). Accordingly, since $L(V_{t-1}) > 0$ for $\tau \ge t > 1$, we have $V_t - \beta V_t > 0$ for $\tau \ge t > 1$ due to (20.2.9(p.189)) or equivalently $V_t > \beta V_t$ for $\tau \ge t > 1$, from which we have $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$. Hence, from (7) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0.$$

Accordingly, we obtain $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\text{Ⓢ}_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(d) Let $x_L < \rho \cdots$ **(8)**, hence $x_L < V_0 \cdots$ **(9)** from (6.5.27(p.39)). Thus, if $s = (\!(>)\!)\ 0$, then $L(V_0) = (\!(<)\!)\ 0 \cdots$ **(10)** from Lemma 10.2.1(p.55) (d(\!(e1)\!)), hence $V_1 - \beta V_0 = (\!(<)\!)\ 0$ from (4) or equivalently $V_1 = (\!(<)\!)\ \beta V_0 \cdots$ **(11)**.

(d1) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $L(V_0) < 0$ from (10), hence $V_1 < \beta V_0 \cdots$ **(12)** from (20.2.9(p.189)). Now, since $x_L = x_K$ due to Lemma 10.2.3(p.56) (b), from (5) we have $V_{t-1} \ge x_L$ for $t > 0$, hence $L(V_{t-1}) \le 0$

for $t>0$ due to Lemma 10.2.1(p.55) (e1), thus $V_t-\beta V_{t-1}\le 0$ for $t>0$ from (20.2.9(p.189)). Then, since $V_t-\beta V_{t-1}\le 0$ for $\tau\ge t>0$, we have $V_t\le\beta V_{t-1}$ for $\tau\ge t>0$, leading to

$$V_\tau\le\beta V_{\tau-1}\le\cdots\le\beta^{\tau-1}V_1\le\beta^\tau V_0.$$

Hence we have $t^*_\tau=0$ for $\tau>0$, i.e., ❶$_{\tau>0}\langle 0\rangle_\vartriangle$.

(d2) Let $\beta<1$ and $s=0$ ⟪$s>0$⟫.

(d2i) Let $b\le 0$ ⟪$\kappa\le 0$⟫. Then $x_L\le x_K$ due to Lemma 10.2.3(p.56) (c ⟪d⟫). Hence, from (5) we have $V_{t-1}\ge x_L$ for $t>0$, hence $L(V_{t-1})\le 0$ for $t>0$ due to Corollary 10.2.1(p.55) (a), so $V_t-\beta V_{t-1}\le 0$ for $t>0$ from (20.2.9(p.189)). Then, since $V_t-\beta V_{t-1}\le 0$ for $\tau\ge t>0$, we have $V_t\le\beta V_{t-1}$ for $\tau\ge t>0$, leading to

$$V_\tau\le\beta V_{\tau-1}\le\cdots\le\beta^{\tau-1}V_1\le\beta^\tau V_0.$$

Due to (11) the inequality can be rewritten as

$$V_\tau\le\beta V_{\tau-1}\le\cdots\le\beta^{\tau-1}V_1=⟪<⟫\ \beta^\tau V_0,$$

hence $t^*_\tau=0$ for $\tau>0$, i.e., ❶$_{\tau>0}\langle 0\rangle_\vartriangle$ ⟪❶$_{\tau>0}\langle 0\rangle_\blacktriangle$⟫.

(d2ii) Let $b>0$ ⟪$\kappa>0$⟫. Then $x_L>x_K>0\cdots$(13) from Lemma 10.2.3(p.56) (c ⟪d⟫). Hence, from (3) and (9) and from the nonincreasingness of V_t and the convergency of V_t to $V=x_K$ due to (a) we see that there exists $t^\bullet>0$ such that

$$V_0>V_1\ge V_2\ge\cdots\ge V_{t^\bullet-1}\ge x_L>V_{t^\bullet}\ge V_{t^\bullet+1}\ge\cdots\to x_K\cdots(14)$$

or equivalently $V_0>x_L$, $V_{t-1}\ge x_L$ for $t^\bullet\ge t>1$, and $x_L>V_{t-1}$ for $t>t^\bullet$. Hence, we have

$$\begin{aligned}&L(V_{t-1})>0,\quad t>t^\bullet, && \text{due to Corollary 10.2.1(p.55) (a)},\\ &L(V_{t-1})\le 0,\quad t^\bullet\ge t>1, && \text{due to Corollary 10.2.1(p.55) (a)},\\ &L(V_0)=⟪<⟫\ 0 && \text{due to Lemma 10.2.1(p.55) (d⟪e1⟫)}.\end{aligned}$$

Hence, from (20.2.9(p.189)) we have

$$V_t>\beta V_{t-1}\cdots(15),\quad t>t^\bullet,\qquad V_t\le\beta V_{t-1}\cdots(16),\quad t^\bullet\ge t>1,\qquad V_1=⟪<⟫\ \beta V_0\cdots(17).$$

⟨A⟩ Let $\tau=1$. Then, since $V_1=⟪<⟫\ \beta V_0$ due to (17), we have ❶$_{\tau=1}\langle 0\rangle_{\parallel}$ (❶$_{\tau=1}\langle 0\rangle_\blacktriangle$), hence (1) of $\mathtt{S}_9$ holds.

⟨B⟩ Let $t^\bullet\ge\tau>1$. Then, since $V_t\le\beta V_{t-1}$ for $\tau\ge t>1$ from (16), we have

$$V_\tau\le\beta V_{\tau-1}\le\cdots\le\beta^{\tau-1}V_1,$$

hence

$$V_\tau\le\beta V_{\tau-1}\le\cdots\le\beta^{\tau-1}V_1=⟪<⟫\ \beta^\tau V_0\cdots(18),\quad t^\bullet\ge\tau>0,$$

from (17) or equivalently

$$I_\tau^\tau\le I_\tau^{\tau-1}\le\cdots\le I_\tau^1=⟪<⟫\ I_\tau^0\cdots(19),\quad t^\bullet\ge\tau>0.$$

Thus $t^*_\tau=0$ for $t^\bullet\ge\tau>0$, i.e., ❶$_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle_\vartriangle$ ⟪❶$_{t^\bullet_\tau\ge\tau>1}\langle 0\rangle_\blacktriangle$⟫, hence (2) of $\mathtt{S}_9$ holds. Now, from (18) with $\tau=t^\bullet$ we have

$$V_{t^\bullet}\le\beta V_{t^\bullet-1}\le\cdots\le\beta^{t^\bullet-1}V_1=⟪<⟫\ \beta^{t^\bullet}V_0\cdots(20).$$

⟨C⟩ Let $\tau>t^\bullet$ (>0), hence $\tau>1$. From (15) with $\tau\ge t>t^\bullet$ we have

$$V_\tau>\beta V_{\tau-1}>\cdots>\beta^{\tau-t^\bullet-1}V_{t^\bullet+1}>\beta^{\tau-t^\bullet}V_{t^\bullet_\tau}\cdots(21),\quad\tau>t^\bullet.$$

Combining (21) and (20) leads to

$$V_\tau>\beta V_{\tau-1}>\cdots>\beta^{\tau-t^\bullet-1}V_{t^\bullet+1}>\beta^{\tau-t^\bullet}V_{t^\bullet}\le\beta^{\tau-t^\bullet+1}V_{t^\bullet-1}\le\cdots\le\beta^{\tau-1}V_1=⟪<⟫\ \beta^\tau V_0,\quad\tau>t^\bullet,$$

or equivalently

$$I_\tau^\tau>I_\tau^{\tau-1}>I_\tau^{\tau-2}>\cdots>I_\tau^{t^\bullet+1}>I_\tau^{t^\bullet}\le I_\tau^{t^\bullet-1}\le\cdots\le I_\tau^1=⟪<⟫\ I_\tau^0\cdots(22),\quad\tau>t^\bullet.$$

Hence we have Ⓢ$_{\tau>t^\bullet}\langle\tau\rangle$ or ❶$_{\tau>t^\bullet}\langle 0\rangle$, thus (3) of $\mathtt{S}_9$ holds. ▮

20.2.3.3 Market Restriction

20.2.3.3.1 Positive Restriction

20.2.3.3.1.1 Case of $\beta=1$ and $s=0$

□ `Pom` **20.2.1** ($\mathscr{A}$\{M:2[ℝ][E]$^+$\}) *Suppose $a>0$. Let $\beta=1$ and $s=0$.*

(a) *V_t is nondecreasing in $t\ge 0$.*

(b) *Let $\rho\ge b$. Then* ❶$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let $\rho<b$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$.

● *Proof* The same as `Tom` 20.2.1(p.189) due to Lemma 17.4.4(p.116). ▮

20.2.3.3.1.2 Case of $\beta < 1$ or $s > 0$

□ Pom **20.2.2** ($\mathscr{A}${M:2[ℝ][E]$^+$}) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho <\ x_K$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t \geq 0$ *if* $\lambda < 1$ *or* $a \leq \rho$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.

(b) *Let* $x_L \leq \rho$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.

(c) *Let* $\rho <\ x_L$.

1. Ⓢ$_1\langle 1\rangle_{\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$.
   - i. *Let* $a \leq \rho$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
   - ii. *Let* $\rho < a$.
     1. *Let* $(\lambda\mu - s)/\lambda \leq a$.
        - i. *Let* $\lambda = 1$. *Then* ◎$_{\tau>1}\langle 1\rangle_{\|}$.
        - ii. *Let* $\lambda < 1$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
     2. *Let* $(\lambda\mu - s)/\lambda > a$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
4. *Let* $\beta < 1$ *and* $s > 0$.
   - i. *Let* $a \leq \rho$.
     1. *Let* $\lambda\beta\mu \geq s$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.*IvsD*
     2. *Let* $\lambda\beta\mu < s$. *Then* $\mathtt{S}_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*
   - ii. *Let* $\rho < a$.
     1. *Let* $(\lambda\beta\mu - s)/\delta \leq a$.
        - i. *Let* $\lambda = 1$.
          1. *Let* $\beta\mu > s$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
          2. *Let* $\beta\mu \leq s$. *Then* ◎$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
        - ii. *Let* $\lambda < 1$.
          1. *Let* $\lambda\beta\mu \geq s$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
          2. *Let* $\lambda\beta\mu < s$. *Then* $\mathtt{S}_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*
     2. *Let* $(\lambda\beta\mu - s)/\delta > a$.
        - i. *Let* $\lambda\beta\mu \geq s$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
        - ii. *Let* $\lambda\beta\mu < s$. *Then* $\mathtt{S}_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.57) (a).

(a-c2ii2) The same as (a-c2ii2) of `Tom` 20.2.2(p.189) .

(c3) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c3i1,c3ii1i1,c3ii1ii1,c3ii2i) of `Tom` 20.2.2(p.189) .

(c4-c4ii2ii) Let $\beta < 1$ and $s < 0$. Then, due to (2) it suffices to consider only (c3-c3ii2ii) of `Tom` 20.2.2(p.189) . ∎

□ Pom **20.2.3** ($\mathscr{A}${M:2[ℝ][E]$^+$}) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho =\ x_K$.

(a) $V_t =\ x_K\ = \rho$ *for* $t > 0$.

(b) *Let* $\beta = 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(d) *Let* $\beta < 1$ *and* $s > 0$.

1. *Let* $\lambda\beta\mu > s$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
2. *Let* $\lambda\beta\mu \leq s$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.57) (a).

(a,b) The same as (a,b) of `Tom` 20.2.3(p.192) .

(c) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (c1) of `Tom` 20.2.3(p.192) .

(d,d2) Let $\beta < 1$ and $s > 0$. Then, due to (2) it suffices to consider only (c1,c2) of `Tom` 20.2.3(p.192) . ∎

□ Pom **20.2.4** ($\mathscr{A}${M:2[ℝ][E]$^+$}) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho >\ x_K$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.

(b) *Let* $\rho <\ x_L$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$ $\to$ $\to$ Ⓢ

(c) *Let* $\rho =\ x_L$. *Then* ⓓ$_1\langle 0\rangle_{\|}$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ *for* $\tau > 1$.

(d) *Let* $\rho >\ x_L$.

1. *Let* $\beta = 1$. *Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\vartriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* $\mathtt{S}_9$(p.193) [Ⓢ$\vartriangle$ | ●$\vartriangle$ | ●$\blacktriangle$] *is true.*
3. *Let* $\beta < 1$ *and* $s > 0$.
   i. *Let* $\lambda\beta\mu \le s$. *Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\blacktriangle}$.
   ii. *Let* $\lambda\beta\mu > s$. *Then* $\mathtt{S}_9$(p.193) [Ⓢ$\vartriangle$ | ●$\vartriangle$ | ●$\blacktriangle$] *is true* (see *Numerical Example* 6(p.216))

● *Proof* Suppose $a > 0$. Then $b > a > 0 \cdots (1)$. We have $\kappa = \lambda\beta\mu - s \cdots (2)$ from Lemma 10.3.1(p.57) (a).

(a-d1) The same as (a-d1) of `Tom` 20.2.4(p.193).

(d2) Let $\beta < 1$ and $s = 0$. Then, due to (1) it suffices to consider only (d2ii) of `Tom` 20.2.4(p.193).

(d3,d3ii) Let $\beta < 1$ and $s > 0$. Then, due to (2) it suffices to consider only (d2i,d2ii) of `Tom` 20.2.4(p.193). ∎

20.2.3.3.2 Mixed Restriction

Omitted.

20.2.3.3.3 Negative Restriction

Omitted.

20.2.4 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}$

Due to Lemma 20.2.1(p.189) (a), we see that the following `Tom`'s 20.2.5(p.196) − 20.2.8(p.197) can be obtained by applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128))) to `Tom`'s 20.2.1(p.189) − 20.2.4(p.193) (see Theorem 20.2.1(p.188)).

20.2.4.1 Analysis

20.2.4.1.1 Case of $\beta = 1$ and $s = 0$

□ `Tom` **20.2.5** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \ge 0$.

(b) *Let* $\rho \le a$. *Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let* $\rho > a$. *Then* $\text{Ⓢ}_{\tau>0}\langle \tau\rangle_{\blacktriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 20.2.1(p.189). ∎

20.2.4.1.2 Case of $\beta < 1$ or $s > 0$

□ `Tom` **20.2.6** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t \ge 0$ *if* $\lambda < 1$ *or* $b > \rho$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $x_{\tilde{L}} \ge \rho$. *Then* $\text{ⓓ}_{\tau>0}\langle 0\rangle_{\vartriangle}$.

(c) *Let* $\rho > x_{\tilde{L}}$.

1. $\text{Ⓢ}_1\langle 1\rangle_{\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$.
   i. *Let* $b > \rho$. *Then* $\text{Ⓢ}_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
   ii. *Let* $\rho \ge b$.
      1. *Let* $(\lambda\mu + s)/\lambda \ge b$.
         i. *Let* $\lambda = 1$. *Then* $\text{◎}_{\tau>1}\langle 1\rangle_{\parallel}$.
         ii. *Let* $\lambda < 1$. *Then* $\text{Ⓢ}_{\tau>0}\langle \tau\rangle_{\blacktriangle}$.
      2. *Let* $(\lambda\mu + s)/\lambda < b$. *Then* $\text{Ⓢ}_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$ ($s > 0$).
   i. *Let* $b > \rho$.
      1. *Let* $a \le 0$ ($\tilde{\kappa} \le 0$). *Then* $\text{Ⓢ}_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
      2. *Let* $a > 0$ ($\tilde{\kappa} > 0$). *Then* $\mathtt{S}_8$ [Ⓢ$\blacktriangle$ | ◎ $\parallel$ | ◎ $\vartriangle$ | ◎ $\blacktriangle$] *is true.*
   ii. *Let* $\rho \ge b$.
      1. *Let* $(\lambda\beta\mu + s)/\delta \ge b$.
         i. *Let* $\lambda = 1$.
            1. *Let* $a < 0$ ($\tilde{\kappa} < 0$). *Then* $\text{Ⓢ}_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
            2. *Let* $a \ge 0$ ($\tilde{\kappa} \ge 0$). *Then* $\text{◎}_{\tau>1}\langle 1\rangle_{\vartriangle}$.
         ii. *Let* $\lambda < 1$.
            1. *Let* $a \le 0$ ($\tilde{\kappa} \le 0$). *Then* $\text{Ⓢ}_{\tau>1}\langle \tau\rangle_{\blacktriangle}$.
            2. *Let* $a > 0$ ($\tilde{\kappa} > 0$). *Then* $\mathtt{S}_8$ [Ⓢ$\blacktriangle$ | ◎ $\parallel$ | ◎ $\vartriangle$ | ◎ $\blacktriangle$] *is true.*

2. *Let* $(\lambda\beta\mu + s)/\delta < b$.
   i. *Let* $a \le 0$ ⟅$\tilde{\kappa} \le 0$⟆. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   ii. *Let* $a > 0$ ⟅$\tilde{\kappa} > 0$⟆. *Then* $\mathtt{S}_8$ [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 20.2.2(p.189). ▮

□ `Tom` **20.2.7** (⊡$\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* 🅓$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0$ ⟅$s > 0$⟆.

  1. *Let* $a < 0$ ⟅$\tilde{\kappa} < 0$⟆. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
  2. *Let* $a \ge 0$ ⟅$\tilde{\kappa} \ge 0$⟆. *Then* 🅓$_{\tau>0}\langle 0\rangle_{\vartriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 20.2.3(p.192). ▮

□ `Tom` **20.2.8** (⊡$\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$\}$) *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.

(b) *Let* $\rho > x_{\tilde{L}}$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c) *Let* $\rho = x_{\tilde{L}}$. *Then* 🅓$_1\langle 0\rangle_{\parallel}$ *and* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(d) *Let* $\rho < x_{\tilde{L}}$.

  1. *Let* $\beta = 1$. *Then* 🅓$_{\tau>0}\langle 0\rangle_{\vartriangle}$.
  2. *Let* $\beta < 1$ *and* $s = 0$ ⟅$s > 0$⟆.
     i. *Let* $a \ge 0$ ⟅$\tilde{\kappa} \ge 0$⟆. *Then* 🅓$_{\tau>0}\langle 0\rangle_{\vartriangle}$ ⟅ 🅓$_{\tau>0}\langle 0\rangle_{\blacktriangle}$⟆ .
     ii. *Let* $a < 0$ ⟅$\tilde{\kappa} < 0$⟆. *Then* $\mathtt{S}_9$ [ⓢ△ | ●△ | ●▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ to `Tom` 20.2.4(p.193). ▮

20.2.4.2 Market Restriction

20.2.4.2.1 Positive Restriction

20.2.4.2.1.1 Case of $\beta = 1$ and $s = 0$

□ `Pom` **20.2.5** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$^+\}$) *Suppose* $a > 0$. *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \ge 0$.

(b) *Let* $\rho \le a$. *Then* 🅓$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let* $\rho > a$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

● *Proof* The same as `Tom` 20.2.5(p.196) due to Lemma 17.4.4(p.116). ▮

20.2.4.2.1.2 Case of $\beta < 1$ or $s > 0$

□ `Pom` **20.2.6** ($\mathscr{A}\{\tilde{\mathsf{M}}$:2[ℝ][E]$^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t \ge 0$ *if* $\lambda < 1$ *or* $b \ge \rho$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $x_{\tilde{L}} \ge \rho$. *Then* 🅓$_{\tau}\langle 0\rangle_{\vartriangle}$.

(c) *Let* $\rho > x_{\tilde{L}}$.

  1. ⓢ$_1\langle 1\rangle_{\blacktriangle}$. *Below let* $\tau > 1$.
  2. *Let* $\beta = 1$.
     i. *Let* $b \ge \rho$. *Then* ⓢ$_{\tau}\langle\tau\rangle_{\blacktriangle}$.
     ii. *Let* $\rho > b$.
        1. *Let* $(\lambda\mu + s)/\lambda \ge b$.
           i. *Let* $\lambda = 1$. *Then* ◎$_{\tau>1}\langle 1\rangle_{\parallel}$.
           ii. *Let* $\lambda < 1$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
        2. *Let* $(\lambda\mu + s)/\lambda < b$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
  3. *Let* $\beta < 1$ *and* $s = 0$. *Then we have* $\mathtt{S}_8$(p.189) [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲].
  4. *Let* $\beta < 1$ *and* $s > 0$.
     i. *Let* $b > \rho$. *Then* $\mathtt{S}_8$(p.189) [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*
     ii. *Let* $\rho \ge b$.

1. *Let* $(\lambda\beta\mu + s)/\delta \geq b$.
   i. *Let* $\lambda = 1$. *Then* ◎$_{\tau>1}\langle 1\rangle_{\triangle}$.
   ii. *Let* $\lambda < 1$. *Then* $\mathtt{S}_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*
2. *Let* $(\lambda\beta\mu + s)/\delta < b$. *Then* $\mathtt{S}_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* □

● *Proof* Suppose $a > 0 \cdots$ (1), hence $b > a > 0 \cdots$ (2). Then $\tilde{\kappa} = s \cdots$ (3) from Lemma 12.6.6(p.81) (a).

(a-c2ii2) The same as (a-c2ii2) of `Tom` 20.2.6(p.196).

(c3) Let $\beta < 1$ and $s = 0$. Assume $(\lambda\beta\mu + s)/\delta \geq b$. Then, since $\lambda\beta\mu/\delta \geq b$, we have $\lambda\beta\mu \geq \delta b$ from (10.2.2 (1) (p.54)), hence $\lambda\beta\mu \geq \delta b \geq \lambda b$ due to (10.2.2 (1) (p.54)), so $\beta\mu \geq b$, which contradicts [3(p.116)]. Thus, it must be that $(\lambda\beta\mu + s)/\delta < b$. From this it suffices to consider only (c3i2,c3ii2ii) of `Tom` 20.2.6(p.196).

(c4-c4ii2) Let $\beta < 1$ and $s > 0$. Then, since $\tilde{\kappa} > 0$ due to $\tilde{\kappa} > 0$ due to (3), it suffices to consider only (c3i2,c3ii1i2,c3ii1ii2,c3ii2ii) of `Tom` 20.2.2(p.189). ■

□ `Pom` **20.2.7** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$.

(b) *Let* $\beta = 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\beta < 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\triangle}$.

● *Proof* Suppose $a > 0 \cdots$ (1). Then $\tilde{\kappa} = s \cdots$ (2) from Lemma 12.6.6(p.81) (a).

(a,b) The same as `Tom` 20.2.7(p.197) (a,b).

(c) If $s = 0$, then due to (1) it suffices to consider only (c2) of `Tom` 20.2.7(p.197) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of `Tom` 20.2.7(p.197) with $\tilde{\kappa}$. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ■

□ `Pom` **20.2.8** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]^+\}$) *Suppose* $a > 0$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.

(b) *Let* $\rho > x_{\tilde{L}}$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c) *Let* $\rho = x_{\tilde{L}}$. *Then* ⓓ$_1\langle 0\rangle_{\|}$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(d) *Let* $\rho < x_{\tilde{L}}$.
   1. *Let* $\beta = 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\triangle}$.
   2. *Let* $\beta < 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\triangle}$ (ⓓ$_{\tau>0}\langle 0\rangle_{\blacktriangle}$).

● *Proof* Suppose $a > 0 \cdots$ (1). Then $\tilde{\kappa} = s \cdots$ (2) from Lemma 12.6.6(p.81) (a).

(a-d1) The same as `Tom` 20.2.8(p.197) (a-d1).

(d2) If $s = 0$, then due to (1) it suffices to consider only (d2i) of `Tom` 20.2.8(p.197) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (d2i) of `Tom` 20.2.8(p.197) (d2i) with $\tilde{\kappa}$. Accordingly, whether $s = 0$ or $s > 0$, we have the same result. ■

20.2.4.2.2 Mixed Restriction

Omitted.

20.2.4.2.3 Negative Restriction

Omitted.

20.2.5 Derivation of $\mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]\}$

20.2.5.1 Preliminary

From (6.5.33(p.39)) and from (5.1.21(p.24)) and (5.1.20(p.24)) we have

$$V_t - \beta V_{t-1} = K(V_{t-1}) + (1-\beta)V_{t-1} = L(V_{t-1}), \quad t > 1. \tag{20.2.12}$$

From (6.5.32(p.39)) we have

$$V_1 - \beta V_0 = V_1 - \beta\rho = \lambda\beta\max\{0, a-\rho\} - s. \tag{20.2.13}$$

20.2.5.2 Analysis

20.2.5.2.1 Case of $\beta = 1$ and $s = 0$

Let $\beta = 1$ and $s = 0$. Then, from (20.2.12(p.198)) and (5.1.20(p.24)) we have

$$V_t - \beta V_{t-1} = \lambda T(V_{t-1}) \geq 0, \quad t > 1, \tag{20.2.14}$$

due to Lemma 13.2.1(p.91) (g). From (6.5.32(p.39)) we have

$$\begin{aligned} V_1 &= \lambda\max\{0, a-\rho\} + \rho & (20.2.15)\\ &= \max\{\rho, \lambda a + (1-\lambda)\rho\}. & (20.2.16)\end{aligned}$$

20.2.5.2.1.1 Case of $\rho \le a^\star$

In this case, Theorem 20.2.2(p.188) holds due to Lemma 20.2.1(p.189) (c). Hence, Proposition 20.2.1 below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.128))) to Tom 20.2.1(p.189) .

Proposition 20.2.1 ($\rho \le a^\star$) Assume $\rho \le a^\star$. Let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \ge 0$.

(b) ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$. □

● *Proof by analogy* Assume $\rho \le a^\star$. Let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.2.1(p.189) (a).

(b) Since (b,c) of Tom 20.2.1(p.189) have none of a and μ, even if $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied the two assertions, no change occurs (see Lemma 13.6.1(p.97)). However, since $\rho \le a^\star < a < b$ due to the assumption $\rho \le a^\star$ and Lemma 13.2.1(p.91) (n), it follows that only (c) of Tom 20.2.1(p.189) holds. ▮

20.2.5.2.1.2 Case of $b \le \rho$

In this case, Theorem 20.2.2(p.188) holds due to Lemma 20.2.1(p.189) (c). Hence, Proposition 20.2.2 below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ (see (18.0.5(p.128))) to Tom 20.2.1(p.189) .

Proposition 20.2.2 ($b \le \rho$) Assume $b \le \rho$. Let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \ge 0$.

(b) ⓓ$_{\tau>0}\langle 0\rangle_{\|}$. □

● *Proof by analogy* Assume $b \le \rho$. Let $\beta = 1$ and $s = 0$.

(a) The same as Tom 20.2.1(p.189) (a).

(b) Due to the assumption $b \le \rho$, only (b) of Tom 20.2.1(p.189) holds. ▮

20.2.5.2.1.3 Case of $a^\star < \rho < b$

In this case, Theorem 20.2.2(p.188) does not always hold due to Lemma 20.2.1(p.189) (d). Hence, Proposition 20.2.3 below must be directly proven.

Proposition 20.2.3 ($a^\star < \rho < b$) Assume $a^\star < \rho < b$. Let $\beta = 1$ and $s = 0$.

(a) V_t is nondecreasing in $t \ge 0$.

(b) Let $a \le \rho$. Then ⓓ$_1\langle 0\rangle_{\|}$ and ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(c) Let $\rho < a$. Then ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$. □

● *Proof* Assume $a^\star < \rho < b \cdots$**(1)** and let $\beta = 1$ and $s = 0$. Then $L(x) = K(x) = \lambda T(x) \ge 0 \cdots$**(2)** for any x from (5.1.20(p.24)) and (5.1.21(p.24)) and from Lemma 13.2.1(p.91) (g). Since $V_0 < b$ from (1) and (6.5.31(p.39)), we have $L(V_0) = \lambda T(V_0) = \lambda T(\rho) > 0 \cdots$**(3)** from (2) and Lemma 13.2.1(p.91) (g). Then, since $\rho < b$ and $a < b$, from (20.2.16(p.198)) we obtain $V_1 < \max\{b, \lambda b + (1-\lambda)b\} = \max\{b, b\} = b$. Suppose $V_{t-1} < b$. Then, since $a^\star < b$ from (1), we have $V_t < K(b) + b$ from (6.5.33(p.39)) and Lemma 13.2.3(p.94) (h), hence $V_t < \beta b - s$ from (13.2.12 (2) (p.94)), so $V_{t-1} < b$ due to the assumption "$\beta = 1$ and $s = 0$". Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $T(V_{t-1}) > 0 \cdots$**(4)** for $t > 1$ from Lemma 13.2.1(p.91) (g). Thus $V_t - \beta V_{t-1} > 0$ for $t > 1$ from (20.2.14(p.198)) or equivalently $V_t > \beta V_{t-1}$ for $t > 1$. Then, since $V_t > \beta V_{t-1}$ for $\tau \ge t > 1$, we have

$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 \cdots \textbf{(5)}, \qquad \tau > 1.$$

In addition, from (2) we have $L(V_{t-1}) = \lambda T(V_{t-1}) > 0 \cdots$**(6)** for $t > 1$ due to (4), so $L(V_{t-1}) > 0$ for $t > 0$ due to (3).

(a) From (20.2.15(p.198)) and (6.5.31(p.39)) we have $V_1 - V_0 = V_1 - \rho = \lambda \max\{0, a - \rho\} \ge 0$, hence $V_1 \ge V_0 \cdots$**(7)**. From (6.5.33(p.39)) with $t = 2$ we have $V_2 - V_1 = K(V_1) > 0$ due to (6) with $t = 2$, hence $V_2 > V_1$, so $V_2 \ge V_1 \cdots$**(8)**. Suppose $V_t \ge V_{t-1}$. Then from (6.5.33(p.39)) and Lemma 13.2.3(p.94) (e) we have $V_{t+1} = K(V_t) + V_t \ge K(V_{t-1}) + V_{t-1} = V_t$. Hence, by induction $V_t \ge V_{t-1}$ for $t > 1$. From this and (7) we have $V_t \ge V_{t-1}$ for $t > 0$, hence it follows that V_t is nondecreasing in $t \ge 0$.

(b) Let $a \le \rho$, hence $V_1 = \lambda \max\{0, a - \rho\} + \rho = \rho$ from (6.5.32(p.39)), so $V_1 < b$ due to (1). Then, since $V_1 - \beta V_0 = V_1 - V_0 = \rho - \rho = 0$, we have $V_1 = \beta V_0 \cdots$**(9)**, hence $t_1^* = 0$, i.e., ⓓ$_1\langle 0\rangle_{\|}$. Below let $\tau > 1$. Then, from (5) and (9) we have

$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 = \beta^\tau V_0,$$

hence $t_\tau^* = \tau$ for $\tau > 1$, i.e., ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(c) Let $\rho < a$. Then, since $V_1 = \lambda(a - \rho) + \rho$ due to (6.5.32(p.39)), we have $V_1 - \beta V_0 = V_1 - V_0 = V_1 - \rho = \lambda(a - \rho) > 0$, i.e., $V_1 > \beta V_0$, hence $t_1^* = 1 \cdots$**(10)**. Let $\tau > 1$. Then, from (5) we have

$$V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^{\tau-1} V_1 > \beta^\tau V_0, \qquad \tau > 1,$$

hence $t_\tau^* = \tau$ for $\tau > 1$, hence ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$. From this and (10) we have $t_\tau^* = \tau$ for $\tau > 0$, i.e., ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$. ▮

20.2.5.2.1.4 Integration of Propositions 20.2.1(p.199) –20.2.3(p.199)

☐ `Tom` **20.2.9** (■ $\mathscr{A}${M:2[ℙ][E]}) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*

(b) *Let $\rho \leq a^\star$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow$ Ⓢ$_\blacktriangle$

(c) *Let $b \leq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_\parallel \mapsto$ $\rightarrow$ ⓓ$_\parallel$

(d) *Let $a^\star < \rho < b$.*

   1. *Let $a \leq \rho$. Then* ⓓ$_1\langle 0\rangle_\parallel$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow$ ⓓ$_\parallel$/Ⓢ$_\blacktriangle$
   2. *Let $\rho < a$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle \mapsto$ $\rightarrow$ Ⓢ$_\blacktriangle$

● *Proof* (a) The same as Propositions 20.2.1(p.199) (a), 20.2.2(p.199) (a), and 20.2.3(p.199) (a).
(b) The same as Proposition 20.2.1(p.199) (b).
(c) The same as Proposition 20.2.2(p.199) (b).
(d-d2) The same as Proposition 20.2.3(p.199) (b,c). ▮

Corollary 20.2.1 (M:2[ℙ][E]) *Let $\beta = 1$ and $s = 0$. Then, z_t is nondecreasing in $t \geq 0$.* ☐

● *Proof* Immediate from Lemma 20.2.9(p.200) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ▮

20.2.5.2.2 Case of $\beta < 1$ or $s > 0$

20.2.5.2.2.1 Case of $\rho \leq a^\star$

In this case, Theorem 20.2.2(p.188) holds due to Lemma 20.2.1(p.189) (c), hence `Tom`'s 20.2.10(p.200) –20.2.12(p.201) below can be derived by applying $\mathcal{A}_{\mathbb{R}\rightarrow\mathbb{P}}$ (see (18.0.5(p.128))) to `Tom`'s 20.2.2(p.189) –20.2.4(p.193). In the proofs below, let us represent what results from applying $\mathcal{A}_{\mathbb{R}\rightarrow\mathbb{P}}$ to a given `Tom` by `Tom`$'$ (see (20.1.42(p.166))).

☐ `Tom` **20.2.10** (☐ $\mathscr{A}${M:2[ℙ][E]}) *Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as $t \rightarrow \infty$.*

(b) *Let $x_L \leq \rho$. Then* ⓓ$_{\tau>0}\langle 0\rangle_\vartriangle$.

(c) *Let $\rho < x_L$.*

   1. Ⓢ$_1\langle 1\rangle_\blacktriangle$. *Below let $\tau > 1$.*
   2. *Let $\beta = 1$.*
      i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
         1. *Let $\lambda = 1$. Then* ◎$_{\tau>1}\langle 1\rangle_\parallel$.
         2. *Let $\lambda < 1$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.

      ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
   3. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
      i. *Let $(\lambda\beta a - s)/\delta \leq a^\star$.*
         1. *Let $\lambda = 1$.*
            i. *Let $b > 0$ ⦅$\kappa > 0$⦆. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
            ii. *Let $b \leq 0$ ⦅$\kappa \leq 0$⦆. Then* ◎$_{\tau>1}\langle 1\rangle_\vartriangle$.
         2. *Let $\lambda < 1$.*
            i. *Let $b \geq 0$ ⦅$\kappa \geq 0$⦆. Then* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.
            ii. *Let $b < 0$ ⦅$\kappa < 0$⦆. Then* S$_8$ [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

      ii. *Let $(\lambda\beta a - s)/\delta > a^\star$.*
         1. *Let $b \geq 0$ ⦅$\kappa \geq 0$⦆. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$.
         2. *Let $b < 0$ ⦅$\kappa < 0$⦆. Then* S$_8$ [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* ☐

● *Proof by analogy* Consider the `Tom`$'$ resulting from applying $\mathcal{A}_{\mathbb{R}\rightarrow\mathbb{P}}$ to `Tom` 20.2.2(p.189). Then "$a < \rho$" in `Tom` 20.2.2(p.189) (c2i,c3i) changes into "$a^\star < \rho$" in the `Tom`$'$, which contradicts the assumption $\rho \leq a^\star$. Accordingly, removing all assertions with "$a^\star < \rho$" from the `Tom`$'$ leads to `Tom` 20.2.10 above. ▮

Corollary 20.2.2 (M:2[ℙ][E]) *Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$. Then, z_t is nondecreasing in $t \geq 0$.* ☐

● *Proof* Immediate from `Tom` 20.2.10(p.200) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ▮

☐ `Tom` **20.2.11** (☐ $\mathscr{A}${M:2[ℙ][E]}) *Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t \geq 0$.*

(b) *Let $\beta = 1$. Then* ⓓ$_{\tau>0}\langle 0\rangle_\parallel$.

(c) *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*

1. *Let* $b > 0$ ⦅$\kappa > 0$⦆ . *Then* $ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
2. *Let* $b \le 0$ ⦅$\kappa \le 0$⦆ . *Then* $❶_{\tau>0}\langle 0\rangle_{\vartriangle}$. □

● *Proof by analogy* The same as Tom 20.2.3(p.192) due to Lemma 13.6.1(p.97) . ∎

Corollary 20.2.3 (M:2[ℙ][E]) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$. *Then, the optimal price to propose is given by* $z_t = z(\rho)$ *for* $t \ge 0$. □

● *Proof* Immediate from Tom 20.2.11(p.200) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ∎

□ **Tom 20.2.12** (⊡𝒜{M:2[ℙ][E]}) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.

(b) *Let* $\rho < x_L$. *Then* $ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c) *Let* $\rho = x_L$. *Then* $❶_1\langle 0\rangle_{\vartriangle}$ *and* $ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(d) *Let* $\rho > x_L$.

1. *Let* $\beta = 1$. *Then* $❶_{\tau>0}\langle 0\rangle_{\vartriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆ .
   i. *Let* $b \le 0$ ⦅$\kappa \le 0$⦆. *Then* $❶_{\tau>0}\langle 0\rangle_{\vartriangle}$ ⦅ $❶_{\tau>0}\langle 0\rangle_{\blacktriangle}$ ⦆ .
   ii. *Let* $b > 0$ ⦅$\kappa > 0$⦆. *Then* S_9 [ⓢ△ | ●△ | ●▲] *is true.* □

● *Proof by analogy* The same as Tom 20.2.4(p.193) due to Lemma 13.6.1(p.97) . ∎

Corollary 20.2.4 (M:2[ℙ][E]) *Assume* $\rho \le a^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$. *Then, the optimal price to propose* z_t *is nonincreasing in* $t \ge 0$. □

● *Proof* Immediate from Tom 20.2.12(p.201) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ∎

20.2.5.2.2.2 Case of $b \le \rho$

In this case, Theorem 20.2.2(p.188) holds due to Lemma 20.2.1(p.189) (c). Hence Tom's 20.2.13-20.2.15 below can be derived by applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom's 20.2.2(p.189) -20.2.4(p.193) .

□ **Tom 20.2.13** (⊡𝒜{M:2[ℙ][E]}) *Assume* $b \le \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$.

(a) V_t *is nondecreasing in* $t \ge 0$, *is strictly increasing in* $t \ge 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_K$ *as* $t \to \infty$.

(b) *Let* $x_L \le \rho$. *Then* $❶_{\tau>0}\langle 0\rangle_{\vartriangle}$.

(c) *Let* $\rho < x_L$.

1. $ⓢ_1\langle 1\rangle_{\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$. *Then* $ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆ .
   i. *Let* $b \ge 0$ ⦅$\kappa \ge 0$⦆ . *Then* $ⓢ_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   ii. *Let* $b < 0$ ⦅$\kappa < 0$⦆ . *Then* S_8 [ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.* □

● *Proof by analogy* Consider the Tom′ resulting from applying $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ to Tom 20.2.2(p.189) . Then "$\rho \le a$" in (c2ii,c3ii) of Tom 20.2.2(p.189) changes into "$\rho \le a^\star$" in the Tom′, hence $\rho \le a^\star < a < b$ due to Lemma 13.2.1(p.91) (n), which contradicts the assumption $b \le \rho$. Accordingly, removing all assertions with "$\rho \le a$" from the Tom′ leads to Tom 20.2.13 above. ∎

Corollary 20.2.5 (M:2[ℙ][E]) *Assume* $b \le \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_K$. *Then, the optimal price to propose* z_t *is nondecreasing in* $t \ge 0$. □

● *Proof* Immediate from Tom 20.2.13(p.201) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87) . ∎

□ **Tom 20.2.14** (⊡𝒜{M:2[ℙ][E]}) *Assume* $b \le \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $❶_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆ .

1. *Let* $b \ge 0$ ⦅$\kappa \ge 0$⦆ . *Then* $ⓢ_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
2. *Let* $b < 0$ ⦅$\kappa < 0$⦆ . *Then* $❶_{\tau>0}\langle 0\rangle_{\vartriangle}$. □

● *Proof by analogy* The same as Tom 20.2.3(p.192) due to Lemma 13.6.1(p.97) . ∎

Corollary 20.2.6 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$. Then, the optimal price to propose is given by $z_t = z(\rho)$ for $t \ge 0$.* □

• *Proof* Immediate from Tom 20.2.14(p.201) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ∎

□ **Tom 20.2.15** (⊡𝒜{M:2[ℙ][E]}) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*

(b) *Let $\rho = x_L$. Then* ❶$_1\langle 0\rangle_\triangle$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(c) *Let $\rho > x_L$.*

1. *Let $\beta = 1$. Then* ❶$_{\tau>0}\langle 0\rangle_\triangle$.
2. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
   i. *Let $b \le 0$ ⦅$\kappa \le 0$⦆. Then* ❶$_{\tau>0}\langle 0\rangle_\triangle$ ⦅ ❶$_{\tau>0}\langle 0\rangle_\blacktriangle$ ⦆.
   ii. *Let $b > 0$ ⦅$\kappa > 0$⦆. Then* S$_9$ [Ⓢ△ | ●△ | ●▲] *is true.* □

• *Proof by analogy* Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. In this case, even if $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ is applied to Tom 20.2.4(p.193), it can be easily confirmed that no change occurs (see Lemma 13.6.1(p.97)). However, if the condition $\rho < x_L$ is added, we encounter the following contradiction. Then we have $b \le \rho < x_L \cdots$ **(1)**. Now, since $0 = L(x_L) = \lambda\beta T(x_L) - s$ and $T(x_L) = 0$ from Lemma 13.2.1(p.91) (g), we have $0 = -s$, hence $s = 0$, so we have $x_L = b$ due to Lemma 13.2.2(p.94) (d), which is a contradicts (1). Accordingly, the condition $\rho < x_L$ becomes impossible. This result implies that the assertion (b) with $\rho \ge x_L$ in Tom 20.2.4(p.193) must be omitted; accordingly, it follows that we have Tom 20.2.15 above. ∎

Corollary 20.2.7 (M:2[ℙ][E]) *Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$. Then, the optimal price to propose z_t is nonincreasing in $t \ge 0$.* □

• *Proof* Immediate from Tom 20.2.15(p.202) (a) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ∎

20.2.5.2.2.3 Case of $a^\star < \rho < b$

In this case, Theorem 20.2.2(p.188) does not always hold due to Lemma 20.2.1(p.189) (d). Hence, Tom 20.2.16(p.203) below must be directly proven. For explanatory convenience, let us define:

S$_{10}$ [Ⓢ△ | ●△] = {
We have:
(1) Let $\lambda\max\{0, a-\rho\} < s$. Then Ⓢ$_{\tau>1}\langle\tau\rangle_\triangle$ or ❶$_{\tau>0}\langle 0\rangle_\triangle$.
(2) Let $\lambda\max\{0, a-\rho\} \ge s$. Then Ⓢ$_{\tau>1}\langle\tau\rangle_\triangle$.
}

S$_{11}$ [Ⓢ△ | Ⓢ▲ | ◎△ | ●△] = {
There exists $t^\bullet_\tau > 1$ such that:
(1) If $\lambda\beta\max\{0, a-\rho\} < s$, then
  i. Ⓢ$_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle_\triangle$ or ❶$_{t^\bullet_\tau \ge \tau > 1}\langle 0\rangle_\triangle$,
  ii. ◎$_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\triangle$ or ❶$_{\tau > t^\bullet_\tau}\langle 0\rangle_\triangle$.
(2) If $\lambda\beta\max\{0, a-\rho\} \ge s$, then
  i. Ⓢ$_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle_\blacktriangle$,
  ii. ◎$_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\triangle$.
}

S$_{12}$ [Ⓢ△ | Ⓢ▲ | ◎△ | ●△ | ●▲] = {
There exists $t^\bullet_\tau > 1$ such that:
(1) If $\lambda\beta\max\{0, a-\rho\} < s$, then
  i. ❶$_{t^\bullet_\tau \ge \tau > 0}\langle 0\rangle_\blacktriangle$,
  ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_\triangle$ or ❶$_{\tau > t^\bullet_\tau}\langle 0\rangle_\triangle$.
(2) If $\lambda\beta\max\{0, a-\rho\} \ge s$, then
  i. ◎$_{t^\bullet_\tau \ge \tau > 1}\langle 1\rangle_\parallel$,
  ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_\blacktriangle$.
}

S$_{13}$ [Ⓢ△ | ◎△ | ●△ | ●▲] = {
There exists $t^\bullet_\tau > 1$ and $t^\star_\tau > 1$ such that:
(1) If $\lambda\beta\max\{0, a-\rho\} < s$, then
  i. ❶$_{t^\bullet_\tau \ge \tau > 1}\langle 0\rangle_\blacktriangle$,
  ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_\triangle$ or ❶$_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle_\triangle$.
(2) If $\lambda\beta\max\{0, a-\rho\} \ge s$, then
  i. ◎$_{t^\bullet_\tau \ge \tau > 1}\langle 1\rangle_\triangle$,
  ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_\triangle$ or ❶$_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle_\triangle$.
}

For convenience of reference, below let us copy (6.5.32(p.39))

$$V_1 = \lambda\beta\max\{0, a-\rho\} + \beta\rho - s. \tag{20.2.17}$$

□ Tom **20.2.16** (▣$\mathscr{A}${M:2[ℙ][E]}) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta \max\{0, a - \rho\} \le s$, then* ⓓ$_1\langle 0\rangle_{△}$*, or else* Ⓢ$_1\langle 1\rangle_{▲}$*. Below let $\tau > 1$* ↦ → ⓓ$_{△}$/Ⓢ$_{▲}$

(b) *Let $V_1 \le x_K$.*

1. *V_t is nondecreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $V_1 \ge x_L$. If $\lambda\beta \max\{0, a - \rho\} \le s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{△}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{△}$ ↦ → ⓓ$_{△}$/◎$_{△}$
3. *Let $V_1 < x_L$.*
   - i. *Let $\beta = 1$. Then* S$_{10}$ [Ⓢ△ | ●△] *is true* ↦ → Ⓢ$_{△}$/ⓓ$_{△}$
   - ii. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
     1. *Let $b > 0$ ⦅$\kappa > 0$⦆. Then* S$_{10}$ [Ⓢ△ | ●△] *is true* ↦ → Ⓢ$_{△}$/ⓓ$_{△}$
     2. *Let $b = 0$ ⦅$\kappa = 0$⦆. If $\lambda\beta \max\{0, a - \rho\} < s$, then* Ⓢ$_{\tau>1}\langle\tau\rangle_{△}$ *or* ⓓ$_{\tau>1}\langle 0\rangle_{△}$*, or else* Ⓢ$_{\tau>1}\langle\tau\rangle_{△}$ → Ⓢ$_{△}$/ⓓ$_{△}$/Ⓢ$_{△}$
     3. *Let $b < 0$ ⦅$\kappa < 0$⦆. Then* S$_{11}$ [Ⓢ△ | Ⓢ▲ | ◎△ | ●△] *is true* → Ⓢ$_{△}$/Ⓢ$_{▲}$/◎$_{△}$/ⓓ$_{△}$

(c) *Let $V_1 > x_K$.*

1. *V_t is nonincreasing in $t > 0$ and converges to a finite $V = x_K$ as $t \to \infty$.*
2. *Let $\beta = 1$. If $\lambda \max\{0, a - \rho\} < s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{▲}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{△}$ ↦ → ⓓ$_{▲}$/◎$_{△}$
3. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆.*
   - i. *Let $b > 0$ ⦅$\kappa > 0$⦆.*
     1. *Let $V_1 < x_L$. Then* S$_{10}$ [Ⓢ△ | ●△] *is true* ↦ → Ⓢ$_{△}$/ⓓ$_{△}$
     2. *Let $V_1 = x_L$. Then* S$_{12}$ [Ⓢ△ | Ⓢ▲ | ◎△ | ●△ | ●▲] *is true* ↦ → Ⓢ$_{△}$/Ⓢ$_{▲}$/◎$_{△}$/ⓓ$_{△}$/ⓓ$_{▲}$
     3. *Let $V_1 > x_L$. Then* S$_{13}$ [Ⓢ△ | ◎△ | ●△ | ●▲] *is true* ↦ → Ⓢ$_{△}$/◎$_{△}$/ⓓ$_{△}$/ⓓ$_{▲}$
   - ii. *Let $b \le 0$ ⦅$\kappa \le 0$⦆. If $\lambda\beta \max\{0, a - \rho\} \le s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{△}$*, or else* ◎$_{\tau>1}\langle 1\rangle_{△}$ ↦ → ⓓ$_{△}$/◎$_{△}$

● *Proof* Assume $a^\star < \rho < b \cdots$ **(**1**)** and let $\beta < 1$ or $s > 0$.

(a) If $\lambda\beta \max\{0, a - \rho\} \le s$, then $V_1 \le \beta V_0$ from (20.2.13(p.198)) or equivalently $V_1 \le \beta V_0 \cdots$ **(**2**)**, hence $t_1^* = 0$, i.e., ⓓ$_1\langle 0\rangle_{△} \cdots$ **(**3**)**, or else $V_1 > \beta V_0 \cdots$ **(**4**)**, hence $t_1^* = 1$, i.e., Ⓢ$_1\langle 1\rangle_{▲} \cdots$ **(**5**)**. Below let $\tau > 1$.

(b) Let $V_1 \le x_K \cdots$ **(**6**)**, hence $K(V_1) \ge 0 \cdots$ **(**7**)** from Lemma 13.2.3(p.94) (j1).

(b1) From (6.5.33(p.39)) with $t = 2$ we have $V_2 = K(V_1) + V_1 \ge V_1$ due to (7). Suppose $V_t \ge V_{t-1}$. Then $V_{t+1} \ge K(V_{t-1}) + V_{t-1} = V_t$ from Lemma 13.2.3(p.94) (e), hence by induction $V_t \ge V_{t-1}$ for $t > 1$, so V_t is nondecreasing in $t > 0$. Note (6). Suppose $V_{t-1} \le x_K$. Then, from (6.5.33(p.39)) and Lemma 13.2.3(p.94) (e) we have $V_t \le K(x_K) + x_K = x_K$. Hence, by induction $V_t \le x_K \cdots$ **(**8**)** for $t > 0$, i.e., V_t is upper bounded in t, hence V_t converges to a finite V as $t \to \infty$. Then, since $V = K(V) + V$ as $\tau \to \infty$ from (6.5.33(p.39)), we have $V = K(V) + V$, hence $K(V) = 0$ thus $V = x_K$ from Lemma 13.2.3(p.94) (j1).

(b2) Let $V_1 \ge x_L$. Then, since $x_L \le V_{t-1}$ for $t > 1$ due to (b1), we have $L(V_{t-1}) \le 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a), thus $L(V_{t-1}) \le 0$ for $\tau \ge t > 1$. Accordingly, since $V_t \le \beta V_{t-1}$ for $\tau \ge t > 1$ from (20.2.12(p.198)), we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 \cdots \textbf{(}9\textbf{)}, \qquad \tau > 1.$$

(1) Let $\lambda\beta \max\{0, a - \rho\} \le s$. Then, from (2) and (9) we have

$$V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^{\tau-1} V_1 \le \beta^\tau V_0,$$

hence $t_\tau^* = 0$ for $\tau > 1$, i.e., ⓓ$_{\tau>1}\langle 0\rangle_{△}$.

(2) Let $\lambda\beta \max\{0, a - \rho\} > s$. Then, from (4) and (9) we have

$$V_\tau \le \beta V_{\tau-1} \le \beta^2 V_{\tau-2} \le \cdots \le \beta^{\tau-1} V_1 > \beta^\tau V_0,$$

hence $t_\tau^* = 1$ for $\tau > 1$, i.e., ◎$_{\tau>1}\langle 1\rangle_{△}$.

(b3) Let $V_1 < x_L \cdots$ **(**10**)**.

(b3i) Let $\beta = 1 \cdots$ **(**11**)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then $x_L = x_K \cdots$ **(**12**)** from Lemma 13.2.4(p.95) (b), hence $V_{t-1} \le x_L$ for $t > 1$ due to (8). Accordingly, since $V_{t-1} \le x_L$ for $\tau \ge t > 1$, we have $L(V_{t-1}) \ge 0$ for $\tau \ge t > 1$ from Lemma 13.2.2(p.94) (e1), hence $V_t \ge \beta V_{t-1}$ for $\tau \ge t > 1$ from (20.2.12(p.198)), so

$$V_\tau \ge \beta V_{\tau-1} \ge \beta^2 V_{\tau-2} \ge \cdots \ge \beta^{\tau-1} V_1 \cdots \textbf{(}13\textbf{)}, \quad \tau > 1.$$

(A) Let $\lambda \max\{0, a - \rho\} < s$, hence $\lambda\beta \max\{0, a - \rho\} < s$ due to (11). Then $V_1 - \beta V_0 < 0 \cdots$ **(**14**)** from (20.2.13(p.198)) or equivalently $V_1 < \beta V_0 \cdots$ **(**15**)**. Hence, from (13) we have

$$V_\tau \ge \beta V_{\tau-1} \ge \beta^2 V_{\tau-2} \ge \cdots \ge \beta^{\tau-1} V_1 < \beta^\tau V_0 \cdots \textbf{(}16\textbf{)}, \quad \tau > 1.$$

Thus, we have Ⓢ$_{\tau>1}\langle\tau\rangle_{△}$ or ⓓ$_{\tau>0}\langle 0\rangle_{△}$, hence (1) of S$_{10}$ is true.

(B) Let $\lambda \max\{0, a-\rho\} \geq s$, hence $\lambda\beta \max\{0, a-\rho\} \geq s$ due to (11). Then $V_1 - \beta V_0 \geq 0$ from (20.2.13(p.198)) or equivalently $V_1 \geq \beta V_0$ from (20.2.13(p.198)). Then, from (13) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 \geq \beta^\tau V_0,$$

hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., Ⓢ$_{\tau>1}\langle\tau\rangle_\vartriangle$, thus (2) of $\mathtt{S}_{10}$ holds.

(b3ii) Let $\beta < 1 \cdots$ (17) and $s = 0$ ($s > 0$).

(b3ii1) Let $b > 0$ ($\kappa > 0$). Then $x_L > x_K > 0 \cdots$ (18) from Lemma 13.2.4(p.95) (c (d)). Accordingly, from (8) we have $V_{t-1} \leq x_K < x_L$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a), thus $L(V_{t-1}) > 0$ for $\tau \geq t > 1$. Accordingly, since $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$ from (20.2.12(p.198)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (19) \quad \tau > 1.$$

(1) Let $\lambda\beta \max\{0, a-\rho\} < s$. Then for the same reason as in (A) we have (1) of $\mathtt{S}_{10}$.

(2) Let $\lambda\beta \max\{0, a-\rho\} \geq s$. Then for the same reason as in (B) we have (2) of $\mathtt{S}_{10}$.

(b3ii2) Let $b = 0$ ($\kappa = 0$). Then $x_L = x_K$ from Lemma 13.2.4(p.95) (c (d)). Accordingly, from (6) and (b1) we have $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} \leq x_K = x_L$ for $\tau \geq t > 1$. Therefore, from Corollary 13.2.1(p.94) (b) we have $L(V_{t-1}) \geq 0 \cdots$ (20) for $\tau \geq t > 1$, hence $V_t - \beta V_{t-1} \geq 0$ for $\tau \geq t > 1$ from (20.2.12(p.198)) or equivalently $V_t \geq \beta V_{t-1}$ for $\tau \geq t > 1$, leading to

$$V_t \geq \beta V_{t-1} \geq \cdots \geq \beta^{t-1} V_1.$$

(1) Let $\lambda\beta \max\{0, a-\rho\} \leq s$. Then, since $V_1 \leq \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 \leq \beta^\tau V_0,$$

hence Ⓢ$_{\tau>1}\langle\tau\rangle_\vartriangle$ or ❶$_{\tau>1}\langle 0\rangle_\vartriangle$.

(2) Let $\lambda\beta \max\{0, a-\rho\} > s$. Then, since $V_1 > \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \geq \beta V_{\tau-1} \geq \beta^2 V_{\tau-2} \geq \cdots \geq \beta^{\tau-1} V_1 > \beta^\tau V_0,$$

hence Ⓢ$_{\tau>1}\langle\tau\rangle_\vartriangle$.

(b3ii3) Let $b < 0$ ($\kappa < 0$), hence $x_L < x_K \leq 0 \cdots$ (21) from Lemma 13.2.4(p.95) (c (d)). Then, from (10) we have $V_1 < x_L < x_K = V$ due to (b1). Accordingly, due to the nondecreasing of V_t it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 \leq V_2 \leq \cdots \leq V_{t^\bullet_\tau - 1} < x_L \leq V_{t^\bullet_\tau} \leq V_{t^\bullet_\tau + 1} \leq \cdots.$$

Hence $V_{t-1} < x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \leq V_{t-1}$ for $t > t^\bullet_\tau$. Therefore, from Corollary 13.2.1(p.94) (a) we have

$$L(V_{t-1}) > 0 \cdots (22), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \leq 0 \cdots (23), \quad t > t^\bullet_\tau.$$

∘ Let $t^\bullet_\tau \geq \tau > 1$. Then, since $L(V_{t-1}) > 0$ for $\tau \geq t > 1$ from (22), we have $V_t - \beta V_{t-1} >$ for $\tau \geq t > 1$ from (20.2.12(p.198)) or equivalently $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$, so

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (24).$$

(1) Let $\lambda\beta \max\{0, a-\rho\}\rho < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 < \beta^\tau V_0$$

from (24), hence $t^*_\tau = \tau$ or $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., Ⓢ$_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_\vartriangle$ or ❶$_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle_\vartriangle$. Accordingly (1i) of $\mathtt{S}_{11}$ holds.

(2) Let $\lambda\beta \max\{0, a-\rho\}\rho \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \geq \beta^\tau V_0$$

from (24), hence $t^*_\tau = \tau$, i.e., Ⓢ$_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_\blacktriangle$. Accordingly (2i) of $\mathtt{S}_{11}$ holds.

∘ Let $\tau > t^\bullet_\tau$. Since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t^\bullet_\tau$ from (23), we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (20.2.12(p.198)), hence

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (25), \quad \tau > t^\bullet_\tau.$$

From (22) and (20.2.12(p.198)) we have $V_t > \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, hence

$$V_{t^\bullet_\tau} > \beta V_{t^\bullet_\tau - 1} > \cdots > \beta^{t^\bullet_\tau - 1} V_1 \cdots (26).$$

From (25) and (26) we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} > \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} > \cdots > \beta^{\tau-1} V_1. \cdots (27)$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \beta^{\tau-t^\bullet_\tau+2} V_{t^\bullet_\tau-2} > \cdots > \beta^{\tau-1} V_1 < \beta^\tau V_0,$$

Hence, we have $t^*_\tau = t^\bullet_\tau$ or $t^*_\tau = 0$ for $\tau > t^\bullet_\tau$, i.e., $\circledcirc_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\vartriangle$ or ⓓ$_{\tau > t^\bullet_\tau}\langle 0\rangle_\vartriangle$. Accordingly (1ii) of $\mathtt{S}_{11}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \ge s$. Then, since $V_1 \ge \beta V_0$ from (20.2.13(p.198)), from (27) we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} > \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} > \beta^{\tau-t^\bullet_\tau+2} V_{t^\bullet_\tau-2} > \cdots > \beta^{\tau-1} V_1 \ge \beta^\tau V_0,$$

hence $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$, i.e., $\circledcirc_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\vartriangle$. Accordingly (2ii) of $\mathtt{S}_{11}$ holds.

(c) Let $V_1 > x_K \cdots$ (28), hence $K(V_1) < 0 \cdots$ (29) due to Lemma 13.2.3(p.94) (j1).

(c1) From (6.5.33(p.39)) with $t = 2$ we have $V_2 = K(V_1) + V_1 < V_1 \cdots$ (30) due to (29), hence $V_2 \le V_1$. Suppose $V_t \le V_{t-1}$. Then, from Lemma 13.2.3(p.94) (e) we have $V_{t+1} = K(V_t) + V_t \le K(V_{t-1}) + V_{t-1} = V_t$. Hence, by induction $V_t \le V_{t-1}$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Note (28), hence $V_1 \ge x_K$. Suppose $V_{t-1} \ge x_K$. Then, since $V_t \ge K(x_K) + x_K = x_K$ from Lemma 13.2.3(p.94) (e), by induction we have $V_t \ge x_K \cdots$ (31) for $t > 0$, i.e., V_t is lower bounded in t, hence V_t converges to a finite V. Then, we have $V = x_K$ for the same reason as in the proof of (b1).

(c2) Let $\beta = 1$, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$". Then, since $x_L = x_K \cdots$ (32) from Lemma 13.2.4(p.95) (b), we have $V_{t-1} \ge x_L$ for $t > 1$ from (31). Accordingly $L(V_{t-1}) \le 0$ for $t > 1$ from Lemma 13.2.2(p.94) (e1), hence $L(V_{t-1}) \le 0$ for $\tau \ge t > 1$, so $V_t \le \beta V_{t-1}$ for $\tau \ge t > 1$ from (20.2.12(p.198)), leading to $V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1$.

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $\tau > 1$, i.e., ⓓ$_{\tau>1}\langle 0\rangle_\blacktriangle$.

(2) Let $\lambda\beta\max\{0, a-\rho\} \ge s$. Then, since $V_1 \ge \beta V_0$ from (20.2.13(p.198)) we have

$$V_\tau \le \beta V_{\tau-1} \le \cdots \le \beta^{\tau-1} V_1 \ge \beta^\tau V_0,$$

hence $t^*_\tau = 1$ for $\tau > 1$, i.e., $\circledcirc_{\tau>1}\langle 1\rangle_\vartriangle$.

(c3) Let $\beta < 1 \cdots$ (33) and $s = 0$ ($s > 0$).

(c3i) Let $b > 0$ ($\kappa > 0$). Then $x_L > x_K > 0 \cdots$ (34) from Lemma 13.2.4(p.95) (c (d)).

(c3i1) Let $V_1 < x_L$, hence $x_L > V_{t-1}$ for $t > 1$ from (c1). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a), we have $V_t - \beta V_{t-1} > 0$ for $t > 1$ due to (20.2.12(p.198)) or equivalently $V_t > \beta V_{t-1}$ for $t > 1$, hence $V_t > \beta V_{t-1}$ for $\tau \ge t > 1$, leading to

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1 \cdots (35).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then for the same reason as in (A(p.203)) we have (1) of $\mathtt{S}_{10}$.
(2) Let $\lambda\beta\max\{0, a-\rho\} \ge s$. Then for the same reason as in (B(p.204)) we have (2) of $\mathtt{S}_{10}$.

(c3i2) Let $V_1 = x_L$. Then, since $V_1 = x_L > x_K = V$ from (34) and (c1), there exists $t^\bullet_\tau > 1$ such that

$$V_1 = V_2 = \cdots = V_{t^\bullet_\tau-1} = x_L > V_{t^\bullet_\tau} \ge V_{t^\bullet_\tau+1} \ge \cdots,$$

i.e., $V_{t-1} = x_L$ for $t^\bullet_\tau \ge t > 1$ and $x_L > V_{t-1}$ for $t > t^\bullet_\tau$. Hence, from Corollary 13.2.1(p.94) (a) we have

$$L(V_{t-1}) = L(x_L) = 0 \cdots (36), \quad t^\bullet_\tau \ge t > 1, \qquad L(V_{t-1}) > 0 \cdots (37), \quad t > t^\bullet_\tau.$$

Accordingly, from (20.2.12(p.198)) we have $V_t - \beta V_{t-1} = 0$ for $t^\bullet_\tau \ge t > 1$ and $V_t - \beta V_{t-1} > 0$ for $t > t^\bullet_\tau$ or equivalently

$$V_t = \beta V_{t-1} \cdots (38), \quad t^\bullet_\tau \ge t > 1, \qquad V_t > \beta V_{t-1} \cdots (39), \quad t > t^\bullet_\tau.$$

◦ Let $t^\bullet_\tau \ge \tau > 1$. Then, we have $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$ from (38), leading to

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 \cdots (40).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $t^\bullet_\tau \ge \tau > 1$, i.e., ⓓ$_{t^\bullet_\tau \ge \tau > 1}\langle 0\rangle_\blacktriangle$, hence (1i) of $\mathtt{S}_{12}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1 > \beta^\tau V_0$$

for $t^\bullet_\tau \geq \tau > 1$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., ◎$_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle_{\parallel}$, hence (2i) of $\mathtt{S}_{12}$ holds.

From (40) with $\tau = t^\bullet_\tau$ we have

$$V_{t^\bullet_\tau} = \beta V_{t^\bullet_\tau - 1} = \cdots = \beta^{t^\bullet_\tau - 1}V_1 \cdots (41).$$

◦ Let $\tau > t^\bullet_\tau$. Then, we have $V_t > \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (39), leading to

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} \cdots (42).$$

From this and (41) we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1.$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = \tau$ or $t^*_\tau = 0$ for $\tau > t^\bullet_\tau$, i.e., ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_\vartriangle$ or 🅓$_{\tau > t^\bullet_\tau}\langle 0\rangle_\vartriangle$, thus (1ii) of $\mathtt{S}_{12}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} = \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} = \cdots = \beta^{\tau-1}V_1 \geq \beta^\tau V_0$$

for $\tau > t^\bullet_\tau$, hence $t^*_\tau = \tau$ for $\tau > t^\bullet_\tau$, i.e., ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_\blacktriangle$, hence (2ii) of $\mathtt{S}_{12}$ holds.

(c3i3) Let $V_1 > x_L \cdots (43)$. Then, since $V_1 > x_L > x_K = V$ from (34) and (c1), due to the nonincreasingness of V_t it follows that there exists $t^\bullet_\tau > 1$ such that

$$V_1 \geq V_2 \geq \cdots \geq V_{t^\bullet_\tau - 1} > x_L \geq V_{t^\bullet_\tau} \geq V_{t^\bullet_\tau + 1} \geq \cdots,$$

from which $V_{t-1} > x_L$ for $t^\bullet_\tau \geq t > 1$ and $x_L \geq V_{t-1}$ for $t > t^\bullet_\tau$. Hence, from Corollary 13.2.1(p.94) (a) we have

$$L(V_{t-1}) \leq 0 \cdots (44), \quad t^\bullet_\tau \geq t > 1, \qquad L(V_{t-1}) \geq 0 \cdots (45), \quad t > t^\bullet_\tau.$$

◦ Let $t^\bullet_\tau \geq \tau > 1$. Then $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$ from (44), hence $V_t - \beta V_{t-1} \leq 0$ for $\tau \geq t > 1$ from (20.2.12(p.198)), we have $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$. Hence

$$V_\tau \leq \beta V_{\tau-1} \leq \beta^2 V_{\tau-2} \leq \cdots \leq \beta^{\tau-1}V_1 \cdots (46).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Then, since $V_1 < \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 < \beta^\tau V_0,$$

hence $t^*_\tau = 0$ for $t^\bullet_\tau \geq \tau > 1$, i.e., 🅓$_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle_\blacktriangle$, so (1i) of $\mathtt{S}_{13}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then, since $V_1 \geq \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1}V_1 \geq \beta^\tau V_0$$

for $t^\bullet_\tau \geq \tau > 1$, hence $t^*_\tau = 1$ for $t^\bullet_\tau \geq \tau > 1$, i.e., ◎$_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle_\vartriangle$, hence (2i) of $\mathtt{S}_{13}$ holds.

From (46) with $\tau = t^\bullet_\tau$ we have

$$V_{t^\bullet_\tau} \leq \beta V_{t^\bullet_\tau - 1} \leq \cdots \leq \beta^{t^\bullet_\tau - 1}V_1 \cdots (47).$$

◦ Let $\tau > t^\bullet_\tau$. Then $L(V_{t-1}) \geq 0$ for $\tau \geq t > t^\bullet_\tau$ from (45), hence $V_t - \beta V_{t-1} \geq 0$ for $\tau \geq t > t^\bullet_\tau$ from (20.2.12(p.198)) or equivalently $V_t \geq \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$, leading to

$$V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau}.$$

Hence, from (47) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau - t^\bullet_\tau}V_{t^\bullet_\tau} \leq \beta^{\tau - t^\bullet_\tau + 1}V_{t^\bullet_\tau - 1} \leq \cdots \leq \beta^{\tau-1}V_1 \cdots (48).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} < s$. Since $V_1 - \beta V_0 < 0 \cdots$ (49) from (20.2.13(p.198)) or equivalently $V_1 < \beta V_0 \cdots$ (50). Then, from (48) and (50) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \leq \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 < \beta^\tau V_0.$$

hence Thus, we obtain Ⓢ$_\tau\langle\tau\rangle_\vartriangle$ or ⓓ$_\tau\langle 0\rangle_\vartriangle$, hence (1ii) of $\mathtt{S}_{13}$ holds.

(2) Let $\lambda\beta\max\{0, a-\rho\} \geq s$. Then $V_1 - \beta V_0 \geq 0$ from (20.2.13(p.198)), hence $V_1 \geq \beta V_0$. Then, from (48) we have

$$V_\tau \geq \beta V_{\tau-1} \geq \cdots \geq \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \leq \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} \leq \cdots \leq \beta^{\tau-2} V_2 \leq \beta^{\tau-1} V_1 \geq \beta^\tau V_0.$$

Thus, we have Ⓢ$_\tau\langle\tau\rangle_\vartriangle$ or ⓓ$_\tau\langle 0\rangle_\vartriangle$, hence (2ii) of $\mathtt{S}_{13}$ holds.

(c3ii) Let $b \leq 0$ ⟪$\kappa \leq 0$⟫. Then, since $x_L \leq x_K$ from Lemma 13.2.4(p.95) (c ⟪d⟫), we have $V_1 > x_K \geq x_L$ from (28), hence $V_{t-1} \geq x_K \geq x_L$ for $t > 1$ due to (c1). Accordingly $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary 13.2.1(p.94) (a), hence $V_t - \beta V_{t-1} \leq 0$ for $t > 1$ from (20.2.12(p.198)) or equivalently $V_t \leq \beta V_{t-1}$ for $t > 1$. Accordingly, since $V_t \leq \beta V_{t-1}$ for $\tau \geq t > 1$, we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 \cdots (51).$$

(1) Let $\lambda\beta\max\{0, a-\rho\} \leq s$. Then, since $V_1 \leq \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 \leq \beta^\tau V_0,$$

from (51), hence $t^*_\tau = 0$ for $\tau > 1$, i.e., ⓓ$_{\tau>1}\langle 0\rangle_\vartriangle$.

(2) Let $\lambda\beta\max\{0, a-\rho\} > s$. Then, since $V_1 > \beta V_0$ from (20.2.13(p.198)), we have

$$V_\tau \leq \beta V_{\tau-1} \leq \cdots \leq \beta^{\tau-1} V_1 > \beta^\tau V_0,$$

from (51), hence $t^*_\tau = 1$ for $\tau > 1$, i.e., ◎$_{\tau>1}\langle 1\rangle_\vartriangle$. ∎

Corollary 20.2.8 (M:2[ℙ][E]) *Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$. :*

(a) *Let $x_K \geq V_1$. Then z_t is nondecreasing in $t > 0$.*
(b) *Let $x_K < V_1$. Then z_t is nonincreasing in $t > 0$.* ∎

● *Proof* Immediate from Tom 20.2.16(p.203) (b1,c1) and from (6.2.94(p.33)) and Lemma 13.1.3(p.87). ∎

20.2.5.3 Market Restriction

20.2.5.3.1 Positive Restriction

20.2.5.3.1.1 Case of $\beta = 1$ and $s = 0$

□ **Pom 20.2.9** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nondecreasing in $t \geq 0$.*
(b) *Let $\rho \leq a^\star$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$.*
(c) *Let $b \leq \rho$. Then ⓓ$_{\tau>0}\langle 0\rangle_\parallel$.*
(d) *Let $a^\star < \rho < b$.*
  1. *Let $a \leq \rho$. Then ⓓ$_1\langle 0\rangle_\parallel$ and Ⓢ$_{\tau>1}\langle\tau\rangle_\blacktriangle$.*
  2. *Let $\rho < a$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$.*

● *Proof* The same as Tom 20.2.9(p.200) due to Lemma 17.4.4(p.116). ∎

20.2.5.3.1.2 Case of $\beta < 1$ or $s > 0$

20.2.5.3.1.2.1 Case of $\rho \leq a^\star$

□ **Pom 20.2.10** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose $a > 0$. Assume $\rho \leq a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t \geq 0$ if $\lambda < 1$ or $a^\star < \rho$, and converges to a finite $V = x_K$ as $t \to \infty$.*
(b) *Let $x_L \leq \rho$. Then ⓓ$_{\tau>0}\langle 0\rangle_\vartriangle$.*
(c) *Let $\rho < x_L$.*
  1. *Ⓢ$_1\langle 1\rangle_\blacktriangle$. Below let $\tau > 1$.*
  2. *Let $\beta = 1$.*
    i. *Let $(\lambda a - s)/\lambda \leq a^\star$.*
      1. *Let $\lambda = 1$. Then ◎$_{\tau>1}\langle 1\rangle_\parallel$.*
      2. *Let $\lambda < 1$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$.*
    ii. *Let $(\lambda a - s)/\lambda > a^\star$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$.*
  3. *Let $\beta < 1$ and $s = 0$. Then Ⓢ$_{\tau>0}\langle\tau\rangle_\blacktriangle$.*

4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $(\lambda\beta a - s)/\delta \le a^\star$.*
      1. *Let $\lambda = 1$.*
         i. *Let $s < \lambda\beta T(0)$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
         ii. *Let $s \ge \lambda\beta T(0)$. Then* ◎$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
      2. *Let $\lambda < 1$.*
         i. *Let $s \le \lambda\beta T(0)$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
         ii. *Let $s > \lambda\beta T(0)$. Then* $\mathbf{S}_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true*
   ii. *Let $(\lambda\beta a - s)/\delta > a^\star$.*
      1. *Let $s \le \lambda\beta T(0)$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
      2. *Let $s > \lambda\beta T(0)$. Then* $\mathbf{S}_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.24)).

(a-c2ii) The same as (a-c2ii) of Tom 20.2.10(p.200).

(c3) Due to (1) it suffices to consider only (c3i1i,c3i2i,c3ii1) of Tom 20.2.10(p.200).

(c4-c4ii2) Immediate from (2) and (c3-c3ii2) of Tom 20.2.10(p.200). ∎

□ Pom **20.2.11** ($\mathscr{A}$\{M:2[ℙ][E]$^+$\}) *Suppose $a > 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_K$.*

(a) *$V_t = x_K = \rho$ for $t \ge 0$.*

(b) *Let $\beta = 1$. Then* 🅓$_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\beta < 1$ and $s = 0$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(d) *Let $\beta < 1$ and $s > 0$.*
   1. *Let $s < \lambda\beta T(0)$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
   2. *Let $s \ge \lambda\beta T(0)$. Then* 🅓$_{\tau>0}\langle 0\rangle_{\vartriangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.24)).

(a,b) The same as (a,b) of Tom 20.2.11(p.200).

(c) Due to (1) it suffices to consider only (c1) of Tom 20.2.11(p.200).

(d-d2) Immediate from (2) and (c1,c2) of Tom 20.2.11(p.200). ∎

□ Pom **20.2.12** ($\mathscr{A}$\{M:2[ℙ][E]$^+$\}) *Suppose $a > 0$. Assume $\rho \le a^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_K$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_K$ as to $t \to \infty$.*

(b) *Let $\rho < x_L$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c) *Let $\rho = x_L$. Then* 🅓$_1\langle 0\rangle_{\vartriangle}$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(d) *Let $\rho > x_L$.*
   1. *Let $\beta = 1$. Then* 🅓$_{\tau>0}\langle 0\rangle_{\vartriangle}$.
   2. *Let $\beta < 1$ and $s = 0$. Then* $\mathbf{S}_9$(p.193) [Ⓢ△ | ●△ | ●▲] *is true.*
   3. *Let $\beta < 1$ and $s > 0$.*
      i. *Let $s \ge \lambda\beta T(0)$. Then* 🅓$_{\tau>0}\langle 0\rangle_{\vartriangle}$ (🅓$_{\tau>0}\langle 0\rangle_{\blacktriangle}$).
      ii. *Let $s < \lambda\beta T(0)$. Then* $\mathbf{S}_9$(p.193) [Ⓢ△ | ●△ | ●▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.24)).

(a-d1) The same as (a-d1) of Tom 20.2.12(p.201).

(d2) Due to (1) it suffices to consider only (d2ii) of Tom 20.2.12(p.201).

(d3i,d3ii) Immediate from (2) and (d2i,d2ii) of Tom 20.2.12(p.201). ∎

20.2.5.3.1.2.2 Case of $b \le \rho$

□ Pom **20.2.13** ($\mathscr{A}$\{M:2[ℙ][E]$^+$\}) *Suppose $a > 0$. Assume $b \le \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_K$.*

(a) *V_t is nondecreasing in $t \ge 0$, is strictly increasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V = x_K$ as $t \to \infty$.*

(b) *Let $x_L \le \rho$. Then* 🅓$_{\tau>0}\langle 0\rangle_{\vartriangle}$.

(c) *Let $\rho < x_L$.*
   1. Ⓢ$_1\langle 1\rangle_{\blacktriangle}$. *Below let $\tau > 1$.*
   2. *Let $\beta = 1$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
   3. *Let $\beta < 1$ and $s = 0$. Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
   4. *Let $\beta < 1$ and $s > 0$.*

i. *Let* $s \le \lambda\beta T(0)$. *Then* $Ⓢ_{\tau>0}\langle\tau\rangle_{▲}$.

ii. *Let* $s > \lambda\beta T(0)$. *Then* S$_8$(p.189) [Ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.24)).

(a-c2) The same as (a-c2) of Tom 20.2.13(p.201).

(c3) Due to (1) it suffices to consider only (c3i) of Tom 20.2.13(p.201).

(c4-c4ii) Immediate from (2) and Tom 20.2.13(p.201) (c3i,c3ii). ∎

☐ **Pom 20.2.14** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose* $a > 0$. *Assume* $b \le \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_K$.

(a) $V_t = x_K = \rho$ *for* $t \ge 0$.

(b) *Let* $\beta = 1$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let* $\beta < 1$ *and* $s = 0$. *Then* $Ⓢ_{\tau>0}\langle\tau\rangle_{▲}$.

(d) *Let* $\beta < 1$ *and* $s > 0$.

1. *Let* $s < \lambda\beta T(0)$. *Then* $Ⓢ_{\tau>0}\langle\tau\rangle_{▲}$.
2. *Let* $s \ge \lambda\beta T(0)$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{△}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.24)).

(a,b) The same as (a,b) of Tom 20.2.14(p.201).

(c) Due to (1) it suffices to consider only (c1) of Tom 20.2.14(p.201).

(d-d2) Immediate from (2) and (c1,c2) of Tom 20.2.14(p.201). ∎

☐ **Pom 20.2.15** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose* $a > 0$. *Assume* $b \le \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_K$.

(a) V_t *is nonincreasing in* $t \ge 0$, *is strictly decreasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_K$ *as to* $t \to \infty$.

(b) *Let* $\rho = x_L$. *Then* $ⓓ_1\langle 0\rangle_{△}$ *and* $Ⓢ_{\tau>1}\langle\tau\rangle_{▲}$.

(c) *Let* $\rho > x_L$.

1. *Let* $\beta = 1$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{△}$.
2. *Let* $\beta < 1$ *and* $s = 0$. *Then* S$_9$(p.193) [Ⓢ△ | ●△ | ●▲] *is true.*
3. *Let* $\beta < 1$ *and* $s > 0$.

   i. *Let* $s \ge \lambda\beta T(0)$. *Then* $ⓓ_{\tau>0}\langle 0\rangle_{△}$ ($ⓓ_{\tau>0}\langle 0\rangle_{▲}$).

   ii. *Let* $s < \lambda\beta T(0)$. *Then* S$_9$(p.193) [Ⓢ△ | ●△ | ●▲] *is true.*

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.24)).

(a-c1) The same as (a-c1) of Tom 20.2.15(p.202).

(c2) Due to (1) it suffices to consider only (c2ii) of Tom 20.2.15(p.202).

(c3-c3ii) Immediate from (2) and (c2i,c2ii) of Tom 20.2.15(p.202). ∎

20.2.5.3.1.2.3 Case of $a^\star < \rho < b$

☐ **Pom 20.2.16** ($\mathscr{A}${M:2[ℙ][E]$^+$}) *Suppose* $a > 0$. *Assume* $a^\star \le \rho < a$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *If* $\lambda\beta\max\{0, a-\rho\} < s$, *then* $ⓓ_1\langle 0\rangle_{△}$, *or else* $Ⓢ_1\langle 1\rangle_{▲}$. *Below let* $\tau > 1$.

(b) *Let* $V_1 \le x_K$.

1. V_t *is nondecreasing in* $t \ge 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$
2. *Let* $V_1 \ge x_L$. *If* $\lambda\beta\max\{0, a-\rho\} \le s$, *then* $ⓓ_{\tau>1}\langle 0\rangle_{△}$, *or else* $◎_{\tau>1}\langle 1\rangle_{△}$.
3. *Let* $V_1 < x_L$.

   i. *Let* $\beta = 1$. *Then* S$_{10}$(p.202) [Ⓢ△ | ●△] *is true.*

   ii. *Let* $\beta < 1$ *and* $s = 0$. *Then* S$_{10}$(p.202) [Ⓢ△ | ●△] *is true.*

   iii. *Let* $\beta < 1$ *and* $s > 0$.

      1. *Let* $s < \lambda\beta T(0)$. *Then* S$_{10}$(p.202) [Ⓢ△ | ●△] *is true.*
      2. *Let* $s = \lambda\beta T(0)$. *If* $\lambda\beta\max\{0, a-\rho\} < s$, *then* $Ⓢ_{\tau>1}\langle\tau\rangle_{△}$ *or* $ⓓ_{\tau>1}\langle 0\rangle_{△}$, *or else* $Ⓢ_{\tau>1}\langle\tau\rangle_{△}$.
      3. *Let* $s > \lambda\beta T(0)$. *Then* S$_{11}$(p.202) [Ⓢ△ | Ⓢ▲ | ◎ △ | ●△] *is true.*

(c) *Let* $V_1 > x_K$.

1. V_t *is nonincreasing in* $t \ge 0$ *and converges to a finite* $V = x_K$ *as* $t \to \infty$.
2. *Let* $\beta = 1$. *If* $\lambda\beta\max\{0, a-\rho\} < s$, *then* $ⓓ_{\tau>1}\langle 0\rangle_{▲}$, *or else* $◎_{\tau>1}\langle 1\rangle_{△}$.
3. *Let* $\beta < 1$ *and* $s = 0$.

   i. *Let* $V_1 < x_L$. *Then* S$_{10}$(p.202) [Ⓢ△ | ●△] *is true.*

   ii. *Let* $V_1 = x_L$. *Then* S$_{12}$(p.202) [Ⓢ△ | Ⓢ▲ | ◎ △ | ●△ | ●▲] *is true.*

iii. *Let* $V_1 > x_L$. *Then* $\mathtt{S}_8$(p.189) | ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲ | *is true.*

4. *Let* $\beta < 1$ *and* $s > 0$.

   i. *Let* $s < \lambda\beta T(0)$.

      1. *Let* $V_1 < x_L$. *Then* $\mathtt{S}_{10}$(p.202) | ⓢ△ | ●△ | *is true.*
      2. *Let* $V_1 = x_L$. *Then* $\mathtt{S}_{12}$(p.202) | ⓢ△ | ⓢ▲ | ◎ △ | ●△ | ●▲ | *is true.*
      3. *Let* $V_1 > x_L$. *Then* $\mathtt{S}_8$(p.189) | ⓢ▲ | ◎ ‖ | ◎ △ | ◎ ▲ | *is true.*

   ii. *Let* $s \geq \lambda\beta T(0)$. *If* $\lambda\beta \max\{0, a-\rho\} < s$, *then* ⓓ$_{\tau>1}\langle 0\rangle_{\triangle}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\triangle}$.

● *Proof* Suppose $a > 0$, hence $b > a > 0 \cdots (1)$. Then, we have $\kappa = \lambda\beta T(0) - s \cdots (2)$ from (5.1.23(p.24)).

(a-b3i) The same as (a-b3i) of Tom 20.2.16(p.203) .

(b3ii) Due to (1) it suffices to consider only (b3ii1) of Tom 20.2.16(p.203) .

(b3iii-b3iii3) The same as (b3ii1-b3ii3) of Tom 20.2.16(p.203) .

(c-c2) Immediate from (2) and (c-c2) of Tom 20.2.16(p.203) .

(c3-c3iii) Due to (1) it suffices to consider only (c3i1-c3i3) of Tom 20.2.16(p.203) .

(c4-c4ii) Immediate from (2) and (c3i-c3ii) of Tom 20.2.16(p.203) . ∎

20.2.5.3.2 Mixed Restriction

Omitted.

20.2.5.3.3 Negative Restriction

Omitted.

20.2.6 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathrm{E}]\}$

20.2.6.1 Preliminary

Since Theorem 20.2.3(p.188) holds due to Lemma 20.2.1(p.189) (b), we can derive $\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathrm{E}]\}$ by applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.128))) to $\mathscr{A}\{\mathsf{M}:2[\mathbb{P}][\mathrm{E}]\}$.

20.2.6.2 Analysis

20.2.6.2.1 Case of $\beta = 1$ and $s = 0$

□ **Tom 20.2.17** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathrm{E}]\}$) *Let* $\beta = 1$ *and* $s = 0$.

(a) V_t *is nonincreasing in* $t \geq 0$.

(b) *Let* $\rho \geq b^\star$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c) *Let* $a \geq \rho$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\|}$.

(d) *Let* $b^\star > \rho > a$.

   1. *Let* $b \geq \rho$. *Then* ⓓ$_1\langle 0\rangle_{\|}$ *and* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   2. *Let* $\rho > b$. *Then* ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$. ▯

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Lemma 20.2.9(p.200) . ∎

Corollary 20.2.9 ($\tilde{\mathsf{M}}:2[\mathbb{P}][\mathrm{E}]$) *Let* $\beta = 1$ *and* $s = 0$. *Then,* z_t *is nonincreasing in* $t \geq 0$. ▯

● *Proof* Immediate from Tom 20.2.17(p.210) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244) . ∎

20.2.6.2.2 Case of $\beta < 1$ or $s > 0$

20.2.6.2.2.1 Case of $\rho \geq b^\star$

□ **Tom 20.2.18** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}:2[\mathbb{P}][\mathrm{E}]\}$) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\triangle}$.

(c) *Let* $\rho > x_{\tilde{L}}$.

   1. ⓢ$_1\langle 1\rangle_{\blacktriangle}$. *Below let* $\tau > 1$.
   2. *Let* $\beta = 1$.

      i. *Let* $(\lambda b + s)/\lambda \geq b^\star$.

         1. *Let* $\lambda = 1$. *Then* ◎$_{\tau>1}\langle 1\rangle_{\|}$.
         2. *Let* $\lambda < 1$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

      ii. *Let* $(\lambda b + s)/\lambda < b^\star$. *Then* ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

   3. *Let* $\beta < 1$ *and* $s = 0$ ⦅$s > 0$⦆ .

      i. *Let* $(\lambda\beta b + s)/\delta \geq b^\star$.

1. *Let* $\lambda = 1$.
   i. *Let* $a < 0 ⦅\tilde{\kappa} < 0⦆$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   ii. *Let* $a \geq 0 ⦅\tilde{\kappa} \geq 0⦆$. *Then* ⊚$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
2. *Let* $\lambda < 1$.
   i. *Let* $a \leq 0 ⦅\tilde{\kappa} \leq 0⦆$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   ii. *Let* $a > 0 ⦅\tilde{\kappa} > 0⦆$. *Then* $\texttt{S}_8$ [Ⓢ▲ | ⊚ ‖ | ⊚ △ | ⊚ ▲] *is true.*

ii. *Let* $(\lambda\beta b + s)/\delta < b^\star$.
1. *Let* $a \leq 0 ⦅\tilde{\kappa} \leq 0⦆$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
2. *Let* $a > 0 ⦅\tilde{\kappa} > 0⦆$. *Then* $\texttt{S}_8$ [Ⓢ▲ | ⊚ ‖ | ⊚ △ | ⊚ ▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.10(p.200) . ∎

Corollary 20.2.10 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$. *Then,* z_t *is nonincreasing in* $t \geq 0$. □

● *Proof* Immediate from `Tom` 20.2.18(p.210) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244) . ∎

□ `Tom` **20.2.19** (⧈$\mathscr{A}$\{$\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]\}) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$.

(a) $V_t = x_{\tilde{K}} = \rho$ *for* $t \geq 0$.

(b) *Let* $\beta = 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.

(c) *Let* $\beta < 1$ *and* $s = 0 ⦅s > 0⦆$.
1. *Let* $a < 0 ⦅\tilde{\kappa} < 0⦆$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.
2. *Let* $a \geq 0 ⦅\tilde{\kappa} \geq 0⦆$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.11(p.200) . ∎

Corollary 20.2.11 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho = x_{\tilde{K}}$. *Then,* $z_t = z(\rho)$ *for* $t \geq 0$. □

● *Proof* Immediate from `Tom` 20.2.19(p.211) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244) . ∎

□ `Tom` **20.2.20** (⧈$\mathscr{A}$\{$\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]\}) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$.

(a) V_t *is nondecreasing in* $t \geq 0$, *is strictly increasing in* $t > 0$ *if* $\lambda < 1$, *and converges to* $V = x_{\tilde{K}}$ *as to* $t \to \infty$.

(b) *Let* $\rho > x_{\tilde{L}}$. *Then* Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.

(c) *Let* $\rho = x_{\tilde{L}}$. *Then* ⓓ$_1\langle 0\rangle_{\vartriangle}$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(d) *Let* $\rho < x_{\tilde{L}}$.
1. *Let* $\beta = 1$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0 ⦅s > 0⦆$.
   i. *Let* $a \geq 0 ⦅\tilde{\kappa} \geq 0⦆$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$ ⦅ ⓓ$_{\tau>0}\langle 0\rangle_{\blacktriangle}$ ⦆ .
   ii. *Let* $a < 0 ⦅\tilde{\kappa} < 0⦆$. *Then* $\texttt{S}_9$ [Ⓢ△ | ●△ | ●▲] *is true.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.12(p.201) . ∎

Corollary 20.2.12 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume* $\rho \geq b^\star$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho < x_{\tilde{K}}$. *Then,* z_t *is nondecreasing in* $t \geq 0$. □

● *Proof* Immediate from `Tom` 20.2.20(p.211) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244) . ∎

20.2.6.2.2.2 Case of $a \geq \rho$

□ `Tom` **20.2.21** (⧈$\mathscr{A}$\{$\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]\}) *Assume* $a \geq \rho$. *Let* $\beta < 1$ *or* $s > 0$ *and let* $\rho > x_{\tilde{K}}$.

(a) V_t *is nonincreasing in* $t \geq 0$, *is strictly decreasing in* $t \geq 0$ *if* $\lambda < 1$, *and converges to a finite* $V = x_{\tilde{K}}$ *as* $t \to \infty$.

(b) *Let* $x_{\tilde{L}} \geq \rho$. *Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.

(c) *Let* $\rho > x_{\tilde{L}}$.
1. Ⓢ$_1\langle 1\rangle_{\blacktriangle}$. *Below let* $\tau > 1$.
2. *Let* $\beta = 1$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
3. *Let* $\beta < 1$ *and* $s = 0 ⦅s > 0⦆$.
   i. *Let* $a \leq 0 ⦅\tilde{\kappa} \leq 0⦆$. *Then* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.
   ii. *Let* $a > 0 ⦅\tilde{\kappa} > 0⦆$. *Then* $\texttt{S}_8$ [Ⓢ▲ | ⊚ ‖ | ⊚ △ | ⊚ ▲] *is true.* □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.13(p.201) .[†] ∎

[†] $\texttt{S}_8$ does not change by the application of the operation.

Corollary 20.2.13 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$. Then, z_t is nonincreasing in $t \geq 0$.*

● *Proof* Immediate from `Tom` 20.2.21(p.211) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244) . ∎

☐ `Tom` **20.2.22** (⧈$\mathscr{A}${$\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]}) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \geq 0$.*

(b) *Let $\beta = 1$. Then ⓓ$_{\tau>0}\langle 0\rangle_{\parallel}$.*

(c) *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆ .*

1. *Let $a \leq 0$ ⦅$\tilde{\kappa} \leq 0$⦆ . Then Ⓢ$_{\tau>0}\langle\tau\rangle_{\blacktriangle}$.*
2. *Let $a > 0$ ⦅$\tilde{\kappa} > 0$⦆ . Then ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.* □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.14(p.201) . ∎

Corollary 20.2.14 ($\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E]) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$. Then, $z_t = z(\rho)$ for $t \geq 0$.* □

● *Proof* Immediate from `Tom` 20.2.22(p.212) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244) . ∎

☐ `Tom` **20.2.23** (⧈$\mathscr{A}${M:2[$\mathbb{P}$][E]}) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*

(b) *Let $\rho = x_{\tilde{L}}$. Then ⓓ$_1\langle 0\rangle_{\vartriangle}$ and Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.*

(c) *Let $\rho < x_{\tilde{L}}$.*

1. *Let $\beta = 1$. Then ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.*
2. *Let $\beta < 1$ and $s = 0$ ⦅$s > 0$⦆ .*
   - i. *Let $a \geq 0$ ⦅$\tilde{\kappa} \geq 0$⦆. Then ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$ ⦅ ⓓ$_{\tau>0}\langle 0\rangle_{\blacktriangle}$ ⦆ .*
   - ii. *Let $a < 0$ ⦅$\tilde{\kappa} < 0$⦆. Then* $\mathtt{S}_9$ [Ⓢ△ | ●△ | ●▲] *is true.* □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to `Tom` 20.2.15(p.202) .[‡] ∎

Corollary 20.2.15 (M:2[$\mathbb{P}$][E]) *Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$. Then, z_t is nondecreasing in $t \geq 0$.* □

● *Proof* Immediate from `Tom` 20.2.23(p.212) (a) and from (6.2.111(p.34)) and Lemma A 3.3(p.244) . ∎

20.2.6.2.2.3 Case of $b^\star > \rho > a$

By applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ in Theorem 20.2.3(p.188) , we see that $\mathtt{S}_{10}$(p.202) $-\mathtt{S}_{13}$ change as follows respectively:

$\mathtt{S}_{14}$ [Ⓢ△ | ●△] = {
We have:
(1) Let $\lambda\min\{0, \rho - b\} > -s$. Then Ⓢ$_{\tau>1}\langle\tau\rangle_{\vartriangle}$ or ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.
(2) Let $\lambda\min\{0, \rho - b\} \leq -s$. Then Ⓢ$_{\tau>1}\langle\tau\rangle_{\vartriangle}$.
}

$\mathtt{S}_{15}$ [Ⓢ△ | Ⓢ▲ | ◎ △ | ●△] = {
There exists $t^\bullet_\tau > 1$ such that:
(1) If $\lambda\beta\min\{0, \rho - b\} > -s$, then
 i. Ⓢ$_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_{\vartriangle}$ or ⓓ$_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle_{\vartriangle}$,
 ii. ◎$_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_{\vartriangle}$ or ⓓ$_{\tau > t^\bullet_\tau}\langle 0\rangle_{\vartriangle}$.
(2) If $\lambda\beta\min\{0, \rho - b\} \leq -s$, then
 i. Ⓢ$_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_{\blacktriangle}$,
 ii. ◎$_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_{\vartriangle}$.
}

$\mathtt{S}_{16}$ [Ⓢ△ | Ⓢ▲ | ◎ △ | ●△ | ●▲] = {
There exists $t^\bullet_\tau > 1$ such that:
(1) If $\lambda\beta\min\{0, \rho - b\} > -s$, then
 i. ⓓ$_{t^\bullet_\tau \geq \tau > 0}\langle 0\rangle_{\blacktriangle}$,
 ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_{\vartriangle}$ or ⓓ$_{\tau > t^\bullet_\tau}\langle 0\rangle_{\vartriangle}$.
(2) If $\lambda\beta\min\{0, \rho - b\} \leq -s$, then
 i. ◎$_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle_{\parallel}$,
 ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_{\blacktriangle}$.
}

$\mathtt{S}_{17}$ [Ⓢ△ | ◎ △ | ●△ | ●▲] = {
There exists $t^\bullet_\tau > 1$ and $t^\star_\tau > 1$ such that:
(1) If $\lambda\beta\min\{0, \rho - b\} > -s$, then
 i. ⓓ$_{t^\bullet_\tau \geq \tau > 1}\langle 0\rangle_{\blacktriangle}$,
 ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_{\vartriangle}$ or ⓓ$_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle_{\vartriangle}$.
(2) If $\lambda\beta\min\{0, \rho - b\} \leq -s$, then
 i. ◎$_{t^\bullet_\tau \geq \tau > 1}\langle 1\rangle_{\vartriangle}$,
 ii. Ⓢ$_{\tau > t^\bullet_\tau}\langle\tau\rangle_{\vartriangle}$ or ⓓ$_{\tau > t^\bullet_\tau}\langle t^\star_\tau\rangle_{\vartriangle}$.
}

Moreover, note that (20.2.17(p.202)) can be changed into

$$V_1 = \lambda\beta\min\{0, \rho - b\} + \beta\rho + s. \tag{20.2.18}$$

[‡] $\mathtt{S}_9$ does not change by the application of the operation.

□ Tom **20.2.24** ($\boxminus\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}$) *Assume $b^\star \ge \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\min\{0, \rho - b\} \ge -s$, then $ⓓ_1\langle 0\rangle_\vartriangle$, or else $\circledS_1\langle 1\rangle_\blacktriangle$. Below let $\tau > 1$.*

(b) *Let $V_1 \ge x_{\tilde{K}}$.*[†]

1. *V_t is nonincreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $V_1 \le x_{\tilde{L}}$. If $\lambda\beta\min\{0, \rho - b\} \ge -s$, then $ⓓ_{\tau>1}\langle 0\rangle_\vartriangle$, or else $\circledcirc_{\tau>1}\langle 1\rangle_\vartriangle$.*
3. *Let $V_1 > x_{\tilde{L}}$.*
   i. *Let $\beta = 1$. Then $\mathtt{S}_{14}$ $\boxed{\circledS_\vartriangle \mid \bullet_\vartriangle}$ is true.*
   ii. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
      1. *Let $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. Then $\mathtt{S}_{14}$ $\boxed{\circledS_\vartriangle \mid \bullet_\vartriangle}$ is true.*
      2. *Let $a = 0\,(\!(\tilde{\kappa} = 0)\!)$. If $\lambda\beta\min\{0, \rho - b\} > -s$, then $\circledS_{\tau>1}\langle\tau\rangle_\vartriangle$ or $ⓓ_{\tau>1}\langle 0\rangle_\vartriangle$, or else $\circledS_{\tau>1}\langle\tau\rangle_\vartriangle$.*
      3. *Let $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. Then $\mathtt{S}_{15}$ $\boxed{\circledS_\vartriangle \mid \circledS_\blacktriangle \mid \circledcirc_\vartriangle \mid \bullet_\vartriangle}$ is true.*

(c) *Let $V_1 < x_{\tilde{K}}$.*

1. *V_t is nondecreasing in $t \ge 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $\beta = 1$. If $\lambda\beta\min\{0, \rho - b\} > -s$, then $ⓓ_{\tau>1}\langle 0\rangle_\vartriangle$, or else $\circledcirc_{\tau>1}\langle 1\rangle_\blacktriangle$.*
3. *Let $\beta < 1$ and $s = 0\,(\!(s > 0)\!)$.*
   i. *Let $a < 0\,(\!(\tilde{\kappa} < 0)\!)$.*
      1. *Let $V_1 \ge x_{\tilde{L}}$. Then $\mathtt{S}_{14}$ $\boxed{\circledS_\vartriangle \mid \bullet_\vartriangle}$ is true.*
      2. *Let $V_1 = x_{\tilde{L}}$. Then $\mathtt{S}_{16}$ $\boxed{\circledS_\vartriangle \mid \circledS_\blacktriangle \mid \circledcirc_\vartriangle \mid \bullet_\vartriangle \mid \bullet_\blacktriangle}$ is true.*
      3. *Let $V_1 < x_{\tilde{L}}$. Then $\mathtt{S}_{17}$ $\boxed{\circledS_\vartriangle \mid \circledcirc_\vartriangle \mid \bullet_\vartriangle \mid \bullet_\blacktriangle}$ is true.*
   ii. *Let $a \ge 0\,(\!(\tilde{\kappa} \ge 0)\!)$. If $\lambda\beta\min\{0, \rho - b\} > -s$, then $ⓓ_{\tau>1}\langle 0\rangle_\vartriangle$, or else $\circledcirc_{\tau>1}\langle 1\rangle_\vartriangle$.* □

● *Proof by symmetry* Immediate from $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ to Tom 20.2.16(p.203). ▮

Corollary 20.2.16 ($\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]$) *Assume $b^\star \ge \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *Let $V_1 \ge x_{\tilde{K}}$. Then z_t is nonincreasing in $t > 0$.*

(b) *Let $V_1 < x_{\tilde{K}}$. Then z_t is nondecreasing in $t > 0$.* □

● *Proof* Immediate from Tom 20.2.24(p.213) (b1,c1) and from (6.2.111(p.34)) and Lemma A 3.3(p.244). ▮

20.2.6.3 Market Restriction

20.2.6.3.1 Positive Restriction

20.2.6.3.1.1 $\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]^+\}$

20.2.6.3.1.1.1 Case of $\beta = 1$ and $s = 0$

□ Pom **20.2.17** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t \ge 0$.*

(b) *Let $\rho \ge b^\star$. Then $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$.*

(c) *Let $a \ge \rho$. Then $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.*

(d) *Let $b^\star > \rho > a$.*

1. *Let $b \ge \rho$. Then $ⓓ_1\langle 0\rangle_{\|}$ and $\circledS_{\tau>1}\langle\tau\rangle_\blacktriangle$.*
2. *Let $\rho > b$. Then $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$.*

● *Proof* The same as Tom 20.2.17(p.210) due to Lemma 17.4.4(p.116). ▮

20.2.6.3.1.1.2 Case of $\beta < 1$ or $s > 0$

20.2.6.3.1.1.2.1 Case of $\rho \ge b^\star$

□ Pom **20.2.18** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]^+\}$) *Suppose $a > 0$. Assume $\rho \ge b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \ge 0$, is strictly decreasing in $t \ge 0$ if $\lambda < 1$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \ge \rho$. Then $ⓓ_{\tau>0}\langle 0\rangle_\vartriangle$.*

(c) *Let $\rho > x_{\tilde{L}}$.*

1. *$\circledS_1\langle 1\rangle_\blacktriangle$ and $\mathtt{Conduct}_{1\blacktriangle}$. Below let $\tau > 1$.*
2. *Let $\beta = 1$.*
   i. *Let $(\lambda b + s)/\lambda \ge b^\star$.*
      1. *Let $\lambda = 1$. Then $\circledcirc_{\tau>1}\langle 1\rangle_{\|}$.*
      2. *Let $\lambda < 1$. Then $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$.*
   ii. *Let $(\lambda b + s)/\lambda < b^\star$. Then $\circledS_{\tau>0}\langle\tau\rangle_\blacktriangle$.*

[†] $V_1 = \lambda\beta\min\{0, b - \rho\} + \beta\rho + s$ (see (6.5.25(p.39))).

3. *Let $\beta < 1$ and $s = 0$. Then* $\mathtt{S}_8$ $\boxed{Ⓢ\blacktriangle \mid ◎\ \| \mid ◎\ \triangle \mid ◎\ \blacktriangle}$ *is true.*
4. *Let $\beta < 1$ and $s > 0$.*
   i. *Let $(\lambda\beta b + s)/\delta \geq b^\star$.*
      1. *Let $\lambda = 1$. Then* $◎_\tau\langle 1\rangle_\triangle$.
      2. *Let $\lambda < 1$. Then* $\mathtt{S}_8$(p.189) $\boxed{Ⓢ\blacktriangle \mid ◎\ \| \mid ◎\ \triangle \mid ◎\ \blacktriangle}$ *is true.*
   ii. *Let $(\lambda\beta b + s)/\delta < b^\star$. Then* $\mathtt{S}_8$(p.189) $\boxed{Ⓢ\blacktriangle \mid ◎\ \| \mid ◎\ \triangle \mid ◎\ \blacktriangle}$ *is true.*

• *Proof* Suppose $a > 0 \cdots (1)$, hence $b^\star > b > a > 0 \cdots (2)$ from Lemma 14.6.1(p.105) (n). Then we have $\tilde{\kappa} = s \cdots (3)$ from Lemma 14.6.6(p.106) (a).

(a-c2ii) The same as (a-c2ii) of `Tom` 20.2.18(p.210).

(c3) Let $\beta < 1$ and $s = 0$, hence $\tilde{\kappa} = 0$ due to (3). Assume $(\lambda\beta b + s)/\delta \geq b^\star$. Then since $\lambda\beta b/\delta \geq b^\star$, we have $\lambda\beta b \geq \delta b^\star$ from (10.2.2 (1) (p.54)), hence $\lambda\beta b \geq \delta b^\star \geq \lambda b^\star$ due to (10.2.2 (1) (p.54)), so $\beta b \geq b^\star$, which contradicts [7(p.116)]. Thus it must be that $(\lambda\beta b + s)/\delta < b^\star$. From this it suffices to consider only (c3ii2) of `Tom` 20.2.18(p.210).

(c4-c4ii) Let $\beta < 1$ and $s > 0$. Then $\tilde{\kappa} > 0$ from (3), hence it suffices to consider only (c3i1ii,c3i2ii,c3ii2) of `Tom` 20.2.18(p.210). ∎

☐ `Pom` **20.2.19** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \geq 0$.*

(b) *We have* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.

• *Proof* Let $a > 0 \cdots (1)$, then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.106) (a).

(a) The same as (a) of `Tom` 20.2.19(p.211).

(b) Let $\beta = 1$. First, we have (a) of `Tom` 20.2.19(p.211). Let $\beta < 1$. Then, if $s = 0$, due to (1) it suffices to consider only (c2) of `Tom` 20.2.19(p.211) and if $s > 0$, then $\tilde{\kappa} > 0$ from (2), hence it suffices to consider only (c2 of `Tom` 20.2.19(p.211). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ `Pom` **20.2.20** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $\rho \geq b^\star$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*

(b) *Let $\rho > x_{\tilde{L}}$. Then* $Ⓢ_{\tau>0}\langle\tau\rangle_\blacktriangle$.

(c) *Let $\rho = x_{\tilde{L}}$. Then* $ⓓ_1\langle 0\rangle_\triangle$ *and* $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$.

(d) *Let $\rho < x_{\tilde{L}}$.*
   1. *Let $\beta = 1$. Then* $ⓓ_{\tau>0}\langle 0\rangle_\triangle$.
   2. *Let $\beta < 1$. Then* $ⓓ_{\tau>0}\langle 0\rangle_\triangle$ ($ⓓ_{\tau>0}\langle 0\rangle_\blacktriangle$).

• *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ due to Lemma 14.6.6(p.106) (a).

(a-d1) The same as `Tom` 20.2.20(p.211) (a-d1).

(d2) If $s = 0$, due to (1) it suffices to consider only (d2i) of `Tom` 20.2.20(p.211) and if $s > 0$, we have $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (d2i) of `Tom` 20.2.20(p.211). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.2.6.3.1.1.2.2 Case of $a \geq \rho$

☐ `Pom` **20.2.21** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho > x_{\tilde{K}}$.*

(a) *V_t is nonincreasing in $t \geq 0$, is strictly decreasing in $t \geq 0$ if $\lambda < 1$, and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $x_{\tilde{L}} \geq \rho$. Then* $ⓓ_{\tau>0}\langle 0\rangle_\triangle$.

(c) *Let $\rho > x_{\tilde{L}}$.*
   1. $Ⓢ_1\langle 1\rangle_\blacktriangle$. *Below let $\tau > 1$.*
   2. *Let $\beta = 1$. Then* $Ⓢ_{\tau>1}\langle\tau\rangle_\blacktriangle$.
   3. *Let $\beta < 1$. Then* $\mathtt{S}_8$(p.189) $\boxed{Ⓢ\blacktriangle \mid ◎\ \| \mid ◎\ \triangle \mid ◎\ \blacktriangle}$ *is true.*

• *Proof* Suppose $a > 0 \cdots (1)$. Then $\tilde{\kappa} = s \cdots (2)$ from Lemma 14.6.6(p.106) (a).

(a-c2) The same as (a-c2) of `Tom` 20.2.21(p.211).

(c3) If $s = 0$, due to (1) it suffices to consider only (c3ii) of `Tom` 20.2.21(p.211) and if $s > 0$, we have $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c3ii) of `Tom` 20.2.21(p.211). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

☐ `Pom` **20.2.22** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^+\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho = x_{\tilde{K}}$.*

(a) *$V_t = x_{\tilde{K}} = \rho$ for $t \geq 0$.*

(b) *Let $\beta = 1$. Then we have* $ⓓ_{\tau>0}\langle 0\rangle_{\|}$.

(c) *Let $\beta < 1$. Then we have* $ⓓ_{\tau>0}\langle 0\rangle_\triangle$.

● *Proof* Suppose $a > 0 \cdots$ **(1)**. Then $\tilde{\kappa} = s \cdots$ **(2)** from Lemma 14.6.6(p.106) (a).

(a,b) The same as (a,b) of Tom 20.2.22(p.212).

(c) Let $\beta < 1$. If $s = 0$, due to (1) it suffices to consider only (c2) of Tom 20.2.22(p.212) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2) of Tom 20.2.22(p.212). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

□ Pom **20.2.23** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^{+}\}$) *Suppose $a > 0$. Assume $a \geq \rho$. Let $\beta < 1$ or $s > 0$ and let $\rho < x_{\tilde{K}}$.*

(a) *V_t is nondecreasing in $t \geq 0$, is strictly increasing in $t > 0$ if $\lambda < 1$, and converges to $V = x_{\tilde{K}}$ as to $t \to \infty$.*

(b) *Let $\rho = x_{\tilde{L}}$. Then* ⓓ$_1\langle 0\rangle_{\vartriangle}$ *and* Ⓢ$_{\tau>1}\langle\tau\rangle_{\blacktriangle}$.

(c) *Let $\rho < x_{\tilde{L}}$.*

1. *Let $\beta = 1$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$.
2. *Let $\beta < 1$ and let $s = 0 (s > 0)$. Then* ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$ (ⓓ$_{\tau>0}\langle 0\rangle_{\blacktriangle}$).

● *Proof* Suppose $a > 0 \cdots$ **(1)**. Then $\tilde{\kappa} = s \cdots$ **(2)** due to Lemma 14.6.6(p.106) (a).

(a,b) The same as (a,b) of Tom 20.2.23(p.212).

(c) Let $\rho < x_{\tilde{L}}$.

(c1) Let $\beta = 1$. Then we have ⓓ$_{\tau>0}\langle 0\rangle_{\vartriangle}$ from (c1) of Tom 20.2.23(p.212).

(c2) Let $\beta < 1$. If $s = 0$, then due to (2) it suffices to consider only (c2i) of Tom 20.2.23(p.212) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (c2i) of Tom 20.2.23(p.212). Thus, whether $s = 0$ or $s > 0$, we have the same result. ∎

20.2.6.3.1.1.2.3 Case of $b^{\star} > \rho > a$

□ Pom **20.2.24** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathsf{E}]^{+}\}$) *Suppose $a > 0$. Assume $b^{\star} \geq \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *If $\lambda\beta\max\{0, \rho - b\} \leq s$, then* ⓓ$_1\langle 0\rangle_{\vartriangle}$, *or else* Ⓢ$_1\langle 1\rangle_{\blacktriangle}$. *Below let $\tau > 1$.*

(b) *Let $V_1 \geq x_{\tilde{K}}$.*[†]

1. *V_t is nonincreasing in $t \geq 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *Let $V_1 \geq x_{\tilde{L}}$. If $\lambda\beta\max\{0, \rho - b\} \leq s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\vartriangle}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\vartriangle}$.
3. *Let $V_1 > x_{\tilde{L}}$.*
   i. *Let $\beta = 1$. Then* $\mathtt{S}_{14}$(p.212) [Ⓢ△ | ●△] *is true.*
   ii. *Let $\beta < 1$. Then* $\mathtt{S}_{15}$(p.212) [Ⓢ△ | Ⓢ▲ | ◎△ | ●△] *is true.*

(c) *Let $V_1 < x_{\tilde{K}}$.*

1. *V_t is nondecreasing in $t \geq 0$ and converges to a finite $V = x_{\tilde{K}}$ as $t \to \infty$.*
2. *If $\lambda\beta\max\{0, \rho - b\} < s$, then* ⓓ$_{\tau>1}\langle 0\rangle_{\blacktriangle}$, *or else* ◎$_{\tau>1}\langle 1\rangle_{\vartriangle}$.

● *Proof* Suppose $a > 0 \cdots$ **(1)**, hence $b > a > 0$. Then $\tilde{\kappa} = s \cdots$ **(2)** due to Lemma 14.6.6(p.106) (a).

(a-b3i) The same as (a-b3i) of Tom 20.2.24(p.213).

(b3ii) Let $\beta < 1$. If $s = 0$, then due to (1) it suffices to consider only (b3ii3) of Tom 20.2.24(p.213) and if $s > 0$, then $\tilde{\kappa} > 0$ due to (2), hence it suffices to consider only (b3ii3) of Tom 20.2.24(p.213). Thus, whether $s = 0$ or $s > 0$, we have the same result.

(c1) The same as (c1) of Tom 20.2.24(p.213).

(c2) If $\beta = 1$, then it suffices to consider only (c2) of Tom 20.2.24(p.213) and if $\beta < 1$, whether $s = 0$ or $s > 0$, it suffices to consider only (c3ii) of Tom 20.2.24(p.213). Accordingly, whether $\beta = 1$ or $\beta < 1$, we have the same result. ∎

20.2.6.3.2 Mixed Restriction

Omitted.

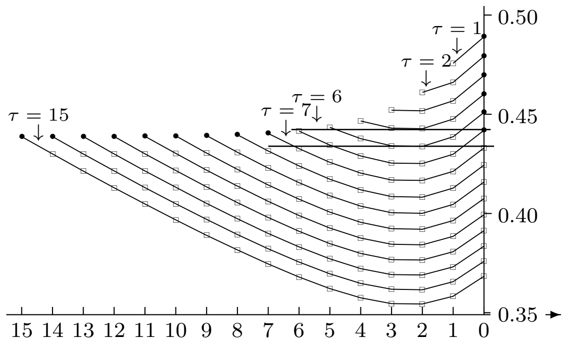
20.2.6.3.3 Negative Restriction

Omitted.

[†] $V_1 = \lambda\beta\min\{0, b - \rho\} + \beta\rho + s$ (see (6.5.25(p.39))).

20.2.6.4 Numerical Calculation

Numerical Example **6** ($\mathscr{A}\{$M:2[ℝ][E]$^+\}$ **(selling model)** This example is for the assertion in Pom 20.2.4(p.195) (d3ii) in which $a > 0$, $\rho > x_K$, $\rho > x_L$, $\beta < 1$, $s > 0$, and $\lambda\beta\mu > s$. As an example let $a = 0.01$, $b = 1.00$, $\lambda = 0.7$, $\beta = 0.98$, $s = 0.1$, and $\rho = 0.5$.[†] where $x_L = 0.462767$ and $x_K = 0.439640$. The symbols • in the figure below shows the optimal initiating times $t^*_{15 \geq \tau \geq 1}$ (see the t^*_τ-column in the table of Figure 20.2.2(p.216) below).



Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $15 \geq \tau > 0$ and $\tau \geq t \geq 0$

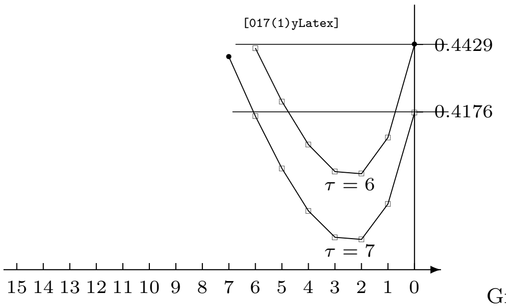
t	V_t	$\Delta_\beta V_t$	t^*_τ
0	0.5000000		
1	0.4766162	−0.0133838	1
2	0.4619911	−0.0050927	1
3	0.4530367	+0.0002854	1
4	0.4476274	+0.0036514	1
5	0.4443866	+0.0057117	1
6	0.4424547	+0.0069558	1
7	0.4413065	+0.0077009	7
8	0.4406253	+0.0081449	8
9	0.4402216	+0.0084088	9
10	0.4399825	+0.0085653	10
11	0.4398410	+0.0086581	11
12	0.4397572	+0.0087130	12
13	0.4397076	+0.0087456	13
14	0.4396783	+0.0087648	14
15	0.4396609	+0.0087762	15

$\Delta_\beta V_t = V_t - \beta V_{t-1}$

[TAB05218]

Figure 20.2.2: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $15 \geq \tau \geq 2$ and $\tau \geq t \geq 1$

Scaling up the graphs for $\tau = 6$ and $\tau = 7$ in the above figure, we have the figure below. This figure demonstrates that the optimal initiating time *shifts* from 0 to 7 when the starting time changes from $\tau = 6$ to $\tau = 7$.



Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ with $\tau = 6, 7$

$\tau = 6$		$\tau = 7$	
t	$\beta^{6-t}V_t$	t	$\beta^{7-t}V_t$
0	0.4429212	0	0.4340628
1	0.4308233	1	0.4222069
2	0.4261259	2	0.4176034
3	0.4263946	3	0.4178667
4	0.4299014	4	0.4213034
5	0.4354989	5	0.4267889
6	0.4424547	6	0.4336056
		7	0.4413065

Graphs for $\tau = 6$ and $\tau = 7$ [TAB05224]

Figure 20.2.3: Graphs of $I^t_\tau = \beta^{\tau-t}V_t$ for $\tau = 6$ and $\tau = 7$

20.2.6.5 Conclusion 4 (Search-Enforced-Model 2)

C1. Mental Conflict

On $\mathscr{F}^+$, we have (see (7.3.1(p.45)) and (7.3.2(p.45)) for the definitions of opt-ℝ-price and opt-ℙ-price below):

a. Let $\beta = 1$ and $s = 0$.

1. The opt-ℝ-price V_t in M:2[ℝ][E] (selling model) is nondecreasing in t▴[a] (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).
2. The opt-ℙ-price z_t in M:2[ℙ][E] (selling model) is nondecreasing in t▴[b] (see Figure 7.3.1(p.45) (I)), hence we have the normal conflict (see Remark 7.3.1(p.45)).
3. The opt-ℝ-price V_t in M̃:2[ℝ][E] (buying model) is nonincreasing in t▾[c] (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)).
4. The opt-ℙ-price z_t in M̃:2[ℙ][E] (buying model) is nonincreasing in t (see Figure 7.3.1(p.45) (II)), hence we have the normal conflict (see Remark 7.3.1(p.45)),▾[d].

· ▴[a] ← Tom 20.2.1(p.189) (a)
· ▴[b] ← Corollary 20.2.1(p.200)
· ▾[c] ← Tom 20.2.5(p.196) (a)
· ▾[d] ← Corollary 20.2.9(p.210).

b. Let $\beta < 1$ or $s > 0$.

1. The opt-ℝ-price V_t in M:2[ℝ][E] (selling model) is nondecreasing in t▴[a], constant ‖[a], or nonincreasing in t▾[a] (see Figure 7.3.2(p.45) (I)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).

[†] We have $\rho = 0.5 > 0.462767 = x_L$, $\beta = 0.98 < 1$, and $s = 0.1 > 0$. Since $\mu = (0.01 + 1.00)/2 = 0.505$, we have $\lambda\beta\mu = 0.7 \times 0.98 \times 0.505 = 0.34634 > 0.1 = s$. Thus the condition of this assertion is confirmed.

2. The opt-$\mathbb{P}$-price z_t in M:2[$\mathbb{P}$][E] (selling model) is nondecreasing in $t^{\blacktriangle\text{b}}$, constant$^{\|\text{b}}$, or nonincreasing in $t^{\blacktriangledown\text{b}}$ (see Figure 7.3.2(p.45) (I)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).
3. The opt-$\mathbb{R}$-price V_t in $\tilde{\mathsf{M}}$:2[$\mathbb{R}$][E] (buying model) is nondecreasing in $t^{\blacktriangle\text{c}}$, constant$^{\|\text{c}}$, or nonincreasing in $t^{\blacktriangledown\text{c}}$ (see Figure 7.3.2(p.45) (II)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).
4. The opt-$\mathbb{P}$-price z_t in $\tilde{\mathsf{M}}$:2[$\mathbb{P}$][E] (buying model) is nondecreasing in $t^{\blacktriangle\text{d}}$, constant$^{\|\text{d}}$, or nonincreasing in $t^{\blacktriangledown\text{d}}$ (see Figure 7.3.2(p.45) (II)), hence we have the abnormal conflict (see Remark 7.3.2(p.45)).

- $^{\blacktriangle\text{a}}$ ← Tom 20.2.1(p.189) (a), 20.2.2(p.189) (a).
  $^{\|\text{a}}$ ← Tom 20.2.3(p.192) (a)).
  $^{\blacktriangledown\text{a}}$ ← Tom 20.2.4(p.193) (a).
- $^{\blacktriangle\text{b}}$ ← Corollary 20.2.1(p.200) , 20.2.2(p.200) , 20.2.5(p.201) ,20.2.8(p.207) (a).
  $^{\|\text{b}}$ ← Corollary 20.2.3(p.201) , 20.2.6(p.202) .
  $^{\blacktriangledown\text{b}}$ ← Corollary 20.2.4(p.201) , 20.2.7(p.202) , 20.2.8(p.207) (b).
- $^{\blacktriangle\text{c}}$ ← Tom 20.2.8(p.197) (a).
  $^{\|\text{c}}$ ← Tom 20.2.7(p.197) (a).
  $^{\blacktriangledown\text{c}}$ ← Tom 20.2.5(p.196) (a), 20.2.6(p.196) (a).
- $^{\blacktriangle\text{d}}$ ← Corollary 20.2.12(p.211) , 20.2.15(p.212) , 20.2.16(p.213) (b).
  $^{\|\text{c}}$ ← Corollary 20.2.11(p.211) , 20.2.14(p.212) .
  $^{\blacktriangledown\text{d}}$ ← Corollary 20.2.9(p.210) , 20.2.10(p.211) , 20.2.13(p.212) , 20.2.16(p.213) (a).

The above results can be summarized as below.

A. If $\beta = 1$ and $s = 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the normal mental conflict in *Examples* 1.4.1(p.6) - 1.4.4(p.6) .

B. If $\beta < 1$ or $s > 0$, then, on $\mathscr{F}^+$, whether selling problem or buying problem and whether $\mathbb{R}$-model or $\mathbb{P}$-model, we have the abnormal mental conflict.

C2. Symmetry

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$. Then we have:

Pom 20.2.5(p.197) $\sim$ Pom 20.2.1(p.194) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$,
Pom 20.2.17(p.213) $\sim$ Pom 20.2.9(p.207) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathrm{E}]\}^+ \sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{P}][\mathrm{E}]\}^+)$.

b. Let $\beta < 1$ or $s > 0$. Then we have:

Pom 20.2.6(p.197) $\not\sim$ Pom 20.2.2(p.195) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.7(p.198) $\not\sim$ Pom 20.2.3(p.195) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.8(p.198) $\not\sim$ Pom 20.2.4(p.195) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.18(p.213) $\not\sim$ Pom 20.2.10(p.207) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.19(p.214) $\not\sim$ Pom 20.2.11(p.208) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.20(p.214) $\not\sim$ Pom 20.2.12(p.208) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.21(p.214) $\not\sim$ Pom 20.2.13(p.208) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.22(p.214) $\not\sim$ Pom 20.2.14(p.209) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.23(p.215) $\not\sim$ Pom 20.2.15(p.209) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.24(p.215) $\not\sim$ Pom 20.2.16(p.209) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\sim \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$

The above results can be summarized as below.

A. Let $\beta = 1$ and $s = 0$. Then the symmetry is always <u>inherited</u> (see C2a(p.217)).

B. Let $\beta < 1$ or $s > 0$. Then the symmetry always <u>collapses</u> (see C2b(p.217)).

C3. Analogy

a. On $\mathscr{F}^+$, for any $\beta \le 1$ and $s \ge 0$ we have:

Pom 20.2.9(p.207) $\not\bowtie$ Pom 20.2.1(p.194) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.10(p.207) $\not\bowtie$ Pom 20.2.2(p.195) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.11(p.208) $\bowtie$ Pom 20.2.3(p.195) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+) \cdots (*)$
Pom 20.2.12(p.208) $\not\bowtie$ Pom 20.2.4(p.195) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.17(p.213) $\not\bowtie$ Pom 20.2.5(p.197) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.18(p.213) $\not\bowtie$ Pom 20.2.6(p.197) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.19(p.214) $\not\bowtie$ Pom 20.2.7(p.198) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.20(p.214) $\not\bowtie$ Pom 20.2.8(p.198) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.21(p.214) $\not\bowtie$ Pom 20.2.6(p.197) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$
Pom 20.2.22(p.214) $\bowtie$ Pom 20.2.7(p.198) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+) \cdots (**)$
Pom 20.2.23(p.215) $\not\bowtie$ Pom 20.2.8(p.198) $(\mathscr{A}\{\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathrm{E}]\}^+ \not\bowtie \mathscr{A}\{\mathsf{M}{:}2[\mathbb{R}][\mathrm{E}]\}^+)$

The above results can be summarized as below.

A. The analogy collapses except (∗) and (∗∗).

C4. Optimal initiating time (OIT)

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$. Then, from

Pom 20.2.1(p.194), Pom 20.2.5(p.197), Pom 20.2.9(p.207), Pom 20.2.17(p.213),

we obtain the following table.

Table 20.2.3: Possible OIT on $\mathscr{F}^+$ ($\beta = 1$ and $s = 0$)

	$\mathscr{A}\{\mathsf{M{:}2}[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}\mathsf{{:}2}[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}\mathsf{{:}2}[\mathbb{P}][\mathsf{E}]^+\}$
Ⓢ$_\tau\langle\tau\rangle_{\|}$				
Ⓢ$_\tau\langle\tau\rangle_{\vartriangle}$				
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○	○
◎$_\tau\langle t_\tau^\bullet\rangle_{\|}$				
◎$_\tau\langle t_\tau^\bullet\rangle_{\vartriangle}$				
◎$_\tau\langle t_\tau^\bullet\rangle_{\blacktriangle}$				
ⓓ$_\tau\langle 0\rangle_{\|}$	○	○	○	○
ⓓ$_\tau\langle 0\rangle_{\vartriangle}$				
ⓓ$_\tau\langle 0\rangle_{\blacktriangle}$				

b. Let $\beta < 1$ or $s > 0$. Then, from

Pom 20.2.4(p.195), Pom 20.2.12(p.208), Pom 20.2.15(p.209), Pom 20.2.16(p.209), Pom 20.2.24(p.215),
Pom 20.2.2(p.195), Pom 20.2.3(p.195), Pom 20.2.4(p.195), Pom 20.2.6(p.197), Pom 20.2.8(p.198),
Pom 20.2.10(p.207), Pom 20.2.11(p.208), Pom 20.2.13(p.208), Pom 20.2.14(p.209), Pom 20.2.16(p.209),
Pom 20.2.18(p.213), Pom 20.2.20(p.214), Pom 20.2.23(p.215), Pom 20.2.16(p.209), Pom 20.2.21(p.214),
Pom 20.2.7(p.198), Pom 20.2.19(p.214), Pom 20.2.22(p.214), Pom 20.2.22(p.214),

we obtain the following table:

Table 20.2.4: Possible OIT on $\mathscr{F}^+$ ($\beta < 1$ or $s > 0$)

	$\mathscr{A}\{\mathsf{M{:}2}[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}\mathsf{{:}2}[\mathbb{R}][\mathsf{E}]^+\}$	$\mathscr{A}\{\mathsf{M{:}1}[\mathbb{P}][\mathsf{E}]^+\}$	$\mathscr{A}\{\tilde{\mathsf{M}}\mathsf{{:}2}[\mathbb{P}][\mathsf{E}]^+\}$
Ⓢ$_\tau\langle\tau\rangle_{\|}$				
Ⓢ$_\tau\langle\tau\rangle_{\vartriangle}$	○	○	○	○
Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$	○	○	○	○
◎$_\tau\langle t_\tau^\bullet\rangle_{\|}$	○	○	○	○
◎$_\tau\langle t_\tau^\bullet\rangle_{\vartriangle}$	○	○	○	○
◎$_\tau\langle t_\tau^\bullet\rangle_{\blacktriangle}$	○	○	○	○
ⓓ$_\tau\langle 0\rangle_{\|}$	○	○	○	○
ⓓ$_\tau\langle 0\rangle_{\vartriangle}$	○	○	○	○
ⓓ$_\tau\langle 0\rangle_{\blacktriangle}$	○	○	○	○

c. The table below is the list of the occurrence rates of Ⓢ, ◎, and ⓓ (Def. 11.2.4(p.61)) on $\mathscr{F}$ (see Tom's 20.2.1(p.189) (▣), 20.2.2(p.189) (▣), 20.2.3(p.192) (▣), 20.2.4(p.193) (▣), 20.2.9(p.200) (▣), and Tom 20.2.16(p.203) (▣)).

Table 20.2.5: Occurance rates of Ⓢ, ◎, and ⓓ on $\mathscr{F}$

Ⓢ			◎			ⓓ		
41.6 % / 30			23.6 % / 17			34.7 % / 25		
Ⓢ$_{\|}$	Ⓢ$_{\vartriangle}$	Ⓢ$_{\blacktriangle}$	◎$_{\|}$	◎$_{\vartriangle}$	◎$_{\blacktriangle}$	ⓓ$_{\|}$	ⓓ$_{\vartriangle}$	ⓓ$_{\blacktriangle}$
—	possible	possible	possible	possible	possible	possible	possible	possible
– % / –	12.5 % / 9	29.1 % / 21	5.5 % / 4	13.8 % / 10	4.2 % / 3	7.0 % / 5	20.8 % / 15	7.0 % / 5

C5. Null-time-zone and deadline-engulfing

From Table 20.2.5(p.218) above we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.43) for the implication of the symbol "▲" representing the strict optimality of the initiating time t_τ^*.

b. As a whole, Ⓢ, ◎, and 🅓 are possible at 41.6%, 23.6%, and 34.7% respectively where

1. Ⓢ‖ cannot be defined (see Remark ??(p.??)).
2. ◎‖ is possible (5.5 %).
3. 🅓‖ is possible (7.0 %).
4. Ⓢ△ is possible (12.5 %).
5. ◎△ is possible (13.8 %).
6. 🅓△ is possible (20.8 %).
7. Ⓢ▲ is possible (29.1 %).
8. ◎▲ is possible (4.2 %).
    - `Tom` 20.2.2(p.189) (c3i2,c3ii1ii2,c3ii2i).
9. 🅓▲ is possible (7.0 %).
    - `Tom` 20.2.4(p.193) (d2i,d2ii).
    - `Tom` 20.2.16(p.203) (c2,c3i2,c3i3).

From the above results we see that:

A. ◎‖ and 🅓‖ causing the `null-time-zone` are possible at 58.6% (= 24.3% + 34.3%).

B. ◎▲ *strictly* causing the `null-time-zone` is possible at 4.3%.

C. 🅓▲ *strictly* causing the `deadline-engulfing` are possible at 7.1%.

20.3 Conclusions of Model 2

Conclusions 3(p.185) and 4(p.216) can be summed up as below.

C1. Mental Conflict

On $\mathscr{F}^+$, from C1A(p.186) and C1B(p.186) and from C1A(p.217) and C1B(p.217). we have:

a. If $\beta = 1$ and $s = 0$, then, whether selling problem or buying problem, whether $\mathbb{R}$-model or $\mathbb{P}$-model, and whether search-**A**llowed-model or search-**E**nforced-model, we have the normal mental conflict in *Example*s 1.4.1(p.6) - 1.4.4(p.6).

b. If $\beta < 1$ or $s > 0$, then, whether selling problem or buying problem, whether $\mathbb{R}$-model or $\mathbb{P}$-model, and whether search-**A**llowed-model or search-**E**nforced-model, we have the abnormal mental conflict.

C2. Symmetry

On $\mathscr{F}^+$, we have:

a. If $\beta = 1$ and $s = 0$, the symmetry is always inherited (see C2A(p.186) and C2A(p.217)).

b. if $\beta < 1$ or $s > 0$, the symmetry always collapses (see C2B(p.186) and C2B(p.217)).

C3. Analogy

On $\mathscr{F}^+$, we have:

a. For any $\beta \leq 1$ and $s \geq 0$, the analogy collapse (see C3A(p.186) and C3A(p.218)) except (∗) and (∗∗) of C3(p.217).

C4. Optimal Initiating Time (`OIT`)

On $\mathscr{F}^+$, we have:

a. Let $\beta = 1$ and $s = 0$. Then we have Ⓢ▲ and 🅓‖ (see Tables 20.1.1(p.186) and 20.2.3(p.218)).

b. Let $\beta < 1$ or $s > 0$.
1. For search-**A**llowed-model we have Ⓢ▲, ◎‖, and 🅓‖ (see Table 20.1.2(p.187)).
2. For search-**E**nforced-model we have Ⓢ△,Ⓢ▲, ◎‖, ◎△, ◎▲, 🅓‖, 🅓△, and 🅓▲ (see Table 20.2.4(p.218)).

c. Joining Tables 20.1.3(p.187) and 20.2.5(p.218) produces the following table:

Table 20.3.1: Occurence rates of Ⓢ, ◎, and 🅓 on $\mathscr{F}$

Ⓢ			◎			🅓		
43.9 % / 58			22.7 % / 30			33.3 % / 44		
Ⓢ‖	Ⓢ△	Ⓢ▲	◎‖	◎△	◎▲	🅓‖	🅓△	🅓▲
−	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`	`possible`
−%/−	6.8 %/ 9	37.1 %/ 49	12.8 %/ 17	7.5 %/ 10	2.3 %/ 3	18.1 %/ 24	11.4 %/ 15	3.8 %/ 5

C5. Null-time-zone and deadline-engulfing

On $\mathscr{F}^+$, from Table 20.3.1(p.219) above we see that:

a. See Remark 7.2.2(p.43) for the implication of the symbol "▲" representing the strict optimality of the initiating time t^*_τ.

b. As a whole, ⓢ, ◎, and ⓓ are possible at 43.9%, 22.7%, and 33.3% respectively where

1. ⓢ$_{||}$ cannot be defined (see Remark ??(p.??)).
2. ◎$_{||}$ is possible (12.8 %).
3. ⓓ$_{||}$ is possible (18.1 %).
4. ⓢ$_{\triangle}$ is possible (6.8 %).
5. ◎$_{\triangle}$ is possible (7.5 %).
6. ⓓ$_{\triangle}$ is possible (11.4 %).
7. ⓢ$_{\blacktriangle}$ is possible (37.1 %).
8. ◎$_{\blacktriangle}$ is possible (2.3 %).
   - `Tom` 20.2.2(p.189) (c3i2,c3ii1ii2,c3ii2i).
9. ⓓ$_{\blacktriangle}$ is possible (3.8 %).
   - `Tom` 20.2.4(p.193) (d2i,d2ii).
   - `Tom` 20.2.16(p.203) (c2,c3i2,c3i3).

From the above results we see that:

A. ◎$_{||}$ and ⓓ$_{||}$ causing the `null-time-zone` are possible at 55.8% (= 23.0% + 32.8%).

B. ◎$_{\blacktriangle}$ *strictly* causing the `null-time-zone` is possible at 2.3%.

C. ⓓ$_{\blacktriangle}$ *strictly* causing the `deadline-engulfing` is possible at 3.8%.

Chapter 21

Analysis of Model 3

21.1 Reduction

Definition 21.1.1 (reduction)

(a) If it is always optimal to reject the intervening quitting penalty price ρ in Model 3, then it follows that Model 3 is substantively reduced to Model 2 in which the ρ is not defined, schematized as

$$\text{Model 3} \twoheadrightarrow \text{Model 2}. \tag{21.1.1}$$

Let us refer to this model reduction as the `model-running-back`.

(b) Let us define

$$\texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop} \overset{\text{def}}{\equiv} \{\text{Accept the intervening quitting penalty price } \rho \text{ at every time } t \geq 0 \} \text{ and stop the process}\}. \tag{21.1.2}$$

Let us refer to the reduction of this optimal decision rule as `odr-reduction`.

(c) Let us schematize the above two reductions as

$$\text{Reduction} \begin{cases} \texttt{model-running-back} \rightarrow \text{Model 3} \twoheadrightarrow \text{Model 2} \\ \texttt{odr-reduction} \rightarrow \texttt{odr} \mapsto \texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop} \end{cases} \tag{21.1.3}$$

Lemma 21.1.1 *Suppose that* $\texttt{odr} \mapsto \texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop}$ *holds. Then*

(a) *00Let* $\beta = 1$. *Then we have* 🅓$_{\parallel}$ *for any* ρ.

(b) *Let* $\beta < 1$ *and* $\rho < 0$. *Then we have* 🅓$_{\blacktriangle}$.

(c) *Let* $\beta < 1$ *and* $\rho = 0$. *Then we have* 🅓$_{\parallel}$.

(d) *Let* $\beta < 1$ *and* $\rho > 0$. *Then we have* ⓢ$_{\blacktriangle}$.

(e) *Let* $\rho \geq 0$. *Then we have* ⓢ$_{\vartriangle}$. □

• *Proof* If $\texttt{Accept}_{t\geq 0}(\rho) \triangleright \texttt{Stop}$ holds, then we have $V_t = \rho$ for $t > 0$ (see (6.5.38(p.39)), (6.5.44(p.39)), (6.5.52(p.39)), and (6.5.58(p.39))), hence we have $I_\tau^t = \beta^{\tau-t}\rho$ for $t > 0$ from (7.2.4(p.43)). Then, for a selling problem we have:

(a) Let $\beta = 1$. Then $\beta^0\rho = \beta^1\rho = \cdots = \beta^\tau\rho = \rho$ for any ρ, hence $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^0 = \rho$, so $t_\tau^* = 0$, i.e., 🅓$_{\parallel}$.

(b) Let $\beta < 1$ and $\rho < 0$. Then $\beta^0\rho < \beta^1\rho < \cdots < \beta^\tau\rho$, hence $I_\tau^\tau < I_\tau^{\tau-1} < \cdots < I_\tau^0$, so $t_\tau^* = 0$, i.e., 🅓$_{\blacktriangle}$.

(c) Let $\beta < 1$ and $\rho = 0$. Then $\beta^0\rho = \beta^1\rho = \cdots = \beta^\tau\rho = 0$, hence $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^0$, so $t_\tau^* = \tau = 0$, i.e., 🅓$_{\parallel}$.

(d) Let $\beta < 1$ and $\rho > 0$. Then $\beta^0\rho > \beta^1\rho > \cdots > \beta^\tau\rho$, hence $I_\tau^\tau > I_\tau^{\tau-1} > \cdots > I_\tau^0$, so $t_\tau^* = \tau$, i.e., ⓢ$_{\blacktriangle}$.

(e) Let $\rho \geq 0$. Then $\beta^0\rho \geq \beta^1\rho \geq \cdots \geq \beta^\tau\rho$ for any $0 < \beta \leq 1$, hence $I_\tau^\tau \geq I_\tau^{\tau-1} \geq \cdots \geq I_\tau^0$, so $t_\tau^* = \tau$, i.e., ⓢ$_{\vartriangle}$.

The same as the above hold also for a buying problem except that the directions of inequality reverse. ▮

21.2 Search-Allowed-Model 3: $\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{A}]\} = \{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}], \tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}], \mathsf{M}{:}3[\mathbb{P}][\mathtt{A}], \tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$

21.2.1 Theorems

As ones corresponding to Theorems 12.5.1(p.78), 13.3.1(p.96), and 14.5.1(p.104) let us consider the following three theorems:

Theorem 21.2.1 (symmetry$[\mathbb{R} \to \tilde{\mathbb{R}}]$) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{21.2.1}$$

Theorem 21.2.2 (analogy$[\mathbb{R} \to \mathbb{P}]$) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}]. \quad \square \tag{21.2.2}$$

Theorem 21.2.3 (symmetry$(\mathbb{P} \to \tilde{\mathbb{P}}]$) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}]. \quad \square \tag{21.2.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{21.2.4}$$
$$\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{21.2.5}$$
$$\mathtt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathtt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}], \tag{21.2.6}$$

corresponding to (12.5.34(p.75)), (13.2.4(p.91)), and (14.5.4(p.104)). Now, from the comparison of (I) and (II) and of (III) and (IV) in Table 6.5.5(p.39) it can be easily shown that (21.2.4(p.222)) and (21.2.6(p.222)) hold. However, from the comparison of (I) and (III) in Table 6.5.5(p.39) we can immediately see that (21.2.5(p.222)) does not always hold, hence it follows that also Theorem 21.2.2(p.222) does not always hold.

21.2.2 A Lemma

The following lemma determines if Theorem 21.2.2(p.222) holds by testing whether or not each of (21.2.5(p.222)) is true.

Lemma 21.2.1

(a) Theorem 21.2.1(p.222) *always hold.*

(b) Theorem 21.2.3(p.222) *always hold.*

(c) *Let $\rho \le a^\star$ or $b \le \rho$. Then* Theorem 21.2.2(p.222) *holds.*

(d) *Let $a^\star < \rho < b$. Then* Theorem 21.2.2(p.222) *does not always hold.* $\square$

● *Proof* Almost the same as the proof of Lemma 20.1.1(p.151). ∎

21.2.3 Proof of $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$

□ **Tom 21.2.1** ($\blacksquare\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]\}$)

(a) *Let $\rho \le x_K$ or $\rho \le 0$. Then* $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$.

(b) *Let $\rho \ge x_K$ and $\rho \ge 0$. Then we have* $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$. $\square$

● *Proof* From (6.5.39(p.39)) with $t = 1$ and (6.5.37(p.39)) we have $U_1 = \max\{K(V_0)+\rho, \beta V_0\} = \max\{K(\rho)+\rho, \beta\rho\} \cdots$ **(1)**, hence $U_1 - \rho = \max\{K(\rho), -(1-\beta)\rho\} \cdots$ **(2)**. From (6.5.38(p.39)) with $t = 1$ we have $V_1 \ge \rho = V_0$. Then, from (6.5.39(p.39)) with $t = 2$ and Lemma 10.2.2(p.55) (e) we have $U_2 = \max\{K(V_1) + V_1, \beta V_1\} = \max\{K(V_0) + V_0, \beta V_0\} = U_1$. Suppose $U_{t-1} \ge U_{t-2}$, hence from (6.5.38(p.39)) we have $V_{t-1} = \max\{\rho, U_{t-1}\} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$. Then, from (6.5.39(p.39)) we have $U_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = U_{t-1}$ due to Lemma 10.2.2(p.55) (e). Thus, by induction we have $U_t \ge U_{t-1}$ for $t > 1$, i.e., we have that U_t is nondecreasing in $t > 0 \cdots$ **(3)**.

(a) Let $\rho \le x_K$ or $\rho \le 0$. Suppose $\rho \le x_K$, hence $K(\rho) \ge 0 \cdots$ **(4)** from Corollary 10.2.2(p.56) (b). Then, from (1) we have $U_1 \ge K(\rho) + \rho \ge \rho$. Hence $U_t \ge \rho$ for $t > 0$ due to (3). Suppose $\rho \le 0$, hence $-(1-\beta)\rho \ge 0$. Then, noting (4), from (2) we have $U_1 - \rho \ge 0$, i.e., $U_1 \ge \rho$, so $U_t \ge \rho$ for $t > 0$ due to (3). Accordingly, whether $\rho \le x_K$ or $\rho \le 0$, we have $U_t \ge \rho$ for $t > 0$, meaning that it is always optimal to reject the intervening quitting penalty price ρ for any $t > 0$. This fact is the same as the event "the intervening quitting penalty price ρ does not exist on any time $t > 0$"; in other words, it follows that $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]$ is substantially reduced to $\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$ which has not an intervening quitting penalty price ρ, i.e., $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$.

(b) Let $\rho \ge x_K$ and $\rho \ge 0 \cdots$ **(5)**, hence $K(\rho) \le 0 \cdots$ **(6)** from Corollary 10.2.2(p.56) (a) and $-(1-\beta)\rho \le 0$. Then, since $U_1 - \rho \le 0$ from (2), we have $U_1 \le \rho \cdots$ **(7)**. Suppose $U_{t-1} \le \rho$. Then $V_{t-1} = \rho$ from (6.5.38(p.39)), hence from (6.5.39(p.39)) we have $U_t = \max\{K(\rho) + \rho, \beta\rho\} = U_1 \le \rho$ due to (1) and (7). Accordingly, by induction $U_t \le \rho$ for $t > 0$, meaning that it is always optimal to accept the intervening quitting penalty price ρ at all time $t \ge 0$ and stop the process. Hence we have $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$. ∎

21.2.4 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$

□ `Tom` **21.2.2** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]\}$)

(a) *Let* $\rho \ge x_{\tilde{K}}$ *or* $\rho \ge 0$. *Then* $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]$.

(b) *Let* $\rho \ge x_{\tilde{K}}$ *and* $\rho \ge 0$. *Then we have* $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$. □

● *Proof by symmetry* Immediately from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128))) to `Tom` 21.2.1(p.222) due to Lemma 21.2.1(p.222) (a). ∎

21.2.5 Derivation of $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$

21.2.5.1 Case of $\rho \le a^\star$ or $b \le \rho$

□ `Tom` **21.2.3** (⊡ $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume* $\rho \le a^\star$ *or* $b \le \rho$. *Then*:

(a) *Let* $\rho \le x_K$ *or* $\rho \le 0$. *Then* $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$.

(b) *Let* $\rho \ge x_K$ *and* $\rho \ge 0$. *Then we have* $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$. □

● *Proof by analogy* The same as `Tom` 21.2.1(p.222) due to Lemma 13.6.1(p.97). ∎

21.2.5.2 Case of $a^\star < \rho < b$

□ `Tom` **21.2.4** (⊡ $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume* $a^\star < \rho < b$. *Let* $\beta = 1$ *and* $s = 0$. *Then* $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$. □

● *Proof by analogy* Assume $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.24)) we have $K(x) = \lambda T(x) \ge 0 \cdots \mathbf{(1)}$ for any x due to Lemma 13.2.1(p.91) (g). From (6.5.45(p.39)) we have $U_1 \ge \beta\rho = \rho$. Suppose $U_{t-1} \ge \rho$. Then, from (6.5.44(p.39)) we have $V_{t-1} = U_{t-1} \ge \rho$, hence from (6.5.46(p.39)) we obtain $U_t \ge \beta V_{t-1} = V_{t-1} \ge \rho$. Thus, by induction $U_t \ge \rho$ for $t > 0$. Accordingly, for the same reason as in the proof of `Tom` 21.2.1(p.222) (a) we have $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$. ∎

□ `Tom` **21.2.5** (▣ $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume* $a^\star < \rho < b$. *Let* $\beta < 1$ *or* $s > 0$.

(a) *Let* $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$ *or* $-(1-\beta)\rho \ge 0$. *Then* $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$.

(b) *Let* $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \le s$ *and* $-(1-\beta)\rho \le 0$.

1. *Let* $\tau = 1$. *Then we have* $\mathtt{odr} \mapsto \mathtt{Accept}_1(\rho) \triangleright \mathtt{Stop}$.†
2. *Let* $\tau > 1$. *Then*:
   - i. *Let* $\rho \le x_K$. *Then* $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$
   - ii. *Let* $\rho \ge x_K$. *Then we have* $\mathtt{odr} \mapsto \mathtt{Accept}_{t\ge0}(\rho) \triangleright \mathtt{Stop}$.† □

● *Proof* Assume $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$. From (6.5.45(p.39)) we have

$$U_1 - \rho = \max\{\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho - s, -(1-\beta)\rho\} \cdots \mathbf{(1)}.$$

(a) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \ge s$ or $-(1-\beta)\rho \ge 0$, hence $U_1 - \rho \ge 0$ from (1) or equivalently $U_1 \ge \rho \cdots \mathbf{(2)}$. Then, since $V_1 = U_1 \cdots \mathbf{(3)}$ from (6.5.44(p.39)) with $t = 1$, from (6.5.46(p.39)) with $t = 2$ we have $U_2 = \max\{K(V_1) + V_1, \beta V_1\} = \max\{K(U_1) + U_1, \beta U_1\} \cdots \mathbf{(4)}$. Hence, from (2), Lemma 13.2.3(p.94) (e), and (5.1.21(p.24)) we have

$$\begin{aligned} U_2 &\ge \max\{K(\rho) + \rho, \beta\rho\} \\ &= \max\{\lambda\beta T(\rho) - (1-\beta)\rho - s + \rho, \beta\rho\} \\ &= \max\{\lambda\beta T(\rho) + \beta\rho - s, \beta\rho\}. \end{aligned}$$

Then, from Lemma 13.2.1(p.91) (h) we have $U_2 \ge \max\{\lambda\beta\max\{0, a-\rho\} + \beta\rho - s, \beta\rho\} = U_1$ due to (6.5.45(p.39)). Suppose $U_{t-1} \ge U_{t-2}$, so $V_{t-1} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.5.44(p.39)). Hence, from (6.5.46(p.39)) and Lemma 13.2.3(p.94) (e) we have $U_t \ge \max\{K(V_{t-2}) + V_{t-2}, \beta V_{t-2}\} = U_{t-1}$. Accordingly, by induction $U_t \ge U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0$. Hence, from (2) we have $U_t \ge \rho$ for $t > 0$. Therefore, for almost the same reason as in the proof of `Tom` 21.2.1(p.222) (a) we have $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$.

(b) Let $\lambda\beta\max\{0, a-\rho\} - (1-\beta)\rho \le s$ and $-(1-\beta)\rho \le 0 \cdots \mathbf{(5)}$. Then $U_1 - \rho \le 0$ from (1), i.e., $U_1 \le \rho \cdots \mathbf{(6)}$.

(b1) Let $\tau = 1$. Then (6) implies that it is optimal to accept the intervening quitting penalty price ρ at $t = 1$ and stop the process, i.e., $\mathtt{odr} \mapsto \mathtt{Accept}_1(\rho) \triangleright \mathtt{Stop}$.

(b2) Let $\tau > 1$. Due to (6) we have $V_1 = \rho$ from (6.5.44(p.39)) with $t = 1$, hence $U_2 = \max\{K(\rho) + \rho, \beta\rho\} \cdots \mathbf{(7)}$ from (6.5.46(p.39)) with $t = 2$.

(b2i) Let $\rho \le x_K$. Then $K(\rho) \ge 0$ from Lemma 13.2.3(p.94) (j1), hence from (7) we have $U_2 \ge K(\rho) + \rho \ge \rho$. Suppose $U_{t-1} \ge \rho$, hence $V_{t-1} = U_{t-1} = \rho$ from (6.5.44(p.39)). Then, from (6.5.46(p.39)) and Lemma 13.2.3(p.94) (e) we have $U_t \ge \max\{K(\rho) + \rho, \beta\rho\} \ge$

---

†In this case, we have four possibilities for the optimal initiating time (`OIT`): ⓓ$_{\|}$, ⓓ$_{\blacktriangle}$, ⓢ$_{\blacktriangle}$, and ⓢ$_{\triangle}$.

$K(\rho)+\rho \geq \rho$. Accordingly, by induction we have $U_t \geq \rho$ for $t > 1$. Thus the assertion holds for the same reason as in the proof of Lemma 21.2.1(p.222) (a).

(b2ii) Let $\rho \geq x_K$, hence $K(\rho) < 0$ from Lemma 13.2.3(p.94) (j1). Then, from (7) we have $U_2 \leq \max\{\rho, \beta\rho\} \cdots$ (8). If $\beta < 1$, then $\rho \geq 0$ from (5), hence $U_2 \leq \max\{\rho,\rho\} = \rho$ and if $\beta = 1$, then $U_2 \leq \max\{\rho,\rho\} = \rho$. Accordingly, whether $\beta < 1$ or $\beta = 1$, we have $U_2 \leq \rho$ for $t > 0$. Suppose $U_{t-1} \leq \rho$, hence $V_{t-1} = \rho$ from (6.5.44(p.39)). Then, from (6.5.46(p.39)) we have $U_t = \max\{K(\rho)+\rho, \beta\rho\} = U_2 \leq \rho$. Accordingly, by induction we have $U_t \leq \rho$ for $t > 1$. Hence, from (6) we have $U_t \leq \rho$ for $t > 0$. Thus, for the same reason as in the proof of Tom 21.2.1(p.222) (b) it follows that the assertion holds. ∎

21.2.6 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$

21.2.6.1 Case of $\rho \geq b^\star$ or $a \geq \rho$

□ **Tom 21.2.6** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume $\rho \geq b^\star$ or $a \geq \rho$.*

(a) *Let $\rho \geq x_{\tilde{K}}$ or $\rho \geq 0$. Then $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$.*

(b) *Let $\rho \leq x_{\tilde{K}}$ and $\rho \leq 0$. Then we have $\mathtt{odr} \mapsto \mathtt{Accept}_\tau(\rho) \triangleright \mathtt{Stop}$.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.128))) due to Lemma 21.2.1(p.222) (b). ∎

21.2.6.2 Case of $b^\star > \rho > a$

□ **Tom 21.2.7** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume $b^\star > \rho > b$. Let $\beta = 1$ and $s = 0$. Then $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.128))) due to Lemma 21.2.1(p.222) (b). ∎

□ **Tom 21.2.8** ($\boxdot\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]\}$) *Assume $b^\star > \rho > a$. Let $\beta < 1$ or $s > 0$.*

(a) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho \geq 0$ or $(1-\beta)\rho \geq 0$. Then $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$.*

(b) *Let $-\lambda\beta\min\{0, \rho - b\} + (1-\beta)\rho < s$ and $(1-\beta)\rho < 0$.*

1. *Let $\tau = 1$. Then we have $\mathtt{odr} \mapsto \mathtt{Accept}_1(\rho) \triangleright \mathtt{Stop}$.*
2. *Let $\tau > 1$.*
   i. *Let $\rho > x_{\tilde{K}}$. Then $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$.*
   ii. *Let $\rho \leq x_{\tilde{K}}$. Then we have $\mathtt{odr} \mapsto \mathtt{Accept}_{t\geq 0}(\rho) \triangleright \mathtt{Stop}$.* □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.3(p.128))) due to Lemma 21.2.1(p.222) (b). ∎

21.2.7 Conclusion 5 (Search-Allowed-Model 3)

The search-Allowed-model 3 (whether selling model or buying model and whether $\mathbb{R}$-model or $\mathbb{P}$-model) is reduced to either of the following two cases (see (21.1.3(p.221))):

Case A $\mathsf{M}/\tilde{\mathsf{M}}{:}3[\mathbb{R}/\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathsf{M}/\tilde{\mathsf{M}}{:}2[\mathbb{R}/\mathbb{P}][\mathtt{A}]$ (`model-running-back`) where

1. for $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \mathrm{r}\mathsf{M}{:}2[\mathbb{R}][\mathtt{A}]$, see Tom 21.2.1(p.222) (a),
2. for $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}] \twoheadrightarrow \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathtt{A}]$, see Tom 21.2.2(p.223) (a),
3. for $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathrm{r}\mathsf{M}{:}2[\mathbb{P}][\mathtt{A}]$, see Tom 21.2.3(p.223) (a), 21.2.4(p.223), and 21.2.5(p.223) (a,b2i),
4. for $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}] \twoheadrightarrow \mathrm{r}\tilde{\mathsf{M}}{:}2[\mathbb{P}][\mathtt{A}]$, see Tom 21.2.6(p.224) (a), 21.2.7(p.224), and 21.2.8(p.224) (a,b2i).

Case B $\mathtt{odr} \mapsto \mathtt{Accept}_{t\geq 0}(\rho) \triangleright \mathtt{Stop}$ (`odr-reduction`) where

1. for $\mathsf{M}{:}3[\mathbb{R}][\mathtt{A}]$, see Tom 21.2.1(p.222) (b),
2. for $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{A}]$, see Tom 21.2.2(p.223) (b),
3. for $\mathsf{M}{:}3[\mathbb{P}][\mathtt{A}]$, see Tom 21.2.3(p.223) (b),21.2.5(p.223) (b1,b2ii),
4. for $\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{A}]$, see Tom 21.2.6(p.224) (b),21.2.8(p.224) (b1.b2ii).

21.3 Search-Enforced-Model 3: $\mathcal{Q}\{\mathsf{M}{:}3[\mathtt{E}]\} = \{\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}], \tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}], \mathsf{M}{:}3[\mathbb{P}][\mathtt{E}], \tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathtt{E}]\}$

21.3.1 Preliminary

As the ones corresponding to Theorems 21.2.1(p.222), 21.2.2(p.222), and 21.2.3(p.222) let us consider the following three theorems:

Theorem 21.3.1 (symmetry[$\mathbb{R} \to \mathbb{R}$]) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathtt{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]\}]. \quad \square \tag{21.3.1}$$

Theorem 21.3.2 (analogy[$\mathbb{R} \to \mathbb{P}$]) *Let $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$. Then $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{E}]\}$ holds on $\mathscr{P} \times \mathscr{F}$ where*

$$\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathtt{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathtt{E}]\}]. \quad \square \tag{21.3.2}$$

Theorem 21.3.3 (symmetry[$\mathbb{P} \to \mathbb{P}$])) *Let* $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$. *Then* $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}]\}$ *holds on* $\mathscr{P} \times \mathscr{F}$ *where*

$$\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}]\}]. \quad \square \tag{21.3.3}$$

In order for the above three theorems to hold, the following three relations *must* be satisfied:

$$\texttt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}[\texttt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathsf{E}]\}], \tag{21.3.4}$$

$$\texttt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}]\} = \mathcal{A}_{\mathbb{R}\to\mathbb{P}}[\texttt{SOE}\{\mathsf{M}{:}3[\mathbb{R}][\mathsf{E}]\}], \tag{21.3.5}$$

$$\texttt{SOE}\{\tilde{\mathsf{M}}{:}3[\mathbb{P}][\mathsf{E}]\} = \mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}[\texttt{SOE}\{\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}]\}], \tag{21.3.6}$$

corresponding to (21.2.4(p.222)), (21.2.5(p.222)), and (21.2.6(p.222)). Now, from the comparison of (I) and (II) and of (III) and (IV) in Table 6.5.6(p.39) it can be easily shown that (21.3.4(p.225)) and (21.3.6(p.225)) hold. However, from the comparison of (I) and (III) in Table 6.5.6(p.39) we can immediately see that (21.3.5(p.225)) does not hold, hence it follows that also Theorem 21.3.2(p.224) does not always hold.

21.3.2 A Lemma

Lemma 21.3.1

(a) Theorem 21.3.1(p.224) *always hold.*

(b) Theorem 21.3.3(p.225) *always hold.*

(c) *Let* $\rho \le a^\star$ *or* $b \le \rho$. *Then* Theorem 21.3.2(p.224) *holds.*

(d) *Let* $a^\star < \rho < b$. *Then* Theorem 21.3.2(p.224) *does not always hold.* □

• *Proof* Almost the same as the proof of Lemma 20.1.1(p.151). ▮

21.3.3 Proof of $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathsf{E}]\}$

□ `Tom` **21.3.1** (▣ $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{R}][\mathsf{E}]\}$)

(a) *Let* $\rho \le x_K$. *Then* $\mathsf{M}{:}3[\mathbb{R}][\mathsf{E}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{R}][\mathsf{E}]$.

(b) *Let* $\rho \ge x_K$. *Then we have* $\texttt{odr} \mapsto \texttt{Accept}_{\tau\ge t\ge 0}(\rho) \triangleright \texttt{Stop}$.† □

• *Proof* From (6.5.53(p.39)) with $t = 1$ and (6.5.51(p.39)) we have $U_1 = K(\rho) + \rho \cdots (1)$ and from (6.5.52(p.39)) with $t = 1$ we have $V_1 \ge \rho = V_0$. Then, from (6.5.53(p.39)) with $t = 2$ and Lemma 10.2.2(p.55)(e) we have $U_2 = K(V_1) + V_1 \ge K(\rho) + \rho = U_1$. Suppose $U_{t-1} \ge U_{t-2}$, hence $V_{t-1} \ge \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.5.52(p.39)). Then from (6.5.53(p.39)) we have $U_t = K(V_{t-1}) + V_{t-1} \ge K(V_{t-2}) + V_{t-2} = U_{t-1}$ due to Lemma 10.2.2(p.55)(e) Thus, by induction we have $U_t \ge U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0 \cdots (2)$.

(a) Let $\rho \le x_K$, hence $K(\rho) \ge 0$ from Corollary 10.2.2(p.56)(b). Then, from (1) we have $U_1 \ge \rho$. Hence $U_t \ge \rho$ for $t > 0$ due to (2). Accordingly, for almost the same reason as in the proof of `Tom` 21.2.1(p.222)(a) we have $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathsf{E}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]$.

(b) Let $\rho \ge x_K$, hence $K(\rho) \le 0 \cdots (3)$ from Corollary 10.2.2(p.56)(a). Then, from (1) we have $U_1 \le \rho$. Suppose $U_{t-1} \le \rho$. Then $V_{t-1} = \rho$ from (6.5.52(p.39)), hence from (6.5.53(p.39)) we have $U_t = K(\rho) + \rho \le \rho$ due to (3). Accordingly, by induction $U_t \le \rho$ for $t > 0$, so we have $\texttt{odr} \mapsto \texttt{Accept}_{\tau\ge t\ge 0}(\rho) \triangleright \texttt{Stop}$ for the same reason as in `Tom` 21.2.1(p.222)(b).

21.3.4 Derivation of $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathsf{E}]\}$

□ `Tom` **21.3.2** (⊡ $\mathscr{A}\{\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathsf{E}]\}$) *For any* $\beta \le 1$ *and* $s \ge 0$ *we have*:

(a) *Let* $\rho \le x_{\tilde{K}}$. *Then* $\tilde{\mathsf{M}}{:}3[\mathbb{R}][\mathsf{E}] \twoheadrightarrow \tilde{\mathsf{M}}{:}2[\mathbb{R}][\mathsf{E}]$.

(b) *Let* $\rho \le x_{\tilde{K}}$. *Then we have* $\texttt{odr} \mapsto \texttt{Accept}_{\tau\ge t\ge 0}(\rho) \triangleright \texttt{Stop}$. □

• *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ (see (18.0.1(p.128))) due to Lemma 21.3.1(p.225)(a). ▮

21.3.5 Derivation of $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}]\}$

21.3.5.1 Case of $\rho \le a^\star$ or $b \le \rho$

In this case, we can use Lemma 21.3.1(p.225)(c) to prove `Tom` 21.3.3(p.225) below.

□ `Tom` **21.3.3** (⊡ $\mathscr{A}\{\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}]\}$) *Assume* $\rho \le a^\star$ *or* $b \le \rho$.

(a) *Let* $\rho \le x_K$. *Then* $\mathsf{M}{:}3[\mathbb{P}][\mathsf{E}] \twoheadrightarrow \mathsf{M}{:}2[\mathbb{P}][\mathsf{E}]$.

(b) *Let* $\rho \ge x_K$. *Then we have* $\texttt{odr} \mapsto \texttt{Accept}_{\tau\ge t\ge 0}(\rho) \triangleright \texttt{Stop}$. □

• *Proof by analogy* The same as `Tom` 21.3.1(p.225) due to Lemma 13.6.1(p.97). ▮

†In this case, we have four possibilities for the optimal initiating time (`OIT`): ⓓ$_{\|}$, ⓓ$_{\blacktriangle}$, ⓢ$_{\blacktriangle}$, and ⓢ$_{\triangle}$ (see Lemma 21.1.1(p.221)).

21.3.5.2 Case of $a^\star < \rho < b$

In this case, Tom's 21.3.4(p.226) and 21.3.5(p.226) below must be directly proven due to Lemma 21.3.1(p.225) (d).

□ Tom **21.3.4** (▣$\mathscr{A}${M:3[ℙ][E]}) *Assume $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$. Then we have* M:3[ℙ][E] ↠ M:2[ℙ][E]. □

● *Proof* Suppose $a^\star < \rho < b$ and let $\beta = 1$ and $s = 0$. From (5.1.21(p.24)) we have $K(x) = \lambda T(x) \geq 0 \cdots$ **(1)** for any x due to Lemma 13.2.1(p.91) (g). Now, from (6.5.59(p.39)) we have $U_1 = \lambda \max\{0, a - \rho\} + \rho \geq \rho$ due to $\max\{0, a - \rho\} \geq 0$. Suppose $U_{t-1} \geq \rho$. Then, since $V_{t-1} = U_{t-1}$ due to (6.5.58(p.39)), from (6.5.60) we have $U_t = K(U_{t-1}) + U_{t-1} \geq U_{t-1}$ due to (1), hence $U_t \geq \rho$. Accordingly, by induction $U_t \geq \rho$ for $t > 0$, implying that it is optimal to reject the intervening quitting penalty price ρ for any $t > 1$. Thus, for almost the same as in the proof of Tom 21.2.1(p.222) (a) we have M:3[ℙ][E] ↠ M:2[ℙ][E]. ▮

□ Tom **21.3.5** (▣$\mathscr{A}${M:3[ℙ][E]}) *Assume $a^\star < \rho < b$ and let $\beta < 1$ or $s > 0$.*

(a) *Let $\lambda\beta \max\{0, a - \rho\} - (1 - \beta)\rho \geq s$. Then* M:3[ℙ][E] ↠ M:2[ℙ][E].
(b) *Let $\lambda\beta \max\{0, a - \rho\} - (1 - \beta)\rho \leq s$.*
  1. *Let $\tau = 1$. Then we have* odr ↦ Accept$_{t=1}(\rho)$ ▷ Stop.
  2. *Let $\tau > 1$. Then*
     i. *Let $\rho \leq x_K$. Then* M:3[ℙ][E] ↠ M:2[ℙ][E].
     ii. *Let $\rho \geq x_K$. Then we have* odr ↦ Accept$_{\tau \geq t \geq 0}(\rho)$ ▷ Stop.

● *Proof* Suppose $a^\star < \rho < b$. Let $\beta < 1$ or $s > 0$. From (6.5.59(p.39)) we have

$$U_1 - \rho = \lambda\beta \max\{0, a - \rho\} - (1 - \beta)\rho - s \cdots \textbf{(1)}.$$

(a) Let $\lambda\beta \max\{0, a - \rho\} - (1 - \beta)\rho \geq s$, hence $U_1 \geq \rho \cdots$ **(2)** from (1). Then, since $V_1 = U_1 \cdots$ **(3)** from (6.5.58(p.39)) with $t = 1$, we have $U_2 = K(U_1) + U_1 \cdots$ **(4)** from (6.5.60(p.39)) with $t = 2$. Hence, from (2), Lemma 13.2.3(p.94) (e), and (5.1.21(p.24)) we have $U_2 \geq K(\rho) + \rho = \lambda\beta T(\rho) - (1 - \beta)\rho - s + \rho = \lambda\beta T(\rho) + \beta\rho - s$. Then, from Lemma 13.2.1(p.91) (h) we have $U_2 \geq \lambda\beta \max\{0, a - \rho\} + \beta\rho - s = U_1$ due to (6.5.59(p.39)). Suppose $U_{t-1} \geq U_{t-2}$, hence $V_{t-1} \geq \max\{\rho, U_{t-2}\} = V_{t-2}$ from (6.5.58(p.39)). Then, from Lemma 13.2.3(p.94) (e) we have $U_t \geq K(V_{t-2}) + V_{t-2} = U_{t-1}$. Accordingly, by induction $U_t \geq U_{t-1}$ for $t > 1$, i.e., U_t is nondecreasing in $t > 0$. Hence, from (2) we have $U_t \geq \rho$ for $t > 0$, implying that it is optimal to reject the intervening quitting penalty price ρ for any $t > 1$. Therefore, for the same as in the proof of Tom 21.2.1(p.222) (a) we have M:3[ℙ][E] ↠ M:2[ℙ][E].

(b) Let $\lambda\beta \max\{0, a - \rho\} - (1 - \beta)\rho \leq s \cdots$ **(5)**. Then $U_1 - \rho \leq 0$ from (1), i.e., $U_1 \leq \rho \cdots$ **(6)**.

(b1) Let $\tau = 1$. Now (6) implies that it is optimal to accept the intervening quitting penalty price ρ at the starting time $t = 1$ and the process stops, hence we have odr ↦ Accept$_{t=1}(\rho)$ ▷ Stop.

(b2) Let $\tau > 1$. Now, due to (6) we have $V_1 = \rho$ from (6.5.58(p.39)) with $t = 1$, thus $U_2 = K(\rho) + \rho \cdots$ **(7)** from (6.5.60(p.39)) with $t = 2$.

(b2i) Let $\rho \leq x_K$, hence $K(\rho) \geq 0$ from Lemma 13.2.3(p.94) (j1). Then, from (7) we have $U_2 \geq \rho$. Suppose $U_{t-1} \geq \rho$, hence $V_{t-1} = U_{t-1}$ from (6.5.58(p.39)). Then, from (6.5.60(p.39)) and Lemma 13.2.3(p.94) (e) we have $U_t = K(U_{t-1}) + U_{t-1} \geq K(\rho) + \rho \geq \rho$. Hence, by induction $U_t \geq \rho$ for $t > 1$, implying that it is optimal to reject the intervening quitting penalty price ρ for any $t > 1$. Thus, for almost the same as in the proof of Lemma 21.2.1(p.222) (a) we have M:3[ℙ][E] ↠ M:2[ℙ][E].

(b2ii) Let $\rho \geq x_K$. Then $K(\rho) \leq 0 \cdots$ **(8)** from Lemma 13.2.3(p.94) (j1). Hence $U_2 \leq \rho$ from (7). Suppose $U_{t-1} \leq \rho$, hence $V_{t-1} = \rho$ from (6.5.58(p.39)). Then, from (6.5.60(p.39)) we have $U_t = K(\rho) + \rho \leq \rho \cdots$ **(9)** due to (8). Thus, by induction $U_t \leq \rho$ for $t > 1$. From this and (6) we have $U_t \leq \rho$ for $t > 0$, hence we have odr ↦ Accept$_{\tau \geq t \geq 0}(\rho)$ ▷ Stop for the same reason as in the proof of Tom 21.2.1(p.222) (b) we have that the assertion holds. ▮

21.3.6 Derivation of $\mathscr{A}${M̃:3[ℙ][E]}

21.3.6.1 Case of $\rho \geq b^\star$ or $a \geq \rho$

□ Tom **21.3.6** (◻$\mathscr{A}${M̃:3[ℙ][E]}) *Assume $\rho \geq b^\star$ or $a \geq \rho$ and let $\beta \leq 1$ and $s \geq 0$.*

(a) *Let $\rho \geq x_{\tilde{K}}$. Then* M̃:3[ℙ][E] ↠M̃:2[ℙ][E].
(b) *Let $\rho \leq x_{\tilde{K}}$. Then we have* odr ↦ Accept$_{\tau \geq t \geq 0}(\rho)$ ▷ Stop.

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\rightarrow\tilde{\mathbb{P}}}$ (see (18.0.2(p.128))) to Tom 21.3.3(p.225). ▮

21.3.6.2 Case of $b^\star > \rho > a$

□ Tom **21.3.7** (◻$\mathscr{A}${M̃:3[ℙ][E]}) *Assume $b^\star > \rho \geq b$ and let $\beta = 1$ and $s = 0$. Then* M̃:3[ℙ][E] ↦M̃:2[ℙ][E]. □

● *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\rightarrow\tilde{\mathbb{P}}}$ (see (18.0.2(p.128))) to Tom 21.3.4(p.226). ▮

□ Tom **21.3.8** (◻$\mathscr{A}${M̃:3[ℙ][E]}) *Assume $b^\star > \rho > a$ and let $\beta < 1$ or $s > 0$.*

(a) *Let* $-\lambda\beta\min\{0,\rho-b\}+(1-\beta)\rho\geq s$. *Then* $\tilde{\mathsf{M}}$:3[ℙ][E] ↠$\tilde{\mathsf{M}}$:2[ℙ][E].

(b) *Let* $-\lambda\beta\min\{0,\rho-b\}+(1-\beta)\rho\leq s$.

1. *Let* $\tau=1$. *Then we have* `odr` ↦ $\texttt{Accept}_{t=1}(\rho)\triangleright$`Stop`.
2. *Let* $\tau>1$. *Then*
   i. *Let* $\rho>x_{\tilde{K}}$. *Then* $\tilde{\mathsf{M}}$:3[ℙ][E] ↠$\tilde{\mathsf{M}}$:2[ℙ][E]
   ii. *Let* $\rho\leq x_{\tilde{K}}$. *Then* `odr` ↦ $\texttt{Accept}_{\tau\geq t\geq 0}(\rho)\triangleright$`Stop`.

• *Proof by symmetry* Immediate from applying $\mathcal{S}_{\mathbb{P}\to\tilde{\mathbb{P}}}$ (see (18.0.2(p.128))) to `Tom` 21.3.5(p.226). ∎

21.3.7 Conclusion 6 (Search-Enforced-Model 3)

The search-Enforced-model 3 (whether selling model or buying model and whether ℝ-model or ℙ-model) is reduced to either of the following two cases (see (21.1.3(p.221))):

Case A M/$\tilde{\mathsf{M}}$:3[ℝ/ℙ][E] ↠ M/$\tilde{\mathsf{M}}$:2[ℝ/ℙ][E] (`model-running-back`) where

1. for M:3[ℝ][E] ↠ rM:2[ℝ][E], see `Tom` 21.3.1(p.225) (a),
2. for $\tilde{\mathsf{M}}$:3[ℝ][E] ↠ r$\tilde{\mathsf{M}}$:2[ℝ][E], see `Tom` 21.3.2(p.225) (a),
3. for M:3[ℙ][E] ↠ rM:2[ℙ][E], see `Tom` 21.3.3(p.225) (a), 21.3.4(p.226), and 21.3.5(p.226) (a,b2i),
4. for $\tilde{\mathsf{M}}$:3[ℙ][E] ↠ r$\tilde{\mathsf{M}}$:2[ℙ][E], see `Tom` 21.3.6(p.226) (a), 21.3.7(p.226), and 21.3.8(p.226) (a,b2i).

Case B `odr` ↦ $\texttt{Accept}_{\tau\geq t\geq 0}(\rho)\triangleright$`Stop` (`odr-reduction`) where

1. for M:3[ℝ][E], see `Tom` 21.3.1(p.225) (b),
2. for $\tilde{\mathsf{M}}$:3[ℝ][E], see `Tom` 21.3.2(p.225) (b),
3. for M:3[ℙ][E], see `Tom` 21.3.3(p.225) (b),21.3.5(p.226) (b1,b2ii),
4. for $\tilde{\mathsf{M}}$:3[ℙ][E], see `Tom` 21.3.6(p.226) (b),21.3.8(p.226) (b1,b2ii).

21.4 Conclusions of Model 3

From Conclusions 5 (p.224) and 6 (p.227) we see that the model 3 (whether selling model or buying model, whether ℝ-model or ℙ-model, and whether search-Enforced-model or search-Allowed-model) is reduced to either of the following two cases:

a. M/$\tilde{\mathsf{M}}$:3[ℝ/ℙ][A/E] ↠ M/$\tilde{\mathsf{M}}$:2[ℝ/ℙ][A/E] (`model-running-back`).

b. `odr` ↦ $\texttt{Accept}_{\tau\geq t\geq 0}(\rho)\triangleright$`Stop` (`odr-reduction`).

Chapter 22

Conclusions of Part 3 (Analyses of Models)

Below is the summary of Sections 19.3(p.148), 20.3(p.219), and 21.4(p.227).

■ Models 1/2

C1. Mental Conflict

Here "always" means "whether selling model or buying model, whether $\mathbb{R}$-model or $\mathbb{P}$-model, and whether search-**A**llowed-model or search-**E**nforced-model". Then, on $\mathscr{F}^+$ we have:

a. For Model 1 (see C1(p.148)):
   Let $\beta \leq 1$ and $s \geq 0$. Then, we always have the *normal* mental conflict in *Examples* 1.4.1(p.6) - 1.4.4(p.6).

b. For Model 2 (see C1(p.219)):
   1. Let $\beta = 1$ and $s = 0$. Then we always have the *normal* mental conflict in *Examples* 1.4.1(p.6) - 1.4.4(p.6).
   2. Let $\beta < 1$ or $s > 0$. Then we always have the *abnormal* mental conflict.

C2. Symmetry

a. On $\mathscr{F}^+$:
   1. Let $\beta = 1$ and $s = 0$. Then, the symmetry is inherited for Models 1/2 (see C2b(p.148) and C2a(p.219)).
   2. Let $\beta < 1$ or $s > 0$. Then, the symmetry *may* collapse for Model 1 (see C2c(p.148)), but *always* collapse for Model 2 (see C2b(p.219)).

C3. Analogy

On $\mathscr{F}^+$ we have:

a. For Model 1:
   1. Let $\beta = 1$ and $s = 0$. Then the analogy is inherited (see C3b(p.148)).
   2. Let $\beta < 1$ or $s > 0$. Then analogy is *may* collapses (see C3c(p.148)).

b. For Model 2:
   1. For any $\beta \leq 1$ and $s \geq 0$, the analogy *may* collapse (see C3a(p.219)).

C4. Optimal Initiating Time (OIT)

On $\mathscr{F}^+$:

a. Let $\beta = 1$ and $s = 0$.
   1. For Model 1, only Ⓢ$_\blacktriangle$ is possible (see Tables 19.1.1(p.133) and 19.2.1(p.147)).
   2. For Model 2, only Ⓢ$_\blacktriangle$ and ⓓ$_{\|}$ are possible (see Tables 20.1.1(p.186) and 20.2.3(p.218)).
   3. What is remarkable here is that ⓓ$_{\|}$ (deadline-engulfing) occurs even in the simplest case of "$\beta = 1$ and $s = 0$" (see C4a(p.219)).

b. Let $\beta < 1$ or $s > 0$.
   1. For Model 1, Ⓢ$_\blacktriangle$, ◎$_{\|}$, and ⓓ$_{\|}$ are possible (see Tables 19.1.2(p.133) and 19.2.2(p.147)).
   2. For Model 2, Ⓢ$_\vartriangle$,Ⓢ$_\blacktriangle$, ◎$_{\|}$, ◎$_\vartriangle$, ◎$_\blacktriangle$, ⓓ$_{\|}$, ⓓ$_\vartriangle$, and ⓓ$_\blacktriangle$ are possible (see Tables 20.1.2(p.187) and 20.2.4(p.218)).

c. Joining Tables 19.3.1(p.148) and 20.3.1(p.219) produces the following table:

Table 22.0.1: Occurance rates of Ⓢ, ◎, and ⓓ on $\mathscr{F}$

Ⓢ			◎			ⓓ		
44.4 % / 68			22.2 % / 34			33.4 % / 52		
Ⓢ$_{\|}$	Ⓢ$_\vartriangle$	Ⓢ$_\blacktriangle$	◎$_{\|}$	◎$_\vartriangle$	◎$_\blacktriangle$	ⓓ$_{\|}$	ⓓ$_\vartriangle$	ⓓ$_\blacktriangle$
—	possible	possible	possible	possible	possible	possible	possible	possible
– %/ –	5.9 %/ 9	38.6 %/ 59	12.4 %/ 19	7.2 %/ 11	2.6 %/ 4	19.0 %/ 30	11.1 %/ 17	3.2 %/ 5

C5. Null-time-zone and deadline-engulfing

From Table 22.0.1(p.229) above, we see that on $\mathscr{F}$:

a. See Remark 7.2.2(p.43) for the implication of the symbol "▲" representing the strict optimality of the initiating time t^*_τ.

b. As a whole, Ⓢ, ◎, and ⓓ are possible at 44.4%, 22.2%, and 33.4% respectively where

1. Ⓢ$_{||}$ cannot be defined (see Remark **??**(p.??)).
2. ◎$_{||}$ is possible (12.4 %).
3. ⓓ$_{||}$ is possible (19.0 %).
4. Ⓢ$_{\triangle}$ never occur (5.9 %).
5. ◎$_{\triangle}$ is possible (7.2 %).
6. ⓓ$_{\triangle}$ is possible (11.1 %).
7. Ⓢ$_{\blacktriangle}$ is possible (38.6%),
8. ◎$_{\blacktriangle}$ is possible(2.6%).
   - Tom 19.2.2(p.135) (c2iii2)
   - Tom 20.2.2(p.189) (c3i2,c3ii1ii2,c3ii2i).
9. ⓓ$_{\blacktriangle}$ is possible (3.2%).
   - Tom 20.2.4(p.193) (d2i,d2ii).
   - Tom 20.2.16(p.203) (c2,c3i2,c3i3).

The following three are especially noteworthy findings:

A. ◎and ⓓcausing the *null-time-zone* occur at 55.6% (= 22.2% + 33.4%).

B. ⓓcausing the *deadline-engulfing* occurs at 33.4%.

C. ◎$_{\blacktriangle}$ and ⓓ$_{\blacktriangle}$ *strictly* causing the deadline-engulfing occurs at 2.6% and 3.2% respectively.

D. ⓓ$_{||}$ causing the *deadline-engulfing* occurs even in the simplest case of "$\beta = 1$ and $s = 0$" (see C4a3(p.229)).

C6. `C⇝S` (`Conduct⇝Skip`) (see Def. 2.2.1(p.12) and Remark 7.2.1(p.42))

It is *only* for M:2[ℝ][A]$^+$ and M:2[ℙ][A]$^+$ with $\beta < 1$ or $s > 0$ (see Table 20.1.4(p.188)) that we have observed `C⇝S`. It is usual to assume that once conducting a search is optimal, it will become optimal to continue conducting the search afterward. However, in this paper we demonstrated that this expectation does not always hold. In other words, it can become optimal to skip the search after having continued the search for a while.

■ **Models 3**

C7. Reduction (see Section 21.4(p.227))

Model 3 is reduced to either of the following two cases (see Conclusions 5 (p.224) and 6 (p.227)):

a. M/M̃:3[ℝ/ℙ][A/E] ↠ M/M̃:2[ℝ/ℙ][A/E] (`model-running-back`).

b. `odr` ↦ `Accept`$_{\tau \geq t \geq 0}(\rho)$ ▷ `Stop` (`odr-reduction`).

Part 4

Conclusions and Future Studies

In Chap. 23(p.233) the main points of the conclusions that were obtained in Parts 1,2, and 3 are summarized and in Chap. 24(p.235) subjects of future studies are listed.

Chapter 23

Overall Conclusion

Below are the essential points distilled from the conclusions which were summarized in Chaps. 8(p.47), 18(p.127), and 22(p.229).

C1. **Two Motives**

This study was triggered by the following two naive motives (see Section 1.2(p.4)):

Motive 1: Is a buying problem always symmetrical to a selling problem?

Motive 2: Does a general theory integrating quadruple-asset-trading-problems exist?

C2. **Philosophical Background**

On *March 31, 1965*, the original theme of this paper was proposed by the academic supervisor Prof. Shizuo Senju who has PhD (Eng.) (see the episode in Section 1.3.2(p.4)). Enlightened by his thought background, before long I (Ikuta, the first author of this paper) obtained PhD (Eng.) under his research guidance, and more than 20 years later since then, also Kang (the second author of this paper) obtained PhD (Mgt. Sci.&Eng.) under my research guidance. In time, we, who have the same *physical recognition*, found ourself down the middle of the philosophy of "decision theory as physics" (see Section 1.3.3(p.4)). This background exerted considerable influence on the whole writing of this paper — It is not an exaggeration that this study would never complete without it. However, we should never forget that the following two temptations always hide there.

a. Knowledge of Ignorance

First, let us confirm that a physical perspective is to observe things as they happen, which stems from a mental process involving unfiltered observation of things, which is free from any preconceived premises, assumptions, hypotheses, biases, and so on. Looking back at the history of science, we will immediately see that this task was very often quite ungovernably difficult. For instance, prior to Galileo's era (pre-1600s), no one would have questioned the belief that the heaven revolves around the Earth (Ptolemaic system). However, it should be registered that it took thousands of years to acknowledge the transition to the sun-centered theory (Copernican system) and that even in this modern age a situation is not so different from around that time in the sense that very many things still remain not-yet-unknown also at present. Taking into account this fact, we should not forget "the knowledge of ignorance" at any time. Without this recognition, even modern people could repeat the same mistake, not only in the natural science but also in the business and economic sciences.

b. Overconfidence in Mathematics

Next, what should be kept in mind is that mathematics-oriented researchers who get fixated on the conviction of "the total truth of this world is completely included within the truth of mathematics" exist at considerable ratio. Those familiar with physics will quickly grasp the essence of the cautionary word of Albert Einstein "*As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality*". However, for those without this experience (physics), the understanding of his apothegm may require significant time or might never fully materialize. Recall here historical cases in physics that, even if being how highly mathematical elaborated theory, bases of conventional principles can be drastically rewritten by only one simple discovery from an observance using a cheap experimental instrument and by only one elementary result from an experiment conducted in a small laboratory.

C3. **Structured-Unit-of-Problems**

In this paper we defined the *quadruple-asset-trading-problems* (see Section 1.4.5(p.7)) and the *structured-unit-of-problems* (see Section 3.3(p.18)). Our main concern is not to *one-by-one* and *independently* treat these problems but to clarify the *interconnectedness* among them.

C4. **Integrated Theory**

In this paper we succeeded in constructing the theory which integrates the *quadruple-asset-trading-problems*. In this theory we first derived the two symmetry theorems, Theorems 12.5.1(p.78) and 12.8.1(p.85), which connect a selling problem with $\mathbb{R}$-mechanism and a buying problem with $\mathbb{R}$-mechanism through the operation $\mathcal{S}_{\mathbb{R}\to\tilde{\mathbb{R}}}$ defined by (12.5.29(p.75)), called the *symmetry transformation operation*. On the other hand, at the earlier stage of this study we did not anticipate at all the existence of a relationship between trading problem with $\mathbb{R}$-mechanism and trading problem with $\mathbb{P}$-mechanism. However, through countless arrangements and rearrangements, as if solving a jigsaw puzzle, we noticed similarities between the above two problems and finally reached the two lemmas, Lemmas 10.1.1(p.53) and 13.2.1(p.91), which are connected by the operation $\mathcal{A}_{\mathbb{R}\to\mathbb{P}}$ defined by (13.2.1(p.91)), called the *analogy replacement operation*. This finding led to the derivation of the two analogy theorems, Theorems 13.3.1(p.96) and 13.3.2(p.96), which combine the above two problems. Through the long discussions over more than twenty years, we finally reached the integrated theory, schematized by the quadrangular bi-directional connection of the above four theorems (see Figure 16.2.1(p.113)). Thus, it follows that the answer to the question in Motive 2(p.4) is "Yes!".

C5. **Collapse of Symmetry and Analogy**

When we started this study, we were grappling with a conflict between mathematical thinking and physical thinking; "Should the price ξ be defined on which of $(-\infty,\infty)$ and $(0,\infty)$". It goes without saying that defining on $(-\infty,\infty)$ makes the mathematical treatment easier than on $(0,\infty)$, so we tried to construct the integrated theory on $(-\infty,\infty)$ and fortunately succeeded in the construction of the integrated theory under this premise. However, the price ξ should be defined on $(0,\infty)$ in the usual transaction market of the actual world so that a negative price does not occur. Then, we brought the solution to this problem by formulating the methodology of transforming results obtained on $(-\infty,\infty)$ into ones on $(0,\infty)$, i.e., the market restriction (see Chap. 17(p.115)). However, the market restriction naturally leads to the possibility that the symmetry and analogy which are guaranteed under the integrated theory constructed on $(-\infty,\infty)$ may collapse. In Parts 3(p.129) we demonstrated that the collapse can occur in fact at significant frequency. Thus, it follows that the answer to the question in Motive 1(p.4) is "No!". For more detailed results of the inheritance and collapse of symmetry and analogy.

C6. **Null-Time-Zone and Deadline-Engulfing**

Our physical recognition (see C2(p.233)) led to the time concepts of *recognizing time*, *starting time*, *initiating time*, *stopping time*, and *deadline* (see H1(p.8) and Section 7.1(p.41)), and the concept of the "*initiating time*" further inevitably yields the concept of "*optimal initiating time*" (see (7.2.5(p.43))). Then, we found out that there exists the three types of optimal initiating time, the *starting time* (Ⓢ), *non-degenerate time* (◎), and the *deadline* (ⓓ) (see Section 7.2.4.3(p.43)), and that ◎ and ⓓ yielded the two unfamiliar phenomena, *null-time-zone* (see Section 7.2.4.4(p.44)) and *deadline-engulfing* (see Section 7.2.4.5(p.44) and Alice 3(p.44)). The two phenomena are the most significant discoveries in this paper in the sense that they strongly press for the comprehensive re-examination and rewriting of almost all results obtained in conventional researches in which the concept of initiating time has not been introduced. Now, we see that ◎ and ⓓ causing the above two singular properties are not rare (seeTable 22.0.1(p.229)); in fact it can occur at the rather high occurrence rates of 22.2% and 33.4% respectively. What is furthermore amazing is that the strictly optimal initiating times, ◎$_\blacktriangle$ and ⓓ$_\blacktriangle$, are possible although at the very small occurrence rates of 2.6% and 3.2% (see *Example* 7.2.1(p.44) and C5C(p.230)). What is moreover striking is that both ◎$_\blacktriangle$ and ⓓ$_\blacktriangle$ are possible even in the simplest case of $\beta = 0$ and $s = 0$ (see *Example* 7.2.2(p.44) and C5D(p.230)).

C7. **Others**

a. Mental Conflict

It is only for Model 2 with $\beta < 1$ or $s > 0$ that we have the abnormal mental conflict (see C1b2(p.229)). This was an unpredictable phenomenon at the beginning of this study.

b. `C⇝S (Conduct⇝Skip)`

It is *only* for M:2[ℝ/ℙ][`A`]$^+$ with $\beta < 1$ or $s > 0$ that `C⇝S` is possible (see C6(p.230)). This phenomenon was also what is not predictable at the beginning of this study.

c. Reduction (see C7(p.230))

Model 3 is reduced to either of the following two cases:

1. M/M̃:3[ℝ/ℙ][`A/E`] ↠ M/M̃:2[ℝ/ℙ][`A/E`] (`model-running-back`), i.e., Model 3 is reduced to Model 2. This reduction implies that all discussions for Model 3 can be reduced to those of Model 2 which have already been completed in Chap. 19(p.131), hence it follows that it becomes unnecessary to discuss any more for Model 3.
2. `odr` ↦ `Accept`$_{\tau\geq t\geq 0}(\rho)$ ▷ `Stop` (`odr-reduction`), i.e., the process stops by accepting the intervening quitting price ρ.

Chapter 24

Future Studies

In the final chapter we present the subjects of study to be tackled in the future. In the present paper we examined the 24 no-recall-models (see Table 3.2.1(p.17) and Chap. 4(p.19)). For these models we can consider the following different variations.

V1. *Limited search budget* [6,Ikuta] This model involves a limited total budget that can be allocated for search activities. The challenge lies in determining how to distribute this limited budget among search activities at every time point throughout the planning horizon.

V2. *Price mechanism switching* [2,Ee] [3,Ee] This model allows for the switching of price mechanisms between $\mathbb{R}$-mechanism and $\mathbb{P}$-mechanism (see Section 1.1(p.3)) at each time point during the planning horizon.

V3. *Several search areas* [7,Ikuta] For instance, consider Tokyo, Kyoto, and Osaka as potential areas where the leading-trader can search for counter-traders. If the leading-trader is in Tokyo today, the decision arises tomorrow whether to stay in Tokyo or to move to which of Kyoto and Osaka.

V4. *Uncertain deadline* [4,Ee] In *Example* 1.4.1(p.6), the return home date is not yet definite; it could be imminent or one week later, or the directive itself might be rescinded.

The four variations posed in the above references are all *basic models*. Then we can consider the structured-unit-of-models (see Section 3.3(p.18)) corresponding to each basic model. Since each structured-unit-of-models consists of 24 models, it follows that we have $96 = 4 \times 24$ in all. Furthermore, the following different mixed variations can be considered:

- Model with several search areas and limited search budget,
- Model with uncertain deadline and mechanism switching,
- Model with limited search budget, uncertain deadline, and mechanism switching,
- Model with several search areas, limited search budget, uncertain deadline, and mechanism switching,

$\vdots$

Taking into account the existence of these mixed variations, it follows that the number of models to be tackled will dramatically increases. In addition, the introduction of the concept of the optimal initiating time will make the study situation moreover complicated. In ⋆A10(p.13) we presented that the asset trading problems can be largely classified into the two types, "no-recall-model" and "recall-model". In the present paper we exclusively examined only the no-recall-model. The investigation for the recall-model is left as the subject of future study.

Appendix

A 1 Direct Proof of Underlying Functions of Type $\mathbb{R}$

In this appendix we provide the direct proofs for all lemmas in Section 12.6(p.78) in which they were proven by using Theorem 12.5.1(p.78) (symmetry theorem).

A 1.1 $\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$

For convenience of reference, below let us copy Lemma 12.6.1(p.79).

Lemma A 1.1 ($\mathscr{A}\{\tilde{T}_{\mathbb{R}}\}$) *For any $F \in \mathscr{F}$:*

(a) *$\tilde{T}(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$\tilde{T}(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$.*
(d) *$\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(e) *$\tilde{T}(x) + x$ strictly increasing on $(-\infty, b]$.*
(f) *$\tilde{T}(x) = \mu - x$ on $[b, \infty)$ and $\tilde{T}(x) < \mu - x$ on $(-\infty, b)$.*
(g) *$\tilde{T}(x) < 0$ on (a, ∞) and $\tilde{T}(x) = 0$ on $(-\infty, a]$.*
(h) *$\tilde{T}(x) \le \min\{0, \mu - x\}$ on $x \in (-\infty, \infty)$.*
(i) *$\tilde{T}(0) = 0$ if $a > 0$ and $\tilde{T}(0) = \mu$ if $b < 0$.*
(j) *$\beta\tilde{T}(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*
(k) *$\beta\tilde{T}(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*
(l) *If $x > y$ and $b > y$, then $\tilde{T}(x) + x > \tilde{T}(y) + y$.*
(m) *$\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s$ is nondecreasing in s and is strictly increasing in s if $\lambda\beta < 1$.*
(n) *$b > \mu$.* □

● *Proof* First, for any x and y let us prove the following two inequalities:

$$-(x-y)F(y) \ge \tilde{T}(x) - \tilde{T}(y) \ge -(x-y)F(x) \cdots \mathbf{(1)},$$

$$(x-y)(1-F(y)) \ge \tilde{T}(x) + x - \tilde{T}(y) - y \ge (x-y)(1-F(x)) \cdots \mathbf{(2)}.$$

Then, let $\tilde{T}(x,y) \overset{\text{def}}{=} \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)]$ for any x and y.[‡] Since $1 \ge I(\boldsymbol{\xi} < y) \ge 0$ and since $\min\{\boldsymbol{\xi} - x, 0\} \le 0$ and $\min\{\boldsymbol{\xi} - x, 0\} \le \boldsymbol{\xi} - x$, we have $\min\{\boldsymbol{\xi} - x, 0\} \le \min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} < y) \le (\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)$, hence from (5.1.11(p.23)) we get $\tilde{T}(x) \le \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)] = \tilde{T}(x,y)$. Accordingly, for any x and y we have

$$\tilde{T}(x) - \tilde{T}(y) \le \tilde{T}(x,y) - \tilde{T}(y) = \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} < y)] - \mathbf{E}[(\boldsymbol{\xi} - y)I(\boldsymbol{\xi} < y)] = -(x-y)\,\mathbf{E}[I(\boldsymbol{\xi} < y)].$$

Since $I(\boldsymbol{\xi} \ge y) + I(\boldsymbol{\xi} < y) = 1$, we have $\tilde{T}(x) - \tilde{T}(y) \le -(x-y)(\mathbf{E}[1 - I(\boldsymbol{\xi} \ge y)]) = -(x-y)(1 - \mathbf{E}[I(\boldsymbol{\xi} \ge y)])$. Then, since

$$\mathbf{E}[I(\boldsymbol{\xi} \ge y)] = \int_{-\infty}^{\infty} I(\xi \ge y)f(\xi)d\xi = \int_y^{\infty} 1 \times f(\xi)d\xi = \int_y^{\infty} f(\xi)d\xi = \Pr\{\boldsymbol{\xi} > y\} = 1 - \Pr\{\boldsymbol{\xi} \le y\} = 1 - F(y),$$

[‡] If a given statement S is true, then $I(S) = 1$, or else $I(S) = 0$.

we have $\tilde{T}(x) - \tilde{T}(y) \le -(x-y)F(y)$, hence the far left inequality of (1) holds. Multiplying both sides of the inequality by -1 leads to $-\tilde{T}(x) + \tilde{T}(y) \ge (x-y)F(y)$ or equivalently $\tilde{T}(y) - \tilde{T}(x) \ge -(y-x)F(y)$. Then, interchanging the notations x and y yields $\tilde{T}(x) - \tilde{T}(y) \ge -(x-y)F(x)$, hence the far right inequality of (1) holds. (2) is immediate from adding $x-y$ to (1). Let us note here that $\tilde{T}(x)$ defined by (5.1.11(p.23)) can be rewritten as

$$\begin{aligned}\tilde{T}(x) &= \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(b \ge \boldsymbol{\xi})] + \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} > b)]. \cdots \mathbf{(3)} \\ &= \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(\boldsymbol{\xi} \ge a)] + \mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(a > \boldsymbol{\xi})]. \cdots \mathbf{(4)}.\end{aligned}$$

(a,b) Immediate from the fact that $\min\{\boldsymbol{\xi} - x, 0\}$ is continuous and nonincreasing in $x \in (-\infty, \infty)$ for any given $\boldsymbol{\xi}$.

(c) Let $x > y > a$. Then, since $-(x-y) < 0$ and $F(y) > 0$ due to (2.2.1 (2,3) (p.13)), we have $-(x-y)F(y) < 0$, hence $0 > \tilde{T}(x) - \tilde{T}(y)$ from (1), i.e., $\tilde{T}(y) > \tilde{T}(x)$, so $\tilde{T}(x)$ is *strictly* decreasing on $(a, \infty) \cdots \mathbf{(5)}$. Suppose $\tilde{T}(a) = \tilde{T}(x)$ for any $x > a$, hence $x - a > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $x - a > 2\varepsilon > 0$ we have $a < a + \varepsilon < x - \varepsilon < x$, hence $\tilde{T}(a) = \tilde{T}(x) < \tilde{T}(a+\varepsilon) \le \tilde{T}(a)$ due to (5) and (b), which is a contradiction. Thus it must be that $\tilde{T}(a) \ne \tilde{T}(x)$ for any $x > a$, i.e., $\tilde{T}(a) > \tilde{T}(x)$ or $\tilde{T}(a) < \tilde{T}(x)$ for any $x > a$. Since the latter is impossible due to (b), it follows that $\tilde{T}(a) > \tilde{T}(x)$ for any $x > a$. From this and (5) it eventually follows that $\tilde{T}(x)$ is strictly decreasing on $[a, \infty)$ instead of (a, ∞).

(d) Evident from the fact that $\tilde{T}(x) + x = \mathbf{E}[\min\{\boldsymbol{\xi}, x\}]$ from (5.1.11(p.23)) and that $\min\{\boldsymbol{\xi}, x\}$ is nondecreasing in x for any $\boldsymbol{\xi}$.

(e) Let $b > x > y$, hence $F(x) < 1$ due to (2.2.1 (1,2) (p.13)). Then, since $(x-y)(1-F(x)) > 0$, we have $\tilde{T}(x) + x > \tilde{T}(y) + y$ from (2), i.e., $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b) \cdots \mathbf{(6)}$. Suppose $\tilde{T}(b) + b = \tilde{T}(x) + x$ for any $x < b$. Then, for any sufficiently small $\varepsilon > 0$ such that $b - x > \varepsilon$ we have $x < x + \varepsilon < b$, hence $\tilde{T}(b) + b = \tilde{T}(x) + x < \tilde{T}(x+\varepsilon) + x + \varepsilon \le \tilde{T}(b) + b$ due to (6) and (d), which is a contradiction. Thus, $\tilde{T}(x) + x \ne \tilde{T}(b) + b$ for $x < b$, i.e., $\tilde{T}(x) + x > \tilde{T}(b) + b$ or $\tilde{T}(x) + x < \tilde{T}(b) + b$ for $x < b$. Since the former is impossible due to (d), it must be that $\tilde{T}(x) + x < \tilde{T}(b) + b$ for $x < b$. From this and (6) it follows that $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b]$.

(f) Let $x \ge b$. If $b \ge \boldsymbol{\xi}$, then $x \ge \boldsymbol{\xi}$, hence $\min\{\boldsymbol{\xi} - x, 0\} = \boldsymbol{\xi} - x$, and if $\xi > b$, then $f(\xi) = 0$ due to (2.2.3 (3) (p.13)). Thus, from (3) we have $\tilde{T}(x) = \mathbf{E}[(\boldsymbol{\xi} - x)I(b \ge \boldsymbol{\xi})] + 0 = \mathbf{E}[(\boldsymbol{\xi} - x)I(b \ge \boldsymbol{\xi})] + \mathbf{E}[(\boldsymbol{\xi} - x)I(\boldsymbol{\xi} > b)] = \mathbf{E}[(\boldsymbol{\xi} - x)(I(b \ge \boldsymbol{\xi}) + I(\boldsymbol{\xi} > b))] = \mathbf{E}[\boldsymbol{\xi} - x] = \mu - x$,[†] hence the former half is true. Then, since $\tilde{T}(b) = \mu - b$ or equivalently $\tilde{T}(b) + b = \mu$, if $b > x$, from (e) we have $\tilde{T}(x) + x < \tilde{T}(b) + b = \mu$, hence $\tilde{T}(x) < \mu - x$, so the latter half is true.

(g) Let $a \ge x$. If $\boldsymbol{\xi} \ge a$, then $\boldsymbol{\xi} \ge x$, hence $\min\{\boldsymbol{\xi} - x, 0\} = 0$ and if $a > \boldsymbol{\xi}$, then $f(\boldsymbol{\xi}) = 0$ due to (2.2.3 (1) (p.13)), hence $\mathbf{E}[\min\{\boldsymbol{\xi} - x, 0\}I(a > \boldsymbol{\xi})] = 0$. Accordingly, we have $\tilde{T}(x) = 0$ from (4), hence the latter half is true. Let $x > a$. Then, since $\tilde{T}(x) < \tilde{T}(a)$ from (c) and since $\tilde{T}(a) = 0$ from the fact stated just above, we have $\tilde{T}(x) < 0$ for $x > a$, hence the former half is true.

(h) From (f) we have $\tilde{T}(x) \le \mu - x$ for any x and from (g) we have $\tilde{T}(x) \le 0$ for any x, thus it follows that $\tilde{T}(x) \le \min\{0, \mu - x\}$ for any x.

(i) From (5.1.11(p.23)) we have $\tilde{T}(0) = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}] = \mathbf{E}[\min\{\boldsymbol{\xi}, 0\}I(a \le \boldsymbol{\xi} \le b)]$. If $a > 0$, then $0 \le \boldsymbol{\xi}$, hence $\min\{\boldsymbol{\xi}, 0\} = 0$, so $\tilde{T}(0) = \mathbf{E}[0] = 0$, and if $b < 0$, then $\boldsymbol{\xi} < 0$, hence $\min\{\boldsymbol{\xi}, 0\} = \boldsymbol{\xi}$, so $\tilde{T}(0) = \mathbf{E}[\boldsymbol{\xi}] = \mu$.

(j) If $\beta = 1$, then $\beta\tilde{T}(x) + x = \tilde{T}(x) + x$, hence the assertion is true from (d).

(k) Since $\beta\tilde{T}(x) + x = \beta(\tilde{T}(x) + x) + (1-\beta)x$, if $\beta < 1$, then $(1-\beta)x$ is strictly increasing in x, hence the assertion is true from (d).

(l) Let $x > y$ and $b > y$. If $x \ge b$, then $\tilde{T}(x) + x \ge \tilde{T}(b) + b > \tilde{T}(y) + y$ due to (d,e), and if $b > x$, then $b > x > y$, hence $\tilde{T}(x) + x > \tilde{T}(y) + y$ due to (e).

(m) From (5.1.11(p.23)) we have

$$\begin{aligned}\lambda\beta\tilde{T}(\lambda\beta\mu + s) + s &= \lambda\beta\,\mathbf{E}[\min\{\boldsymbol{\xi} - \lambda\beta\mu - s, 0\}] + s \\ &= \mathbf{E}[\min\{\lambda\beta\boldsymbol{\xi} - (\lambda\beta)^2\mu - \lambda\beta s, 0\}] + s \\ &= \mathbf{E}[\min\{\lambda\beta\boldsymbol{\xi} - (\lambda\beta)^2\mu + (1-\lambda\beta)s, s\}],\end{aligned}$$

which is nondecreasing in s and strictly increasing in s if $\lambda\beta < 1$.

(n) Evident from (2.2.2(p.13)). ∎

A 1.2 $\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$, $\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$, $\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$, and $\tilde{\kappa}_{\mathbb{R}}$

From (5.1.13(p.23)) and (5.1.14(p.23)) and from Lemma A 1.1(p.236) (f) we obtain, noting (10.2.1(p.54)),

$$\tilde{L}(x)\begin{cases} = \lambda\beta\mu + s - \lambda\beta x \ \text{ on } [b, -\infty) & \cdots(1), \\ < \lambda\beta\mu + s - \lambda\beta x \ \text{ on } (-\infty, b) & \cdots(2), \end{cases} \tag{A 1.1}$$

$$\tilde{K}(x)\begin{cases} = \lambda\beta\mu + s - \delta x \ \text{ on } [b, \infty) & \cdots(1), \\ < \lambda\beta\mu + s - \delta x \ \text{ on } (-\infty, b) & \cdots(2). \end{cases} \tag{A 1.2}$$

[†] $I(b \ge \boldsymbol{\xi}) + I(\boldsymbol{\xi} > b) = 1$.

In addition, from (5.1.14(p.23)) and Lemma A 1.1(p.236) (g) we have

$$\tilde{K}(x)\begin{cases} < -(1-\beta)x+s & \text{on} \quad (a,\infty) \quad \cdots(1), \\ = -(1-\beta)x+s & \text{on} \quad (-\infty,a] \quad \cdots(2), \end{cases} \tag{A 1.3}$$

hence we obtain

$$\tilde{K}(x)+x \le \beta x+s \quad \text{on} \quad (-\infty,\infty). \tag{A 1.4}$$

Then, from (A 1.2 (1) (p.237)) and (A 1.3 (2) (p.238)) we get

$$\tilde{K}(x)+x = \begin{cases} \lambda\beta\mu+s+(1-\lambda)\beta x \ \text{on} & [b,\infty) \quad \cdots(1), \\ \beta x+s \quad \text{on} & (-\infty,a] \quad \cdots(2). \end{cases} \tag{A 1.5}$$

Since $\tilde{K}(x)=\tilde{L}(x)-(1-\beta)x$ from (5.1.14(p.23)) and (5.1.13(p.23)), if $x_{\tilde{L}}$ and $x_{\tilde{K}}$ exist, then

$$\tilde{K}(x_{\tilde{L}}) = -(1-\beta)\,x_{\tilde{L}}\cdots(1), \qquad \tilde{L}(x_{\tilde{K}}) = (1-\beta)\,x_{\tilde{K}}\cdots(2). \tag{A 1.6}$$

Lemma A 1.2 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) *$\tilde{L}(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$\tilde{L}(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$\tilde{L}(x)$ is strictly decreasing on $[a,\infty)$.*
(d) *Let $s=0$. Then $x_{\tilde{L}}=a$ where $x_{\tilde{L}}<(\ge)\,x \Leftrightarrow \tilde{L}(x)<(=)\,0 \Rightarrow \tilde{L}(x)<(\ge)\,0$.*
(e) *Let $s>0$.*

1. *$x_{\tilde{L}}$ uniquely exists with $x_{\tilde{L}}>a$ where $x_{\tilde{L}}<(=(>))\,x \Leftrightarrow \tilde{L}(x)<(=(>))\,0$.*
2. *$(\lambda\beta\mu+s)/\lambda\beta \ge (<)\,b \Leftrightarrow x_{\tilde{L}}=(<)\,(\lambda\beta\mu+s)/\lambda\beta \ge (<)\,b$.* □

• *Proof* (a-c) Immediate from (5.1.13(p.23)) and Lemma A 1.1(p.236) (a-c).

(d) Let $s=0$. Then, since $\tilde{L}(x)=\lambda\beta\tilde{T}(x)$, from Lemma A 1.1(p.236) (g) we have $\tilde{L}(x)=0$ for $a\ge x$ and $\tilde{L}(x)<0$ for $x>a$, hence $x_{\tilde{L}}=a$ by the definition of $x_{\tilde{L}}$ (see Section 5.2(p.25) (b)), so $x_{\tilde{L}}<(\ge)\,x \Rightarrow \tilde{L}(x)<(=)\,0$. The inverse is true by contraposition. In addition, since $\tilde{L}(x)=0 \Rightarrow \tilde{L}(x)\ge 0$, we have $\tilde{L}(x)<(=)\,0 \Rightarrow \tilde{L}(x)<(\ge)\,0$.

(e) Let $s>0$.

(e1) From (A 1.1 (1) (p.237)) and from $\lambda>0$ and $\beta>0$ we have $\tilde{L}(x)<0$ for a sufficiently large $x>0$ such that $x\ge b$. In addition, we have $\tilde{L}(a)=\lambda\beta\tilde{T}(a)+s=s>0$ from Lemma A 1.1(p.236) (g). Hence, from (c) it follows that $x_{\tilde{L}}$ uniquely exists. The inequality $x_{\tilde{L}}>a$ is immediate from $\tilde{L}(a)>0$ and (c). The latter half is evident.

(e2) If $(\lambda\beta\mu+s)/\lambda\beta \ge (<)\,b$, from (A 1.1(p.237)) we have $\tilde{L}((\lambda\beta\mu+s)/\lambda\beta)=(<)\,\lambda\beta\mu+s-\lambda\beta(\lambda\beta\mu+s)/\lambda\beta=0$, hence $x_{\tilde{L}}=(<)\,(\lambda\beta\mu+s)/\lambda\beta\ge(<)\,b$ from (e1). ∎

Corollary A 1.1 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}\}$)

(a) $x_{\tilde{L}}<(\ge)\,x \Leftrightarrow \tilde{L}(x)<(\ge)\,0$.
(b) $x_{\tilde{L}}\le(\ge)\,x \Rightarrow \tilde{L}(x)\le(\ge)\,0$. □

• *Proof* (a) "$\Rightarrow$" is immediate from Lemma A 1.2(p.238) (d,e1). "$\Leftarrow$" is evident by contraposition.

(b) Since $x_{\tilde{L}}<(\ge)\,x \Rightarrow \tilde{L}(x)<(\ge)\,0$ due to (a) and since $\tilde{L}(x)<(\ge)\,0 \Rightarrow \tilde{L}(x)\le(\ge)\,0$, we have $x_{\tilde{L}}<(\ge)\,x \Rightarrow \tilde{L}(x)\le(\ge)\,0$. In addition, if $x_{\tilde{L}}=x$, then $\tilde{L}(x)=\tilde{L}(x_{\tilde{L}})=0$ or equivalently $x_{\tilde{L}}=x\Rightarrow\tilde{L}(x)=0$, hence $x_{\tilde{L}}=x\Rightarrow\tilde{L}(x)\le 0$. Accordingly, it follows that $x_{\tilde{L}}\le(\ge)\,x\Rightarrow\tilde{L}(x)\le(\ge)\,0$. ∎

Lemma A 1.3 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) *$\tilde{K}(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$\tilde{K}(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$\tilde{K}(x)$ is strictly decreasing on $[a,\infty)$.*
(d) *$\tilde{K}(x)$ is strictly decreasing on $(-\infty,\infty)$ if $\beta<1$.*
(e) *$\tilde{K}(x)+x$ is nondecreasing on $(-\infty,\infty)$.*
(f) *$\tilde{K}(x)+x$ is strictly increasing on $(-\infty,\infty)$ if $\lambda<1$.*
(g) *$\tilde{K}(x)+x$ is strictly increasing on $(-\infty,b]$.*
(h) *If $x>y$ and $b>y$, then $\tilde{K}(x)+x>\tilde{K}(y)+y$.*
(i) *Let $\beta=1$ and $s=0$. Then $x_{\tilde{K}}=a$ where $x_{\tilde{K}}<(\ge)\,x \Leftrightarrow \tilde{K}(x)<(=)\,0 \Rightarrow \tilde{K}(x)<(\ge)\,0$.*
(j) *Let $\beta<1$ or $s>0$.*

1. *There uniquely exists $x_{\tilde{K}}$ where $x_{\tilde{K}}<(=(>))\,x \Leftrightarrow \tilde{K}(x)<(=(>))\,0$.*
2. *$(\lambda\beta\mu+s)/\delta \ge (<)\,b \Leftrightarrow x_{\tilde{K}}=(<)\,(\lambda\beta\mu+s)/\delta$.*
3. *Let $\tilde{\kappa}<(=(>))\,0$. Then $x_{\tilde{K}}<(=(>))\,0$.* □

• *Proof* (a-c) Immediate from (5.1.14(p.23)) and Lemma A 1.1(p.236) (a-c).

(d) Immediate from (5.1.14(p.23)) and Lemma A 1.1(p.236) (b).

(e) From (5.1.14(p.23)) we have

$$\tilde{K}(x) + x = \lambda\beta\tilde{T}(x) + \beta x + s = \lambda\beta(\tilde{T}(x) + x) + (1-\lambda)\beta x + s \cdots \textbf{(1)},$$

hence the assertion holds from Lemma A 1.1(p.236) (d).

(f) Obvious from (1) and Lemma A 1.1(p.236) (d).

(g) Clearly from (1) and Lemma A 1.1(p.236) (e).

(h) Let $x > y$ and $b > y$. If $x \geq b$, then $\tilde{K}(x) + x \geq \tilde{K}(b) + b > \tilde{K}(y) + y$ due to (e,g), and if $b > x$, then $b > x > y$, hence $\tilde{K}(x) + x > \tilde{K}(y) + y$ due to (g). Thus, whether $x \geq b$ or $b > x$, we have $\tilde{K}(x) + x > \tilde{K}(y) + y$

(i) Let $\beta = 1$ and $s = 0$. Then, since $\tilde{K}(x) = \lambda\tilde{T}(x)$ due to (5.1.14(p.23)), from Lemma A 1.1(p.236) (g) we have $\tilde{K}(x) = 0$ for $a \geq x$ and $\tilde{K}(x) < 0$ for $x > a$, so $x_{\tilde{K}} = a$ by the definition of $x_{\tilde{K}}$ (see Section 5.2(p.25) (b)). Hence $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (=)\ 0$. The inverse holds by contraposition. In addition, since $\tilde{K}(x) = 0 \Rightarrow \tilde{K}(x) \geq 0$, we have $\tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) This proof consists of the following six steps:

- First note (A 1.3 (2) (p.238)). If $\beta < 1$, then $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ with $x \geq a$ and if $s > 0$, then, whether $\beta < 1$ or $\beta = 1$, we have $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ with $x \leq a$. Hence, whether $\beta < 1$ or $s > 0$, we have $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ with $x \leq a$.
- Next note (A 1.2 (1) (p.237)). Then, since $\delta > 0$ from (10.2.2 (1) (p.54)), whether $\beta < 1$ or $s > 0$ we have $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$.
- Hence, whether $\beta < 1$ or $s > 0$, it follows that there exists the solution x_K.

◦ Let $\beta < 1$. Then, the solution $x_{\tilde{K}}$ is unique from (d).

◦ Let $s > 0$. If $\beta < 1$, the solution $x_{\tilde{K}}$ is unique for the reason just above. If $\beta = 1$, we have $\tilde{K}(a) = s > 0$ from (A 1.3 (2) (p.238)), hence $x_{\tilde{K}} > a$ due to (c), so $\tilde{K}(x)$ is strictly decreasing on the neighbourhood of $x = x_{\tilde{K}}$ due to (c), thus the solution $x_{\tilde{K}}$ is unique. Therefore, whether $\beta < 1$ or $\beta = 1$, it follows that the solution $x_{\tilde{K}}$ is unique.

◦ Hence, whether $\beta < 1$ or $s > 0$, it follows that the solution $x_{\tilde{K}}$ is unique.

From all the above, whether $\beta < 1$ or $s > 0$, it eventually follows that the solution $x_{\tilde{K}}$ uniquely exists.

(j2) Let $(\lambda\beta\mu + s)/\delta \geq (<)\ b$. Then, from (A 1.2 (1(2)) (p.237)) we have $\tilde{K}((\lambda\beta\mu + s)/\delta) = (<)\ \lambda\beta\mu + s - \delta(\lambda\beta\mu + s)/\delta = 0$, hence $x_{\tilde{K}} = (<)\ (\lambda\beta\mu + s)/\delta$ due to (j1). The inverse is true by contraposition.

(j3) If $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$ from (5.1.17(p.23)), hence $x_{\tilde{K}} < (= (>))\ 0$ from (j1). ▮

Corollary A 1.2 ($\mathscr{A}\{\tilde{K}_{\mathbb{R}}\}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0$.

(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. □

• *Proof* (a) Clearly $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to Lemma A 1.3(p.238) (i,j1). The inverse holds by contraposition.

(b) Since $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to (a) and since $\tilde{K}(x) < (\geq)\ 0 \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$, we have $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{K}} = x$, then $\tilde{K}(x) = \tilde{K}(x_{\tilde{K}}) = 0$ or equivalently $x_{\tilde{K}} = x \Rightarrow \tilde{K}(x) = 0$, hence $x_{\tilde{K}} = x \Rightarrow \tilde{K}(x) \leq 0$. Accordingly, it follows that $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. ▮

Lemma A 1.4 ($\mathscr{A}\{\tilde{L}_{\mathbb{R}}/\tilde{K}_{\mathbb{R}}\}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_{\tilde{L}} = x_{\tilde{K}} = a$.*

(b) *Let $\beta = 1$ and $s > 0$. Then $x_{\tilde{L}} = x_{\tilde{K}}$.*

(c) *Let $\beta < 1$ and $s = 0$. Then $a < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (=))\ 0$.*

(d) *Let $\beta < 1$ and $s > 0$. Then $\tilde{\kappa} < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (>))\ 0$.* □

• *Proof* (a) If $\beta = 1$ and $s = 0$, then $x_{\tilde{L}} = a$ from Lemma A 1.2(p.238) (d) and $x_{\tilde{K}} = a$ from Lemma A 1.3(p.238) (i), hence $x_{\tilde{L}} = x_{\tilde{K}} = a$.

(b) Let $\beta = 1$ and $s > 0$. Then $\tilde{K}(x_{\tilde{L}}) = 0$ from (A 1.6 (1) (p.238)), hence $x_{\tilde{K}} = x_{\tilde{L}}$ from Lemma A 1.3(p.238) (j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_{\tilde{L}} = a \cdots \textbf{(1)}$ from Lemma A 1.2(p.238) (d).

◦ If $a < 0$, then $x_{\tilde{L}} < 0$, hence $\tilde{K}(x_{\tilde{L}}) > 0$ from (A 1.6 (1) (p.238)), hence $x_{\tilde{L}} < x_{\tilde{K}}$ from Lemma A 1.3(p.238) (j1), and if $a = (>)\ 0$, then $x_{\tilde{L}} = (>)\ 0$, hence $\tilde{K}(x_L) = (<)\ 0$ from (A 1.6 (1) (p.238)), so $x_{\tilde{L}} = (>)\ x_{\tilde{K}}$ from Lemma A 1.3(p.238) (j1). Accordingly, we have "⇒" holds and its inverse "⇐" is immediate by contraposition. Thus the *first relation* "⇔" holds.

◦ If $a < 0$, from (5.1.17(p.23)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) < 0$ due to Lemma A 1.1(p.236) (g), hence $x_{\tilde{K}} < 0 \cdots \textbf{(2)}$ from Lemma A 1.3(p.238) (j1), and if $a = (>)\ 0$, from (5.1.17(p.23)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) = 0$ due to Lemma A 1.1(p.236) (g), hence $x_{\tilde{K}} = 0$ from Lemma A 1.3(p.238) (j1) or equivalently $x_{\tilde{K}} = (=)\ 0$. Accordingly, we have the *second relation* "⇒".

(d) Let $\beta < 1$ and $s > 0$. Now, since $\tilde{\kappa} = \tilde{K}(0)$ from (5.1.17(p.23)), if $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$, thus $x_{\tilde{K}} < (= (>))\ 0 \cdots (3)$ from Lemma A 1.3(p.238) (j1). Accordingly $\tilde{L}(x_{\tilde{K}}) < (= (>))\ 0$ from (A 1.6 (2) (p.238)), hence $x_{\tilde{L}} < (= (>)\ x_{\tilde{K}}$ from Lemma A 1.2(p.238) (e1). Thus "⇒" holds and its inverse "⇐" is immediate by contraposition. The last "⇒" is immediate from (3). ▮

Lemma A 1.5 ($\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{R}}\}$)

(a) *$\tilde{\mathcal{L}}(s)$ is nondecreasing in s and strictly increasing in s if $\lambda\beta < 1$.*

(b) *Let $\lambda\beta\mu \le a$.*

1. *$x_{\tilde{L}} \ge \lambda\beta\mu + s$.*
2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_{\tilde{L}} > \lambda\beta\mu + s$.*

(c) *Let $\lambda\beta\mu > a$. Then, there exists a $s_{\tilde{\mathcal{L}}} > 0$ such that if $s_{\tilde{\mathcal{L}}} > (\le)\ s$, then $x_{\tilde{L}} < (\ge)\ \lambda\beta\mu + s$.* □

• *Proof* (a) From (5.1.15(p.23)) and (5.1.13(p.23)) we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta\mu + s) = \lambda\beta\tilde{T}(\lambda\beta\mu + s) + s \cdots (1)$, hence the assertion holds from Lemma A 1.1(p.236) (m).

(b) Let $\lambda\beta\mu \le a$. Then, from (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta\mu) = 0 \cdots (2)$ due to Lemma A 1.1(p.236) (g).

(b1) Since $s \ge 0$, from (a) we have $\tilde{\mathcal{L}}(s) \ge \tilde{\mathcal{L}}(0) = 0$ due to (2) or equivalently $\tilde{L}(\lambda\beta\mu + s) \ge 0$ due to (1), hence $x_{\tilde{L}} \ge \lambda\beta\mu + s$ from Corollary A 1.1(p.238) (a).

(b2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (a) we have $\tilde{\mathcal{L}}(s) > \tilde{\mathcal{L}}(0) = 0 \cdots (3)$ due to (2) or equivalently $\tilde{L}(\lambda\beta\mu + s) > 0$, hence $x_{\tilde{L}} > \lambda\beta\mu + s$ from Lemma A 1.2(p.238) (e1).

(c) Let $\lambda\beta\mu > a$. From (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta\mu) < 0$ due to Lemma A 1.1(p.236) (g). Noting (A 1.1 (1) (p.237)), for any sufficiently large $s > 0$ such that $\lambda\beta\mu + s \ge b$ and $\lambda\beta\mu + s > 0$ we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta\mu + s) = \lambda\beta\mu + s - \lambda\beta(\lambda\beta\mu + s) = (1 - \lambda\beta)(\lambda\beta\mu + s) \ge 0$. Accordingly, due to (a) it follows that there exists the solution $s_{\tilde{\mathcal{L}}} > 0$ of $\tilde{\mathcal{L}}(s) = 0$. Then $\tilde{\mathcal{L}}(s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{\mathcal{L}}(s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$ or equivalently $\tilde{L}(\lambda\beta\mu + s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{L}(\lambda\beta\mu + s) \ge 0$ for $s \ge s_{\tilde{\mathcal{L}}}$. Hence, from Corollary A 1.1(p.238) (a) we get $x_{\tilde{L}} < \lambda\beta\mu + s$ for $s < s_{\tilde{\mathcal{L}}}$ and $x_{\tilde{L}} \ge \lambda\beta\mu + s$ for $s \ge s_{\tilde{\mathcal{L}}}$. ▮

Lemma A 1.6 ($\tilde{\kappa}_{\mathbb{R}}$) *We have:*

(a) *$\tilde{\kappa} = s$ if $a > 0$ and $\tilde{\kappa} = \lambda\beta\mu + s$ if $b < 0$.*

(b) *Let $\beta < 1$ or $s > 0$. Then $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{K}} < (= (>))\ 0$.* □

• *Proof* (a) Immediate from (5.1.16(p.23)) and Lemma A 1.1(p.236) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\tilde{\kappa} < (= (>))\ 0$, we have $\tilde{K}(0) < (= (>))\ 0$ from (5.1.17(p.23)), hence $x_{\tilde{K}} < (= (>))\ 0$ from Lemma A 1.3(p.238) (j3). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ▮

A 2 Direct Proof of Underlying Functions of Type $\mathbb{P}$

A 2.1 $\mathscr{A}\{T_{\mathbb{P}}\}$

For convenience of reference, below let us copy Lemma 13.2.1(p.91).

Lemma A 2.1 ($\mathscr{A}\{T_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a) *$T(x)$ is continuous on $(-\infty, \infty)$.*

(b) *$T(x)$ is nonincreasing on $(-\infty, \infty)$.*

(c) *$T(x)$ is strictly decreasing on $(-\infty, b]$.*

(d) *$T(x) + x$ is nondecreasing on $(-\infty, \infty)$.*

(e) *$T(x) + x$ is strictly increasing on $[a^\star, \infty)$.*

(f) *$T(x) = a - x$ on $(-\infty, a^\star]$ and $T(x) > a - x$ on $(a^\star, \infty)$.*

(g) *$T(x) > 0$ on $(-\infty, b)$ and $T(x) = 0$ on $[b, \infty)$.*

(h) *$T(x) \ge \max\{0, a - x\}$ on $(-\infty, \infty)$.*

(i) *$T(0) = a$ if $a^\star > 0$ and $T(0) = 0$ if $b < 0$.*

(j) *$\beta T(x) + x$ is nondecreasing on $(-\infty, \infty)$ if $\beta = 1$.*

(k) *$\beta T(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\beta < 1$.*

(l) *If $x < y$ and $a^\star < y$, then $T(x) + x < T(y) + y$.*

(m) *$\lambda\beta T(\lambda\beta a - s) - s$ is nonincreasing in s and strictly decreasing in s if $\lambda\beta < 1$.*

(n) *$a^\star < a$.* □

A 2.2 $\mathscr{A}\{L_{\mathbb{P}}\}$, $\mathscr{A}\{K_{\mathbb{P}}\}$, $\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$, and $\kappa_{\mathbb{P}}$

Noting Lemma A 2.1(p.240) (f), from (5.1.20(p.24)) and (5.1.21(p.24)) we obtain

$$L(x)\begin{cases} = \lambda\beta a - s - \lambda\beta x & \text{on } (-\infty, a^\star] \quad \cdots (1), \\ > \lambda\beta a - s - \lambda\beta x & \text{on } (a^\star, \infty) \quad \cdots (2), \end{cases} \tag{A 2.1}$$

$$K(x)\begin{cases} = \lambda\beta a - s - \delta x & \text{on } (-\infty, a^\star] \quad \cdots (1), \\ > \lambda\beta a - s - \delta x & \text{on } (a^\star, \infty) \quad \cdots (2). \end{cases} \tag{A 2.2}$$

In addition, from (5.1.21(p.24)) and Lemma A 2.1(p.240) (g) we have

$$K(x)\begin{cases} > -(1-\beta)x - s & \text{on } (-\infty, b) \quad \cdots (1), \\ = -(1-\beta)x - s & \text{on } [b, \infty) \quad \cdots (2), \end{cases} \tag{A 2.3}$$

from which we obtain

$$K(x) + x \geq \beta x - s \quad \text{on} \quad (-\infty, \infty). \tag{A 2.4}$$

Then, from (A 2.2 (1) (p.241)) and (A 2.3 (2) (p.241)) we get

$$K(x) + x = \begin{cases} \lambda\beta a - s + (1-\lambda)\beta x \text{ on } & (-\infty, a^\star] \quad \cdots (1), \\ \beta x - s & \text{on } [b, \infty) \quad \cdots (2). \end{cases} \tag{A 2.5}$$

Since $K(x) = L(x) - (1-\beta)x$ from (5.1.21(p.24)) and (5.1.20(p.24)), if x_L and x_K exist, then

$$K(x_L) = -(1-\beta)\, x_L \cdots (1), \quad L(x_K) = (1-\beta)\, x_K \cdots (2). \tag{A 2.6}$$

Lemma A 2.2 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$L(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$L(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$L(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *Let $s = 0$. Then $x_L = b$ where $x_L > (\leq)\, x \Leftrightarrow L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.*
(e) *Let $s > 0$.*

1. *x_L uniquely exists with $x_L < b$ where $x_L > (= (<))\, x \Leftrightarrow L(x) > (= (<))\, 0$.*
2. *$(\lambda\beta a - s)/\lambda\beta \leq (>)\, a^\star \Leftrightarrow x_L = (>)\, (\lambda\beta a - s)/\lambda\beta > (\leq)\, a^\star$.* □

● *Proof* (a-c) Immediate from (5.1.20(p.24)) and Lemma A 2.1(p.240) (a-c).

(d) Let $s = 0$. Then, since $L(x) = \lambda\beta T(x)$, from Lemma A 2.1(p.240) (g) we have $L(x) > 0$ for $x < b$ and $L(x) = 0$ for $b \leq x$, hence $x_L = b$ by the definition of x_L (see Section 5.2(p.25) (a)), thus $x_L > (\leq)\, x \Rightarrow L(x) > (=)\, 0$. The inverse is true by contraposition. In addition, since $L(x) = 0 \Rightarrow L(x) \leq 0$, we have $L(x) > (=)\, 0 \Rightarrow L(x) > (\leq)\, 0$.

(e) Let $s > 0$.

(e1) From (A 2.1 (1) (p.241)) and from $\lambda > 0$ and $\beta > 0$ we have $L(x) > 0$ for a sufficiently small $x < 0$ such that $x \leq a^\star$. In addition, we have $L(b) = \lambda\beta T(b) - s = -s < 0$ from Lemma A 2.1(p.240) (g). Hence, from (a,c) it follows that x_L uniquely exists. The inequality $x_L < b$ is immediate from $L(b) < 0$. The latter half is evident.

(e2) If $(\lambda\beta a - s)/\lambda\beta \leq (>)\, a^\star$, from (A 2.1 (1(2)) (p.241)) we have $L((\lambda\beta a - s)/\lambda\beta) = (>)\, \lambda\beta a - s - \lambda\beta(\lambda\beta a - s)/\lambda\beta = 0$, hence $x_L = (>)\, (\lambda\beta a - s)/\lambda\beta$ from (e1). ∎

Corollary A 2.1 ($\mathscr{A}\{L_{\mathbb{P}}\}$)

(a) *$x_L > (\leq)\, x \Leftrightarrow L(x) > (\leq)\, 0$.*
(b) *$x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$.* □

● *Proof* (a) "⇒" is immediate from Lemma A 2.2(p.241) (d,e2). "⇐" is evident by contraposition.

(b) Since $x_L > (\leq)\, x \Rightarrow L(x) > (\leq)\, 0$ due to (a) and since $L(x) > (\leq)\, 0 \Rightarrow L(x) \geq (\leq)\, 0$, we have $x_L > (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. In addition, if $x_L = x$, then $L(x) = L(x_L) = 0$ or equivalently $x_L = x \Rightarrow L(x) = 0$, hence $x_L = x \Rightarrow L(x) \geq 0$. Accordingly, it follows that $x_L \geq (\leq)\, x \Rightarrow L(x) \geq (\leq)\, 0$. ∎

Lemma A 2.3 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) *$K(x)$ is continuous on $(-\infty, \infty)$.*
(b) *$K(x)$ is nonincreasing on $(-\infty, \infty)$.*
(c) *$K(x)$ is strictly decreasing on $(-\infty, b]$.*
(d) *$K(x)$ is strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(e) *$K(x) + x$ is nondecreasing on $(-\infty, \infty)$.*
(f) *$K(x) + x$ is strictly increasing on $(-\infty, \infty)$ if $\lambda < 1$.*
(g) *$K(x) + x$ is strictly increasing on $[a^\star, \infty)$.*

(h) *If $x < y$ and $a^\star < y$, then $K(x) + x < K(y) + y$.*
(i) *Let $\beta = 1$ and $s = 0$. Then $x_K = b$ where $x_K > (\leq)\ x \Leftrightarrow K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.*
(j) *Let $\beta < 1$ or $s > 0$.*

1. *There uniquely exists x_K where $x_K > (= (<))\ x \Leftrightarrow K(x) > (= (<))\ 0$.*
2. *$(\lambda\beta a - s)/\delta \leq (>)\ a^\star \Leftrightarrow x_K = (>)\ (\lambda\beta a - s)/\delta$.*
3. *Let $\kappa > (= (<))\ 0$. Then $x_K > (= (<))\ 0$.* □

• *Proof* (a-c) Immediate from (5.1.21(p.24)) and Lemma A 2.1(p.240) (a-c).

(d) Immediate from (5.1.21(p.24)) and Lemma A 2.1(p.240) (b).

(e) From (5.1.21(p.24)) we have $K(x) + x = \lambda\beta T(x) + \beta x - s = \lambda\beta(T(x) + x) + (1-\lambda)\beta x - s \cdots$ **(1)**, hence the assertion holds from Lemma A 2.1(p.240) (d).

(f) Obvious from (1) and Lemma A 2.1(p.240) (d).

(g) Clearly from (1) and Lemma A 2.1(p.240) (e).

(h) Let $x < y$ and $a^\star < y$. If $x \leq a^\star$, then $K(x) + x \leq K(a^\star) + a^\star < K(y) + y$ due to (e,g). If $a^\star < x$, then $a^\star < x < y$, hence $K(x) + x < K(y) + y$ due to (g). Thus, whether $x \leq a^\star$ or $a^\star < x$, we have $K(x) + x < K(y) + y$

(i) Let $\beta = 1$ and $s = 0$. Then, since $K(x) = \lambda T(x)$ due to (5.1.21(p.24)), from Lemma A 2.1(p.240) (g) we have $K(x) = 0$ for $b \leq x$ and $K(x) > 0$ for $x < b$, so that $x_K = b$ due to the definition in Section 5.2(p.25) (a). Hence $x_K > (\leq)\ x \Rightarrow K(x) > (=)\ 0$. The inverse holds by contraposition. In addition, since $K(x) = 0 \Rightarrow K(x) \leq 0$, we have $K(x) > (=)\ 0 \Rightarrow K(x) > (\leq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) This proof consists of the following six steps:

- First note (A 2.3 (2) (p.241)). If $\beta < 1$, then $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$ and if $s > 0$, then, whether $\beta < 1$ or $\beta = 1$, we have $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$. Hence, whether $\beta < 1$ or $s > 0$, we have $K(x) < 0$ for any sufficiently large $x > 0$ with $x \geq b$.
- Next note (A 2.2 (1) (p.241)). Then, since $\delta > 0$ from (10.2.2 (1) (p.54)), whether $\beta < 1$ or $s > 0$ we have $K(x) > 0$ for any sufficiently small $x < 0$ with $x \leq a^\star$.
- Hence, whether $\beta < 1$ or $s > 0$, it follows that there exists the solution x_K.

∘ Let $\beta < 1$. Then, the solution x_K is unique from (d).

∘ Let $s > 0$. If $\beta < 1$, the solution x_K is unique for the reason just above. If $\beta = 1$, we have $K(b) = -s < 0$ from (A 2.3 (2) (p.241)), hence $x_K < b$ due to (c), so $K(x)$ is strictly decreasing on the neighbourhood of $x = x_K$ due to (c), thus the solution x_K is unique. Therefore, whether $\beta < 1$ or $\beta = 1$, it follows that the solution x_K is unique.

∘ Hence, whether $\beta < 1$ or $s > 0$, it follows that the solution x_K is unique.

From all the above, whether $\beta < 1$ or $s > 0$, it eventually follows that the solution x_K uniquely exists.

(j2) Let $(\lambda\beta a - s)/\delta \leq (>)\ a^\star$. Then, from (A 2.2 (1(2)) (p.241)) we have $K((\lambda\beta a - s)/\delta) = (>)\ \lambda\beta a - s - \delta(\lambda\beta a - s)/\delta = 0$, hence $x_K = (>)\ (\lambda\beta a - s)/\delta$ due to (j1). The inverse is true by contraposition.

(j3) If $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$ from (5.1.24(p.24)), hence $x_K > (= (<))\ 0$ from (j1). ∎

Corollary A 2.2 ($\mathscr{A}\{K_{\mathbb{P}}\}$)

(a) $x_K > (\leq)\ x \Leftrightarrow K(x) > (\leq)\ 0$.
(b) $x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. □

• *Proof* (a) Clearly $x_K > (\leq)\ x \Rightarrow K(x) > (\leq)\ 0$ due to Lemma A 2.3(p.241) (i,j1). The inverse holds by contraposition.

(b) Since $x_K > (\leq)\ x \Rightarrow K(x) > (\leq)\ 0$ due to (a) and since $K(x) > (\leq)\ 0 \Rightarrow K(x) \geq (\leq)\ 0$, we have $x_K > (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. In addition, if $x_K = x$, then $K(x) = K(x_K) = 0$ or equivalently $x_K = x \Rightarrow K(x) = 0$, hence $x_K = x \Rightarrow K(x) \geq 0$. Accordingly, it follows that $x_K \geq (\leq)\ x \Rightarrow K(x) \geq (\leq)\ 0$. ∎

Lemma A 2.4 ($\mathscr{A}\{L_{\mathbb{P}}/K_{\mathbb{P}}\}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_L = x_K = b$.*
(b) *Let $\beta = 1$ and $s > 0$. Then $x_L = x_K$.*
(c) *Let $\beta < 1$ and $s = 0$. Then $b > (= (<))\ 0 \Rightarrow x_L > (= (<))\ x_K > (= (=))\ 0$.*
(d) *Let $\beta < 1$ and $s > 0$. Then $\kappa > (= (<))\ 0 \Rightarrow x_L > (= (<))\ x_K > (= (<))\ 0$.* □

• *Proof* (a) If $\beta = 1$ and $s = 0$, then $x_L = b$ from Lemma A 2.2(p.241) (d) and $x_K = b$ from Lemma A 2.3(p.241) (i), hence $x_L = x_K = b$.

(b) Let $\beta = 1$ and $s > 0$. Then $K(x_L) = 0$ from (A 2.6 (1) (p.241)), hence $x_K = x_L$ from Lemma A 2.3(p.241) (j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_L = b \cdots$ **(1)** from Lemma A 2.2(p.241) (d).

∘ If $b > 0$, then $x_L > 0$, hence $K(x_L) < 0$ from (A 2.6 (1) (p.241)), so $x_L > x_K$ from Lemma A 2.3(p.241) (j1), and if $b = (<)\ 0$, then $x_L = (<)\ 0$, hence $K(x_L) = (>)\ 0$ from (A 2.6 (1) (p.241)), so $x_L = (<)\ x_K$ from Lemma A 2.3(p.241) (j1). Accordingly, we have "⇒" holds and its inverse "⇐" is immediate by contraposition. Thus the *first relation* "⇔" holds.

- If $b > 0$, from (5.1.24(p.24)) we have $K(0) = \lambda\beta T(0) > 0$ due to Lemma A 2.1(p.240) (g), hence $x_K > 0 \cdots (2)$ from Lemma A 2.3(p.241) (j1), and if $b = (<)\ 0$, from (5.1.24(p.24)) we have $K(0) = \lambda\beta T(0) = 0$ due to Lemma A 2.1(p.240) (g), hence $x_K = 0$ from Lemma A 2.3(p.241) (j1) or equivalently $x_K = (=)\ 0$. Accordingly, we have the *second relation* "$\Rightarrow$".

(d) Let $\beta < 1$ and $s > 0$. Now, from (5.1.24(p.24)) and (5.1.23(p.24)), if $\kappa > (= (<))\ 0$, then $K(0) > (= (<))\ 0$, thus $x_K > (= (<))\ 0$ from Lemma A 2.3(p.241) (j1). Accordingly $L(x_K) > (= (<))\ 0$ from (A 2.6 (2) (p.241)), hence $x_L > (= (<))\ x_K$ from Lemma A 2.2(p.241) (e1). ∎

Lemma A 2.5 ($\mathscr{A}\{\mathcal{L}_{\mathbb{P}}\}$)
(a) *$\mathcal{L}(s)$ is nonincreasing in s and is strictly decreasing in s if $\lambda\beta < 1$.*
(b) *Let $\lambda\beta a \geq b$.*
 1. *$x_L \leq \lambda\beta a - s$.*
 2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_L < \lambda\beta a - s$.*

(c) *Let $\lambda\beta a < b$. Then, there exists a $s_{\mathcal{L}} > 0$ such that if $s_{\mathcal{L}} > (\leq)\ s$, then $x_L > (\leq)\ \lambda\beta a - s$.* □

● *Proof* (a) From (5.1.22(p.24)) and (5.1.20(p.24)) we have $\mathcal{L}(s) = L(\lambda\beta a - s) = \lambda\beta T(\lambda\beta a - s) - s$, hence the assertion holds from Lemma A 2.1(p.240) (m).

(b) Let $\lambda\beta a \geq b$. Then, from (5.1.22(p.24)) and (5.1.20(p.24)) we have $\mathcal{L}(0) = L(\lambda\beta a) = \lambda\beta T(\lambda\beta a) = 0 \cdots (1)$ due to Lemma A 2.1(p.240) (g).

(b1) Since $s \geq 0$, from (a) we have $\mathcal{L}(s) \leq \mathcal{L}(0) = 0$ due to (1) or equivalently $L(\lambda\beta a - s) \leq 0$, hence $x_L \leq \lambda\beta a - s$ from Corollary A 2.1(p.241) (a).

(b2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (a) we have $\mathcal{L}(s) < \mathcal{L}(0) = 0$ due to (1) or equivalently $L(\lambda\beta a - s) < 0$, thus $x_L < \lambda\beta a - s$ from Lemma A 2.2(p.241) (e1).

(c) Let $\lambda\beta a < b$. From (5.1.22(p.24)) we have $\mathcal{L}(0) = \lambda\beta T(\lambda\beta a) > 0$ due to Lemma A 2.1(p.240) (g). Noting (A 2.1 (1) (p.241)), for any sufficiently large $s > 0$ such that $\lambda\beta a - s \leq a^\star$ and $\lambda\beta a - s < 0$ we have $\mathcal{L}(s) = L(\lambda\beta a - s) = \lambda\beta a - s - \lambda\beta(\lambda\beta a - s) = (1 - \lambda\beta)(\lambda\beta a - s) \leq 0$. Accordingly, due to (a) it follows that there exists the solution $s_{\mathcal{L}} > 0$ of $\mathcal{L}(s) = 0$. Then $\mathcal{L}(s) > 0$ for $s < s_{\mathcal{L}}$ and $\mathcal{L}(s) \leq 0$ for $s \geq s_{\mathcal{L}}$ or equivalently $L(\lambda\beta a - s) > 0$ for $s < s_{\mathcal{L}}$ and $L(\lambda\beta a - s) \leq 0$ for $s \geq s_{\mathcal{L}}$. Hence, from Corollary A 2.1(p.241) (a) we get $x_L > \lambda\beta a - s$ for $s < s_{\mathcal{L}}$ and $x_L \leq \lambda\beta a - s$ for $s \geq s_{\mathcal{L}}$. ∎

Lemma A 2.6 ($\mathscr{A}\{\kappa_{\mathbb{P}}\}$) *We have*:
(a) *$\kappa = \lambda\beta a - s$ if $a^\star > 0$ and $\kappa = -s$ if $b < 0$.*
(b) *Let $\beta < 1$ or $s > 0$, Then $\kappa > (= (<))\ 0 \Leftrightarrow x_K > (= (<))\ 0$.* □

● *Proof* (a) Immediate from (5.1.23(p.24)) and Lemma A 2.1(p.240) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\kappa > (= (<))\ 0$, we have $K(0) > (= (<))\ 0$ from (5.1.24(p.24)), hence $x_K > (= (<))\ 0$ from Lemma A 2.3(p.241) (j1). Thus "$\Rightarrow$" was proven. Its inverse "$\Leftarrow$" is immediate by contraposition. ∎

A 3 Direct Proof of Underlying Functions of $\tilde{\text{T}}$ype $\mathbb{P}$

A 3.1 $\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$

Lemma A 3.1
(a) *Let $x \leq a$. Then $z(x) = a$*
(b) *Let $a < x$. Then $a < z(x) < x$.*
(c) *$z(x) \leq b$ for any x.* □

● *Proof* (a) Let $x \leq a$. If $a < z \cdots$ (II), then $x < z$, hence $\tilde{p}(z)(z - x) > 0$ due to (5.1.41 (2) (p.25)), and if $z \leq a \cdots$ (I), then $\tilde{p}(z)(z - x) = 0$ due to (5.1.41 (1) (p.25)) (see Figure A 3.1(p.243) below). Hence $z(x) = a$ due to Def. 5.1.2(p.25).

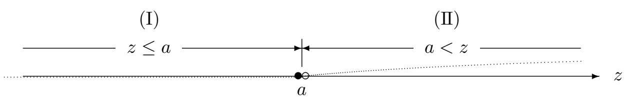


Figure A 3.1: Case $x \leq a$

(b) Let $a < x$. If $x \leq z \cdots$ (III), then $\tilde{p}(z)(z - x) \geq 0$, if $a < z < x \cdots$ (II), then $\tilde{p}(z)(z - x) < 0$ due to (5.1.41 (2) (p.25)), and if $z \leq a \cdots$ (I), then $\tilde{p}(z)(z - x) = 0$ due to (5.1.41 (1) (p.25)) (see Figure A 3.2(p.243) below). Hence, $z(x)$ is given by z on $a < z < x$, i.e., $a < z(x) < x$.

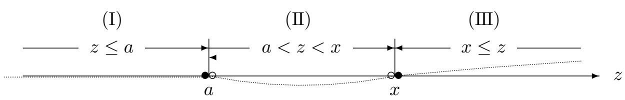


Figure A 3.2: Case $a < x$

(c) Assume that $z(x) > b$ for a certain x. Then, since $\tilde{p}(z(x)) = 1 = \tilde{p}(b)$ due to (5.1.42 (2) (p.25)), from (5.1.38(p.25)) we have $\tilde{T}(x) = z(x) - x > b - x = \tilde{p}(b)(b - x) \geq \tilde{T}(x)$, which is a contradiction. Hence, it must be that $z(x) \leq b$ for any x. ∎

Corollary A 3.1 *$a \leq z(x) \leq b$ for any x.* □

● *Proof* Evident from Lemma A 3.1(p.243). ∎

Lemma A 3.2 *$\tilde{p}(z)$ is nondecreasing on $(-\infty,\infty)$ and strictly increasing in $z \in [a,b]$.* □

● *Proof* The former half is immediate from (5.1.31(p.24)). For $a \leq z' < z \leq b$ we have $\tilde{p}(z) - \tilde{p}(z') = \Pr\{\boldsymbol{\xi} \leq z\} - \Pr\{\boldsymbol{\xi} \leq z'\} = \Pr\{z' < \boldsymbol{\xi} \leq z\} = \int_{z'}^{z} f(\xi)d\xi > 0$ (See (2.2.3 (2) (p.13))), hence $p(z) > p(z')$, i.e., $p(z)$ is strictly increasing on $[a,b]$. ▮

Lemma A 3.3 *$z(x)$ is nondecreasing on $(-\infty,\infty)$.* □

● *Proof* From (5.1.38(p.25)), for any x and y we have

$$\begin{aligned}
\tilde{T}(x) &= \tilde{p}(z(x))(z(x) - x)\\
&= \tilde{p}(z(x))(z(x) - y) - (x - y)\tilde{p}(z(x))\\
&\geq \tilde{T}(y) - (x - y)\tilde{p}(z(x))\\
&= \tilde{p}(z(y))(z(y) - y) - (x - y)\tilde{p}(z(x))\\
&= \tilde{p}(z(y))\big(z(y) - x + (x - y)\big) - (x - y)\tilde{p}(z(x))\\
&= \tilde{p}(z(y))(z(y) - x) + (x - y)(\tilde{p}(z(y)) - \tilde{p}(z(x)))\\
&\geq \tilde{T}(x) + (x - y)(\tilde{p}(z(y)) - \tilde{p}(z(x))).
\end{aligned}$$

Hence $0 \geq (x - y)(\tilde{p}(z(y)) - \tilde{p}(z(x)))$. Let $x > y$. Then $0 \geq \tilde{p}(z(y)) - \tilde{p}(z(x))$ or equivalently $\tilde{p}(z(x)) \geq \tilde{p}(z(y)) \cdots$ **(1)**. Since $a \leq z(x) \leq b$ and $a \leq z(y) \leq b$ from Corollary A 3.1(p.243), if $z(x) < z(y)$, then $\tilde{p}(z(x)) < \tilde{p}(z(y))$ from Lemma A 3.2(p.244), which contradicts (1). Hence, it must be that $z(x) \geq z(y)$, i.e., $z(x)$ is nondecreasing in $x \in (-\infty,\infty)$. ▮

Lemma A 3.4

(a) *$\tilde{T}(x)$ is continuous on $(-\infty,\infty)$.*
(b) *$\tilde{T}(x)$ is nonincreasing on $(-\infty,\infty)$.*
(c) *$\tilde{T}(x)$ is strictly decreasing on $[a,\infty)$.*
(d) *$\tilde{T}(x) < 0$ on (a,∞) and $\tilde{T}(x) = 0$ on $(-\infty,a]$.*
(e) *$\tilde{T}(x) \leq b - x$ on $(-\infty,\infty)$.*
(f) *$\tilde{T}(x) + x$ is nondecreasing on $(-\infty,\infty)$.*
(g) *$\beta\tilde{T}(x) + x$ is nondecreasing on $(-\infty,\infty)$ if $\beta = 1$.*
(h) *$\beta\tilde{T}(x) + x$ is strictly increasing on $(-\infty,\infty)$ if $\beta < 1$.*
(i) *$\tilde{T}(x) \leq \min\{0, b - x\}$ for any $x \in (-\infty,\infty)$.*
(j) *$\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ is nondecreasing in s and is strictly increasing in s if $\lambda\beta < 1$.* □

● *Proof* (a,b) Immediate from the fact that $\tilde{p}(z)(z - x)$ in (5.1.32(p.24)) is continuous and nonincreasing in $x \in (-\infty,\infty)$ for any z.

(c) Let $x' > x > a$. Then $z(x) > a$ from Lemma A 3.1(p.243) (b). Accordingly, since $\tilde{p}(z(x)) > 0$ due to (5.1.41 (2) (p.25)) and since $z(x) - x > z(x) - x'$, from (5.1.38(p.25)) we have $\tilde{T}(x) = \tilde{p}(z(x))(z(x) - x) > \tilde{p}(z(x))(z(x) - x') \geq \tilde{T}(x')$, i.e., $\tilde{T}(x)$ is *strictly decreasing* on $(a,\infty) \cdots$ **(1)**. Assume $\tilde{T}(a) = \tilde{T}(x)$ for a given $x > a$, so $x - a > 0$. Then, for any sufficiently small $\varepsilon > 0$ such that $x - a > 2\varepsilon > 0$ we have $a < a + \varepsilon < x - \varepsilon < x$, hence $\tilde{T}(a) = \tilde{T}(x) < \tilde{T}(a + \varepsilon) \leq \tilde{T}(a)$ due to the *strict* unceasingness shown just above and the nonincreasingness in (b), which is a contradiction. Thus, since $\tilde{T}(x) \neq \tilde{T}(a)$ for any $x > a$, we have $\tilde{T}(x) < \tilde{T}(a)$ or $\tilde{T}(x) > \tilde{T}(a)$ for any $x > a$. However, the latter is impossible due to (b), hence only the former holds, i.e., $\tilde{T}(x) < \tilde{T}(a)$ for any $x > a$. From this and (1) it eventually follows that $\tilde{T}(x)$ is strictly decreasing on $[a,\infty)$ instead of on (a,∞).

(d) Let $x \leq a$. Then, since $z(x) = a$ from Lemma A 3.1(p.243) (a), we have $\tilde{p}(z(x)) = 0$ due to (5.1.41 (1) (p.25)), hence $\tilde{T}(x) = \tilde{p}(z(x))(z(x) - x) = 0$ on $(-\infty,a]$, so $\tilde{T}(a) = 0$. Let $x > a$. Then, from (c) we have $\tilde{T}(x) < \tilde{T}(a) = 0$, i.e., $\tilde{T}(x) < 0$ on (a,∞).

(e) From (5.1.32(p.24)) and (5.1.42 (2) (p.25)) we see that $\tilde{T}(x) \leq \tilde{p}(b)(b - x) = b - x$ for any x on $(-\infty,\infty)$.

(f) For $x' < x$ we have, from (5.1.38(p.25)),

$$\begin{aligned}
\tilde{T}(x) + x &= \tilde{p}(z(x))(z(x) - x) + x\\
&= \tilde{p}(z(x))z(x) + (1 - \tilde{p}(z(x)))x\\
&\geq \tilde{p}(z(x))z(x) + (1 - \tilde{p}(z(x)))x'\\
&= \tilde{p}(z(x))(z(x) - x') + x' \geq \tilde{T}(x') + x',
\end{aligned}$$

hence it follows that $\tilde{T}(x) + x$ is nondecreasing in x on $(-\infty,\infty)$,

(g) If $\beta = 1$, then $\beta\tilde{T}(x) + x = T(x) + x$, hence the assertion is true from (f).

(h) Since $\beta\tilde{T}(x) + x = \beta(\tilde{T}(x) + x) + (1 - \beta)x$, if $\beta < 1$, then $(1 - \beta)x$ is strictly increasing in x, hence the assertion is true from (f).

(i) Since $\tilde{T}(x) \leq b - x$ for any x from (e) and $\tilde{T}(x) \leq 0$ for any x from (d), we have $\tilde{T}(x) \leq \min\{0, b - x\}$ for any $x \in (-\infty,\infty)$.

(j) From (5.1.32(p.24)) we have

$$\lambda\beta\tilde{T}(\lambda\beta b+s)+s \;=\; \lambda\beta\min_z \tilde{p}(z)(z-\lambda\beta b-s)+s$$
$$=\; \min_z \tilde{p}(z)(\lambda\beta z-(\lambda\beta)^2 b-\lambda\beta s)+s.$$

Then, for $s > s'$ we have

$$\begin{aligned}
&\lambda\beta\tilde{T}(\lambda\beta b+s)+s-\lambda\beta\tilde{T}(\lambda\beta b+s')-s' \\
&\quad= \min_z p(z)(\lambda\beta z-(\lambda\beta)^2 b-\lambda\beta s)-\min_z p(z)(\lambda\beta z-(\lambda\beta)^2 b-\lambda\beta s')+(s-s') \\
&\quad\ge \min_z -p(z)\lambda\beta(s-s')+(s-s')^\dagger \\
&\quad\ge \min_z -(s-s')\lambda\beta+(s-s') \quad \text{(due to } p(z)\le 1 \text{ and } s-s'>0) \\
&\quad= -(s-s')\lambda\beta+(s-s') \\
&\quad= (s-s')(1-\lambda\beta)\ge(>)\,0 \text{ if } \lambda\beta\le(<)\,1.
\end{aligned}$$

Hence, since $\lambda\beta\tilde{T}(\lambda\beta b+s)+s\ge(>)\,\lambda\beta\tilde{T}(\lambda\beta b+s')+s'$ if $\lambda\beta\le(<)\,1$, it follows that $\lambda\beta\tilde{T}(\lambda\beta b+s)+s$ is nondecreasing in s and strictly increasing in s if $\lambda\beta<1$. ∎

Let us define

$$\tilde{h}(z)=\tilde{p}(z)(z-b)/(1-\tilde{p}(z)), \quad z<b,$$
$$\tilde{h}^\star=\inf_{z<b}\tilde{h}(z),$$

Below, for any x let us define the following successive four assertions:

$$A_1(x)=\langle\!\langle\, z(x)<b \,\rangle\!\rangle,$$
$$A_2(x)=\langle\!\langle\, \tilde{T}(b,x)>\tilde{T}(z',x,) \text{ for at least one } z'<b \,\rangle\!\rangle,$$
$$A_3(x)=\langle\!\langle\, b-\tilde{h}(z')>x \text{ for at least one } z'<b \,\rangle\!\rangle,$$
$$A_4(x)=\langle\!\langle\, \sup_{z<b}\{b-\tilde{h}(z)\}>x \,\rangle\!\rangle.$$

Proposition A 3.1 For any x we have $A_1(x)\Leftrightarrow A_2(x)\Leftrightarrow A_3(x)\Leftrightarrow A_4(x)$. □

● *Proof* Letting $\tilde{T}(z,x)\overset{\text{def}}{=}\tilde{p}(z)(z-x)$, we can rewrite (5.1.32(p.24)) as $\tilde{T}(x)=\min_z\tilde{T}(z,x)=\tilde{T}(z(x),x)$ (see (5.1.38(p.25))).

1. Let $A_1(x)$ be true for any x. Suppose $\tilde{T}(b,x)\le\tilde{T}(z',x)$ for all $z'<b$. Then the minimum of $\tilde{T}(z,x)$ is attained at $z=b$ (see Def. 5.1.2(p.25)), i.e., $z(x)=b$, which contradicts $A_1(x)$. Hence it must be that $\tilde{T}(b,x)>\tilde{T}(z',x)$ for at least one $z'<b$, thus $A_2(x)$ becomes true. Accordingly, we have $A_1(x)\Rightarrow A_2(x)$. Suppose $A_2(x)$ is true for any x. Then, if $z(x)=b$, we have $\tilde{T}(b,x)>\tilde{T}(z',x)\ge\tilde{T}(x)=\tilde{T}(z(x),x)=\tilde{T}(b,x)$, which is a contradiction, hence it must be that $z(x)<b$ due to Lemma A 3.1(p.243) (c); accordingly, we have $A_2(x)\Rightarrow A_1(x)$. Thus, it follows that we have $A_1(x)\Leftrightarrow A_2(x)$ for any given x.
2. Since $\tilde{p}(b)=1$ from (5.1.42 (2) (p.25)), for $z'<b$ we have

$$\begin{aligned}
&\tilde{T}(b,x)-\tilde{T}(z',x) \\
&\quad= \tilde{p}(b)(b-x)-\tilde{p}(z')(z'-x) \\
&\quad= b-x-\tilde{p}(z')(z'-x) \\
&\quad= b-x-\tilde{p}(z')(b-x+z'-b) \\
&\quad= b-x-\tilde{p}(z')(b-x)-\tilde{p}(z')(z'-b) \\
&\quad= (1-\tilde{p}(z'))(b-x)-\tilde{p}(z')(z'-b) \\
&\quad= (1-\tilde{p}(z'))\big(b-x-\tilde{p}(z')(z'-b)/(1-\tilde{p}(z'))\big) \\
&\quad= (1-\tilde{p}(z'))(b-x-\tilde{h}(z')) \\
&\quad= (1-\tilde{p}(z'))(b-\tilde{h}(z')-x).
\end{aligned}$$

   Accordingly, noting $1>\tilde{p}(z')$ due to (5.1.42 (1) (p.25)), we immediately see that $A_2(x)\Leftrightarrow A_3(x)$ for any given x.
3. Let $A_3(x)$ be true for any x. Then clearly $A_4(x)$ is also true, i.e., $A_3(x)\Rightarrow A_4(x)$. Let $A_4(x)$ be true for any x. Then evidently $b-\tilde{h}(z')>x$ for at least one $z'<b$, hence $A_3(x)$ is true, so we have $A_4(x)\Rightarrow A_3(x)$. Accordingly, it follows that $A_3(x)\Leftrightarrow A_4(x)$ for any given x.

From all the above we have $A_1(x)\Leftrightarrow A_2(x)\Leftrightarrow A_3(x)\Leftrightarrow A_4(x)$. ∎

Lemma A 3.5

(a) $-\infty<\tilde{h}^\star<0$.
(b) $\tilde{x}^\star=b-\tilde{h}^\star>b$.
(c) $\tilde{x}^\star>(\le)\,x\Leftrightarrow z(x)<(=)\,b$.
(d) $b^\star>b$. □

---

† $\min a(x)-\min b(x)\ge\min\{a(x)-b(x)\}$.

● *Proof* (a) For any infinitesimal $\varepsilon > 0$ such that $a < a+\varepsilon < b \cdots$ (II) we have $0 < \tilde{p}(a+\varepsilon) < 1$ from (5.1.41 (2) (p.25)) and (5.1.42 (1) (p.25)), hence, $\tilde{h}(a+\varepsilon) = \tilde{p}(a+\varepsilon)(a+\varepsilon-b)/(1-\tilde{p}(a+\varepsilon)) < 0$. If $z \le a \cdots$ (I), then $\tilde{p}(z) = 0$ due to (5.1.41 (1) (p.25)), hence $\tilde{h}(z) = 0$ for $z \le a$. From the above we have $\tilde{h}^\star < 0$ (finite) or $\tilde{h}^\star = -\infty$.

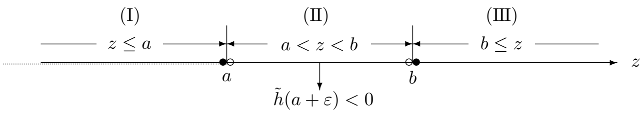


Figure A 3.3: $\tilde{h}(z) = 0$ for $z \le a$ and $\tilde{h}(a+\varepsilon) < 0$

Assume that $\tilde{h}^\star = -\infty$. Then, there exists at least one z' on $a < z' < b$ such that $\tilde{h}(z') \le -N$ for any given $N > 0$. Hence, if the N is given by $M/\underline{f}$ (see (2.2.4(p.13))) with any $M > 1$, i.e., $N = M/\underline{f}$, we have $\tilde{h}(z') \le -M/\underline{f}$, so $\tilde{p}(z')(z'-b)/(1-\tilde{p}(z')) \le -M/\underline{f}$. Hence, noting (5.1.31(p.24)), we have

$$\tilde{p}(z')(z'-b) \;\le\; -(1-\tilde{p}(z'))M/\underline{f} = -(1-\Pr\{\boldsymbol{\xi} \le z'\})M/\underline{f} = -\Pr\{z' < \boldsymbol{\xi}\}M/\underline{f} \cdots (*)$$

where $\Pr\{z' < \boldsymbol{\xi}\} = \int_{z'}^{b} f(w)dw \ge \int_{z'}^{b} \underline{f}dw = (b-z')\underline{f}$. Accordingly, since $\tilde{p}(z')(z'-b) \le -(b-z')\underline{f}M/\underline{f} = (z'-b)M$, we have $\tilde{p}(z') \ge M > 1$ due to $z'-b < 0$, which is a contradiction. Hence, it must follow that $\tilde{h}^\star > -\infty$.

(b) Since $A_1(x) \Rightarrow A_4(x)$ due to Proposition A 3.1, we can rewrite (5.1.40(p.25)) as

$$\begin{aligned}\tilde{x}^\star &= \sup\{x \mid \sup_{z<b}\{b-\tilde{h}(z)\} > x\}\\ &= \sup_{z<b}\{b-\tilde{h}(z)\} \cdots (1)\\ &= b - \inf_{z<b}\tilde{h}(z) = b - \tilde{h}^\star > b\end{aligned}$$

due to (a), hence (b) holds.

(c) Let $\tilde{x}^\star > x$, hence $\sup_{z<b}\{b-\tilde{h}(z)\} > x$ from (1), so $z(x) < b$ due to $A_4(x) \Rightarrow A_1(x)$. Let $\tilde{x}^\star \le x$, hence $\sup_{z<b}\{b-\tilde{h}(z)\} \le x$ from (1). Now, since $\sup_{z<b}\{b-\tilde{h}(z)\} \le x \Rightarrow z(x) \ge b$ due to the contraposition of $A_4(x) \Leftrightarrow A_1(x)$, we obtain $z(x) = b$ due to Lemma A 3.1(p.243) (c).

(d) First note $\tilde{T}(x) \le \tilde{p}(z')(z'-x)$ for any x and z'. Accordingly, for any sufficiently small $\varepsilon > 0$ such that $a < b-\varepsilon$ we have $\tilde{p}(b-\varepsilon) > 0$ from (5.1.41 (2) (p.25)), hence $\tilde{T}(b) \le \tilde{p}(b-\varepsilon)(b-\varepsilon-b) = -\tilde{p}(b-\varepsilon)\varepsilon < 0$, so adding b to the both sides of this inequality yields $\tilde{T}(b)+b < b$, so $\tilde{T}(x)+x \le \tilde{T}(b)+b < b$ for $x \le b$ due to Lemma A 3.4(p.244) (f). Accordingly, if $b^\star \le b$, we have $\tilde{T}(b^\star)+b^\star \le \tilde{T}(b)+b < b$, hence from Lemma A 3.4(p.244) (a) we have $\tilde{T}(b^\star+\varepsilon)+b^\star+\varepsilon < b$ for any sufficiently small $\varepsilon > 0$, so $\tilde{T}(b^\star+\varepsilon) < b-(b^\star+\varepsilon)$, which contradicts the definition of $b^\star$ (see (5.1.39(p.25))). Therefore, it must follow that $b^\star > b$. ∎

Lemma A 3.6

(a) *$\tilde{T}(x)+x$ is strictly increasing on $(-\infty, b^\star]$.*
(b) *$\tilde{T}(x) = b-x$ on $[b^\star, \infty)$ and $\tilde{T}(x) < b-x$ on $(-\infty, b^\star)$.*
(c) *$\tilde{T}(0) = b$ if $b^\star < 0$ and $\tilde{T}(0) = 0$ if $a > 0$.*
(d) *If $x > y$ and $b^\star > y$, then $\tilde{T}(x)+x > \tilde{T}(y)+y$.* □

● *Proof* (a) From (5.1.38(p.25)) we have

$$\tilde{T}(x)+x \;=\; \tilde{p}(z(x))(z(x)-x)+x = \tilde{p}(z(x))z(x)+(1-\tilde{p}(z(x)))x. \cdots (1)$$

- Let $\tilde{x}^\star > x$. Then $z(x) < b$ from Lemma A 3.5(p.245) (c), hence $\tilde{p}(z(x)) < 1$ due to (5.1.42 (1) (p.25)), so $1-\tilde{p}(z(x)) > 0$. If $x > x'$, from (1) we have

$$\tilde{T}(x)+x > \tilde{p}(z(x))z(x)+(1-\tilde{p}(z(x)))x' = \tilde{p}(z(x))(z(x)-x')+x' \ge \tilde{T}(x')+x',$$

  i.e., $\tilde{T}(x)+x$ is strictly increasing on $(-\infty, \infty)$, hence understandably so also on $(-\infty, b^\star]$.

- Let $\tilde{x}^\star \le x$. Then $z(x) = b$ from Lemma A 3.5(p.245) (c), hence $\tilde{p}(z(x)) = 1$ from (5.1.42 (2) (p.25)), so $\tilde{T}(x) = \tilde{p}(z(x))(z(x)-x) = b-x \cdots$ (2). Suppose $b^\star > \tilde{x}^\star$. Then, since $b^\star > b^\star - 2\varepsilon > \tilde{x}^\star$ for an infinitesimal $\varepsilon > 0$, we have $b^\star > b^\star-\varepsilon > \tilde{x}^\star+\varepsilon > \tilde{x}^\star$ or equivalently $\tilde{x}^\star < b^\star-\varepsilon$; accordingly, due to (2) we obtain $\tilde{T}(b^\star-\varepsilon) = b-(b^\star-\varepsilon) \cdots$ (3). Now, due to (5.1.39(p.25)) we have $\tilde{T}(b^\star-\varepsilon) < b-(b^\star-\varepsilon)$, which contradicts (3). Accordingly, it must be that $\tilde{x}^\star \ge b^\star$. Let $x' < x < b^\star$. Then, since $\tilde{x}^\star > x$, we have $z(x) < b$ Lemma A 3.5(p.245) (c), hence $\tilde{p}(z(x)) < 1$ due to (5.1.42 (1) (p.25)) or equivalently $1-\tilde{p}(z(x)) > 0$. Thus, from (1) we have

$$\tilde{T}(x)+x > \tilde{p}(z(x))z(x)+(1-\tilde{p}(z(x)))x' = \tilde{p}(z(x))(z(x)-x')+x' \ge \tilde{T}(x')+x',$$

  implying that $\tilde{T}(x)+x$ is strictly increasing on $(-\infty, b^\star) \cdots$ (4). Now let us assume $\tilde{T}(b^\star)+b^\star = \tilde{T}(x)+x$ for any $x < b^\star$. Then, for any sufficiently small $\varepsilon > 0$ such that $b^\star - x > 2\varepsilon > 0$ we have $x < x+\varepsilon < b^\star-\varepsilon < b^\star$, hence $\tilde{T}(b^\star)+b^\star = \tilde{T}(x)+x < \tilde{T}(x+\varepsilon)+x+\varepsilon \le \tilde{T}(b^\star)+b^\star$ due to (4) and Lemma A 3.4(p.244) (f), which is a contradiction. Thus, $\tilde{T}(x)+x \ne \tilde{T}(b^\star)+b^\star$ for $x < b^\star$, i.e., $\tilde{T}(x)+x > \tilde{T}(b^\star)+b^\star$ or $\tilde{T}(x)+x < \tilde{T}(b^\star)+b^\star$ for $x < b^\star$; however, the former is impossible due to the nondecreasing in Lemma A 3.4(p.244) (f), hence it follows that $\tilde{T}(x)+x < \tilde{T}(b^\star)+b^\star$ for $x < b^\star$. From this and (4) it inevitably follows that $\tilde{T}(x)+x$ is strictly increasing on $(-\infty, b^\star]$ instead of $(-\infty, b^\star)$.

Accordingly, whether $\tilde{x}^\star > x$ or $\tilde{x}^\star \le x$, it follows that $\tilde{T}(x) + x$ is strictly increasing on $(-\infty, b^\star]$.

(b) Due to (5.1.39(p.25)) we have $\tilde{T}(x) < b - x$ for $x < b^\star$, i.e., $\tilde{T}(x) < b - x$ on $(-\infty, b^\star)$, hence the latter half is true. Since $\tilde{T}(x) \le b - x$ on $(-\infty, \infty)$ due to Lemma A 3.4(p.244) (e), we have $\tilde{T}(x) + x \le b \cdots$ **(5)** on $(-\infty, \infty)$. Suppose $\tilde{T}(b^\star) + b^\star < b$. Then, for an infinitesimal $\varepsilon > 0$ we have $\tilde{T}(b^\star + \varepsilon) + b^\star + \varepsilon < b$ due to Lemma A 3.4(p.244) (a), i.e., $\tilde{T}(b^\star + \varepsilon) < b - (b^\star + \varepsilon)$, which contradicts the definition of $b^\star$ (see (5.1.39(p.25))). Consequently, it must be that $\tilde{T}(b^\star) + b^\star = b \cdots$ **(6)** or equivalently $\tilde{T}(b^\star) = b - b^\star$. Let $x > b^\star$. Then, from Lemma A 3.4(p.244) (f) we have $\tilde{T}(x) + x \ge \tilde{T}(b^\star) + b^\star = b$. From this and (5) it must be that $\tilde{T}(x) + x = b$ on $(b^\star, \infty)$, hence $\tilde{T}(x) = b - x$ on $(b^\star, \infty)$. From this and (6) it follows that $\tilde{T}(x) = b - x$ on $[b^\star, \infty)$. Hence the former half is true.

(c) Let $b^\star < 0$. Then, since $0 \in [b^\star, \infty)$, we have $\tilde{T}(0) = b$ from the former half of (b). Now we have $\tilde{T}(0) = \min_z \tilde{p}(z)z \cdots$ **(7)** from (5.1.32(p.24)). Let $a > 0$. Then, if $z \le a$, we have $\tilde{p}(z)z = 0$ from (5.1.41 (1) (p.25)) and if $z > a\,(> 0)$, then $\tilde{p}(z)z > 0$ from (5.1.41 (2) (p.25)). Hence it follows that $\tilde{T}(0) = 0$ due to (7).

(d) Let $x > y$ and $b^\star > y$. If $x \ge b^\star$, then $\tilde{T}(x) + x \ge \tilde{T}(b^\star) + b^\star > \tilde{T}(y) + y$ due to Lemma A 3.4(p.244) (f) and (a), and if $b^\star > x$, then $b^\star \ge x > y$, hence $\tilde{T}(x) + x > \tilde{T}(y) + y$ due to (a). Thus, whether $x \ge b^\star$ or $b^\star > x$, we have $\tilde{T}(x) + x > \tilde{T}(y) + y$. ∎

All the results obtained above (see Lemmas A 3.1(p.243) -A 3.6(p.246)) can be complied into Lemma A 3.7(p.247) below.

Lemma A 3.7 ($\mathscr{A}\{\tilde{T}_{\mathbb{P}}\}$) *For any $F \in \mathscr{F}$ we have:*

(a)	$\tilde{T}(x)$ *is continuous on* $(-\infty, \infty)$ ←	← Lemma A 3.4(p.244) (a)
(b)	$\tilde{T}(x)$ *is nonincreasing on* $(-\infty, \infty)$ ←	← Lemma A 3.4(p.244) (b)
(c)	$\tilde{T}(x)$ *is strictly decreasing on* $[a, \infty)$ ←	← Lemma A 3.4(p.244) (c)
(d)	$\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ ←	← Lemma A 3.4(p.244) (f)
(e)	$\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, b^\star]$ ←	← Lemma A 3.6(p.246) (a)
(f)	$\tilde{T}(x) = b - x$ *on* $[b^\star, \infty)$ *and* $T(x) < b - x$ *on* $(-\infty, b^\star)$ ←	← Lemma A 3.6(p.246) (b)
(g)	$\tilde{T}(x) < 0$ *on* (a, ∞) *and* $T(x) = 0$ *on* $(-\infty, a]$ ←	← Lemma A 3.4(p.244) (d)
(h)	$\tilde{T}(x) \le \min\{0, b - x\}$ *on* $(-\infty, \infty)$ ←	← Lemma A 3.4(p.244) (i)
(i)	$\tilde{T}(0) = b$ *if* $b^\star < 0$ *and* $T(0) = 0$ *if* $a > 0$ ←	← Lemma A 3.6(p.246) (c)
(j)	$\beta\tilde{T}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$ *if* $\beta = 1$ ←	← Lemma A 3.4(p.244) (g)
(k)	$\beta\tilde{T}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$ ←	← Lemma A 3.4(p.244) (h)
(l)	*If* $x > y$ *and* $b^\star > y$, *then* $T(x) + x > T(y) + y$ ←	← Lemma A 3.6(p.246) (d)
(m)	$\lambda\beta\tilde{T}(\lambda\beta b + s) + s$ *is nondecreasing in* s *and strictly increasing in* s *if* $\lambda\beta < 1$ ←	← Lemma A 3.4(p.244) (j)
(n)	$b^\star > b$ ←	← Lemma A 3.5(p.245) (d)

A 3.2 $\mathscr{A}\{\tilde{L}_{\mathbb{P}}\}$, $\mathscr{A}\{\tilde{K}_{\mathbb{P}}\}$, $\mathscr{A}\{\tilde{\mathcal{L}}_{\mathbb{P}}\}$, and $\tilde{\kappa}_{\mathbb{P}}$

From (5.1.33(p.25)) and (5.1.34(p.25)) and from Lemma A 3.7(p.247) (f) we obtain, noting (10.2.1(p.54)),

$$\tilde{L}\,(x) \begin{cases} = \lambda\beta b + s - \lambda\beta x \text{ on } [b^\star, -\infty) & \cdots (1), \\ < \lambda\beta b + s - \lambda\beta x \text{ on } (-\infty, b^\star) & \cdots (2), \end{cases} \tag{A 3.1}$$

$$\tilde{K}\,(x) \begin{cases} = \lambda\beta b + s - \delta x \text{ on } [b^\star, \infty) & \cdots (1), \\ < \lambda\beta b + s - \delta x \text{ on } (-\infty, b^\star) & \cdots (2). \end{cases} \tag{A 3.2}$$

In addition, from (5.1.34(p.25)) and Lemma A 3.7(p.247) (g) we have

$$\tilde{K}\,(x) \begin{cases} < -(1-\beta)x + s \text{ on } (a, \infty) & \cdots (1), \\ = -(1-\beta)x + s \text{ on } (-\infty, a] & \cdots (2), \end{cases} \tag{A 3.3}$$

hence we obtain

$$\tilde{K}\,(x) + x \le \beta x + s \quad \text{on} \quad (-\infty, \infty). \tag{A 3.4}$$

Then, from (A 3.2 (1) (p.247)) and (A 3.3 (2) (p.247)) we get

$$\tilde{K}\,(x) + x = \begin{cases} \lambda\beta b + s + (1-\lambda)\beta x \text{ on } [b^\star, \infty) & \cdots (1), \\ \beta x + s \quad \text{on } (-\infty, a] & \cdots (2). \end{cases} \tag{A 3.5}$$

Since $\tilde{K}\,(x) = \tilde{L}\,(x) - (1-\beta)x$ from (5.1.34(p.25)) and (5.1.33(p.25)), if $x_{\tilde{L}}$ and $x_{\tilde{K}}$ exist, then

$$\tilde{K}\,(x_{\tilde{L}}) = -(1-\beta)\,x_{\tilde{L}} \cdots (1), \quad \tilde{L}\,(x_{\tilde{K}}) = (1-\beta)\,x_{\tilde{K}} \cdots (2). \tag{A 3.6}$$

Lemma A 3.8 ($\tilde{L}_{\mathbb{P}}$)

(a) $\tilde{L}\,(x)$ *is continuous on* $(-\infty, \infty)$.

(b) $\tilde{L}\,(x)$ *is nonincreasing on* $(-\infty, \infty)$.

(c) $\tilde{L}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) *Let* $s = 0$. *Then* $x_{\tilde{L}} = a$ *where* $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.
(e) *Let* $s > 0$.

1. $x_{\tilde{L}}$ *uniquely exists with* $x_{\tilde{L}} > a$ *where* $x_{\tilde{L}} < (= (>))\ x \Leftrightarrow \tilde{L}(x) < (= (>))\ 0$.
2. $(\lambda\beta b + s)/\lambda\beta \geq (<)\ b^\star \Leftrightarrow x_{\tilde{L}} = (<)\ (\lambda\beta b + s)/\lambda\beta < (\geq)\ b^\star$. □

● *Proof* (a-c) Immediate from (5.1.33(p.25)) and Lemma A 3.7(p.247) (a-c).

(d) Let $s = 0$. Then, since $\tilde{L}(x) = \lambda\beta\tilde{T}(x)$, from Lemma A 3.7(p.247) (g) we have $\tilde{L}(x) = 0$ for $a \geq x$ and $\tilde{L}(x) < 0$ for $x > a$, hence $x_{\tilde{L}} = a$ by definition (see Section 5.2(p.25) (b)), so $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (=)\ 0$. The inverse is true by contraposition. In addition, since $\tilde{L}(x) = 0 \Rightarrow \tilde{L}(x) \geq 0$, we have $\tilde{L}(x) < (=)\ 0 \Rightarrow \tilde{L}(x) < (\geq)\ 0$.

(e) Let $s > 0$.

(e1) From (A 3.1 (1) (p.247)) and the assumption of $\lambda > 0$ and $\beta > 0$ we have $\tilde{L}(x) < 0$ for a sufficiently large $x > 0$ such that $x > b^\star$. In addition, we have $\tilde{L}(a) = \lambda\beta\tilde{T}(a) + s = s > 0$ from Lemma A 3.7(p.247) (g). Hence, from (a,c) it follows that $x_{\tilde{L}}$ uniquely exists. The inequality $x_{\tilde{L}} > a$ is immediate from $\tilde{L}(a) > 0$ and (c). The latter half is evident.

(e2) If $(\lambda\beta b + s)/\lambda\beta \geq (<)\ b^\star$, from (A 3.1(p.247)) we have $\tilde{L}((\lambda\beta b + s)/\lambda\beta) = (<)\ \lambda\beta b + s - \lambda\beta(\lambda\beta b + s)/\lambda\beta = 0$, hence $x_{\tilde{L}} = (<)\ (\lambda\beta b + s)/\lambda\beta$ from (e1). ∎

Corollary A 3.2 ($\tilde{L}_{\mathbb{P}}$)

(a) $x_{\tilde{L}} < (\geq)\ x \Leftrightarrow \tilde{L}(x) < (\geq)\ 0$.
(b) $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. □

● *Proof* (a) Clearly $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (\geq)\ 0$ from Lemma A 3.8(p.247) (d,e1). The inverse is true by contraposition.

(b) Since $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) < (\geq)\ 0$ due to (a) and since $\tilde{L}(x) < (\geq)\ 0 \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$, we have $x_{\tilde{L}} < (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{L}} = x$, then $\tilde{L}(x) = \tilde{L}(x_{\tilde{L}}) = 0 \leq 0$ or equivalently $x_{\tilde{L}} = x \Rightarrow \tilde{L}(x) \leq 0$, hence it follows that $x_{\tilde{L}} \leq (\geq)\ x \Rightarrow \tilde{L}(x) \leq (\geq)\ 0$. ∎

Lemma A 3.9 ($\tilde{K}_{\mathbb{P}}$)

(a) $\tilde{K}(x)$ *is continuous on* $(-\infty, \infty)$.
(b) $\tilde{K}(x)$ *is nonincreasing on* $(-\infty, \infty)$.
(c) $\tilde{K}(x)$ *is strictly decreasing on* $[a, \infty)$.
(d) $\tilde{K}(x)$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\beta < 1$.
(e) $\tilde{K}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.
(f) $\tilde{K}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$ *if* $\lambda < 1$.
(g) $K(x) + x$ *is strictly increasing on* $(-\infty, b^\star]$.
(h) *If* $x > y$ *and* $b^\star > y$, *then* $\tilde{K}(x) + x > \tilde{K}(y) + y$.
(i) *Let* $\beta = 1$ *and* $s = 0$. *Then* $x_{\tilde{K}} = a$ *where* $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.
(j) *Let* $\beta < 1$ *or* $s > 0$.

1. *There uniquely exists* $x_{\tilde{K}}$ *where* $x_{\tilde{K}} < (= (>))\ x \Leftrightarrow \tilde{K}(x) < (= (>))\ 0$.
2. $(\lambda\beta b + s)/\delta \geq (<)\ b^\star \Leftrightarrow x_{\tilde{K}} = (<)\ (\lambda\beta b + s)/\delta$.
3. *Let* $\tilde{\kappa} < (= (>))\ 0$. *Then* $x_{\tilde{K}} < (= (>))\ 0$. □

● *Proof* (a-c) Evident from (5.1.34(p.25)) and Lemma A 3.7(p.247) (a-c).

(d) Evident from Lemma A 3.7(p.247) (b) and (5.1.34(p.25)).

(e) From (5.1.34(p.25)) we have

$$\tilde{K}(x) + x = \lambda\beta\tilde{T}(x) + \beta x + s = \lambda\beta(\tilde{T}(x) + x) + (1 - \lambda)\beta x + s \cdots \mathbf{(1)},$$

hence the assertion is immediate from Lemma A 3.7(p.247) (d).

(f) Evident from (1) and Lemma A 3.7(p.247) (d).

(g) Evident from (1) and Lemma A 3.7(p.247) (e).

(h) Let $x > y$ and $b^\star > y$. If $x \geq b^\star$, then $\tilde{K}(x) + x \geq \tilde{K}(b^\star) + b^\star > \tilde{K}(y) + y$ due to (e,g), and if $b^\star > x$, then $b^\star > x > y$, hence $\tilde{K}(x) + x > \tilde{K}(y) + y$ due to (g).

(i) Let $\beta = 1$ and $s = 0$. Then, since $\tilde{K}(x) = \lambda\tilde{T}(x)$ due to (5.1.34(p.25)), from Lemma A 3.7(p.247) (g) we have $\tilde{K}(x) = 0$ for $a \geq x$ and $\tilde{K}(x) < 0$ for $x > a$, so $x_{\tilde{K}} = a$ by the definition of $x_{\tilde{K}}$ (See Section 5.2(p.25) (b)). Hence $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (=)\ 0$. The inverse is immediate by contraposition. In addition, since $\tilde{K}(x) = 0 \Rightarrow \tilde{K}(x) \geq 0$, we have $\tilde{K}(x) < (=)\ 0 \Rightarrow \tilde{K}(x) < (\geq)\ 0$.

(j) Let $\beta < 1$ or $s > 0$.

(j1) First note (A 3.3 (2) (p.247)). Then, if $\beta = 1$, then $s > 0$, hence $\tilde{K}(x) = s > 0$ for any $x \leq a$ and if $\beta < 1$, then $\tilde{K}(x) > 0$ for any sufficiently small $x < 0$ such that $x < a$. Hence, whether $\beta = 1$ or $\beta < 1$ (for any $0 < \beta \leq 1$), we have $\tilde{K}(x) > 0$ for any sufficiently small x. Next, for any sufficiently large $x > 0$ such that $x \geq b^\star$, from (A 3.2 (1) (p.247)) we have $\tilde{K}(x) < 0$ since to $\delta > 0$ due to (10.2.2 (1) (p.54)). Hence, it follows that there exists the solution $x_{\tilde{K}}$ for any $0 < \beta \leq 1$. Let $\beta < 1$. Then, the solution is unique due to (d). Let $\beta = 1$, hence $s > 0$. Then, since $\tilde{K}(a) = s > 0$ from (A 3.3 (2) (p.247)), we have $x_{\tilde{K}} > a$, hence

$\tilde{K}(x)$ is strictly decreasing on the neighbourhood of $x = x_{\tilde{K}}$ due to (c), implying that the solution $x_{\tilde{K}}$ is unique. Therefore, for any $0 < \beta \leq 1$ the solution is unique. Thus, the latter half is immediate.

(j2) Let $(\lambda\beta b + s)/\delta \geq (<)\ b^\star$. Then, from (A 3.2 (1(2)) (p.247)) we have $\tilde{K}((\lambda\beta b + s)/\delta) = (<)\ \lambda\beta b + s - \delta(\lambda\beta b + s)/\delta = 0$, hence $x_{\tilde{K}} = (<)\ (\lambda\beta b + s)/\delta$ due to (j1). Its inverse is also true by contraposition.

(j3) If $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$ from (5.1.37(p.25)), hence $x_{\tilde{K}} < (= (>))\ 0$ from (j1). ∎

The corollary below is used when it is not specified whether $s > 0$ or $s = 0$.

Corollary A 3.3 ($\tilde{K}_{\mathbb{P}}$)

(a) $x_{\tilde{K}} < (\geq)\ x \Leftrightarrow \tilde{K}(x) < (\geq)\ 0$.

(b) $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. □

• *Proof* (a) Clearly $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to Lemma A 3.9(p.248) (i,j1). The inverse is immediate by contraposition.

(b) Since $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) < (\geq)\ 0$ due to (a) and since $\tilde{K}(x) < (\geq)\ 0 \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$, we have $x_{\tilde{K}} < (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. In addition, if $x_{\tilde{K}} = x$, then $\tilde{K}(x) = \tilde{K}(x_{\tilde{K}}) = 0 \leq 0$, hence it follows that $x_{\tilde{K}} \leq (\geq)\ x \Rightarrow \tilde{K}(x) \leq (\geq)\ 0$. ∎

Lemma A 3.10 ($\tilde{L}_{\mathbb{P}}/\tilde{K}_{\mathbb{P}}$)

(a) *Let $\beta = 1$ and $s = 0$. Then $x_{\tilde{L}} = x_{\tilde{K}} = a$.*

(b) *Let $\beta = 1$ and $s > 0$. Then $x_{\tilde{L}} = x_{\tilde{K}}$.*

(c) *Let $\beta < 1$ and $s = 0$. Then $a < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (=))\ 0$.*

(d) *Let $\beta < 1$ and $s > 0$. Then $\tilde{\kappa} < (= (>))\ 0 \Rightarrow x_{\tilde{L}} < (= (>))\ x_{\tilde{K}} < (= (>))\ 0$.* □

• *Proof* (a) If $\beta = 1$ and $s = 0$, then $x_{\tilde{L}} = a$ from Lemma A 3.8(p.247) (d) and $x_{\tilde{K}} = a$ from Lemma A 3.9(p.248) (i), hence $x_{\tilde{L}} = x_{\tilde{K}} = a$.

(b) Let $\beta = 1$ and $s > 0$. Then $\tilde{K}(x_{\tilde{L}}) = 0$ from (A 3.6 (1) (p.247)), hence $x_{\tilde{K}} = x_{\tilde{L}}$ from Lemma A 3.9(p.248) (j1).

(c) Let $\beta < 1$ and $s = 0$. Then $x_{\tilde{L}} = a \cdots$ **(1)** from Lemma A 3.8(p.247) (d). Suppose $a < 0$. Then, since $x_{\tilde{L}} < 0$, we have $\tilde{K}(x_{\tilde{L}}) > 0$ from (A 3.6 (1) (p.247)), hence $x_{\tilde{K}} > x_{\tilde{L}}$ from Lemma A 3.9(p.248) (j1). Furthermore, from (5.1.37(p.25)) and (5.1.36(p.25)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) < 0$ due to Lemma A 3.7(p.247) (g), hence $x_{\tilde{K}} < 0$ from Lemma A 3.9(p.248) (j1). Suppose $a = (>)\ 0$. Then, since $x_{\tilde{L}} = (>)\ 0$ from (1), we have $\tilde{K}(x_{\tilde{L}}) = (<)\ 0$ due to (A 3.6 (1) (p.247)), hence $x_{\tilde{L}} = (>)\ x_{\tilde{K}}$ from Lemma A 3.9(p.248) (j1). Furthermore, from (5.1.37(p.25)) and (5.1.36(p.25)) we have $\tilde{K}(0) = \lambda\beta\tilde{T}(0) = 0$ due to Lemma A 3.7(p.247) (g), hence $x_{\tilde{K}} = (=)\ 0$ from Lemma A 3.9(p.248) (j1).

(d) Let $\beta < 1$ and $s > 0$. Since $\tilde{\kappa} = \tilde{K}(0)$ from (5.1.37(p.25)), if $\tilde{\kappa} < (= (>))\ 0$, then $\tilde{K}(0) < (= (>))\ 0$, hence $x_{\tilde{K}} < (= (>))\ 0$ from Lemma A 3.9(p.248) (j1). Accordingly $\tilde{L}(x_{\tilde{K}}) < (= (>))\ 0$ from (A 3.6 (2) (p.247)), so $x_{\tilde{L}} < (= (>))\ x_{\tilde{K}}$ from Lemma A 3.8(p.247) (e1). ∎

Lemma A 3.11 ($\tilde{\mathcal{L}}_{\mathbb{P}}$)

(a) *$\tilde{\mathcal{L}}(s)$ is nondecreasing in s.*

(b) *If $\lambda\beta < 1$, then $\tilde{\mathcal{L}}(s)$ is strictly increasing in s.*

(c) *Let $\lambda\beta b \leq a$.*

1. $x_{\tilde{L}} \geq \lambda\beta b + s$.
2. *Let $s > 0$ and $\lambda\beta < 1$. Then $x_{\tilde{L}} > \lambda\beta b + s$.*

(d) *Let $\lambda\beta b > a$. Then, there exists a $s_{\tilde{\mathcal{L}}} > 0$ such that if $s_{\tilde{\mathcal{L}}} > (\leq)\ s$, then $x_{\tilde{L}} < (\geq)\ \lambda\beta b + s$.* □

• *Proof* (a,b) From (5.1.35(p.25)) and (5.1.33(p.25)) we have $\tilde{\mathcal{L}}(s) = \lambda\beta\tilde{T}(\lambda\beta b + s) + s \cdots$ **(1)**, hence the assertions are true from Lemma A 3.7(p.247) (m).

(c) Let $\lambda\beta\mu \leq a$. Then, from (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta b) = 0 \cdots$ **(2)** due to Lemma A 3.7(p.247) (g).

(c1) Since $s \geq 0$, from (a) we have $\tilde{\mathcal{L}}(s) \geq \tilde{\mathcal{L}}(0) = 0$ due to (2) or equivalently $\tilde{L}(\lambda\beta b + s) \geq 0$, hence $x_{\tilde{L}} \geq \beta b + s$ from Corollary A 3.2(p.248) (a).

(c2) Let $s > 0$ and $\lambda\beta < 1$. Then, from (b) we have $\tilde{\mathcal{L}}(s) > \tilde{\mathcal{L}}(0) = 0$ due (2), hence $\tilde{L}(\lambda\beta b + s) > 0$, so $x_{\tilde{L}} > \lambda\beta b + s$ from Lemma A 3.8(p.247) (e1).

(d) Let $\lambda\beta b > a$. From (1) we have $\tilde{\mathcal{L}}(0) = \lambda\beta\tilde{T}(\lambda\beta b) < 0$ due to Lemma A 3.7(p.247) (g). Noting (A 3.1 (1) (p.247)), for any sufficiently large $s > 0$ such that $\lambda\beta b + s \geq b^\star$ and $\lambda\beta b + s > 0$ we have $\tilde{\mathcal{L}}(s) = \tilde{L}(\lambda\beta b + s) = \lambda\beta b + s - \lambda\beta(\lambda\beta b + s) = (1 - \lambda\beta)(\lambda\beta b + s) \geq 0$. Accordingly, due to (a) it follows that there exists a $s_{\tilde{\mathcal{L}}} > 0$ where $\tilde{\mathcal{L}}(s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{\mathcal{L}}(s) \geq 0$ for $s \geq s_{\tilde{\mathcal{L}}}$, or equivalently, $\tilde{L}(\lambda\beta b + s) < 0$ for $s < s_{\tilde{\mathcal{L}}}$ and $\tilde{L}(\lambda\beta b + s) \geq 0$ for $s \geq s_{\tilde{\mathcal{L}}}$. Hence, from Corollary A 3.2(p.248) (a) we have $x_{\tilde{L}} < \beta b + s$ for $s < s_{\tilde{\mathcal{L}}}$ and $x_{\tilde{L}} \geq \beta b + s$ for $s \geq s_{\tilde{\mathcal{L}}}$. ∎

Lemma A 3.12 ($\mathscr{A}\{\tilde{\kappa}_{\mathbb{P}}\}$) *We have*:

(a) *$\tilde{\kappa} = \lambda\beta b + s$ if $b^\star < 0$ and $\tilde{\kappa} = s$ if $a > 0$.*

(b) *Let $\beta < 1$ or $s > 0$. Then $\tilde{\kappa} < (= (>))\ 0 \Leftrightarrow x_{\tilde{K}} < (= (>))\ 0$.* □

• *Proof* (a) Immediate from (5.1.36(p.25)) and Lemma A 3.7(p.247) (i).

(b) Let $\beta < 1$ or $s > 0$. Then, if $\tilde{\kappa} > (= (<))\ 0$, we have $\tilde{K}(0) > (= (<))\ 0$ from (5.1.37(p.25)), hence $x_{\tilde{K}} > (= (<))\ 0$ from Lemma A 3.9(p.248) (j1). Thus "⇒" was proven. Its inverse "⇐" is immediate by contraposition. ∎

A 4 Direct Proof of Assertion Systems

A 4.1 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$

Since $\tilde{K}(x)+(1-\beta)x=\tilde{L}(x)$ for any x due to (5.1.14(p.23)) and (5.1.13(p.23))), from (6.5.4(p.39)) we have

$$V_t-\beta V_{t-1}=\min\{\tilde{L}(V_{t-1}),0\},\quad t>1. \tag{A 4.1}$$

Accordingly:

1. If $\tilde{L}(V_{t-1})\le 0$, then $V_t-\beta V_{t-1}=\tilde{L}(V_{t-1})$, hence

$$V_t=\tilde{L}(V_{t-1})+\beta V_{t-1}=\tilde{K}(V_{t-1})+V_{t-1},\quad t>1. \tag{A 4.2}$$

2. If $\tilde{L}(V_{t-1})\ge 0$, then $V_t-\beta V_{t-1}=0$ or equivalently

$$V_t=\beta V_{t-1},\quad t>1.. \tag{A 4.3}$$

Now, from (6.5.4(p.39)) with $t=2$ we have

$$V_2-V_1=\min\{\tilde{K}(V_1),-(1-\beta)V_1\}. \tag{A 4.4}$$

Finally, from (A 4.1(p.250)) we see that

$$\tilde{L}(V_{t-1})<(>)\,0\Rightarrow \mathtt{Conduct}_{t\blacktriangle}\ (\mathtt{Skip}_{t\blacktriangle})^{\dagger}. \tag{A 4.5}$$

In this model let us note that the search must be necessarily conducted at time $t=1$ (see Remark 4.1.3(p.20)) and that

$$\lambda=1\cdots(1)\quad(\text{see } A2(\text{p.20})),\qquad \delta=1\cdots(2)\quad(\text{see }(10.2.1(\text{p.54}))). \tag{A 4.6}$$

□ **Tom A 4.1** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta=1$ and $s=0$.*

(a) *V_t is nonincreasing in $t>0$.*

(b) *We have $\circledS_{\tau}\langle\tau\rangle_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.* □

● *Proof* Let $\beta=1$ and $s=0$. Then, from (5.1.14(p.23)) we have $\tilde{K}(x)=\tilde{T}(x)\le 0\cdots(1)$ for any x due to Lemma A 1.1(p.236) (g), hence from (6.5.4(p.39)) and (1) we have $V_t=\min\{\tilde{T}(V_{t-1})+V_{t-1},V_{t-1}\}=\min\{\tilde{T}(V_{t-1}),0\}+V_{t-1}=\tilde{T}(V_{t-1})+V_{t-1}\cdots(2)$ for $t>1$.

(a) Since $V_2=\tilde{T}(V_1)+V_1$, we have $V_2\le V_1$ due to (1). Suppose $V_{t-1}\ge V_t$. Then, from Lemma A 1.1(p.236) (d) we have $V_t\ge\tilde{T}(V_t)+V_t=V_{t+1}$. Hence, by induction $V_{t-1}\ge V_t$ for $t>1$, i.e., V_t is nonincreasing in $t>0$.

(b) Since $V_1=\mu$ from (6.5.3(p.39)), we have $V_1>a$. Suppose $V_{t-1}>a$. Then, noting $b>a$, from (2) we have $V_t>\tilde{T}(a)+a=a$ due to Lemma A 1.1(p.236) (l,g). Accordingly, by induction $V_{t-1}>a$ for $t>1$, hence $V_{t-1}>x_{\tilde{L}}$ for $t>1$ due to Lemma A 1.2(p.238) (d), thus $\tilde{L}(V_{t-1})<0$ for $t>1$ due to Lemma A 1.2(p.238) (e1)), so $\tilde{L}(V_{t-1})<0\cdots(3)$ for $\tau\ge t>1$. Hence, from (A 4.1(p.250)) we obtain $V_t-\beta V_{t-1}<0$ for $\tau\ge t>1$, i.e., $V_t<\beta V_{t-1}$ for $\tau\ge t>1$. Accordingly $V_\tau<\beta V_{\tau-1}<\cdots<\beta^{\tau-1}V_1$, hence $t^*_\tau=\tau$ for $\tau>1$, i.e., $\circledS_{\tau}\langle\tau\rangle_{\blacktriangle}$ for $\tau>1$. Then $\mathtt{Conduct}_{t\blacktriangle}$ for $\tau\ge t>1$ due to (3) and (A 4.5(p.250)). ∎

Let us define

$$\mathtt{S}_{18}\boxed{\circledS\blacktriangle \mid \circledcirc\,\|}=\left\{\begin{array}{l}\text{For any }\tau>1\text{ there exists }t^\bullet_\tau>1\text{ such that}\\ (1)\quad \circledS_{t^\bullet_\tau\ge\tau>1}\langle\tau\rangle_{\blacktriangle}\text{ where }\mathtt{Conduct}_{\tau\ge t>1\blacktriangle},\\ (2)\quad \circledcirc_{\tau>t^\bullet_\tau}\langle t^\bullet_\tau\rangle_{\|}\text{ where }\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}.\end{array}\right\}$$

□ **Tom A 4.2** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{R}][\mathtt{A}]\}$) *Let $\beta<1$ or $s>0$.*

(a) *V_t is nonincreasing in $t>0$ and converges to a finite $V\le x_{\tilde{K}}$ as $t\to\infty$.*

(b) *Let $\beta\mu\le a$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.

(c) *Let $\beta\mu>a$.*

1. *Let $\beta=1$.*
   - i. *Let $\mu+s\ge b$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
   - ii. *Let $\mu+s<b$. Then $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.*
2. *Let $\beta<1$ and $s=0$ ⦅$s>0$⦆.*
   - i. *Let $a<0$ ⦅$\tilde{\kappa}<0$⦆. Then $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.*
   - ii. *Let $a=0$ ⦅$\tilde{\kappa}=0$⦆.*
     1. *Let $\beta\mu+s\ge b$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
     2. *Let $\beta\mu+s<b$. Then $\circledS_{\tau>1}\langle\tau\rangle_{\blacktriangle}$ where $\mathtt{Conduct}_{\tau\ge t>1\blacktriangle}$.*
   - iii. *Let $a>0$ ⦅$\tilde{\kappa}>0$⦆.*
     1. *Let $\beta\mu+s\ge b$ or $s_{\tilde{\mathcal{L}}}\le s$. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\|}$.
     2. *Let $\beta\mu+s<b$ and $s_{\tilde{\mathcal{L}}}>s$. Then $\mathtt{S}_{18}$(p.250) $\boxed{\circ\blacktriangle \mid *\|}$ is true.* □

---

†See Section 6.1(p.27).

• *Proof* Let $\beta < 1$ or $s > 0$. Note here (A 4.6 (1,2) (p.250)).

(a) Since $x_{\tilde{K}} \le (\beta\mu + s)/\delta = \beta\mu + s = V_1$ due to Lemma A 1.3(p.238) (j2) and (6.5.3(p.39)), we have $\tilde{K}(V_1) \le 0$ due to Lemma A 1.3(p.238) (j1), hence $V_2 - V_1 \le 0$ from (A 4.4(p.250)), i.e., $V_1 \ge V_2$. Suppose $V_{t-1} \ge V_t$. Then, from (6.5.4(p.39)) and Lemma A 1.3(p.238) (e) we have $V_t \ge \min\{\tilde{K}(V_t)+V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Consider a sufficiently small $M < 0$ such that $\beta\mu + s \ge M$ and $a \ge M$, hence $V_1 \ge M$. Suppose $V_{t-1} \ge M$. Then, from Lemma A 1.3(p.238) (e) and (A 1.5 (2) (p.238)) we have $V_t \ge \min\{\tilde{K}(M) + M, \beta M\} = \min\{\beta M + s, \beta M\} \ge \min\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Hence, by induction $V_t \ge M$ for $t > 0$, i.e., V_t is lower bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.4(p.39)) we have $V = \min\{\tilde{K}(V) + V, \beta V\}$, hence $0 = \min\{\tilde{K}(V), -(1-\beta)\beta V\}$. Thus, since $\tilde{K}(V) \ge 0$, we have $V \le x_{\tilde{K}}$ from Lemma A 1.3(p.238) (j1).

(b) Let $\beta\mu \le a \cdots$ **(1)**. Then $x_{\tilde{L}} \ge \beta\mu + s = V_1$ from Lemma A 1.5(p.240) (b1) with $\lambda = 1$ and $\delta = 1$, hence $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a). Accordingly, since $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Corollary A 1.1(p.238) (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $\tau \ge t > 1$. Hence, from (A 4.3(p.250)) we have $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$. Hence $t^*_\tau = 1$ for $\tau > 1$ (see Preference Rule 7.2.1(p.43)), i.e., ⓓ$_\tau\langle 1\rangle_\parallel$ for $\tau > 1$.

(c) Let $\beta\mu > a$.

(c1) Let $\beta = 1 \cdots$ **(2)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" of the lemma. Then $(\lambda\beta\mu+s)/\delta = \mu+s \cdots$ **(3)** due to (2) and (A 4.6 (1,2) (p.250)). In addition, since $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(4)** from Lemma A 1.4(p.239) (b), we have $\tilde{K}(x_{\tilde{L}}) = \tilde{K}(x_{\tilde{K}}) = 0 \cdots$ **(5)**.

(c1i) Let $\mu + s \ge b$. Then $x_{\tilde{L}} = x_{\tilde{K}} = \mu + s = V_1$ from (4), Lemma A 1.3(p.238) (j2), (3), and (6.5.3(p.39)). Accordingly, since $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Lemma A 1.2(p.238) (e1). Hence, for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_\parallel$ for $\tau > 1$.

(c1ii) Let $\mu + s < b$. Then $x_{\tilde{L}} = x_{\tilde{K}} < \mu + s = V_1 < b$ from (4), Lemma A 1.3(p.238) (j2), and (6.5.3(p.39)), hence $b > V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} > x_{\tilde{L}}$, hence $\tilde{L}(V_{t-1}) < 0$ from Lemma A 1.2(p.238) (e1). Then, from (A 4.2(p.250)), Lemma A 1.3(p.238) (g), and (5) we have $V_t > \tilde{K}(x_{\tilde{L}}) + x_{\tilde{L}} = x_{\tilde{L}}$. Accordingly, by induction $V_{t-1} > x_{\tilde{L}}$ for $t > 1$, hence, $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ from Lemma A 1.2(p.238) (e1). Thus, for the same reason as in the proof of `Tom` A 4.1(p.250) (b) we have ⓢ$_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$ and `Conduct`$_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2) Let $\beta < 1$ and $s = 0$ ⟪$s > 0$⟫ .

(c2i) Let $a < 0$ ⟪$\tilde{\kappa} < 0$⟫ . Then $x_{\tilde{L}} < x_{\tilde{K}} < 0 \cdots$ **(6)** from Lemma A 1.4(p.239) (c ⟪d⟫). Now, since $x_{\tilde{K}} \le \beta\mu + s$ due to Lemma A 1.3(p.238) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_{\tilde{K}} \le V_1$ from (6.5.3(p.39)). Suppose $x_{\tilde{K}} \le V_{t-1}$. Then, from Lemma A 1.3(p.238) (e) we have $V_t \ge \min\{\tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}}, \beta x_{\tilde{K}}\} = \min\{x_{\tilde{K}}, \beta x_{\tilde{K}}\} = x_{\tilde{K}}$ due to $x_{\tilde{K}} < 0$. Accordingly, by induction $V_{t-1} \ge x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} > x_{\tilde{L}}$ for $t > 1$ from (6), thus $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 1.1(p.238) (a). Hence, for the same reason as in the proof of `Tom` A 4.1(p.250) (b) we have ⓢ$_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$ and `CONDUCT`$_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2ii) Let $a = 0$ ⟪$\tilde{\kappa} = 0$⟫ . Then $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(7)** from Lemma A 1.4(p.239) (c ⟪d⟫).

(c2ii1) Let $\beta\mu + s \ge b$. Then, $x_{\tilde{K}} = \beta\mu + s = V_1$ from Lemma A 1.3(p.238) (j2) and (6.5.3(p.39)). Suppose $V_{t-1} = x_{\tilde{K}}$, hence $V_{t-1} = x_{\tilde{L}}$ from (7), thus $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$. Then, from (A 4.2(p.250)) we have $V_t = \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Accordingly, by induction $V_{t-1} = x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} = x_{\tilde{L}}$ for $t > 1$ due to (7). Then, since $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.3(p.250)), hence, for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_\parallel$ for $\tau > 1$.

(c2ii2) Let $\beta\mu + s < b$. Then, since $V_1 < b$ from (6.5.3(p.39)), we have $V_{t-1} < b$ for $t > 1$ due to (a). In addition, we have $x_{\tilde{K}} < \beta\mu + s = V_1$ from Lemma A 1.3(p.238) (j2). Suppose $x_{\tilde{K}} < V_{t-1}$, hence $x_{\tilde{L}} < V_{t-1}$ from (7). Then, since $\tilde{L}(V_{t-1}) < 0$ due to Lemma A 1.2(p.238) (e1), from (A 4.2(p.250)) and Lemma A 1.3(p.238) (g) we have $V_t > \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Hence, by induction $x_{\tilde{K}} < V_{t-1}$ for $t > 1$, thus $x_{\tilde{L}} < V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 1.1(p.238) (a), for the same reason as in the proof of `Tom` A 4.1(p.250) (b) we have ⓢ$_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$ and `Conduct`$_{\tau\blacktriangle}$ for $\tau \ge t > 1$.

(c2iii) Let $a > 0$ ⟪$\tilde{\kappa} > 0$⟫ . Then $x_{\tilde{L}} > x_{\tilde{K}} \cdots$ **(8)** from Lemma A 1.4(p.239) (c ⟪d⟫).

(c2iii1) Let $\beta\mu + s \ge b$ or $s_{\tilde{L}} \le s$. First, let $\beta\mu + s \ge b$. Then, since $x_{\tilde{K}} = \beta\mu + s = V_1$ from Lemma A 1.3(p.238) (j2), we have $x_{\tilde{L}} > V_1$ from (8), hence $x_{\tilde{L}} \ge V_1$. Next, let $s_{\tilde{L}} \le s$. Then, since $x_{\tilde{L}} \ge \beta\mu + s$ due to Lemma A 1.5(p.240) (c), we have $x_{\tilde{L}} \ge V_1$ from (6.5.3(p.39)). Accordingly, whether $\beta\mu + s \ge b$ or $s_{\tilde{L}} \le s$, we have $x_{\tilde{L}} \ge V_1$, so $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ due to (a). Hence, since $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ from Corollary A 1.1(p.238) (a), for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_\parallel$ for $\tau > 1$.

(c2iii2) Let $\beta\mu+s < b \cdots$ **(9)** and $s < s_{\tilde{L}}$. Then, from (8) and Lemma A 1.5(p.240) (c) we have $x_{\tilde{K}} < x_{\tilde{L}} < \beta\mu+s = V_1 \cdots$ **(10)**, hence $\tilde{K}(V_1) < 0 \cdots$ **(11)** from Lemma A 1.3(p.238) (j1). In addition, since $V_1 < b$ due to (9) and (6.5.3(p.39)), we have $V_{t-1} < b$ for $t > 0$ from (a). Now, from (A 4.4(p.250)) and (11) we have $V_2 - V_1 < 0$, i.e., $V_2 < V_1$. Suppose $V_{t-1} > V_t$. Then, from (6.5.4(p.39)) and Lemma A 1.3(p.238) (g) we have $V_t > \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} > V_t$ for $t > 1$, i.e., V_t is *strictly decreasing* in $t > 0$. Note that $V_1 > x_{\tilde{L}}$ due to (10), so $V_1 \ge x_{\tilde{L}}$. Assume that $V_{t-1} \ge x_{\tilde{L}}$ for *all* $t > 1$, hence $V \ge x_{\tilde{L}}$. Now, from (8) and $V \le x_{\tilde{K}}$ in (a) we have the contradiction of $V \le x_{\tilde{K}} < x_{\tilde{L}} \le V$. Hence, it is impossible that $V_{t-1} \ge x_{\tilde{L}}$ for all $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 > V_2 > \cdots > V_{t^\bullet_\tau - 1} > x_{\tilde{L}} \ge V_{t^\bullet_\tau} > V_{t^\bullet_\tau + 1} > V_{t^\bullet_\tau + 2} > \cdots , \tag{A 4.7}$$

from which

$$V_{t-1} > x_{\tilde{L}}, \quad t^\bullet_\tau \ge t > 1, \qquad x_{\tilde{L}} \ge V_{t-1}, \quad t > t^\bullet_\tau. \tag{A 4.8}$$

Therefore, from Corollary A 1.1(p.238) (a) we have $\tilde{L}(V_{t-1}) < 0 \cdots$ **(12)** for $t^\bullet_\tau \ge t > 1$ and $\tilde{L}(V_{t-1}) \ge 0 \cdots$ **(13)** for $t > t^\bullet_\tau$.

1. Let $t^\bullet_\tau \geq \tau > 1$. Then, since $\tilde{L}(V_{t-1}) < 0 \cdots (14)$ for $\tau \geq t > 1$ from (12), for the same reason as in the proof of Tom A 4.1(p.250) (b) we have $\circledS_\tau\langle\tau\rangle_\blacktriangle$ for $t^\bullet_\tau \geq \tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$. Hence $\texttt{S}_{18}$(p.250) (1) is true.
2. Let $\tau > t^\bullet$. First, let $\tau \geq t > t^\bullet_\tau$. Then, since $\tilde{L}(V_{t-1}) \geq 0$ for $\tau \geq t > t^\bullet_\tau$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t^\bullet_\tau$ from (A 4.3(p.250)), thus
$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (15).$$
Next, let $t^\bullet_\tau \geq t > 1$. Then, from (12) and (A 4.1(p.250)) we have $V_t - \beta V_{t-1} < 0$ for $t^\bullet_\tau \geq t > 1$, i.e., $V_t < \beta V_{t-1}$ for $t^\bullet_\tau \geq t > 1$, hence
$$V_{t^\bullet_\tau} < \beta V_{t^\bullet_\tau-1} < \beta^2 V_{t^\bullet_\tau-2} < \cdots < \beta^{t^\bullet_\tau-1} V_1 \cdots (16).$$
From (15) and (16) we have
$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau-t^\bullet_\tau} V_{t^\bullet_\tau} < \beta^{\tau-t^\bullet_\tau+1} V_{t^\bullet_\tau-1} < \beta^{\tau-t^\bullet_\tau+2} V_{t^\bullet_\tau-2} < \cdots < \beta^{\tau-1} V_1,$$
hence we obtain $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$ due to Preference Rule 7.2.1(p.43), i.e., $\circledcirc_\tau\langle t^\bullet_\tau\rangle_\parallel$ for $\tau > 1$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet_\tau \geq t > 1$ due to (12) and (A 4.5(p.250)). Hence $\texttt{S}_{18}$(p.250) (2) is true. ∎

A 4.2 $\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$

Since $K(x) + (1-\beta)x = L(x)$ for any x due to (5.1.21(p.24)) and (5.1.20(p.24))), from (6.5.6(p.39)) we have
$$V_t - \beta V_{t-1} = \max\{L(V_{t-1}), 0\} \geq 0, \quad t > 1. \tag{A 4.9}$$
Accordingly:

1. If $L(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = L(V_{t-1})$, hence
$$V_t = L(V_{t-1}) + \beta V_{t-1} = K(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.10}$$
2. If $L(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently
$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.11}$$

Now, from (6.5.6(p.39)) with $t = 2$ we have
$$V_2 - V_1 = \max\{K(V_1), -(1-\beta)V_1\}. \tag{A 4.12}$$
Finally, from (A 4.9(p.252)) we see that
$$L(V_{t-1}) > (<)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_{t\blacktriangle}). \tag{A 4.13}$$
In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.20)) and that
$$\lambda = 1 \cdots (1) \quad (\text{see } A2(\text{p.20})), \qquad \delta = 1 \cdots (2) \quad (\text{see } (10.2.1(\text{p.54}))). \tag{A 4.14}$$

□ **Tom A 4.3** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*
(a) *V_t is nondecreasing in $t > 0$.*
(b) *$\circledS_\tau\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.* ▯

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.21(p.24)) we have $K(x) = T(x) \geq 0 \cdots (1)$ for any x due to Lemma A 2.1(p.240) (g), hence from (6.5.6(p.39)) and (1) we have
$V_t = \max\{T(V_{t-1}) + V_{t-1}, V_{t-1}\} = \max\{T(V_{t-1}), 0\} + V_{t-1} = T(V_{t-1}) + V_{t-1} \cdots (2)$ for $t > 1$.

(a) Since $V_2 = T(V_1) + V_1$, we have $V_2 \geq V_1$ due to (1). Suppose $V_{t-1} \leq V_t$. Then, from Lemma A 2.1(p.240) (d) we have $V_t \leq T(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$.

(b) Since $V_1 = a$ from (6.5.5(p.39)), we have $V_1 < b$. Suppose $V_{t-1} < b$. Then, noting $a^\star < a < b$ due to Lemma A 2.1(p.240) (n), from (2) we have $V_t < T(b) + b = b$ due to Lemma A 2.1(p.240) (c,g). Accordingly, by induction $V_{t-1} < b$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ due to Lemma A 2.2(p.241) (d), so $L(V_{t-1}) > 0 \cdots (3)$ for $\tau \geq t > 1$. Hence, from (A 4.9(p.252)) we obtain $V_t - \beta V_{t-1} > 0$ for $\tau \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $\tau \geq t > 1$. Accordingly $V_\tau > \beta V_{\tau-1} > \cdots > \beta^{\tau-1} V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., $\circledS_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$ due to (3) and (A 4.13(p.252)). ∎

Let us define

$$\texttt{S}_{19}\boxed{\circledS\blacktriangle \mid \circledcirc\parallel} = \left\{\begin{array}{ll} \multicolumn{2}{l}{\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that}} \\ (1) & \circledS_{t^\bullet_\tau \geq \tau > 1}\langle\tau\rangle_\blacktriangle \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle}, \\ (2) & \circledcirc_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\parallel \text{ where } \texttt{Conduct}_{\tau\geq t>1\blacktriangle}. \end{array}\right\}$$

□ **Tom A 4.4** ($\mathscr{A}\{\mathsf{M}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*
(a) *V_t is nondecreasing in $t > 0$ and converges to a finite $V \geq x_K$ as $t \to \infty$.*

(b) *Let* $\beta a \geq b$*. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.

(c) *Let* $\beta a < b$.

1. *Let* $\beta = 1$.
   i. *Let* $a - s \leq a^\star$*. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
   ii. *Let* $a - s > a^\star$*. Then* Ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
2. *Let* $\beta < 1$ *and* $s = 0$ $(\!(s > 0)\!)$.
   i. *Let* $b > 0$ $(\!(\kappa > 0)\!)$*. Then* Ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   ii. *Let* $b = 0$ $(\!(\kappa = 0)\!)$.
      1. *Let* $\beta a - s \leq a^\star$*. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
      2. *Let* $\beta a - s > a^\star$*. Then* Ⓢ$_{\tau>1}\langle \tau\rangle_{\blacktriangle}$ *where* $\texttt{Conduct}_{\tau\geq t>1\blacktriangle}$.
   iii. *Let* $b < 0$ $(\!(\kappa < 0)\!)$.
      1. *Let* $\beta a - s \leq a^\star$ *or* $s_{\mathcal{L}} \leq s$*. Then* ⓓ$_{\tau>1}\langle 1\rangle_{\parallel}$.
      2. *Let* $\beta a - s > a^\star$ *and* $s_{\mathcal{L}} > s$*. Then* $\texttt{S}_{19}$(p.252) $\boxed{\circ\blacktriangle \mid *\parallel}$ *is true.* □

• *Proof* Let $\beta < 1$ or $s > 0$. First note (A 4.14(p.252))

(a) Since $x_K \geq (\lambda\beta a - s)/\delta = \beta a - s = V_1$ due to Lemma A 2.3(p.241) (j2) and (6.5.5(p.39)), we have $K(V_1) \geq 0$ due to Lemma A 2.3(p.241) (j1), hence $V_2 - V_1 \geq 0$ from (A 4.12(p.252)), i.e., $V_1 \leq V_2$. Suppose $V_{t-1} \leq V_t$. Then, from (6.5.6(p.39)) and Lemma A 2.3(p.241) (e) we have $V_t \leq \max\{K(V_t)+V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \leq V_t$ for $t > 1$, i.e., V_t is nondecreasing in $t > 0$. Consider a sufficiently large $M > 0$ such that $\beta a - s \leq M$ and $b \leq M$, hence $V_1 \leq M$. Suppose $V_{t-1} \leq M$. Then, from Lemma A 2.3(p.241) (e) and (A 2.5 (2) (p.241)) we have $V_t \leq \max\{K(M) + M, \beta M\} = \max\{\beta M - s, \beta M\} \leq \max\{M, M\} = M$ due to $\beta \leq 1$ and $s \geq 0$. Hence, by induction $V_t \leq M$ for $t > 0$, i.e., V_t is upper bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.6(p.39)) we have $V = \max\{K(V) + V, \beta V\}$, hence $0 = \max\{K(V), -(1-\beta)\beta V\}$. Thus, since $K(V) \leq 0$, we have $V \geq x_K$ from Lemma A 2.3(p.241) (j1).

(b) Let $\beta a \geq b \cdots$ **(1)**. Then $x_L \leq \beta a - s = V_1$ from Lemma A 2.5(p.243) (b1) with $\lambda = 1$ and $\delta = 1$, hence $x_L \leq V_{t-1}$ for $t > 1$ from (a). Accordingly, since $L(V_{t-1}) \leq 0$ for $t > 1$ due to Corollary A 2.1(p.241) (a), we have $L(V_{t-1}) \leq 0$ for $\tau \geq t > 1$. Hence, from (A 4.11(p.252)) we have $V_t = \beta V_{t-1}$ for $\tau \geq t > 1$. Thus $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1}V_1$, i.e., $I_\tau^\tau = I_\tau^{\tau-1} = \cdots = I_\tau^1$, hence $t_\tau^* = 1$ for $\tau > 1$ due to Preference Rule 7.2.1(p.43), i.e., ⓓ$_\tau\langle 1\rangle_{\parallel}$ for $\tau > 1$.

(c) Let $\beta a < b$.

(c1) Let $\beta = 1 \cdots$ **(2)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" of the lemma. Then $(\lambda\beta a - s)/\delta = a - s \cdots$ **(3)** due to (2) and (A 4.14 (2) (p.252)). In addition, since $x_L = x_K \cdots$ **(4)** from Lemma A 2.4(p.242) (b), we have $K(x_L) = K(x_K) = 0 \cdots$ **(5)**.

(c1i) Let $a - s \leq a^\star$. Then $x_L = x_K = a - s = V_1$ from (4), Lemma A 2.3(p.241) (j2), (3), and (6.5.5(p.39)). Accordingly, since $x_L \leq V_{t-1}$ for $t > 1$ from (a), we have $L(V_{t-1}) \leq 0$ for $t > 1$ due to Lemma A 2.2(p.241) (e1). Hence, for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_{\parallel}$ for $\tau > 1$.

(c1ii) Let $a - s > a^\star$. Then $x_L = x_K > a - s = V_1 > a^\star$ from (4), Lemma A 2.3(p.241) (j2), and (6.5.5(p.39)), hence $a^\star < V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} < x_L$, hence $L(V_{t-1}) > 0$ from Lemma A 2.2(p.241) (e1). Then, from (A 4.10(p.252)), Lemma A 2.3(p.241) (g), and (4) we have $V_t < K(x_L) + x_L = K(x_K) + x_L = x_L$. Accordingly, by induction $V_{t-1} < x_L$ for $t > 1$, hence $L(V_{t-1}) > 0$ for $t > 1$ from Lemma A 2.2(p.241) (e1). Thus, for the same reason as in the proof of $\texttt{Tom}$ A 4.3(p.252) (b) we have Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$ for $\tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2) Let $\beta < 1$ and $s = 0$ $(\!(s > 0)\!)$.

(c2i) Let $b > 0$ $(\!(\kappa > 0)\!)$. Then $x_L > x_K > 0 \cdots$ **(6)** from Lemma A 2.4(p.242) (c $(\!($d$)\!)$). Now, since $x_K \geq \beta a - s$ due to Lemma A 2.3(p.241) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_K \geq V_1$ from (6.5.5(p.39)). Suppose $x_K \geq V_{t-1}$. Then, from Lemma A 2.3(p.241) (e) we have $V_t \leq \max\{K(x_K) + x_K, \beta x_K\} = \max\{x_K, \beta x_K\} = x_K$ due to $x_K > 0$. Accordingly, by induction $V_{t-1} \leq x_K$ for $t > 1$, hence $V_{t-1} < x_L$ for $t > 1$ from (6), thus $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary A 2.1(p.241) (a). Hence, for the same reason as in the proof of $\texttt{Tom}$ A 4.3(p.252) (b) we have Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$ for $\tau > 1$ and $\texttt{conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$.

(c2ii) Let $b = 0$ $(\!(\kappa = 0)\!)$. Then $x_L = x_K \cdots$ **(7)** from Lemma A 2.4(p.242) (c $(\!($d$)\!)$).

(c2ii1) Let $\beta a - s \leq a^\star$. Then, $x_K = \beta a - s = V_1$ from Lemma A 2.3(p.241) (j2) and (6.5.5(p.39)). Suppose $V_{t-1} = x_K$, hence $V_{t-1} = x_L$ from (7), thus $L(V_{t-1}) = L(x_L) = 0$. Then, from (A 4.10(p.252)) we have $V_t = K(x_K) + x_K = x_K$. Accordingly, by induction $V_{t-1} = x_K$ for $t > 1$, hence $V_{t-1} = x_L$ for $t > 1$ due to (7). Then, since $L(V_{t-1}) = L(x_L) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.11(p.252)), hence, for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_{\parallel}$ for $\tau > 1$.

(c2ii2) Let $\beta a - s > a^\star$. Then, since $V_1 > a^\star$, we have $V_{t-1} > a^\star$ for $t > 1$ due to (a). In addition, we have $x_K > \beta a - s = V_1$ from Lemma A 2.3(p.241) (j2) and (6.5.5(p.39)). Suppose $x_K > V_{t-1}$, hence $x_L > V_{t-1}$ from (7). Then, since $L(V_{t-1}) > 0$ due to Corollary A 2.1(p.241) (a), from (A 4.10(p.252)) and Lemma A 2.3(p.241) (g) we have $V_t < K(x_K) + x_K = x_K$. Hence, by induction $x_K > V_{t-1}$ for $t > 1$, thus $x_L > V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $L(V_{t-1}) > 0$ for $t > 1$ due to Corollary A 2.1(p.241) (a), for the same reason as in the proof of $\texttt{Tom}$ A 4.3(p.252) (b) we have Ⓢ$_\tau\langle\tau\rangle_{\blacktriangle}$ for $\tau > 1$ and $\texttt{Conduct}_{\tau\blacktriangle}$ for $\tau \geq t > 1$.

(c2iii) Let $b < 0$ $(\!(\kappa < 0)\!)$. Then $x_L < x_K \cdots$ **(8)** from Lemma A 2.4(p.242) (c $(\!($d$)\!)$).

(c2iii1) Let $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$. First, let $\beta a - s \leq a^\star$. Then, since $x_K = \beta a - s = V_1$ from Lemma A 2.3(p.241) (j2), we have $x_L < V_1$ from (8), hence $x_L \leq V_1$. Next, let $s_{\mathcal{L}} \leq s$. Then, since $x_L \leq \beta a - s$ due to

Lemma A 2.5(p.243) (c), we have $x_L \leq V_1$ and (6.5.5(p.39)). Accordingly, whether $\beta a - s \leq a^\star$ or $s_{\mathcal{L}} \leq s$, we have $x_L \leq V_1$, so $x_L \leq V_{t-1}$ for $t > 1$ due to (a). Hence, since $L(V_{t-1}) \leq 0$ for $t > 1$ from Corollary A 2.1(p.241) (a), for the same reason as in the proof of (b) we obtain $\text{❶}_\tau\langle 1\rangle_{\|}$ for $\tau > 1$.

(c2iii2) Let $\beta a - s > a^\star \cdots$ **(9)** and $s < s_{\mathcal{L}}$. Then, from (8) and Lemma A 2.5(p.243) (c) we have $x_K > x_L > \beta a - s = V_1 \cdots$ **(10)**, hence $K(V_1) > 0 \cdots$ **(11)** from Lemma A 2.3(p.241) (j1). In addition, since $V_1 > a^\star$ due to (9), we have $V_{t-1} > a^\star$ for $t > 0$ from (a). Now, from (A 4.12(p.252)) and (11) we have $V_2 - V_1 > 0$, i.e., $V_2 > V_1$. Suppose $V_{t-1} < V_t$. Then, from (6.5.6(p.39)) and Lemma A 2.3(p.241) (g) we have $V_t < \max\{K(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} < V_t$ for $t > 1$, i.e., V_t is *strictly increasing* in $t > 0$. Note that $V_1 < x_L$ due to (10). Assume that $V_{t-1} \leq x_L$ for *all* $t > 1$, hence $V \leq x_L$. Now, from (8) and $V \geq x_K$ due to (a) we have the contradiction $V \geq x_K > x_L \geq V$. Hence, it is impossible that $V_{t-1} \leq x_L$ for all $t > 1$, implying that there exists $t_\tau^\bullet > 1$ such that

$$V_1 < V_2 < \cdots < V_{t_\tau^\bullet - 1} < x_L \leq V_{t_\tau^\bullet} < V_{t_\tau^\bullet + 1} < V_{t_\tau^\bullet + 2} < \cdots, \tag{A 4.15}$$

from which

$$V_{t-1} < x_L, \quad t_\tau^\bullet \geq t > 1, \qquad x_L \leq V_{t-1}, \quad t > t_\tau^\bullet. \tag{A 4.16}$$

Therefore, from Corollary A 2.1(p.241) (a) we have $L(V_{t-1}) > 0 \cdots$ **(12)** for $t_\tau^\bullet \geq t > 1$ and $L(V_{t-1}) \leq 0 \cdots$ **(13)** for $t > t_\tau^\bullet$.

1. Let $t_\tau^\bullet \geq \tau > 1$. Then, since $L(V_{t-1}) > 0 \cdots$ **(14)** for $\tau \geq t > 1$ from (12), for the same reason as in the proof of `Tom` A 4.3(p.252) (b) we have $\text{ⓢ}_\tau\langle\tau\rangle_\blacktriangle$ for $t_\tau^\bullet \geq \tau > 1$ and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \geq t > 1$. Hence $\texttt{S}_{19}$(p.252) (1) is true.
2. Let $\tau > t_\tau^\bullet$. Firstly, let $\tau \geq t > t_\tau^\bullet$. Then, since $L(V_{t-1}) \leq 0$ for $\tau \geq t > t_\tau^\bullet$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \geq t > t_\tau^\bullet$ from (A 4.11(p.252)), thus

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t_\tau^\bullet} V_{t_\tau^\bullet} \cdots \textbf{(15)}.$$

Next, let $t_\tau^\bullet \geq t > 1$. Then, from (12) and (A 4.9(p.252)) we have $V_t - \beta V_{t-1} > 0$ for $t_\tau^\bullet \geq t > 1$, i.e., $V_t > \beta V_{t-1}$ for $t^\bullet \geq t > 1$, hence

$$V_{t_\tau^\bullet} > \beta V_{t_\tau^\bullet - 1} > \beta^2 V_{t_\tau^\bullet - 2} > \cdots > \beta^{t_\tau^\bullet - 1} V_1 \cdots \textbf{(16)}.$$

From (15) and (16) we have

$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t_\tau^\bullet} V_{t_\tau^\bullet} > \beta^{\tau - t_\tau^\bullet + 1} V_{t_\tau^\bullet - 1} > \beta^{\tau - t_\tau^\bullet + 2} V_{t_\tau^\bullet - 2} > \cdots > \beta^{\tau - 1} V_1,$$

hence we obtain $t_\tau^* = t_\tau^\bullet$ for $\tau > t_\tau^\bullet$ due to Preference Rule 7.2.1(p.43) , i.e., $\circledcirc_\tau\langle t_\tau^\bullet\rangle_{\|}$ for $\tau > t_\tau^\bullet$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet \geq t > 1$ due to (12) and (A 4.13(p.252)). Hence $\texttt{S}_{19}$(p.252) (2) is true. ∎

A 4.3 $\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$

Since $\tilde{K}(x) + (1-\beta)x = \tilde{L}(x)$ due to (5.1.34(p.25)) and (5.1.33(p.25)), from (6.5.8(p.39)) we have

$$V_t - \beta V_{t-1} = \min\{\tilde{L}(V_{t-1}), 0\} \leq 0, \quad t > 1. \tag{A 4.17}$$

Accordingly:

1. If $\tilde{L}(V_{t-1}) \leq 0$, then $V_t - \beta V_{t-1} = \tilde{L}(V_{t-1})$ or equivalently

$$V_t = \tilde{L}(V_{t-1}) + \beta V_{t-1} = \tilde{K}(V_{t-1}) + V_{t-1}, \quad t > 1. \tag{A 4.18}$$

2. If $\tilde{L}(V_{t-1}) \geq 0$, then $V_t - \beta V_{t-1} = 0$ or equivalently

$$V_t = \beta V_{t-1}, \quad t > 1.. \tag{A 4.19}$$

Now, from (6.5.8(p.39)) with $t = 2$ we have

$$V_2 - V_1 = \min\{\tilde{K}(V_1), -(1-\beta)V_1\}. \tag{A 4.20}$$

Finally, from (A 4.17(p.254)) we see that

$$\tilde{L}(V_{t-1}) < (>)\, 0 \Rightarrow \texttt{Conduct}_{t\blacktriangle}\ (\texttt{Skip}_t). \tag{A 4.21}$$

In this model let us note that the search must be necessarily conducted at time $t = 1$ (see Remark 4.1.3(p.20)) and that

$$\lambda = 1 \cdots (1) \quad (\text{see } A2\text{(p.20)}), \qquad \delta = 1 \quad (\text{see (10.2.1(p.54)))}. \tag{A 4.22}$$

□ `Tom` **A 4.5** ($\mathscr{A}\{\tilde{\mathsf{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta = 1$ and $s = 0$.*

(a) *V_t is nonincreasing in $t > 0$.*

(b) *We have $\text{ⓢ}_\tau\langle\tau\rangle_\blacktriangle$ where $\texttt{Conduct}_{\tau \geq t > 1\blacktriangle}$.* □

● *Proof* Let $\beta = 1$ and $s = 0$. Then, from (5.1.34(p.25)) we have $\tilde{K}(x) = \tilde{T}(x) \le 0 \cdots$ **(1)** for any x due to Lemma A 3.7(p.247) (g), hence from (6.5.8(p.39)) and (1) we have $V_t = \min\{\tilde{T}(V_{t-1}) + V_{t-1}, V_{t-1}\} = \tilde{T}(V_{t-1}) + V_{t-1} \cdots$ **(2)** for $t > 1$.

(a) Since $V_2 = \tilde{T}(V_1) + V_1$ from (2), we have $V_2 \le V_1$ due to (1). Suppose $V_{t-1} \ge V_t$. Then, from (2) and Lemma A 3.7(p.247) (d) we have $V_t \ge \tilde{T}(V_t) + V_t = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$.

(b) Since $V_1 = b$ from (6.5.7(p.39)), we have $V_1 > a$. Suppose $V_{t-1} > a$. Then, noting $b^\star > b > a$ due to Lemma A 3.7(p.247) (n), from (2) we have $V_t > \tilde{T}(a) + a = a$ due to Lemma A 3.7(p.247) (l,g). Accordingly, by induction $V_{t-1} > a$ for $t > 1$, hence $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Lemma A 3.8(p.247) (d), thus $\tilde{L}(V_{t-1}) < 0 \cdots$ **(3)** for $\tau \ge t > 1$. Hence, from (A 4.17(p.254)) we obtain $V_t - \beta V_{t-1} < 0$ for $\tau \ge t > 1$, i.e., $V_t < \beta V_{t-1}$ for $\tau \ge t > 1$. Accordingly $V_\tau < \beta V_{\tau-1} < \cdots < \beta^{\tau-1} V_1$, hence $t^*_\tau = \tau$ for $\tau > 1$, i.e., Ⓢ$_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$. Then $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$ due to (3) and (A 4.21(p.254)). ∎

Let us define

$$\texttt{S}_{20}\boxed{Ⓢ\blacktriangle \mid ◎ \parallel} = \left\{ \begin{array}{ll} \multicolumn{2}{l}{\text{For any } \tau > 1 \text{ there exists } t^\bullet_\tau > 1 \text{ such that}} \\ (1) & Ⓢ_{t^\bullet_\tau \ge \tau > 1}\langle\tau\rangle_\blacktriangle \text{ where } \texttt{Conduct}_{\tau \ge t > 1\blacktriangle}, \\ (2) & ◎_{\tau > t^\bullet_\tau}\langle t^\bullet_\tau\rangle_\parallel \text{ where } \texttt{Conduct}_{\tau \ge t > 1\blacktriangle}. \end{array} \right\}$$

□ **Tom A 4.6** ($\mathscr{A}\{\tilde{\mathtt{M}}{:}1[\mathbb{P}][\mathtt{A}]\}$) *Let $\beta < 1$ or $s > 0$.*

(a) *V_t is nonincreasing in $t > 0$ and converges to a finite $V \le x_{\tilde{K}}$ as $t \to \infty$.*

(b) *Let $\beta b \le a$. Then* ⓓ$_{\tau > 1}\langle 1\rangle_\parallel$.

(c) *Let $\beta b > a$.*

1. *Let $\beta = 1$.*
   - i. *Let $b + s \ge b^\star$. Then* ⓓ$_{\tau > 1}\langle 1\rangle_\parallel$.
   - ii. *Let $b + s < b^\star$. Then* Ⓢ$_{\tau > 1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau \ge t > 1\blacktriangle}$.
2. *Let $\beta < 1$ and $s = 0$ ($s > 0$).*
   - i. *Let $a < 0$ ($\tilde{\kappa} < 0$). Then* Ⓢ$_{\tau > 1}\langle\tau\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau \ge t > 1\blacktriangle}$.
   - ii. *Let $a = 0$ ($\tilde{\kappa} = 0$).*
     1. *Let $\beta b + s \ge b^\star$. Then* ⓓ$_{\tau > 1}\langle 1\rangle_\parallel$.
     2. *Let $\beta b + s < b^\star$. Then* Ⓢ$_\tau\langle\tau > 1\rangle_\blacktriangle$ *where* $\texttt{Conduct}_{\tau \ge t > 1\blacktriangle}$.
   - iii. *Let $a > 0$ ($\tilde{\kappa} > 0$).*
     1. *Let $\beta b + s \ge b^\star$ or $s_{\tilde{L}} \le s$. Then* ⓓ$_{\tau > 1}\langle 1\rangle_\parallel$.
     2. *Let $\beta b + s < b^\star$ and $s < s_{\tilde{L}}$. Then* $\texttt{S}_{20}\boxed{\circ\blacktriangle \mid *\triangle}$ *is true.* □

● *Proof* Let $\beta < 1$ or $s > 0$. First note (A 4.22 (1,2) (p.254)).

(a) Since $x_{\tilde{K}} \le (\beta b + s)/\delta = \beta b + s = V_1$ due to Lemma A 3.9(p.248) (j2) and (6.5.7(p.39)), we have $\tilde{K}(V_1) \le 0$ due to Lemma A 3.9(p.248) (j1), hence $V_2 - V_1 \le 0$ from (A 4.20(p.254)), i.e., $V_1 \ge V_2$. Suppose $V_{t-1} \ge V_t$. Then, from (6.5.8(p.39)) and Lemma A 3.9(p.248) (e) we have $V_t \ge \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Hence, by induction $V_{t-1} \ge V_t$ for $t > 1$, i.e., V_t is nonincreasing in $t > 0$. Consider a sufficiently small $M < 0$ such that $\beta b + s \ge M$ and $a \ge M$, hence $V_1 \ge M$. Suppose $V_{t-1} \ge M$. Then, from Lemma A 3.9(p.248) (e) and (A 3.5 (2) (p.247)) we have $V_t \ge \min\{\tilde{K}(M) + M, \beta M\} = \min\{\beta M + s, \beta M\} \ge \min\{M, M\} = M$ due to $\beta \le 1$ and $s \ge 0$. Hence, by induction $V_t \ge M$ for $t > 0$, i.e., V_t is lower bounded in t. Accordingly V_t converges to a finite V as $t \to \infty$. Then, from (6.5.8(p.39)) we have $V = \min\{\tilde{K}(V) + V, \beta V\}$, hence $0 = \min\{\tilde{K}(V), -(1-\beta)\beta V\}$. Thus, since $\tilde{K}(V) \ge 0$, we have $V \le x_{\tilde{K}}$ from Lemma A 3.9(p.248) (j1).

(b) Let $\beta b \le a \cdots$ **(1)**. Then $x_{\tilde{L}} \ge \beta b + s = V_1$ from Lemma A 3.11(p.249) (c1) with $\lambda = 1$ and $\delta = 1$, hence $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a). Accordingly, since $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Corollary A 3.2(p.248) (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $\tau \ge t > 1$. Hence, from (A 4.19(p.254)) we have $V_t = \beta V_{t-1}$ for $\tau \ge t > 1$. Thus, we have $V_\tau = \beta V_{\tau-1} = \cdots = \beta^{\tau-1} V_1$, i.e., $I^\tau_\tau = I^{\tau-1}_\tau = \cdots = I^1_\tau$, hence $t^*_\tau = 1$ for $\tau > 1$, i.e., ⓓ$_\tau\langle 1\rangle_\parallel$ for $\tau > 1$ due to Preference Rule 7.2.1(p.43).

(c) Let $\beta b > a$.

(c1) Let $\beta = 1 \cdots$ **(2)**, hence $s > 0$ due to the assumption "$\beta < 1$ or $s > 0$" of the lemma. Then, we see that $(\lambda\beta b + s)/\delta = b + s \cdots$ **(3)** due to (2(p.255)) and (A 4.22(p.254)). In addition, since $x_{\tilde{L}} = x_{\tilde{K}} \cdots$ **(4)** from Lemma A 3.10(p.249) (b), we have $\tilde{K}(x_{\tilde{L}}) = \tilde{K}(x_{\tilde{K}}) = 0 \cdots$ **(5)**.

(c1i) Let $b + s \ge b^\star$. Then $x_{\tilde{L}} = x_{\tilde{K}} = b + s = V_1$ from (4), Lemma A 3.9(p.248) (j2, (3), and (6.5.7(p.39)). Accordingly, since $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ from (a), we have $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ due to Corollary A 3.2(p.248) (a). Hence, for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_\parallel$ for $\tau > 1$.

(c1ii) Let $b + s < b^\star$. Then $x_{\tilde{L}} = x_{\tilde{K}} < b + s = V_1 < b^\star$ from (4), Lemma A 3.9(p.248) (j2), and (6.5.7(p.39)), hence $b^\star > V_{t-1}$ for $t > 1$ from (a). Suppose $V_{t-1} > x_{\tilde{L}}$, hence $\tilde{L}(V_{t-1}) < 0$ from Corollary A 3.2(p.248) (a). Then, from (A 4.18(p.254)), Lemma A 3.9(p.248) (g), and (5) we have $V_t > \tilde{K}(x_{\tilde{L}}) + x_{\tilde{L}} = x_{\tilde{L}}$. Accordingly, by induction $V_{t-1} > x_{\tilde{L}}$ for $t > 1$, hence, $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ from Corollary A 3.2(p.248) (a). Thus, for the same reason as in the proof of $\texttt{Tom}$ A 4.5(p.254) (b) we have Ⓢ$_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2) Let $\beta < 1$ and $s = 0$ ($s > 0$).

(c2i) Let $a < 0\,(\!(\tilde{\kappa} < 0)\!)$. Then $x_{\tilde{L}} < x_{\tilde{K}} < 0 \cdots (6)$ from Lemma A 3.10(p.249) (c (d)). Now, since $x_{\tilde{K}} \le \beta b + s$ due to Lemma A 3.9(p.248) (j2) with $\lambda = 1$ and $\delta = 1$, we have $x_{\tilde{K}} \le V_1$ from (6.5.7(p.39)). Suppose $x_{\tilde{K}} \le V_{t-1}$. Then, from Lemma A 3.9(p.248) (e) we have $V_t \ge \min\{\tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}}, \beta x_{\tilde{K}}\} = \min\{x_{\tilde{K}}, \beta x_{\tilde{K}}\} = x_{\tilde{K}}$ due to $x_{\tilde{K}} < 0$. Accordingly, by induction $V_{t-1} \ge x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} > x_{\tilde{L}}$ for $t > 1$ from (6), thus $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 3.2(p.248) (a). Hence, for the same reason as in the proof of `Tom` A 4.5(p.254) (b) we have $\circledS_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$, and $\texttt{CONDUCT}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2ii) Let $a = 0\,(\!(\tilde{\kappa} = 0)\!)$. Then $x_{\tilde{L}} = x_{\tilde{K}} \cdots (7)$ from Lemma A 3.10(p.249) (c (d)).

(c2ii1) Let $\beta b + s \ge b^\star$. Then, $x_{\tilde{K}} = \beta b + s = V_1$ from Lemma A 3.9(p.248) (j2) and (6.5.7(p.39)). Suppose $V_{t-1} = x_{\tilde{K}}$, hence $V_{t-1} = x_{\tilde{L}}$ from (7), thus $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$. Then, from (A 4.18(p.254)) we have $V_t = \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Accordingly, by induction $V_{t-1} = x_{\tilde{K}}$ for $t > 1$, hence $V_{t-1} = x_{\tilde{L}}$ for $t > 1$ due to (7). Then, since $\tilde{L}(V_{t-1}) = \tilde{L}(x_{\tilde{L}}) = 0$ for $t > 1$, we have $V_t = \beta V_{t-1}$ for $t > 1$ from (A 4.19(p.254)), hence, for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_{\parallel}$ for $\tau > 1$.

(c2ii2) Let $\beta b + s < b^\star$. Then, since $V_1 < b^\star$ from (6.5.7(p.39)), we have $V_{t-1} < b^\star$ for $t > 1$ due to (a). In addition, we have $x_{\tilde{K}} < \beta b + s = V_1$ from Lemma A 3.9(p.248) (j2). Suppose $x_{\tilde{K}} < V_{t-1}$, hence $x_{\tilde{L}} < V_{t-1}$ from (7). Then, since $\tilde{L}(V_{t-1}) < 0$ due to Corollary A 3.2(p.248) (a), from (A 4.18(p.254)) and Lemma A 3.9(p.248) (g) we have $V_t > \tilde{K}(x_{\tilde{K}}) + x_{\tilde{K}} = x_{\tilde{K}}$. Hence, by induction $x_{\tilde{K}} < V_{t-1}$ for $t > 1$, thus $x_{\tilde{L}} < V_{t-1}$ for $t > 1$ due to (7). Accordingly, since $\tilde{L}(V_{t-1}) < 0$ for $t > 1$ due to Corollary A 3.2(p.248) (a), for the same reason as in the proof of `Tom` A 4.5(p.254) (b) we have $\circledS_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$.

(c2iii) Let $a > 0\,(\!(\tilde{\kappa} > 0)\!)$. Then $x_{\tilde{L}} > x_{\tilde{K}} \cdots (8)$ from Lemma A 3.10(p.249) (c (d)).

(c2iii1) Let $\beta b + s \ge b^\star$ or $s_{\tilde{L}} \le s$. First let $\beta b + s \ge b^\star$. Then, since $x_{\tilde{K}} = \beta b - s = V_1$ from Lemma A 3.9(p.248) (j2), we have $x_{\tilde{L}} > V_1$ from (8), hence $x_{\tilde{L}} \ge V_1$. Next let $s_{\tilde{L}} \le s$. Then, since $x_{\tilde{L}} \ge \beta b + s$ due to Lemma A 3.11(p.249) (d), we have $x_{\tilde{L}} \ge V_1$. Accordingly, whether $\beta b + s \ge b$ or $s_{\tilde{L}} \le s$, we have $x_{\tilde{L}} \ge V_1$, thus $x_{\tilde{L}} \ge V_{t-1}$ for $t > 1$ due to (a). Hence, since $\tilde{L}(V_{t-1}) \ge 0$ for $t > 1$ from Corollary A 3.2(p.248) (a), for the same reason as in the proof of (b) we obtain ⓓ$_\tau\langle 1\rangle_{\parallel}$ for $\tau > 1$.

(c2iii2) Let $\beta b + s < b^\star \cdots (9)$ and $s < s_{\tilde{L}}$. Then, from (8) and Lemma A 3.11(p.249) (d) we have $x_{\tilde{K}} < x_{\tilde{L}} < \beta b + s = V_1 \cdots (10)$, hence $\tilde{K}(V_1) < 0 \cdots (11)$ from Lemma A 3.9(p.248) (j1). In addition, since $V_1 < b^\star$ due to (9), we have $V_{t-1} < b^\star$ for $t > 0$ from (a). Now, from (A 4.20(p.254)) and (11) we have $V_2 - V_1 < 0$, i.e., $V_2 < V_1$. Suppose $V_{t-1} > V_t$. Then, from Lemma A 3.9(p.248) (g) we have $V_t > \min\{\tilde{K}(V_t) + V_t, \beta V_t\} = V_{t+1}$. Accordingly, by induction $V_{t-1} > V_t$ for $t > 1$, i.e., V_t is *strictly decreasing* in $t > 0$. Note that $V_1 > x_{\tilde{L}}$ due to (10). Assume that $V_{t-1} \ge x_{\tilde{L}}$ for *all* $t > 1$, hence $V \ge x_{\tilde{L}}$ due to (a). Then, from (8) and $V \le x_{\tilde{K}}$ due to (a) we have the contradiction of $V \le x_{\tilde{K}} < x_{\tilde{L}} \le V$. Hence, it is impossible that $V_{t-1} \ge x_{\tilde{L}}$ for all $t > 1$, implying that there exists $t^\bullet_\tau > 1$ such that

$$V_1 > V_2 > \cdots > V_{t^\bullet_\tau - 1} > x_{\tilde{L}} \ge V_{t^\bullet_\tau} > V_{t^\bullet_\tau + 1} > V_{t^\bullet_\tau + 2} > \cdots, \tag{A 4.23}$$

from which

$$V_{t-1} > x_{\tilde{L}}, \quad t^\bullet_\tau \ge t > 1, \qquad x_{\tilde{L}} \ge V_{t-1}, \quad t > t^\bullet_\tau. \tag{A 4.24}$$

Therefore, from Corollary A 3.2(p.248) (a) we have $\tilde{L}(V_{t-1}) < 0 \cdots (12)$ for $t^\bullet_\tau \ge t > 1$ and $\tilde{L}(V_{t-1}) \ge 0 \cdots (13)$ for $t > t^\bullet_\tau$.

1. Let $t^\bullet_\tau \ge \tau > 1$. Then, since $\tilde{L}(V_{t-1}) < 0 \cdots (14)$ for $\tau \ge t > 1$ from (12), for the same reason as in the proof of `Tom` A 4.5(p.254) (b) we have $\circledS_\tau\langle\tau\rangle_\blacktriangle$ for $\tau > 1$, and $\texttt{Conduct}_{t\blacktriangle}$ for $\tau \ge t > 1$. Hence $\texttt{S}_{20}$(p.255) (1) is true.
2. Let $\tau > t^\bullet_\tau$. Firstly, let $\tau \ge t > t^\bullet_\tau$. Then, since $\tilde{L}(V_{t-1}) \ge 0$ for $\tau \ge t > t^\bullet_\tau$ from (13), we have $V_t = \beta V_{t-1}$ for $\tau \ge t > t^\bullet_\tau$ from (A 4.19(p.254)), thus
$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} \cdots (15).$$
Next, let $t^\bullet_\tau \ge t > 1$. Then, from (12) and (A 4.17(p.254)) we have $V_t - \beta V_{t-1} < 0$ for $t^\bullet_\tau \ge t > 1$, i.e., $V_t < \beta V_{t-1}$ for $t^\bullet_\tau \ge t > 1$, hence
$$V_{t^\bullet_\tau} < \beta V_{t^\bullet_\tau - 1} < \beta^2 V_{t^\bullet_\tau - 2} < \cdots < \beta^{t^\bullet_\tau - 1} V_1 \cdots (16).$$
From (15) and (16) we have
$$V_\tau = \beta V_{\tau-1} = \beta^2 V_{\tau-2} = \cdots = \beta^{\tau - t^\bullet_\tau} V_{t^\bullet_\tau} < \beta^{\tau - t^\bullet_\tau + 1} V_{t^\bullet_\tau - 1} < \beta^{\tau - t^\bullet_\tau + 2} V_{t^\bullet_\tau - 2} < \cdots < \beta^{\tau-1} V_1,$$
hence we obtain $t^*_\tau = t^\bullet_\tau$ for $\tau > t^\bullet_\tau$ due to Preference Rule 7.2.1(p.43), i.e., $\circledcirc_\tau\langle t^\bullet_\tau\rangle$ for $\tau > t^\bullet_\tau$. In addition, we have $\texttt{Conduct}_{t\blacktriangle}$ for $t^\bullet_\tau \ge t > 1$ due to (12) and (A 4.21(p.254)). Hence $\texttt{S}_{20}(2)$ is true. ∎

A 5 Optimal Initiating Time of Markovian Decision Processes

This section defines the optimal initiating time (`OIT`) for Markovian decision processes (`MDP`) [5,Howard,1960], which can be regarded as the most general model of decision processes.

A 5.1 Standard Definition of Markovian Decision Processes

A 5.1.1 Maximization `MDP`

Let the process be in a *state* i at a time t (see Figure 2.2.1(p.11)), and if an action x is taken at that time, then a *reward* $r(i,x)$ can be obtained and the present state i changes into j at the next time $t-1$ with a known probability $p(j|i,x)$. By $v_t(i)$ let us denote the maximum of the total expected present discounted *profit* gained over a given planning horizon starting from a time t in a state i. Then we have

$$v_t(i) \;=\; \max_x\bigl\{r(i,x) + \beta \textstyle\sum_j p(j|i,x) v_{t-1}(j)\bigr\}, \quad t > 0, \tag{A 5.1}$$

where $v_0(i)$ is a profit specified for a reason inherent in the process; in many cases, $v_0(i) = \max_x r(i,x)$. Let us call the decision process the *maximization* `MDP`.

A 5.1.2 Minimization `MDP`

This is the inverse of the maximization `MDP` where if an action x is taken at a given time t in a state i, a *cost* $c(i,x)$ must be paid. By $v_t(i)$ let us denote the minimum of the total expected present discounted *cost* over a given planning horizon from starting a time t in a state i. Then we have

$$v_t(i) \;=\; \min_x\big\{c(i,x) + \beta \textstyle\sum_j p(j|i,x)v_{t-1}(j)\big\}, \quad t>0, \tag{A 5.2}$$

where $v_0(i)$ is a cost specified for a reason inherent in the process; in many cases, $v_0(i)=\min_x c(i,x)$. Let us call the decision process the *minimization* `MDP`.

A 5.2 Optimal Initiating Time

A 5.2.1 Initiating State $i_\circ$

By $p_t(i)$ let us represent the probability that process is in state i at time t. Then the maximum (minimum) of the total expected present discounted value of profit (cost) from initiating the process at time t ($\tau \geq t \geq \delta_q$) (see H1d(p.8) †) is given by

$$V_t \overset{\text{def}}{=} \textstyle\sum_{i_t} p_t(i_t)v_t(i_t) \tag{A 5.3}$$

where i_t represents the state at time t.

A 5.2.2 Relationship between $V_{[\tau]}$ and $V_{\beta[\tau]}$ (see Section 7.2.4.2(p.43))

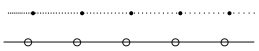
In this section, by using some examples, let us demonstrate that the monotonicity of

$$V_{[\tau]} = \{V_\tau, V_{\tau-1}, V_{\tau-2}, \cdots, V_{\delta_q}\} \quad \text{(original sequence)}$$

is not always inherited to

$$V_{\beta[\tau]} = \{V_\tau, \beta V_{\tau-1}, \beta^2 V_{\tau-2}, \cdots, \beta^\tau V_{\delta_q}\} \quad (\beta\text{-adjusted sequence}).$$

Below let



$V_{[\tau]} \;\rightarrow$ (dotted line with dots)
$V_{\beta[\tau]} \;\rightarrow$ (solid line with circles)
$t^*_\tau \;\rightarrow$ ● (optimal initiating time)

□ *Example* **A 5.1 (maximization `MDP`)** Suppose $V_{[\tau]}$ is strictly increasing in t where

$$V_\tau > V_{\tau-1} > V_{\tau-2} > \cdots > V_0 > 0.$$

In this case, as seen in Figure A 5.1(p.257) below, we have $V_\tau > \beta V_{\tau-1} > \beta^2 V_{\tau-2} > \cdots > \beta^\tau V_0 > 0$, i.e., the monotonicity of $V_{[\tau]}$ is <u>inherited</u> to $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 16$ (ⓢ). ∎

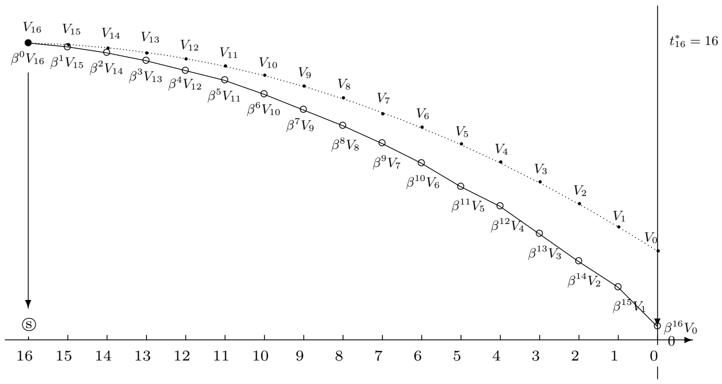


Figure A 5.1: Inheritance of monotonicity

† $\delta_q = 1$ for Model 0 and $\delta_q = 0$ for Models 1/2.

□ *Example* **A 5.2 (maximization `MDP`)** Suppose $V_{[\tau]}$ is strictly increasing in t where

$$V_\tau > \beta V_{\tau-1} > V_{\tau-2} > \cdots > V_{\tau-t'} > 0 > V_{\tau-t'-1} > \cdots > V_0.$$

In this case, as seen in Figure A 5.2(p.258) below, the monotonicity in $V_{[\tau]}$ collapses in $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 16$ (ⓢ). ∎

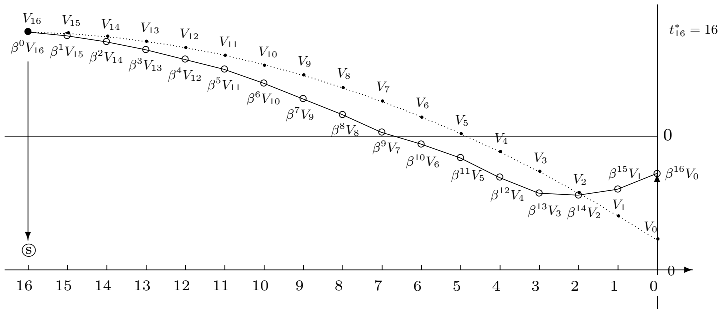


Figure A 5.2: Collapse of monotonicity

□ *Example* **A 5.3 (maximization `MDP`)** Suppose $V_{[\tau]}$ is strictly decreasing in t where

$$0 < V_\tau < \beta V_{\tau-1} < V_{\tau-2} < \cdots < V_0.$$

In this case, as seen in Figure A 5.3(p.258) below, the monotonicity in $V_{[\tau]}$ collapses in $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 6$, i.e., unregenerate (◎). ∎

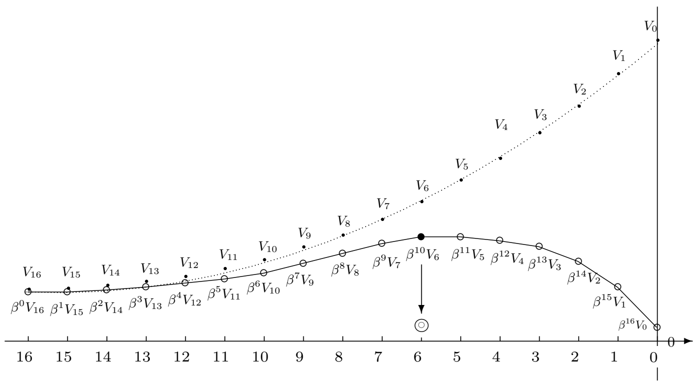


Figure A 5.3: Collapse of monotonicity

□ *Example* **A 5.4 (minimization `MDP`)** Suppose $V_{[\tau]}$ is strictly decreasing in t where

$$0 < V_\tau < \beta V_{\tau-1} < \cdots < V_{\tau-t'} < 0 < V_{\tau-t'-1} < \cdots < V_0.$$

In this case, as seen in Figure A 5.4(p.258) below, the monotonicity in $V_{[\tau]}$ collapses in $V_{\beta[\tau]}$ where the optimal initiating time is given by $t^*_{16} = 16$ (ⓢ). ∎

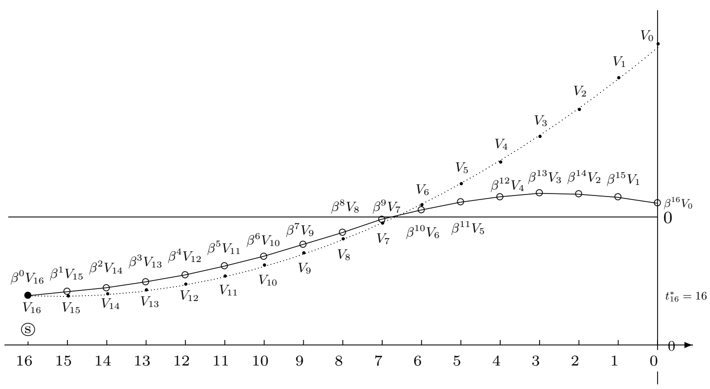


Figure A 5.4: Collapse of monotonicity

A 6 Calculation of Solutions

A 6.1 $x_{K_\mathbb{R}}$, $x_{L_\mathbb{R}}$, and $s_{\mathcal{L}_\mathbb{R}}$

The following lemma is used to numerically calculate the solutions x_K, x_L, and $s_\mathcal{L}$ (see Section 5.2(p.25)).

Lemma A 6.1

(a) $\min\{a, (\lambda\beta\mu - s)/\delta\} \leq x_K \leq \max\{b, 0\}$.

(b) $\min\{a, (\lambda\beta\mu - s)/\lambda\} \leq x_L \leq b$.

(c) $0 \leq s_\mathcal{L} \leq \lambda\beta\mu - \min\{a, 0\}$. □

● *Proof* (a)

- If $x \leq a \cdots$ **(1)**, then from (10.2.4 (1) (p.55)) we have $K(x) = \delta\big((\lambda\beta\mu - s)/\delta - x\big)$, hence $K(x) \geq 0$ for $x \leq (\lambda\beta\mu - s)/\delta$. From this and (1) we have $K(x) \geq 0$ for $x \leq \min\{a, (\lambda\beta\mu - s)/\delta\}$, so $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) \geq 0$.
  1. Let $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) > 0$. Then $\min\{a, (\lambda\beta\mu - s)/\delta\} < x_K \cdots$ **(2)** due to Corollary 10.2.2(p.56) (a).
  2. Let $K(\min\{a, (\lambda\beta\mu - s)/\delta\}) = 0$.
     - If $\beta = 1$ and $s = 0$, then $\min\{a, (\lambda\beta\mu - s)/\delta\} \geq x_K$ due to Lemma 10.2.2(p.55) (i). Since $\min\{a, (\lambda\beta\mu - s)/\delta\} \leq a < b = x_K$ from Lemma 10.2.2(p.55) (i), we have $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K$.
     - If $\beta < 1$ or $s > 0$, then $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K$ due to Lemma 10.2.2(p.55) (j1).

     Accordingly, whether "$\beta = 1$ and $s = 0$" or "$\beta < 1$ or $s > 0$", we have $\min\{a, (\lambda\beta\mu - s)/\delta\} = x_K \cdots$ **(3)**.

  Thus, from (2) and (3) we have $\min\{a, (\lambda\beta\mu - s)/\delta\} \leq x_K \cdots$ **(4)**.
- If $b \leq x \cdots$ **(5)**, then from (10.2.5 (2) (p.55)) we have $K(x) \leq 0$ for $0 \leq x$. From this and (5) we have $K(x) \leq 0$ for $\max\{b, 0\} \leq x$, so $0 \geq K(\max\{b, 0\})$. Accordingly, we have $x_K \leq \max\{b, 0\} \cdots$ **(6)** due to Corollary 10.2.2(p.56) (a).

From (4) and (6) the assertion becomes true.

(b)

- If $x \leq a \cdots$ **(7)**, then from (10.2.3 (1) (p.55)) we have $L(x) = \lambda\beta\big((\lambda\beta\mu - s)/\lambda\beta - x\big)$, hence $L(x) \geq 0$ for $x \leq (\lambda\beta\mu - s)/\lambda\beta$. From this and (7) we have $L(x) \geq 0$ for $x \leq \min\{a, (\lambda\beta\mu - s)/\lambda\beta\}$, so $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) \geq 0$.
  1. Let $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) > 0$. Then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} < x_L$ due to Corollary 10.2.1(p.55) (a), hence $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \leq x_L \cdots$ **(8)**.
  2. Let $L(\min\{a, (\lambda\beta\mu - s)/\lambda\beta\}) = 0$.
     - If $s = 0$, then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \geq x_L$ due to Lemma 10.2.1(p.55) (d). Since $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \leq a < b = x_L$ from Lemma 10.2.1(p.55) (d), we have $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L$.
     - If $s > 0$, then $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L$ due to Lemma 10.2.1(p.55) (e1).

     Accordingly, whether $s = 0$ or $s > 0$, we have $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} = x_L \cdots$ **(9)**.

  Thus, from (8) and (9) we have $\min\{a, (\lambda\beta\mu - s)/\lambda\beta\} \leq x_L \cdots$ **(10)**.
- If $b \leq x \cdots$ **(11)**, then from (5.1.3(p.23)) and Lemma 10.1.1(p.53) (g) we have $L(x) = -s \leq 0$, hence $0 \geq L(b)$. Accordingly, due to Corollary 10.2.1(p.55) (a) we have $x_L \leq b \cdots$ **(12)**.

From (8) and (12) the assertion becomes true.

(c) From (5.1.5(p.23)) and (5.1.3(p.23)) we have $\mathcal{L}(0) = L(\lambda\beta\mu) = \lambda\beta T(\lambda\beta\mu) \geq 0 \cdots$ **(13)** due to Lemma 10.1.1(p.53) (g). Now, consider a sufficiently large $s > 0$ for which $\lambda\beta\mu - s \leq a \cdots$ **(14)** and $\lambda\beta\mu - s \leq 0 \cdots$ **(15)**, so we have $s \geq \lambda\beta\mu - a$ and $s \geq \lambda\beta\mu$, hence $s \geq \max\{\lambda\beta\mu - a, \lambda\beta\mu\} = \lambda\beta\mu + \max\{-a, 0\} = \lambda\beta\mu - \min\{a, 0\} \cdots$ **(16)**. Then, from (5.1.5(p.23)) and (5.1.3(p.23)) we have

$$\begin{aligned}\mathcal{L}(s) &= \lambda\beta T(\lambda\beta\mu - s) - s \\ &= \lambda\beta(\mu - \lambda\beta\mu + s) - s \quad \text{(due to (14) and Lemma 10.1.1(p.53) (f))} \\ &= \lambda\beta\mu - \lambda\beta(\lambda\beta\mu - s) - s \\ &= \lambda\beta\mu - s - \lambda\beta(\lambda\beta\mu - s) \\ &= (1 - \lambda\beta)(\lambda\beta\mu - s).\end{aligned}$$

Here, since $1 \geq \lambda\beta$, due to (15) we have $\mathcal{L}(s) \leq 0$ for $s \geq \lambda\beta\mu - \min\{a, 0\}$ due to (16), hence $\mathcal{L}(\lambda\beta\mu - \min\{a, 0\}) \leq 0$. Thus, from (13) we have $\mathcal{L}(0) \geq 0 \geq \mathcal{L}(\lambda\beta\mu - \min\{a, 0\})$, hence due to Lemma 10.2.4(p.57) (a) we have $0 \leq s_\mathcal{L} \leq \lambda\beta\mu - \min\{a, 0\}$. ∎

A 6.2 $x_{\tilde{K}}$, $x_{\tilde{L}}$, and $s_{\tilde{\mathcal{L}}}$

Lemma A 6.2

(a) $\max\{b, (\lambda\beta\mu + s)/\delta\} \geq x_{\tilde{K}} \geq \min\{a, 0\}$.
(b) $\max\{b, (\lambda\beta\mu + s)/\lambda\beta\} \geq x_{\tilde{L}} \geq a$.
(c) $0 \leq s_{\tilde{\mathcal{L}}} \leq -\lambda\beta\mu + \max\{b, 0\}$. □

• *Proof* Applying the reverse operation $\mathcal{R}$ (see Section 12.1.1(p.67)) to Lemma A 6.1(p.259) leads to

⟨a⟩ $\min\{-\hat{a}, (-\lambda\beta\hat{\mu} - s)/\delta\} \leq -\hat{x}_K \leq \max\{-\hat{b}, 0\}$.
⟨b⟩ $\min\{-\hat{a}, (-\lambda\beta\hat{\mu} - s)/\lambda\}\beta \leq -\hat{x}_L \leq -\hat{b}$.
⟨c⟩ $0 \leq s_{\mathcal{L}} \leq -\lambda\beta\hat{\mu} - \min\{-\hat{a}, 0\}$.

The above can be rewritten as below:

⟨a⟩ $-\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\delta\} \leq -\hat{x}_K \leq -\min\{\hat{b}, 0\}$.
⟨b⟩ $-\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\lambda\}\beta \leq -\hat{x}_L \leq -\hat{b}$.
⟨c⟩ $0 \leq s_{\mathcal{L}} \leq -\lambda\beta\hat{\mu} + \max\{\hat{a}, 0\}$,

which can be rearranged as below:

⟨a⟩ $\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\delta \geq \hat{x}_K \geq \min\{\hat{b}, 0\}$.
⟨b⟩ $\max\{\hat{a}, (\lambda\beta\hat{\mu} + s)/\lambda\}\beta \geq \hat{x}_L \geq \hat{b}$.
⟨c⟩ $0 \leq s_{\mathcal{L}} \leq -\lambda\beta\hat{\mu} + \max\{\hat{a}, 0\}$.

Applying the operation $\mathcal{C}_{\mathbb{R}}$ (see Lemma 12.3.1(p.69) (b,g,h,i) to the above yields

⟨a⟩ $\max\{\check{b}, (\lambda\beta\check{\mu} + s)/\delta\} \geq x_{\check{\tilde{K}}} \geq \min\{\check{a}, 0\}$.
⟨b⟩ $\max\{\check{b}, (\lambda\beta\check{\mu} + s)/\lambda\}\beta \geq x_{\check{\tilde{L}}} \geq \check{a}$.
⟨c⟩ $0 \leq s_{\check{\tilde{\mathcal{L}}}} \leq -\lambda\beta\check{\mu} + \max\{\check{b}, 0\}$.

Finally, applying the operation $\mathcal{I}_{\mathbb{R}}$ (see Lemma 12.3.3(p.71) (b,g,h,i), we obtain (a)-(c) of this lemma. ∎

A 7 Others

A 7.1 Monotonicity of Solution

Proposition A 7.1 The solution x_t of a given equation $g_t(x) = 0$ we have the following monotonicity:

Case A Let $g_t(x)$ is nondecreasing in x for all t.

(I) If $g_t(x)$ is nondecreasing in t for all x, then x_t is nonincreasing in t.

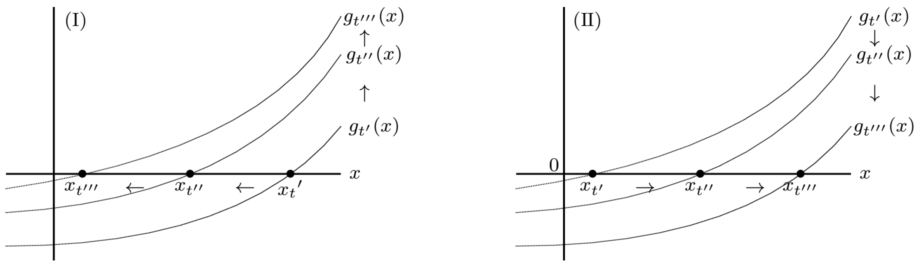
(II) If $g_t(x)$ is nonincreasing in t for all x, then x_t is nondecreasing in t.

Case B Let $g_t(x)$ is nonincreasing in x for all t.

(III) If $g_t(x)$ is nondecreasing in t for all x, then x_t is nondecreasing in t.

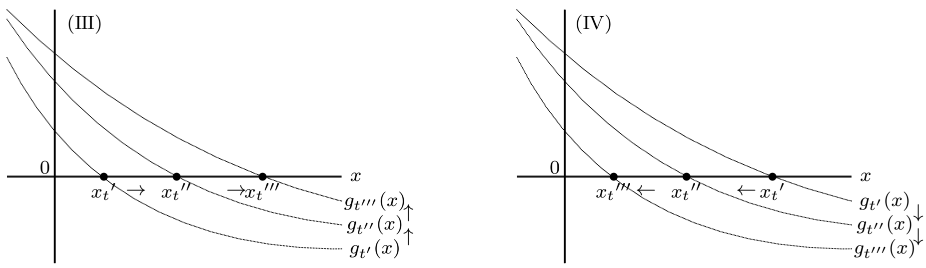
(IV) If $g_t(x)$ is nonincreasing in t for all x, then x_t is nonincreasing in t. □

● *Proof* Evident from Figures A 7.1(p.261) and A 7.2(p.261) below:



$g_t(x)$ is nondecreasing in t (I) $g_t(x)$ is nonincreasing in t (II)

Figure A 7.1: Case A: $g_t(x)$ is nondecreasing in x



$g_t(x)$ is nondecreasing in t (III) $g_t(x)$ is nonincreasing in t (IV)

Figure A 7.2: Case B: $g_t(x)$ is nonincreasing in x

A 7.2 Uniform Probability Density Function

For given a and b with $-\infty < a < b < \infty$ let consider the uniform probability density function:

$$f(x) = \begin{cases} 0, & x < a, \\ 1/(b-a), & a \leq x \leq b, \\ 0, & b < x, \end{cases} \quad \text{(A 7.1)}$$

where the expectation is $\mu = 0.5(a+b)$. Then we have:

$$T(x) = \begin{cases} 0.5(a+b) - x, & x \leq a, & \cdots (1), \\ 0.5(b-x)^2/(b-a), & a \leq x \leq b, & \cdots (2), \\ 0, & b \leq x, & \cdots (3), \end{cases} \quad \text{(A 7.2)}$$

where (1) and (3) are immediate from Lemma 10.1.1(p.53) (f,g). Let $a \leq x \leq b \cdots (2)$. Then, from (5.1.2(p.23)) we have:

$$\begin{aligned} T(x) &= \textstyle\int_a^b \max\{\xi - x, 0\}(b-a)^{-1} d\xi \\ &= \textstyle\int_x^b (\xi - x)(b-a)^{-1} d\xi \\ &= (b-a)^{-1} \textstyle\int_0^{b-x} \eta d\eta \quad (\eta = \xi - x) \\ &= (b-a)^{-1} \tfrac{1}{2} \eta |_0^{b-x} \\ &= 0.5(b-x)^2/(b-a). \end{aligned}$$

A 7.3 Graphs of $T_{\mathbb{R}}(x)$

From Lemma 10.1.1(p.53) (b,f,g) one immediately sees that $T_{\mathbb{R}}(x)$ can be depicted as in Figure A 7.3(p.262) (I) below. Similarly, from Lemma 10.2.2(p.55) (b), (10.2.4 (1) (p.55)), and (10.2.5 (2) (p.55)) we immediately see that $K_{\mathbb{R}}(x)$ can be depicted as in Figure A 7.3(p.262) (II) below.

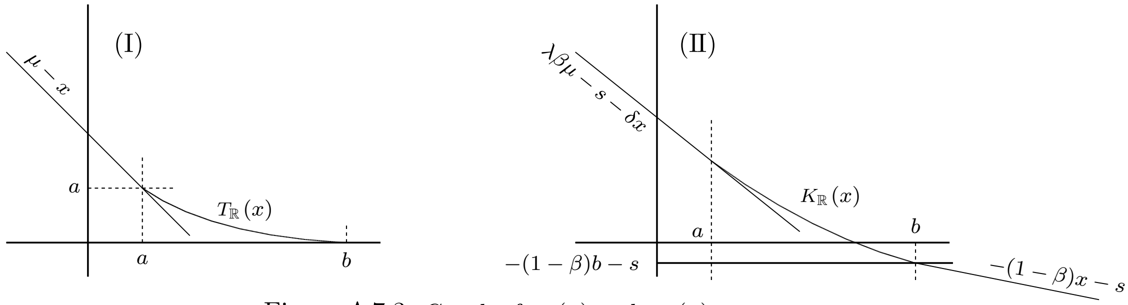


Figure A 7.3: Graph of $T_{\mathbb{R}}(x)$ and $K_{\mathbb{P}}(x)$

A 7.4 Graph of $T_{\mathbb{P}}(x)$

From Lemma 13.2.1(p.91) (b,f,g) we immediately see that $T_{\mathbb{P}}(x)$ can be depicted as in Figure A 7.4 below where $a^\star < a$ due to Lemma 13.2.1(p.91) (n).

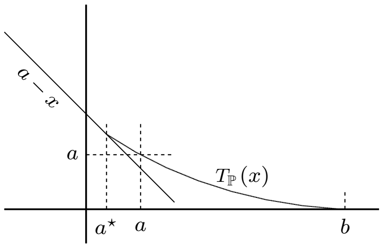


Figure A 7.4: Graph of $T_{\mathbb{P}}(x)$

A 7.5 $a^\star$ and $x^\star$

Lemma A 7.1 ($a^\star$ and $x^\star$) *Let $f(x)$ is the uniform distribution function* (see (A 7.1(p.261))). *Then we have*

$$a^\star = 2a - b, \tag{A 7.3}$$
$$x^\star = 2a - b. \quad \square \tag{A 7.4}$$

● *Proof* First we have:

$p(z) = 1$ for $z \le a$ from (5.1.28 (1) (p.24)),

$p(z) = \int_z^b f(\xi)d\xi = \int_z^b 1/(b-a)d\xi = (b-z)/(b-a)$ for $a \le z \le b$ from (5.1.18(p.24)) and (A 7.1(p.261)),

$p(z) = 0$ for $b \le z$ from (5.1.29 (2) (p.24)).

Hence we get

$$T(z,x) \overset{\text{def}}{=} p(z)(z-x) = \begin{cases} z - x, & z \le a, \\ (b-z)(z-x)/(b-a), & a \le z \le b, \\ 0, & b \le z. \end{cases} \tag{A 7.5}$$

Here, by $z(x)$ let us denote z maximizing $T(z,x)$ for any given x. Then (5.1.19(p.24)) and (5.1.25(p.24)) can be expressed as

$$\begin{aligned} T(x) &= \max_z T(z,x) \\ &= T(z(x),x) \cdots \mathbf{(1)} \\ &= p(z(x))(z(x) - x). \end{aligned}$$

Furthermore let us define

$$g^*(z,x) = (b-z)(z-x)/(b-a), \quad z,x \in (-\infty, \infty),$$

which is a *quadratic expression* of z for any given x. Then (A 7.5) can be rewritten as below.

$$T(z,x) = \begin{cases} z - x, & z \le a & \cdots \mathbf{(2)}, \\ g^*(z,x), & a \le z \le b & \cdots \mathbf{(3)}, \\ 0, & b \le z & \cdots \mathbf{(4)}. \end{cases}$$

†See (5.1.25(p.24))

Here by $z^*(x)$ let us denote z attaining the maximum of $g^*(z,x)$ for a given $x \in (-\infty, \infty)$. Then clearly

$$\bigstar\ z^*(x) = (b+x)/2 \cdots (5).$$

The relationship between $T(z,x)$ and $g^*(z,x)$ can be depicted as in Figure A 7.5(p.263) below.

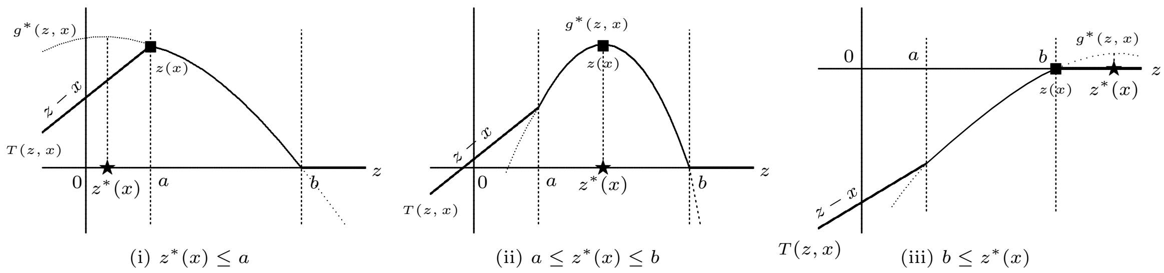


Figure A 7.5: Relationship between $T(z,x)$ (broken curve ——) and $g^*(z,x)$ (smooth curve ·········)

The three curves in Figure A 7.5(p.263) above correspond to the following three cases respectively:

$$\bigstar\ z^*(x) \le a \quad \cdots \text{(i)},$$
$$\bigstar\ a \le z^*(x) \le b \quad \cdots \text{(ii)},$$
$$\bigstar\ b \le z^*(x) \quad \cdots \text{(iii)}.$$

Then, from this figure we see:

1. Let $z^*(x) \le a \cdots (1)$. Then, from (5) we have $(b+x)/2 \le a$, hence $x \le 2a-b$. Thus, from Figure A 7.5(p.263) (i) we have

$$\blacksquare\ z(x) = a \cdots (6), \quad x \le 2a-b.$$

   Hence, from (1) and (2) we have $T(x) = T(a,x) = a - x \cdots (7)$ on $x \le 2a-b$.

2. Let $a < z^*(x) \le b \cdots (2)$. Then, from (5) we have $a < (b+x)/2 \le b$, hence $2a-b < x \le b$. Thus, from Figure A 7.5(p.263) (ii) we have

$$\blacksquare\ z(x) = z^*(x) = (b+x)/2 > a \cdots (8), \quad 2a-b < x \le b.$$

   Hence, from (1) and (3) we have

$$T(x) = T(z^*(x),x) = (b - z^*(x))(z^*(x) - x)/(b-a) = (x-b)^2/4(b-a), \quad 2a-b < x \le b.$$

   Now, since

$$\begin{aligned} m(x) &\overset{\text{def}}{=} T(x) - a + x \\ &= (x-b)^2/4(b-a) - a + x \\ &= ((x-b)^2 + 4(b-a)(x-a))/4(b-a), \end{aligned}$$

   we have

$$m'(x) = (x - 2a + b)/2(b-a) > 0, \quad 2a-b < x \le b,$$

   hence $m(x)$ is strictly increasing on $2a-b < x \le b$. In addition to the fact, since

$$\begin{aligned} m(2a-b) &= ((2a-b-b)^2 + 4(b-a)(2a-b-a))/4(b-a) \\ &= ((2a-2b)^2 + 4(b-a)(a-b))/4(b-a) \\ &= (4(a-b)^2 - 4(a-b)^2)/4(b-a) = 0, \end{aligned}$$

   it follows that $m(x) > 0$ on $2a-b < x \le b$, hence $m(x) = T(x) - a + x > 0$ on $2a-b < x \le b$ or equivalently $T(x) > a - x \cdots (9)$ on $2a-b < x \le b$.

3. Let $b \le z^*(x) \cdots (3)$. Then, from (5) we have $b \le (b+x)/2$, hence $b \le x$. Thus, from Figure A 7.5(p.263) (iii) we have

$$\blacksquare\ z(x) = b > a \cdots (10), \quad b \le x.$$

   Hence, from (1) we have $T(x) = T(b,x) = 0$ due to (4), hence $T(x) = 0 \ge b - x > a - x \cdots (11)$ on $b \le x$.

Collecting up (7), (9), and (11), we have

$$T(x) \begin{cases} = a - x, & x \le 2a - b, \\ > a - x, & 2a - b < x \le b, \\ > a - x, & b \le x. \end{cases} \tag{A 7.6}$$

Accordingly, noting (5.1.26(p.24)) and Figure A 7.4(p.262), from (A 7.6(p.264)) we immediately see that

$$a^\star = 2a - b \cdots (1). \tag{A 7.7}$$

Similarly, collecting up (6), (8), and (10), we have

$$z(x) \begin{cases} = a, & x \le 2a - b, \\ > a, & 2a - b < x \le b, \\ > a, & b \le x. \end{cases} \tag{A 7.8}$$

Accordingly, noting (5.1.27(p.24)), we immediately see that

$$x^\star = 2a - b \cdots (2). \tag{A 7.9}$$

Numerical Experiment **1** (Discontinuity of $z(x)$ (Dr. Mong Shan Ee)) $z(x)$ is not always continuous in $x = x^\star$; in fact we can demonstrate a numerical example in which $z(x)$ is not continuous in $x = x^\star$. For example let us consider $F(\xi)$ with $f(\xi)$ such that $f(\xi) \approx 0.05701$ on $[0.1, 0.599]$, $f(\xi)$ is a triangle on $[0.599, 0.7]$ with its maximum at $\xi = 0.6$, and $f(\xi) \approx 0.06982$ on $[0.7, 3.0]$. Then we have $z(x) \approx 0.599$ for $x \le 0.48568$ and $z(x) \approx 1.7$ for $x < 0.48568$, i.e., $z(x)$ is discontinuous at $x = 0.48568$. □

A 7.6 Economic Implications of Market Partition

The three restricted markets defined in Section 17.2(p.115) implies the following:

- Positive market $\mathscr{F}^+$ In an asset trading problem in the real world, the price is usually positive, i.e., the problem is defined on the positive market $\mathscr{F}^+$, called the *input market* in the sense that all goods are first input in the market.
- Mixed market $\mathscr{F}^\pm$ For example, suppose you must waste a piece of well-worn furniture, say a book cabinet, sofa bed and so on. For such a good, normally the two kinds of receiving-sides (buyers) may appear: One who pays some money on the motive that some profit might be obtained by reselling it and the other who requires some money for the reason that some cost may be incurred for its disposal. This market can be regarded as a market in which the positive market and the negative market are mixed; let us call the market the *secondhand market*.
- Negative market $\mathscr{F}^-$ The trading problem in Section 3.5(p.18) is defined on this market; let us call the market the *junk market*. □

Remark A 7.1 (life of durable goods) A new durable good (automobile, house furnishings, TV and so on) is first placed on the positive market $\mathscr{F}^+$ (*input market*), deteriorates year by year, a while later is drove to the mixed market $\mathscr{F}^\pm$ (*secondhand market*), before long moves into the negative market $\mathscr{F}^-$ (*junk market*), and then finally is recycled or dumped. This deterioration flow implies that the probability density functions of price transfers from right to left as seen in Figure A 7.6(p.264) below. □

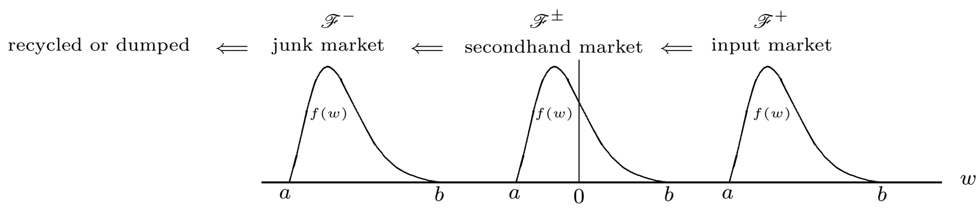


Figure A 7.6: Deterioration transition of goods (life of goods)

Acknowledgment

We express our gratitude to Mr. Kenichro Ikuta for his valuable advice on the use of Fortran and to Ms. Jinxing Iou for her continued encouragement from New Zealand. We are especially grateful to Dr. Mong Shan Ee for her useful comments. Lastly, we extend our heartfelt thanks to our wives, Dr. Xiuling Li and Ms. Samsun Heo, for their unwavering encouragement over the years. Without their thoughtful support, this work would not have been completed.

♡

Many decision theories discussed by researchers have traditionally been framed as mathematical theories. In contrast, this paper approaches "decision" as a subject of study within the natural sciences (see Section 1.3(p.4)). It is important to note that some researchers may have objections to this viewpoint. However, one should recognize that the truth of mathematics resides within mathematics itself, and the truth of physics resides within physics; there is no direct relationship between these two types of truth. To illustrate, physicists sometimes refer to the term "mathematics" as "arithmetic", using it merely as a tool, akin to how carpenters use hammers. While a good hammer is necessary for building a good structure, it would be a mistake to think that a good structure cannot be built without a good hammer. As Albert Einstein famously stated:

As far as the laws of mathematics refer to reality, they are not certain,
and as far as they are certain, they do not refer to reality.

— Albert Einstein —

♢

This paper, which began with a proposition by Dr. Professor Shizuo Senju on March 31, 1966,
concludes with this Einstein's apothegm today, December 12, 2025.

Bibliography

[1] M. H. DeGroot. *Optimal Statistical Decisions.* McGraw-Hill Book Company, New York, 1970.

[2] M. S. Ee. A switching model of optimal asset selling problem. *Discussion paper, No.1098, University of Tsukuba, Institute of Policy and Planning Sciences*, 2004.

[3] M. S. Ee. *Switching model of dynamic asset selling problem.* PhD thesis, University of Tsukuba, 2006.

[4] M. S. Ee. Asset selling problem with an uncertain deadline, quitting offer, and search skipping option. *European Journal of Operational Researc*, 198:215–222, 2009.

[5] R. A. Howard. *Dynamic Programming and Markov Processes.* M.I.T. Press, Cambridge, Massachusetts, 1960.

[6] S. Ikuta. The optimal stopping problem in which the sum of the accepted offer's value and the remaining search budget is an objective function. *Journal of the Operations Research Society of Japan*, 35(2):172–193, 1992.

[7] S. Ikuta. Optimal stopping problem with several search areas. *Journal of the Operations Research Society of Japan*, 38(1):89–106, 1995.

[8] S. Ikuta. An integration of the optimal stopping problem and the optimal pricing problem. `https://commons.sk.tsukuba.ac.jp/discussion` DP 1084 in Discussion Paper Series of University of Tsukuba., 2004.

[9] S. Ikuta. The Integrated Theory of Selling and Buying Problems Based on the Concepts of Symmetry and Analogy (`ver.001`). `https://commons.sk.tsukuba.ac.jp/discussion` DP 1386 in Discussion Paper Series of University of Tsukuba., 2023.

[10] S. Ikuta and B.K. Kang. The Integrated Theory of Selling and Buying Problems Based on the Concepts of Symmetry and Analogy (`ver.003`). `https://commons.sk.tsukuba.ac.jp/discussion` DP 1394 in Discussion Paper Series of University of Tsukuba., 2024.

[11] S. Ikuta and B.K. Kang. The Integrated Theory of Selling and Buying Problems Based on the Concepts of Symmetry and Analogy (`ver.002`). `https://commons.sk.tsukuba.ac.jp/discussion` DP 1393 in Discussion Paper Series of University of Tsukuba., 2024.

[12] S. Ikuta and B.K. Kang. The Integrated Theory of Selling and Buying Problems Based on the Concepts of Symmetry and Analogy (`ver.004`). `https://commons.sk.tsukuba.ac.jp/discussion` DP 1396 in Discussion Paper Series of University of Tsukuba., 2025.

[13] S. Ikuta and B.K. Kang. The Integrated Theory of Selling and Buying Problems Based on the Concepts of Symmetry and Analogy (`ver.005`). `https://commons.sk.tsukuba.ac.jp/discussion` DP 1399 in Discussion Paper Series of University of Tsukuba., 2025.

[14] N. C. Petruzzi and M. Dada. Dynamic pricing and inventory control with learning. *Naval Research Logistics*, 49:303–325, 2002.

[15] M. Sakaguchi. Dynamic programming of some sequential sampling design. *Journal of Mathematical Analysis and Applications*, 2:446–466, 1961.

[16] P. S. You. *Dynamic selling problem with pricing policy and its application to airline seat inventory control problem.* PhD thesis.

[17] P. S. You. Sequential buying policies. *European Journal of Operational Research*, 120:535–544, 2000.