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The Assignment Markets[*)]

by

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Abstract: The purpose of the present paper is to consider markets where commodities have large and indivisible units, e.g., housing markets, car markets, etc. . First, we give a simple model of such a kind of market, and show the nonemptiness of the core of the market and the equivalence of the core and the competitive equilibria. Second, a generalization of the market model is given, and the existences of the core and the competitive equilibria is shown using the result gained in the original model. Furthermore we give a condition under which the equivalence of the core and the competitive equilibria holds. Finally we consider an application of our theory to a housing market , and give the explicit shapes of the competitive rental values of houses. In terms of comparative statics, we consider the effects of changes of some parameters upon the competitive rental values ,i.e., changes of the numbers of households and house-owners, of the income levels of households and of the house-owners's evaluation values of houses.

1 Introduction

The perfect divisibility of commodities is assumed in most usual studies of markets. When they are models of markets where the number of units of commodities which every economic agent consumes or produces is not small, this assumption may not be inadequate even if the commodities are considered to be substantially indivisible. Because the models with perfect divisibility are considered to be approximate ones of such markets. But when some commodities have indivisible and large units and come in small number of units for some economic agents, this 'perfect divisibility' assumption would not be adequate. Hence when we want to consider such a kind of market, we should give a model where such commodities are treated as indivisible ones.

The model given by Shapley and Shubik [1972] is a very adequate one for the study of such a kind of market, because it satisfies the above requirement, though it is restrictive. In the market model, the economic agents consists of sellers and buyers, and indivisible commodities and money which is considered to be 'composite commodity' are traded. Each seller owns one unit of indivisible commodity initially and wants to sell it, but never buys any indivisible commodities. Each buyer wants to purchase at most one unit of the indivisible commodities. This model was generalized by Kaneko [1976a] to the case where sellers are permitted to own more than one unit of indivisible commodities initially.

These models may be considered to be ones of housing markets or car markets, etc. .

But the 'transferable utility' assumption is imposed upon these models, which makes the analysis of them easy. The transferable utility assumption does not admit any decrease of the marginal utility of money and makes it constant. 2) But as the models are ones of markets in which indivisible commodities have large units, the proportions of the prices of indivisible commodities to the initial endowments of money or the income levels of some economic agents can not be neglected. Hence it is inadequate to assume that such economic agents (e.g., buyers in the models of Shapley and Shubik [1972] and Kaneko [1976 a]) have the constant marginal utilities of money.

The purpose of the present paper is to extend the models of Shapley and Shubik [1972] and Kaneko [1976 a] to cases without the transferable utility assumption. In Section 2, the model given by Shapley and Shubik [1972] in which the transferable utility assumption is not made is considered. We show the existence of the core of it by proving the balancedness of the characteristic function of the market game and the equivalence of the core and the competitive equilibria. It is noted that no assumption on the marginal utility of money is necessary to prove these propositions.

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2). The transferable utility assumption is explained in Kaneko [1976 b] and briefly in Section 3 of this paper. The definition of decreasing marginal utility in our model is given in Section 4 .

In Section 3, the model given in Section 2 is extended to a case where sellers may own more than one unit of indivisible commodities initially but the transferable utility assumption is imposed upon the sellers. This model is a generalization of that of Kaneko [1976a].[3)] We also consider the existence of the core and the competitive equilibria, which are proved, and the relationships between them. The equivalence of them does not necessarily hold, but we give a sufficient condition for it, though it is a less general one than that of Kaneko [1976a] in the case with the transferable utility assumption. But as the sufficient condition is very weak, it teaches us that the competitive equilibria can be representatives of the core in the market models, which makes our further analysis easy.

The present paper has another purpose, which is to consider an application of our market model given in Section 3 to a housing market. In Section 4, we give a simple model of a housing market which is represented as a special case of the market model given in Section 3. The competitive rental values of houses (the competitive prices of houses at which households (buyers) rent the houses) is considered. In the model the competitive rental values are given as an explict form, which makes the analysis of

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3). Exactly, this model is neither any generalization of that of Section 2 nor of that of Kaneko [1976a] in the usual mathematical sense. See Section 3 .

the shapes of the competitive rental values possible. In the last section, we consder the effects of changes of some parameters upon the competitive rental values in terms of comparative statics. It should be noted that our consideration is concentrated on a competitive rent vector called a maximal competitive rent vector. We give three causes of rises in rental values, which are increases of the incomes of households , that of the number of households, more precisely, a relative increase of demand with supply, and that of the houseowners' evaluation values of houses.

2. The Assignment Market

We consider a market model (M,N) called an assignment market where s kinds of indivisible commodities are exchanged for one perfectly divisible commodity called money, which is considered to be a 'composite commodity'. The assignment market (M,N) consists of two types of traders, i.e., sellers and buyers. The set of all sellers is $M = \{1', \dots, m'\} = M_1 \cup M_2 \cup \dots \cup M_s$ where $M_k = \{m_{k-1}+1, \dots, m_k\}$ ($k = 1, \dots, s$) and $m_0 = 0' < m_1 < m_2 < \dots < m_s = m'$. The set of all buyers is $N = \{1, \dots, n\}$. Each seller $i \in M_k$ ($k = 1, \dots, s$) owns one unit of the k-th indivisible commodity initially. No buyer owns any indivisible commodities initially. Let a^i be trader i's initial endowment of indivisible commodities. Then if $i \in N$, then $a^i = 0$ and if $i \in M_k$, then $a^i = e^k$,where e^k is the s-vector such that $e^k_k = 1$ and $e^k_{k'} = 0$ if $k' \neq k$. Each trader $i \in M \cup N$ owns I_i ($I_i > 0$) amount of money initially.

Each trader $i \in M \cup N$ has a preference ordering R_i which is defined on the consumption set $X = I^s_+ \times E_+$,where I_+ is the set of all nonnegative integers, I^s_+ the direct product of s number of I_+ and E_+ the set of all nonnegative real numbers. $(x,m) \in X$ means amounts of the indivisible commodities and money to be consumed. We assume that every R_i ($i \in M \cup N$) is a weak ordering. $(x,m_1)R_i(y,m_2)$ means that trader i prefers (x,m_1) to (y,m_2) or is indifferent between (x,m_1) and (y,m_2). We define the strict preference ordering P_i by $(x,m_1)P_i(y,m_2)$ if and only if not (y,m_2) $R_i(x,m_1)$, and the indifferent relation Q_i by $(x,m_1)Q_i(y,m_2)$ if and only if $(x,m_1)R_i(y,m_2)$ and $(y,m_2)R_i(x,m_1)$.

The following three conditions are assumed for every trader $i \in M \cup N$. The meanings of them may be clear and it may not be necessary to give any explanation:

(A) (Monotonicity with respect to Money): if $m_1 > m_2$, then $(x,m_1)P_i(x,m_2)$ for $\forall x \in I_+^s$,

(B) (Monotonicity with respect to the Indivisible Commodities): if $x \geqq y$, then $(x,m)R_i(y,m)$ for $\forall m \geqq 0$;[4)]

(C) (Archimedean Property): if $(x,m_1)P_i(y,m_2)$, then there is an m_3 such that $(x,m_1)Q_i(y,m_3)$.

These assumptions assure the existence of a continuous utility function for every trader $i \in M \cup N$. We introduce the relative topology of the s+1-dimensional Euclidean space E^{s+1} into X.

Theorem 1. For each $i \in M \cup N$, there exists a real valued continuous function $U_i(x,m)$ on X such that

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4). We define $x \geqq y$ by $x_k \geqq y_k$ for $\forall k = 1,\dots,s$, $x \geq y$ by $x \geqq y$ but $x \neq y$, and $x > y$ by $x_k > y_k$ for $\forall k = 1,\dots,s$.

$$(2.1) \quad (x,m_1)R_i(y,m_2) \text{ if and only if } U_i(x,m_1) \geqq U_i(y,m_2) .$$

We call this $U_i(x,m)$ a <u>utility function</u> of trader i .

<u>Proof</u>. By (A) and (B) it holds that $(x,m)R_i(0,0)$ for $\forall(x,m) \in X$. There is a unique real number d by (C) and (A) for each (x,m) such that $(x,m)Q_i(0,d)$. We define $U_i(x,m)$ by

$$(2.2) \quad U_i(x,m) = d .$$

Let us verify that this $U_i(x,m)$ satisfies (2.1). If $(x,m_1)R_i(y,m_2)$, then $(0,d_1)Q_i(x,m_1)R_i(y,m_2)Q_i(0,d_2)$ where $U_i(x,m_1) = d_1$ and $U_i(y,m_2) = d_2$. Hence we have $(0,d_1)R_i(0,d_2)$, which implies $d_1 \geqq d_2$ by (A). Conversely suppose that $U_i(x,m_1) = d_1 \geqq d_2 = U_i(y,m_2)$. This implies $(x,m_1)Q_i(0,d_1)R_i(0,d_2)Q_i(y,m_2)$ by (A) and (2.2) . Hence we have $(x,m_1)R_i(y,m_2)$.

Let us show that $U_i(x,m)$ given by (2.2) is a continuous function. Let $\{(x^s,m^s)\}$ be a sequence in X such that $\lim_{s \to \infty}(x^s,m^s) = (x,m)$. Since I_+^S has the discrete topology, there is an integer k such that $x^s = x$ for $\forall s \geqq k$. Since $\{(x^s,m^s)\}$ is a convergence sequence, there is a $(y,m_1) \in X$ such that $(x^s,m^s) \leqq (y,m_1)$ for $\forall s$, which implies $0 \leqq U_i(x,m^s) \leqq U_i(y,m_1)$ for $\forall s \geqq k$. Hence the sequence $\{U_i(x,m^s)\}$ has at least one limit point in E_+ .

Let the sequence have a limit point not equal to $U_i(x,m)$ and let $\{U_i(x,m^{s^k})\}$ be a converging subsequence such that $\lim_{k \to \infty} U_i(x,m^{s^k}) = u \neq U_i(x,m)$. Let ε be an arbitrary small positive number. Let $u > U_i(x,m)$. Then we have $(0,U_i(x,m)+\varepsilon)P_i(x,m)$ by (A) and (2.2). By (A) and (C) there is a $\delta > 0$ such that $(0,U_i(x,m)+\varepsilon)Q_i(x,m+\delta)$, i.e., $U_i(x,m)+\varepsilon = U_i(x,m+\delta)$. But it holds that $m^{s^k} < m+\delta$ for $\forall k \geqq$ some k_o, which implies $U_i(x,m^{s^k}) \leqq U_i(x,m+\delta) = U_i(x,m)+\varepsilon$. This is a contradiction. Let $u < U_i(x,m)$. Since $U_i(x,m^s) \geqq U_i(x,0)$ by (A), we have $u \geqq U_i(x,0)$. Hence there is an m_o such that $(0,u+\varepsilon)Q_i(x,m_o)$, i.e., $u+\varepsilon = U_i(x,m_o)$. Since ε is sufficiently small, we have $m_o < m$. Since there is a k_o such that $U_i(x,m^{s^k}) < u+\varepsilon = U_i(x,m_o)$ for $\forall k \geqq k_o$, we have $m_o > m^{s^k}$ by (A). This contradicts that $\lim_{k \to \infty} m^{s^k} = m$. Hence we have shown that any limit point coincides with $U_i(x,m)$. Q.E.D.

We assume the following condition (B') and (B") instead of assumption (B), and (D): For each seller $i \in M_k$ ($k = 1,\dots,s$):

(B')(<u>Saturation</u>): for each $(x,m) \in X$, if $x_k = 0$, then $(x,m)Q_i(0,m)$ and if $x_k \geqq 1$, then $(x,m)Q_i(e^k,m)P_i(0,m)$.

For each buyer $i \in N$:

($\underline{B}''$)(Saturation): for each $(x,m) \in X$, if $x_k \geqq 1$ for some k and $(e^k,m)R_i(e^{k'},m)$ for $\forall k'$ with $x_{k'} \geqq 1$, then $(x,m)Q_i(e^k,m)P_i(0\ ,m)$.

($\underline{D}$): For $\forall i \in N$, $(0,I_i)P_i(x,0)$ for $\forall x \in I_+^s$.

It should be noted that these assumptions (B') and (B") are stronger than assumption (B), which implies that Theorem 1 holds yet .

Assumption (B') means that even if seller $i \in M_k$ has indivisible commodities other than the k-th commodity or more than one unit of the k-th commodity, then his utility is not greater than that of having one unit of the k-th commodity. It follows that sellers never become buyers. Assumption (B") means that even if buyer $i \in N$ consumes more than one unit of indivisible commodities , then his utility is not greater than that of consuming one unit of the most preferred commodity. More precisely, if the buyer purchases one unit of a more preferred commodity as the second unit, his utility increases, but it is indifferent for him to purchase the commodity as the first unit. This assumption implies that buyers never purchase more than one unit of indivisible commodities. Assumption (D) means that any buyer purchases no indivisible commodity by paying all his income . When the marginal utility of money at $(x,0)$ is very large and the income I_i is not too small, this assumption is satisfied. It would be a natural assumption.

Since assumption (B') implies that sellers never buy any indivisible commodity, i.e., never decrease their amounts of money, it is not necessary to consider the case where the amounts of money are zeros. This is the reason why we do not impose assumption (D) upon the sellers.

These assumptions (B') and (B") look like strange ones at a first glance. But these never make the modeling of situations which we want to consider inadequate. Now, we consider an example of a market with s kinds of houses for rent.[5] Let us consider two cases where a buyer $i \in N$ rents one unit of the k-th house and where he rents two units of the house. These states are represented by (e^k, m_1) and $(2e^k, m_2)$, where m_1 and m_2 are the amounts of money after paying the rental values. It is natural to suppose that there is a nonnegative real number ε such that

$$(e^k, m_1) Q_i (2e^k, m_1 - \varepsilon) .$$

When the k-th house is not small to live in for him, the real number ε is considered to be small relative to the rental value of the house. Hence it is reasonable and more plausible in order to characterize such kinds of markets that the real number ε is assumed to be zero. Thus a part of (B") is justified. Let $(e^k, m_1) P_i (e^{k'}, m_2)$ and $x = e^k + e^{k'}$. Then it is natural to assume that $(x, m_1) R_i (e^k, m_1)$, and that $(x, m_1 - \varepsilon) Q_i (e^k, m_1)$ for some $\varepsilon \geqq 0$. By the similar reasoning with the above this real

5). We will consider more precisely such a kind of housing market in Sections 4 and 5.

number ε can be assumed to be zero. This is the rest part of assumption (B") . We can justify assumption (B') by the similar reasoning .

For any nonempty coalition $S \subset M \cup N$, we call $(x^S, m^S) = ((x^i, m^i))_{i \in S}$ an S-allocation if

$$(2.3) \quad (x^i, m^i) \in X \quad \text{for } \forall i \in S$$

$$(2.4) \quad \sum_{i \in S} (x^i, m^i) = \sum_{i \in S} (a^i, I_i) \ .$$

We call an $M \cup N$-allocation simply an allocation. We say that a non-empty coalition $S \subset M \cup N$ can improve upon allocation $(x^{M \cup N}, m^{M \cup N})$ if there is an S-allocation (y^S, m_1^S) such that

$$(2.5) \quad (y^i, m_1^i) P_i (x^i, m^i) \quad \text{for } \forall i \in S \ .$$

The core of the assignment market is the set of all allocations can never be improved upon.

The characteristic function V of the assignment market is defined in the usual way as follows:

$$(2.6) \quad V(S) = \left\{ b \in E^{M \cup N} \;\middle|\; \text{for some S-allocation } (x^S, m^S),\ b_i \leqq U_i(x^i, m^i) \quad \text{for } \forall i \in S \right\} \ .$$

Here $E^{M \cup N}$ is the m+n dimensional Euclidean space, the coodinates of which are indexed by members in $M \cup N$. The core of the characteristic function V is the set of all vectors in $V(M \cup N)$ which can never be improved upon, where we say that a nonempty coalition $S \subset M \cup N$ <u>can improve upon</u> $b \in V(M \cup N)$ if there is a vector $c \in V(S)$ with

$$(2.7) \quad c_i > b_i \quad \text{for } \forall i \in S .$$

It is easily verified that for any b in the core of V , there is an allocation $(x^{M \cup N}, m^{M \cup N})$ in the core of the assignment market with $U_i(x^i, m^i) \geq b_i$ for $\forall i \in M \cup N$, and that $(U_i(x^i, m^i))_{i \in M \cup N}$ belongs to the core of V for any allocation $(x^{M \cup N}, m^{M \cup N})$ in the core of the assignment market. Hence, to prove the nonemptiness of the core of the assignment market, it is sufficient to show that the nonemptiness of core of V . In order to show it, it is necessary to define another game V_o in characteristic function form.

Let $\mathcal{D} = \{S \mid |S| = 1 \text{ or } (|S| = 2 , S \cap M \neq \emptyset \text{ and } S \cap N \neq \emptyset)\}$, where $|S|$ denotes the number of members in S. We call $p_S = \{T_1, \ldots, T_k\}$ a <u>$\mathcal{D}$-partition</u> of $S \subset M \cup N$ if

$$(2.8) \quad T_t \in \mathcal{D} \quad \text{for } \forall t = 1, \ldots, k$$

$$(2.9) \quad \bigcup_{t=1}^{k} T_t = S \text{ and } T_t \cap T_{t'} = \emptyset \quad \text{for } \forall t, t' \ (t \neq t') .$$

Let P(S) be the set of all $\mathcal{D}$-partitions of S . We define the game V_o as follows:

$$(2.10) \qquad V_o(S) = \bigcup_{p_S \in P(S)} \bigcap_{T \in p_S} V(T) .$$

We call the game V_o the <u>associated characteristic function</u> of the assignment market. Though V_o is different from V, the following relations (2.11) hold and we can prove the following theorem, which says that we need to prove the nonemptiness of the core of V_o in order to show the nonemptiness of the core of V :

$$(2.11) \qquad V_o(S) \subset V(S) \quad \text{for } \forall S \subset M \cup N$$

$$V_o(S) = V(S) \quad \text{for } \forall S \in \mathcal{D} .$$

<u>Theorem 2</u>. The core of V coincides with the core of V_o .

The following lemma is necessary to prove this theorem. It should be noted that if for any allocation $(x^{M \cup N}, m^{M \cup N})$, a non-empty coalition S satisfies

$$(2.12) \qquad \sum_{i \in S} (a^i, I_i) \geq \sum_{i \in S} (x^i, m^i) \quad \text{and} \quad m^i > 0 \quad \text{for } \forall i \in S ,$$

then S can improve upon the allocation $(x^{M\cup N}, m^{M\cup N})$. In fact, if $\sum_{i\in S} I_i > \sum_{i\in S} m^i$, then S can improve upon $(x^{M\cup N}, m^{M\cup N})$ by the S-allocation (y^S, m_1^S) such that $y^i = x^i$ for $\forall i \in S$ and $m_1^i = m^i + (\sum_{i\in S}(I_i - m^i))/|S|$ for $\forall i \in S$. Let $\sum_{i\in S} I_i = \sum_{i\in S} m^i$. Then there is an $f \leqq s$ such that $\sum_{i\in S} x_f^i < \sum_{i\in S} a_f^i$, which implies that $x_f^{i_o} = 0$ for some $i_o \in S \cap M_f$. By (B') we have $U_{i_o}(e^f, m^{i_o}) > U_{i_o}(x^{i_o}, m^{i_o}) = U_{i_o}(0, m^{i_o})$. If ε is a sufficiently small positive real number, then $U_{i_o}(e^f, m^{i_o} - \varepsilon) > U_{i_o}(x^{i_o}, m^{i_o})$ by Theorem 1. Hence it holds that $U_i(x^i, m^i + \varepsilon/(|S|-1)) > U_i(x^i, m^i)$ for $\forall i \in S - \{i_o\}$, which implies that S can improve upon $(x^{M\cup N}, m^{M\cup N})$.

Lemma 3. Let $(x^{M\cup N}, m^{M\cup N})$ belong to the core of the assignment market. Then there are partitions of M and N such that $M = \{i_1, \ldots, i_k\} \cup M_o$, $N = \{j_1, \ldots, j_k\} \cup N_o$ and

(2.13) $(x^{i_t}, m^{i_t}) + (x^{j_t}, m^{j_t}) = (a^{i_t}, I_{i_t}) + (a^{j_t}, I_{j_t})$
and $x^{i_t} = 0$ for $\forall t = 1, \ldots, k$,

(2.14) $(x^i, m^i) = (a^i, I_i)$ for $\forall i \in M_o \cup N_o$.

It is noted that (2.13) implies $x^{jt} = a^{it}$ for $\forall t = 1,\dots,k$.

Proof. If $m^i = 0$ for some $i \in N$, then $U_i(x^i,m^i) < U_i(a^i,I_i)$ by (D), which contradicts that $(x^{M\cup N},m^{M\cup N})$ is in the core. If $m^i = 0$ for some $i \in M_f$ ($1 \leqq f \leqq s$), then $U_i(0,m^i) < U_i(e^f,0) < U_i(e^f,I_i)$, which is a contradiction. Hence we have $m^i > 0$ for $\forall i \in M \cup N$.

Suppose that there is a buyer $j \in N$ with $x^j \neq 0$. Then by (B") there is an $f \leqq s$ such that $x^j_f > 0$ and $U_j(e^f,m^j) = U_j(x^j,m^j)$. Then there is an seller $i \in M_f$ with $x^i_f = 0$. We show that $m^i + m^j = I_i + I_j$, $x^i = 0$ and $x^j = e^f$. If $m^i + m^j > I_i + I_j$, then the coalition $(M-\{i\}) \cup (N-\{j\})$ can improve upon $(x^{M\cup N},m^{M\cup N})$, because $\sum_{t\neq i,j} a^t \geqq \sum_{t\neq i,j} x^t$ and $\sum_{t\neq i,j} I_t > \sum_{t\neq i,j} m_t$. If $m^i + m^j < I_i + I_j$, then the coalition $\{i,j\}$ can improve upon $(x^{M\cup N},m^{M\cup N})$ by the allocation $((y^t,m^t_1))_{t=i,j}$ such that

$$y^i = 0 , \quad m^i_1 = m^i + (I_i+I_j - m^i - m^j)/2$$

$$y^j = e^f , \quad m^j_1 = m^j + (I_i+I_j - m^i - m^j)/2 \quad .$$

Hence it holds that $m^i + m^j = I_i + I_j$. If $x^i_{f'} \geqq 1$ for some $f' \neq f$ or $x^j \geq e^f$, then the coalition $(M-\{i\}) \cup (N-\{j\})$ can improve upon $(x^{M\cup N}, m^{M\cup N})$, because $\sum_{t\neq i,j} a^t \geq \sum_{t\neq i,j} x^t$ and $\sum_{t\neq i,j} m^t = \sum_{t\neq i,j} I_t$. Hence we have $x^i = 0$ and $x^j = e^f$.

By repeating this argument, we choose such all pairs and denotes the set of them by $\{(i_1,j_1),\dots,(i_k,j_k)\}$. We put $M_o = M - \{i_1,\dots,i_k\}$ and $N_o = N - \{j_1,\dots,j_k\}$. Clearly (2.13) holds for these $\{(i_1,j_1),\dots,(i_k,j_k)\}$. Of course, no buyer $j\in N_o$ has any indivisible commodity, i.e., $x^j = 0$ for $\forall i\in N_o$. For if otherwise, we can find another pair (i_{k+1},j_{k+1}) for which (2.13) holds by the above argument . Since $U_j(0,m^j) \geqq U_j(0,I_j)$ for $\forall j\in N_o$, we have $m^j \geqq I_j$ for $\forall j\in N_o$. If $m^j > I_j$ for some $j\in N_o$, M_o can improve upon the allocation , because $\sum_{i\in M_o} a^i = \sum_{i\in M_o} x^i$ and $\sum_{i\in M_o} I_i > \sum_{i\in M_o} m^i$. Hence $m^j = I_j$ for $\forall j\in N_o$.
If there is a seller $i\in M_o\cap M_f$ such that $x^i_f = 0$, then it must hold by (B') that $m^i > I_i$, because $U_i(x^i,m^i) \geqq U_i(a^i,I_i)$. This implies that there is another $i_o\in M_o$ ($i_o\neq i$) with $m^{i_o} < I_{i_o}$. When $i_o\in M_{f'}$, we have $U_{i_o}(x^{i_o},m^{i_o}) \leqq U_{i_o}(e^{f'},m^{i_o}) < U_{i_o}(e^{f'},I_{i_o})$ by (A) and (B') , which is a contradiction. Hence we have $x^i_f = 1$ for $\forall i\in M_o\cap M_f$ ($f = 1,\dots,s$), which implies $x^i = e^f$ for $\forall i\in M_o\cap M_f$ ($f = 1,\dots,s$). Furthermore it holds

that $U_i(e^f, I_i) \geqq U_i(e^f, m^i)$ for $\forall i \in M_o \cap M_f$ ($f = 1,\ldots,s$), which implies $I_i = m^i$ for $\forall i \in M_o$. Hence we have shown (2.14).

Q.E.D.

The proof of Lemma 3 means the following corollary .

<u>Corollary 4</u>. For $\forall i \in M_f \cap \{i_1,\ldots,i_k\}$ and $j \in \{j_1,\ldots,j_k\}$ with $x^j = e^f$, it holds that $(x^i, m^i) + (x^j, m^j) = (a^i, I_i) + (a^j, I_j)$.

<u>Proof of Theorem 2</u>. Let b belong to the core of V . Since b can not be improved upon in the game V_o by (2.11), it is sufficient to show that b is in $V_o(M \cup N)$. There is an allocation $(x^{M \cup N}, m^{M \cup N})$ such that $U_i(x^i, m^i) \geqq b_i$ for $\forall i \in M \cup N$. Note that $(x^{M \cup N}, m^{M \cup N})$ is in the core of the assignment market . Let the partitions of M and N given in Lemma 3 be $M = \{i_1,\ldots,i_k\} \cup M_o$ and $N = \{j_1,\ldots,j_k\} \cup N_o$. We put $M_o = \{i(1),\ldots,i(g)\}$ and $N_o = \{j(1),\ldots,j(h)\}$.

We put a Ω-partition $p_{M \cup N} = \{T_1,\ldots,T_f\}$ ($f = k+g+h$) of $M \cup N$ as follows:

(2.15) $\quad T_t = \{i_t, j_t\} \quad$ for $\forall t = 1,\ldots,k$

(2.16) $T_{k+t} = \{i(t)\}$ for $\forall t = 1,\ldots,g$

(2.17) $T_{k+g+t} = \{j(t)\}$ for $\forall t = 1,\ldots,h$.

By Lemma 3 it holds that $b \in V(T)$ for $\forall T \in p_{M\cup N}$, i.e., $b \in \bigcap_{T\in p_{M\cup N}} V(T)$. This means that b is in $V_o(M\cup N)$.

Conversely we show that any b in the core of V_o belongs to the core of V . By (2.11) b is in $V(M\cup N)$. We prove that if b can be improved upon by a nonempty coalition S in the game V , then b can be improved upon by a nonempty coalition in the game V_o .

Since b is in $V_o(M\cup N)$, there is a $\mathcal{D}$-partition $p_{M\cup N}$ of $M\cup N$ such that $b \in \bigcap_{T\in p_{M\cup N}} V(T)$. We put the set of pairs $\{T \mid |T| = 2,\ T \in p_{M\cup N}\} = \{\{i_1,j_1\},\ldots,\{i_k,j_k\}\}$ and put $M_o = M - \{i_1,\ldots,i_k\}$ and $N_o = N - \{j_1,\ldots,j_k\}$. Then it holds that

$$b \in V(\{i_t,j_t\}) \quad \text{for } \forall t = 1,\ldots,k$$

$$b \in V(\{i\}) \quad \text{for } \forall i \in M_o \cup N_o .$$

Hence there is an allocation $(x^{M\cup N}, m^{M\cup N})$ such that

$$(x^{i_t},m^{i_t}) + (x^{j_t},m^{j_t}) = (a^{i_t},I_{i_t}) + (a^{j_t},I_{j_t})$$
$$\text{for } \forall t = 1,\ldots,k ,$$

$$(x^i,m^i) = (a^i,I_i) \qquad \text{for } \forall i \in M_o \cup N_o$$

and

$$U_i(x^i,m^i) \geqq b_i \qquad \text{for } \forall i \in M \cup N .$$

Suppose that $U_i(x^i,m^i) > b_i$ for some $i \in M \cup N$. If $i \in M_o \cup N_o$, then $b_i < U_i(x^i,m^i) = \sup \text{pro}_i V(\{i\}) = \sup \text{pro}_i V_o(\{i\})$, which is a contradiction to the supposition that b is in the core of V_o.[6] Let $i = j_t$ ($t \leqq k$). If $m^{j_t} = 0$, then $U_{j_t}(x^{j_t},m^{j_t}) < U_{j_t}(a^{j_t},I_{j_t}) = \sup \text{pro}_{j_t} V_o(\{j_t\})$ by (D), which is a contradiction. We can choose a positive real number ε such that $U_{j_t}(m^{j_t},m^{j_t}-\varepsilon) > b_{j_t}$ and $U_{i_t}(x^{i_t},m^{i_t}+\varepsilon) > b_{i_t}$. Since $(U_{i_t}(x^{i_t},m^{i_t}+\varepsilon), U_{j_t}(x^{j_t},m^{j_t}-\varepsilon)) \times (b_j)_{j \neq i_t, j_t}$ belongs to $V(\{i_t,j_t\}) = V_o(\{i_t, j_t\})$, b can be improved upon in the game V_o, which is a contradiction. Let $i = i_t$ ($t \leqq k$). Then $U_{i_t}(x^{i_t},m^{i_t}) \geqq U_{i_t}(a^{i_t},I_{i_t})$ togather with (A) and (B') implies $m^{i_t} \geqq I_{i_t} > 0$. Then we can get a contradiction similarly with the above. Hence

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6). For any sets $T \subset E^{M \cup N}$ and $S \subset M \cup N$, $\text{pro}_S(T) = \{(b_i)_{i \in S} \mid b \in T\}$.

it must hold that $U_i(x^i,m^i) = b_i$ for $\forall i \in M \cup N$.

Suppose that a nonempty coalition S can improve upon this allocation $(x^{M \cup N}, m^{M \cup N})$ in the assignment market, which is equivalent to that S can improve upon b in the game V. This implies that there is an S-allocation (y^S, m_1^S) such that

$$(2.18) \quad U_i(y^i,m_1^i) > U_i(x^i,m^i) \quad \text{for } \forall i \in S .$$

We choose a buyer $j \in S$ such that $y_f^j \geqq 1$ and $U_j(e^f,m_1^j) = U_j(y^j, m_1^j)$ for some $f \leqq s$, which implies that there is a seller $i \in S \cap M_f$ with $y_f^i = 0$. This choice is always possible because if not, (2.18) can not hold. Of course, it holds that $y^i + y^j \geqq a^i + a^j$. If $m_1^i + m_1^j \leqq I_i + I_j$, then the coalition $\{i,j\}$ can improve upon $(x^{M \cup N}, m^{M \cup N})$, which is a contradiction to the supposition that b is in the core of V_0. Hence we have $m_1^i + m_1^j > I_i + I_j$, which implies

$$\sum_{t \in S-\{i,j\}} (y^t, m_1^t) \leq \sum_{t \in S-\{i,j\}} (a^t, I_t) .$$

This means that there is an $S-\{i,j\}$-allocation $(z^{S-\{i,j\}}, m_2^{S-\{i,j\}})$ such that

$$U_t(z^t,m_2^t) \geqq U_t(y^t,m_1^t) > U_t(x^t,m^t) \qquad \text{for } \forall t \in S-\{i,j\} .$$

Hence if $S-\{i,j\} = \emptyset$, then $S = \{i,j\}$ can improve upon the vector b in the game V_o , which is a contradiction . If $S-\{i,j\} \neq \emptyset$, we reach an T-allocation (w^T, m_o^T) such that $|T| \leqq 2$ and $U_t(w^t, m_o^t) > U_t(x^t, m^t)$ for $\forall t \in T$ by repeating the same argument with the above. It must hold that T contains two traders and $T \cap M \neq \emptyset$ and $T \cap N \neq \emptyset$, because $U_t(w^t, m_o^t) > U_t(x^t, m^t) \geqq \sup \mathrm{pro}_t V(\{t\})$ for $\forall t \in T$. Hence b can be improved upon in the game V_o. This contradicts that b is in the core of V_o . Hence we have shown that b is in the core of V. Q.E.D.

The necessity of Corollary 5 is trivial and the sufficiency has been shown by the latter part of the proof of Theorem 2 .

<u>Corollary 5</u>. An allocation $(x^{M \cup N}, m^{M \cup N})$ belongs to the core of the assignment market if and only if $(x^{M \cup N}, m^{M \cup N})$ can never be improved upon by any $S \in \mathcal{O}$.

It should be noted that as any allocation in the core of the assignment market is the sum of partial allocations as shown in Lemma 3 and Corollary 4 , we do not need to consider the grand coalition $M \cup N$.

This corollary has an important meaning. It is often said that the core-theory neglects costs of coalition-formations and that as costs of making big coalitions are very large, it may be not plausible to assume to able to bargain freely. (See Arrow [1971]). But this corollary says that this criticism has no persuasive power at least in the assignment market. I think that this criticism has a little persuasive power even in the usual market models like perfectly competitive markets, because only small coalitions play important roles substantially, though not in mathematical details, and that even if the costs are taken into account, they only make the space of allocations narrow, e.g., in coalitions traders can only exchanged at common prices.

Now we are in the position to prove the existence of the core of the assignement market. To do it, we show that the core of the game V_o is nonempty, a sufficient condition of which is that V_o is a balanced game.

It is easily verified that the following two conditions hold for $\forall S \subset M \cup N$ ($S \neq \emptyset$):

(2.19) if $b \in V_o(S)$ and $b_i \geqq c_i$ for $\forall i \in M \cup N$, then $c \in V_o(S)$,

(2.20) $\text{pro}_S \left[V_o(S) - \bigcup_{i \in S} (\text{interior of } V_o(\{i\})) \right]$ is bounded and nonempty.

Since the set of all S-allocations is a compact set and each $U_i(x, m)$ is a continuous function, $U_S = \left\{ (U_i(x^i, m^i))_{i \in S} \mid (x^S, m^S) \text{ is an } S\text{-allocation} \right\}$ is a compact set. Since $V(S) = \left\{ b \in E^{M \cup N} \mid \text{for}\right.$

some $(c_i)_{i \in S} \in U_S$, $c_i \geqq b_i$ for $\forall i \in S\}$, V(S) is a closed set. By (2.10), it is easily verified that

(2.21) $V_0(S)$ is a closed set for $\forall S \subset M \cup N$.

Let us call a family T of nonempty coalitions of $M \cup N$ <u>balanced</u> if the system of equations

$$(2.22) \qquad \sum_{S:S \ni j} \delta_S = 1 \quad \text{for } \forall j \in M \cup N ,$$

has a nonnegative solution with $\delta_S = 0$ for $\forall S \notin T$. The numbers $\{\delta_S\}$ are called <u>balancing</u> <u>weights</u> for T . A game v in characteristic function form is said to be <u>balanced</u> if the following inclusion statement:

$$(2.23) \qquad \bigcap_{S \in T} v(S) \subset v(M \cup N)$$

holds for all balanced families T. The fundamental theorem of Scarf [1967] states that the core of a balanced game with (2.19) (2.20) and (2.21) is nonempty. We shall apply this theorem to the game V_0 . Before proving the balancedness of V_0 , we need to show a lemma .

For any nonempty coalition S, if (m+n) by (m+n) zero-one matrix $A_S = (a_{S:ij})$, all rows and columns of which are indexed by members in $M \cup N$, satisfies

$$(2.24)\quad \sum_{i\in M\cup N} a_{S:ij} = 1,\quad \sum_{j\in M\cup N} a_{S:ij} = 1 \quad \text{for } \forall i,j \in S$$

$$\sum_{i\in M\cup N} a_{S:ij} = 0,\quad \sum_{j\in M\cup N} a_{S:ij} = 0 \quad \text{for } \forall i,j \in M\cup N - S,$$

then we call A_S an <u>S-permutation matrix</u>. We define $D_S(b) = (d_{S:ij}(b))$ for each $b \in E^{M\cup N}$ and each nonempty coalition S as follows:

$$(2.25)\quad d_{S:ij}(b) = \begin{cases} 1 & \text{if } (\, i\in S\cap M,\ j\in N,\ b\in V(\{i,j\})) \\ & \text{or } (\, i\in S\cap M,\ j\in M,\ b\in V(\{i\})) \\ & \text{or } (\, i\in S\cap N,\ j\in N,\ b\in V(\{j\})) \\ & \text{or } (\, i\in S\cap N,\ j\in M\,) \\ 0 & \text{otherwise}. \end{cases}$$

<u>Lemma 6</u>. $V_o(S) = \left\{ b \in E^{M\cup N} \mid D_S(b) \geqq A_S \text{ for some S-permuation matrix } A_S \right\}$ for $\forall S \subset M\cup N$.

<u>Proof</u>. Suppose that there is an S-permutation matrix A_S such that $D_S(b) \geqq A_S$. Let $\{(i,j) \mid a_{S:ij} = 1,\ i\in M \text{ and } j\in N\} = \{(i_1,j_1), \ldots, (i_k,j_k)\}$, and let $M_o = M - \{i_1,\ldots,i_k\} = \{i(1),\ldots,i(g)\}$ and $N_o = N - \{j_1,\ldots,j_k\} = \{j(1),\ldots,j(h)\}$. We define a $\mathcal{D}$-partition $p_S = \{T_1,\ldots,T_f\}$ ($f = k+g+h$) by

$$(2.26) \qquad T_t = \{i_t, j_t\} \qquad \text{for } \forall t = 1,\dots,k$$

$$T_{k+t} = \{i(t)\} \qquad \text{for } \forall t = 1,\dots,g$$

$$T_{k+g+t} = \{j(t)\} \qquad \text{for } \forall t = 1,\dots,h \ .$$

It is easily verified that $b \in V(T)$ for $\forall T \in p_S$,i.e., $b \in \bigcap_{T \in p_S} V(T)$.

Let $b \in \bigcap_{T \in p_S} V(T)$ for some $\mathcal{D}$-partition p_S . Let $\{T \mid T \in p_S \text{ and } |T| = 2\} = \{\{i_1, j_1\},\dots,\{i_k, j_k\}\}$, $\{T \mid T \in p_S , |T| = 1 \text{ and } T \cap M \neq \emptyset\} = \{\{i(1)\},\dots,\{i(g)\}\}$ and $\{T \mid T \in p_S , |T| = 1 \text{ and } T \cap N \neq \emptyset\} = \{\{j(1)\},\dots,\{j(h)\}\}$. We assume that $g \leqq h$ but when $g > h$, we cah prove the following similarly .

We define an S-permutation matrix $A_S = (a_{S:ij})$ as follows:

$$(2.27) \qquad \text{for } \forall i_t \ (t = 1,\dots,k)$$

$$a_{S:i_t j} = \begin{cases} 1 & \text{if } j = j_t \\ 0 & \text{otherwise} \end{cases}$$

$$(2.28) \qquad \text{for } \forall i(t) \ (t = 1,\dots,g)$$

$$a_{S:i(t) j} = \begin{cases} 1 & \text{if } j = j(t) \\ 0 & \text{otherwise} \end{cases}$$

(2.29) for $\forall i \in N$

$$a_{S:ij} = \begin{cases} a_{S:ji} & \text{if } i \in \{j_1, \dots, j_k\} \cup \{j(1), \dots, j(g)\} \\ 1 & \text{if } i \in \{j(g+1), \dots, j(h)\},\ i = j \in N \\ 0 & \text{otherwise} . \end{cases}$$

It is easily verified that this A_S is an S-parmutation matrix .

We show that $D_S(b) \geqq A_S$. When $a_{S:ij} = 0$, it is always true that $d_{S:ij}(b) \geqq a_{S:ij}$. Suppose $a_{S:ij} = 1$. If $(i,j) = (i_t, j_t)$ for some $t \leqq k$, then b is in $V(\{i_t, j_t\})$, which implies $d_{S:ij}(b) = 1$. Let $(i,j) = (i(t), j(t))$ ($t \leqq g$) . Then $b \in V(\{i(t)\}) \cap V(\{j(t)\}) \subset V(\{i(t), j(t)\})$, which implies $d_{S:ij}(b) = 1$. When $i = j(t)$ and $i = j(t)$ ($g+1 \leqq t \leqq h$), $b \in V(\{j(t)\})$ by the supposition of p_S . This implies $d_{S:ij}(b) = 1$. When $i \in N \cap S$ and $j \in S \cap M$, we have always $d_{S:ij}(b) = 1$ by (2.25). We have shown that $D_S(b) \geqq A_S$. Q.E.D.

The proof of the following theorem is the almost same as that of Theorem of Shapley and Scarf [1974] that the core is non-empty in the market model where only indivisible commodities are exchanged . But as our market model is different from it, we give the proof of it for mathematical completeness .

Theorem 7. The associated characteristic function V_o is a balanced game . Hence the core of V_o is nonempty .

Proof. Let T be an arbitrary balanced family of coalitions, and let $b \in \bigcap_{S \in T} V_o(S)$. Let $\{\delta_S\}$ be balancing weights for T .

Then it holds that

$$D_{M \cup N}(b) = \sum_{S \in T} \delta_S D_S(S) .$$

For, if $d_{M \cup N:ij}(b) = 1$, then $d_{S:ij}(b) = 1$ for $\forall i \in S$ and $d_{S:ij}(b) = 0$ for $\forall i \notin S$ by (2.25) , which implies $\sum_{S \in T} \delta_S d_{S:ij}(b) = \sum_{S \in T: S \ni i} \delta_S d_{S:ij}(b) = 1$, and if $d_{M \cup N:ij}(b) = 0$, then $d_{S:ij}(b) = 0$ for $\forall S \in T$, which implies $\sum_{S \in T} \delta_S d_{S:ij}(b) = 0$. By Lemma 7 there is an S-permutation matrix A_S such that $D_S(b) \geqq A_S$, and so

$$\sum_{S \in T} \delta_S D_S(b) \geqq \sum_{S \in T} \delta_S A_S .$$

Call the matrix on the right B; then we have

$$D_{M \cup N}(b) \geqq B .$$

The crucial fact about B is that it is doubly stocastic, i.e., it is nonnegative and has all row- and column-sums equal to 1 . This follows directly from the definition of balancing weights; in fact, the j-th column sum is

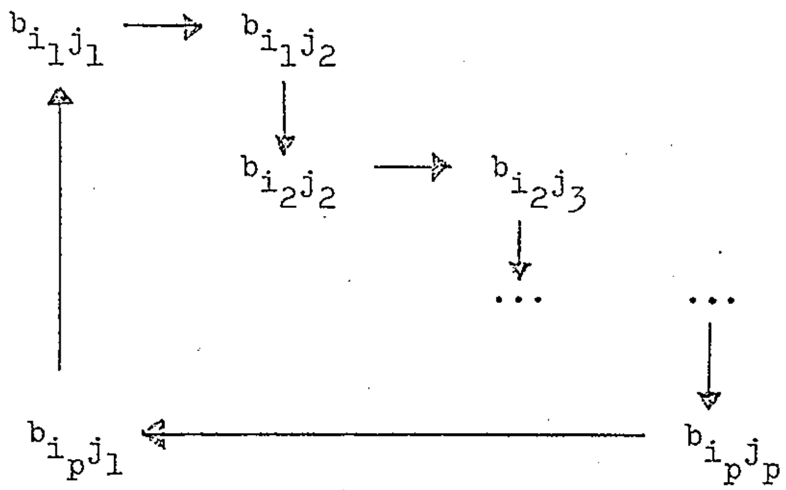
$$\sum_{i \in M \cup N} \sum_{S \in T} \delta_S a_{S:ij} = \sum_{S \in T} \delta_S \sum_{i \in M \cup N} a_{S:ij}$$

$$= \sum_{S \in T} \delta_S \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{if } j \notin S \end{cases} = \sum_{\substack{S \ni j \\ S \in T}} \delta_S = 1 ,$$

and the argument for the row-sums is the same .

The next step is to change B into an $M \cup N$-permutation matrix $A_{M \cup N}$, i.e., to eliminate any fractional entories without changing the row- or column-sums, and to do so without disturbing any entories which are already 0 or 1 . Since all entories of $D_{M \cup N}(b)$ are 0 or 1 , we will thereby ensure that $D_{M \cup N}(b) \geqq A_{M \cup N}$, which implies $b \in V_o(M \cup N)$.

Since a fraction can not occur alone in any row or column , either B is already an $M \cup N$-permutation matrix or there is a closed loop of fractional entries:



Alternately adding and substracting a fixed number ε to the elements of this loop will clearly preserve row- and column-sums. If ε is too large, the negative entories will be created, but making ε as large as possible consistent with nonnegativity will produce a new doubly stocastic matrix B' that has at least one more zero than B, and hence fewer fractional entories . If B' is not yet an $M \cup N$-permutation matrix, we repeat the same operation . Eventually we can obtain what we want- an $M \cup N$-permutation matrix $A_{M \cup N}$ such that $D_{M \cup N}(b) \geqq A_{M \cup N}$. Q.E.D.

Theorem 8 follows Theorem 2 and Theorem 7 .

<u>Theorem 8</u>. The cores of the game V and the assignment market are nonempty .

Now we consider the relationships between the core and the competitive equilibria. We call a pair $(p,(x^{M\cup N},m^{M\cup N}))$ of a price vector $p=(p_1,\ldots,p_s)\in E_+^s$ and an allocation $(x^{M\cup N},m^{M\cup N})$ a competitive equilibrium if

(2.30) for $\forall i\in M\cup N$, $U_i(x^i,m^i)\geqq U_i(y^i,m^i)$ for $\forall(y^i,m_1^i)\in X$ such that $py^i+m_1^i\leqq I_i+pa^i$,

(2.31) for $\forall i\in M\cup N$, $px^i+m^i = I_i+pa^i$.

Note that $(x^{M\cup N},m^{M\cup N})$ is an allocation, i.e., it must satisfy (2.3) and (2.4) for $S=M\cup N$. We call $(x^{M\cup N},m^{M\cup N})$ a competitive allocation if there is a price vector p such that $(p,(x^{M\cup N},m^{M\cup N}))$ is a competitive equilibrium, and p an competitive price vector.

Shapley and Shubik [1972] showed that the core always coincides with the set of all competitive allocations in the assignment market with the transferable utility assumption. This proposition is true even in the assignment market without the transferable utility assumption. Hence in the assignment market, it is sufficient to consider the competitive equilibria, which may make our analysis easier .

Theorem 9. The core coincides with the set of all competitive allocations in the assignment market.

Proof. It can be shown in the well-known way that the competitive allocations belong to the core of the assignment market . We need to show that the core is included by the set of all competitive allocations .

Let $(x^{M \cup N}, m^{M \cup N})$ be in the core . Let the partition given by Lemma 3 be $M = \{i_1, \ldots, i_k\} \cup M_o$ and $N = \{j_1, \ldots, j_k\} \cup N_o$. If there are seller $i \in \{i_1, \ldots, i_k\} \cap M_f$ and buyer $j \in \{j_1, \ldots, j_k\}$ with $x^j = e^f$ such that $m^j = I_j - r$, $m^i = I_i + r'$ and $r \neq r'$, then we have $m^i + m^j \neq I_i + I_j$, which is a contradiction to Corollary 4 . Hence it holds that if there is a buyer $j \in \{j_1, \ldots, j_k\}$ with $x^j = e^f$, there is a real number r_f such that

$$m^i = I_i + r_f \qquad \text{for } \forall i \in M_f \cap \{i_1, \ldots, i_k\}$$

$$m^j = I_j - r_f \qquad \text{for } \forall j \in \{j_1, \ldots, j_k\} \text{ with } x^j = e^f \ .$$

If $r_f \leqq 0$, then $U_i(e^f, I_i) > U_i(0, I_i + r_f)$ for $\forall i \in M_f \cap \{i_1, \ldots, i_k\}$ by (B') , which is a contradiction. Hence we have $r_f > 0$.

We define $p = (p_1, \ldots, p_s)$ by

$$(2.32) \qquad p_f = \begin{cases} r_f & \text{if there is a buyer } j \in \{j_1, \ldots, j_k\} \text{ with } x^j = e^f \\ \min_{i \in M_f} q_f(i) & \text{otherwise ,} \end{cases}$$

where we define $q_f(i)$ by $U_i(e^f, I_i) = U_i(0, I_i + q_f(i))$ for $\forall i \in M_f$ $(f = 1, \ldots, s)$. The existence and uniqueness of $q_f(i)$ are ensured by (A) and (C) . It is clear that $q_f(i) > 0$ by (B'). Hence $p_f > 0$ for $\forall f \leqq s$. It is easily verified that (2.31) holds for $(p, (x^{M \cup N}, m^{M \cup N}))$. Hence we need to show (2.30).

Suppose that there is a $(y^i, m_1^i) \in X$ for some $i \in M \cup N$ such that $U_i(y^i, m_1^i) > U_i(x^i, m^i)$ and $py^i + m_1^i \leqq I_i + pa^i$. First we consider the case where $i \in M_f$ ($1 \leqq f \leqq s$). In this case, by (B') it is sufficient to consider the case where $y^i = 0$. We have $p_f = r_f > \min_{i \in M_f} q_f(i)$, because if $p_f \leqq \min_{i \in M_f} q_f(i)$, then $U_i(y^i, m_1^i) = U_i(a^i, I_i) = U_i(x^i, m^i)$, which is a contradiction to the supposition that $U_i(y^i, m_1^i) > U_i(x^i, m^i)$. If $i \in M_f \cap \{i_1, \ldots, i_k\}$, then the existence of such a (y^i, m_1^i) is impossible by (B') and (2.32). Hence we have $i \in M_0 \cap M_f$. Since $p_f \geqq r_f > \min_{i \in M_f} q_f(i)$, there is a buyer $j \in \{j_1, \ldots, j_k\}$ with $x^j = e^f$. For a sufficiently small positive real number ε , it holds by (A) that

$$U_i(0, I_i + p_f - \varepsilon) > U_i(x^i, m^i)$$

$$U_j(e^f, I_j - p_f + \varepsilon) > U_j(x^j, m^j) \quad ,$$

which is a contradiction . Second, let $i \in N$. Then we can assume that $(y^i, m_1^i) = (e^f, I_i - p_f)$ for some $f \leqq s$. Since $U_i(e^f, I_i - p_f) > U_i(x^i, m^i) \geqq U_i(a^i, I_i)$, we have $I_i - p_f > 0$ by (D). Hence if $M_f \cap \{i_1, \ldots, i_k\} \neq \emptyset$, then for any $j \in M_f \cap \{i_1, \ldots, i_k\}$, it holds that for some $\varepsilon > 0$

$$U_i(e^f, I_i - p_f - \varepsilon) > U_i(x^i, m^i)$$

$$U_i(0, I_j + p_f + \varepsilon) > U_j(0, I_j + p_f) = U_j(x^j, m^j) ,$$

which is a contradiction. If $M_f \cap \{i_1, \ldots, i_k\} = \emptyset$, then for a seller j with $q_f(j) = \min_{t \in M_f} q_f(t)$,

$$U_i(e^f, I_i - p_f - \varepsilon) > U_i(x^i, m^i)$$

$$U_j(0, I_j + p_f + \varepsilon) > U_j(0, I_j + q_f(j)) = U_j(a^j, I_j) = U_j(x^j, m^j) ,$$

which is a contradiction. Q.E.D.

Theorem 10. There exists a competitive equilibrium in the assignment market .

3. The Generalized Assignment Market

In this section we consider a generalization of the assignment market given in the previous section, in which we admit each seller $i \in M_k$ ($k = 1,\dots,s$) to own more than one unit of the k-th indivisible commodity initially. That is, his initial endowment of the indivisible commodities a^i is $w_i e^k$ in the model of this section, where w_i may be any positive integer. By the similar reasoning with that of (B''), we assume: for $\forall i \in M_k$ ($k = 1,\dots,s$),

($\underline{B}$')(Saturation): $(x,m)Q_i(x_k e^k,m)$ for $\forall(x,m) \in X$, and if $x_k \geqq w_i$, then $(x,m)Q_i(w_i e^k,m)P_i((w_i-1)e^k,m)P_i \ \dots \ P_i(e^k,m)P_i(0,m)$.

We impose the transferable utility assumption on the sellers $i \in M$:

($\underline{E}$)(Constant Marginal Utility of Money): if $(x,m_1)Q_i(y,m_2)$, then $(x,m_1+\delta)Q_i(y,m_2+\delta)$ for $\forall \delta > 0$.

Kaneko [1976 b] showed that assumptions (A),(C) and (E) imply

(3.1) there is a real valued function $u_i(x)$ defined on I^s_+ such that $u_i(x) + m_1 \geqq u_i(y) + m_2$ if and only if $(x,m_1)R_i$ (y,m_2) .

This assumption (E) may not necessarily a strong one for the sellers. Each seller's amount of money can be only increased by selling his initial endowment of indivisible commodities . It is not inadequate to assume that the marginal utility of money is constant in the domain where the amount of money consumed is not small. Hence if I_i is not small, then we need to consider only the domain where the marginal utility of money can be assumed to be constant . It is natural to assume that each I_i ($i \in M$) is not too small. Hence this assumption (E) is not inadequate, though it should not be imposed upon the buyers because each buyer's amount of money is decreased by purchasing an indivisible commodity and the proportion of the payment to the income is not negligible.

In Kaneko [1976 a] , the case where $a^i_f > 0$ for $i \in M_k$ ($f \neq k$) was admitted. But the utility function $u_i(x)$ given by (3.1) was assumed to satisfy

$$u_i(x) = \sum_{f=1}^{s} u_i(x_f e^f) \quad .$$

This assumption is inadequate, but rather it is plausible to assume that $u_i(x) = u_i(\sum_{f=1}^{s} x_f)$. Because the indivisible commodities are permitted to be different but they are never substantially different commodities, which is an implication of (B") and ($\bar{B}'$). By this reason we do not generalize the assignment market to such a form . But we note that the following results can be gained without any essential change in the case .

We put $a_i(g) = u_i(ge^k) - u_i((g-1)e^k)$ for $\forall i \in M_k$ ($k = 1,\ldots,s$) and $g \leqq w_i$. We can put $u_i(0) = 0$ for $\forall i \in M$ without loss of generality. For convenience sake , we put $a_i(g) = 0$ for $\forall g > w_i$ ($i \in M$) . Then we have : for $\forall i \in M_k$

$$(3.2) \quad u_i(x) = \sum_{g=1}^{x_k} a_i(g) \; .$$

And assumption ($\bar{B}'$) implies $a_i(g) > 0$ for $\forall g \leqq w_i$ ($i \in M$). We assume that the marginal utility of the k-th indivisible commodity is not increasing for $\forall i \in M_k$ ($k = 1,\ldots,s$), that is, for $\forall i \in M$,

(F)(<u>Nonincreasing Marginal Utility of the Indivisible Commodity</u>): $a_i(g) \geqq a_i(g+1)$ for $\forall g$.

We call this market model (M,N) a <u>generalized assignment market</u>. The market models of Shapley and Shubik [1972] and of Kaneko [1972 a] are special cases of the generalized assignment market.[7)]

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7). Exactly, it is slightly different from a generalization of that of Kaneko [1976 a] as remarked above .

In order to investigate this market model, we shall define another market model (M^*, N), which we call the agent-assignment market.[8] The buyers N are the same as the buyers N of the generalized assignment market (M, N). The sellers M^* consist of $M_1^*, \ldots, M_s^*$, i.e., $M^* = M_1^* \cup \ldots \cup M_s^*$, such that

$$(3.3) \qquad M_k^* = \bigcup_{i \in M_k} \{i(1), \ldots, i(w_i)\} .$$

We assume that each seller $i(g) \in M_k^*$ ($k = 1, \ldots, s$) owns one unit of the k-th indivisible commodity and $I_{i(g)}$ ($I_{i(g)} > 0$) amount of money initially, i.e., $(a^{i(g)}, I_{i(g)}) = (e^f, I_{i(g)})$. The utility function $U_{i(g)}(x,m)$ ($i(g) \in M_k^*$) is given as

$$(3.4) \qquad U_{i(g)}(x,m) = u_{i(g)} + m$$

$$(3.5) \qquad u_{i(g)}(x) = \begin{cases} a_i(g) & \text{if } x_k \geqq 1 \\ 0 & \text{otherwise} . \end{cases}$$

The agent-assignment market is a special case of the as--signment market given in the previous section.

We can show that there is a one-to-one mapping from the set of all competitive allocations in the generalized assignment market to that in the agent-assignment market .

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8). The same procedure was employed in Kaneko [1976 a] .

Theorem 11. If $(p,(x^{M\cup N},m^{M\cup N}))$ is a competitive equilibrium in the generalized assignment market (M,N), then $(p,(x^{M^*\cup N},m^{M^*\cup N}))$ which is defined by

$$(3.6) \quad x^{i(g)} = \begin{cases} e^k & \text{if } x_k^i \geqq g \text{ and } i(g) \in M_k^* \\ 0 & \text{otherwise ,} \end{cases}$$

$$(3.7) \quad m^{i(g)} = \begin{cases} p_k + I_{i(g)} & \text{if } x^{i(g)} = 0 \text{ and } i(g) \in M_k^* \\ I_{i(g)} & \text{otherwise} \end{cases}$$

is a competitive equilibrium in the agent-assignment market (M*,N).[9] Conversely if $(p,(x^{M^*\cup N},m^{M^*\cup N}))$ is a competitive equilibrium in the agent-assignment market, then $(p,(x^{M\cup N},m^{M\cup N}))$ which is defined by

$$(3.8) \quad x^i = \sum_{g=1}^{w_i} x^{i(g)} \qquad \text{for } \forall i \in M .$$

$$(3.9) \quad m^i = I_i + \sum_{g=1}^{w_i} (m^{i(g)} - I_{i(g)}) \quad \text{for } \forall i \in M .$$

is a competitive equilibrium in the generalized assignment market (M,N) .

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9). p, x^j and m^j ($j \in N$) in $(p,(x^{M^*\cup N},m^{M^*\cup N}))$ are the same as those in $(p,(x^{M\cup N},m^{M\cup N}))$.

Proof. Let $(p,(x^{M\cup N},m^{M\cup N}))$ be a competitive equilibrium in the generalized assignment market. Then it is easily verified that $x^i = x_k^i e^k$ for $\forall i \in M_k$ ($k = 1,\ldots,s$) and $x^j = 0$ or e^k for $\forall j \in N$. Since $(x^{M\cup N},m^{M\cup N})$ is an allocation, $\sum_{i\in M\cup N} (x^i,m^i)$ $= \sum_{i\in M\cup N} (a^i,I_i)$.

Let $(p,(x^{M^*\cup N},m^{M^*\cup N}))$ be given by (3.6) and (3.7). Clearly (2.31) holds for $\forall i \in M^*\cup N$. First we show that $(x^{M^*\cup N},m^{M^*\cup N})$ is an allocation. By (3.6) we have

$$\sum_{i\in M} x_k^i = \sum_{i\in M_k} x_k^i = \sum_{i\in M_k} \sum_{g=1}^{x_k^i} x_k^{i(g)} = \sum_{i(g)\in M_k^*} x_k^{i(g)}$$

$$= \sum_{i(g)\in M^*} x_k^{i(g)} \quad ,$$

which implies $\sum_{i\in M^*\cup N} x^i = \sum_{i\in M^*\cup N} a^i$. It is clear that

$\sum_{i\in M} m^i = \sum_{i\in M} I_i + \sum_{k=1}^{s}\sum_{i\in M_k} (w_i - x_k^i)p_k$, which implies $\sum_{j\in N} m^j$ $= \sum_{j\in N} I_j - \sum_{k=1}^{s}\sum_{i\in M_k} (w_i - x_k^i)p_k$. Since we have by (3.7)

$$\sum_{i(g)\in M^*} m^{i(g)} = \sum_{i(g)\in M} I_{i(g)} + \sum_{k=1}^{s}\sum_{i\in M_k} (w_i - x_k^i)p_k \quad ,$$

it holds that $\sum_{i(g)\in M^*} m^{i(g)} + \sum_{j\in N} m^j = \sum_{i(g)\in M^*} I_{i(g)} + \sum_{j\in N} I_j$.

Clearly (2.30) is true for $\forall i \in N$. Hence we need to show that (2.30) also holds for $\forall i \in M^*$. Suppose that there is an i(g) $\in M_k^*$ for whom (2.30) is not true . This means that if $x^{i(g)} = e^k$, then $p_k + I_{i(g)} > a_i(g) + I_{i(g)} = u_{i(g)}(x^{i(g)}) + m^{i(g)}$,i.e., $p_k > a_i(g)$, and that if $x^{i(g)} = 0$, then $a_i(g) + I_{i(g)} > p_k + I_{i(g)} = u_{i(g)}(x^{i(g)}) + m^{i(g)}$, i.e., $a_i(g) > p_k$.

Since $(p,(x^{M\cup N},m^{M\cup N}))$ is a competitive equilibrium, it holds that

$$\sum_{g=1}^{x_k^i} a_i(g) + p_k(w_i - x_k^i) + I_i \geqq \sum_{g=1}^{h} a_i(g) + p_k(w_i - h) + I_i$$

for $\forall h$.

By this and (F) we have $p_k \leqq a_i(g)$ for $\forall g \leqq x_k^i$ and $p_k \geqq a_i(g)$ for $\forall g > x_k^i$. This contradicticts the fact that if $x^{i(g)} = e^k$, then $p_k > a_i(g)$ and if $x^{i(g)} = 0$, then $p_k < a_i(g)$. Hence (2.30) must hold for $\forall i(g) \in M^*$.

Let $(p,(x^{M^*\cup N},m^{M^*\cup N}))$ be a competitive equilibrium in the agent-assignment market, and let $(p,(x^{M\cup N},m^{M\cup N}))$ be given by (3.8) and (3.9). Similarly with the above, we can show (2.31) for $\forall i \in M$ and $(x^{M \cup N}, m^{M \cup N})$ is an allocation. We show (2.30) for $\forall i \in M$. Suppose that there is a seller $i \in M_k$ such that for some y_k^i

$$(3.10) \quad u_i(x^i) + m^i = \sum_{g=1}^{x_k^i} a_i(g) + p_k(w_i - x_k^i) + I_i$$

$$< \sum_{g=1}^{y_k^i} a_i(g) + p_k(w_i - y_k^i) + I_i = u_i(y_k^i e^f) + m_1^i .$$

If $x_k^i > y_k^i$, then we have $\sum_{g=y_k^i+1}^{x_k^i} a_i(g) < p_k(x_k^i - y_k^i)$, which implies that there is an $i(g) \in M_k^*$ such that $p_k > a_i(g)$ and $g \leqq x_k^i$. If $x^{i(g)} = 0$, then by (3.8) $x^{i(g')} = e^k$ for some $g' > g$. For this g', we have $p_k > a_i(g) \geqq a_i(g')$ by (F). Hence we can assume that $p_k > a_i(g)$ and $x^{i(g)} = e^k$. Thus we have $p_k + I_{i(g)} > a_i(g) + I_{i(g)} = u_{i(g)}(x^{i(g)}) + m^{i(g)}$, which is a contradiction. If $x_k^i < y_k^i$, then we have $\sum_{g=x_k^i+1}^{y_k^i} a_i(g) > p_k(y_k^i - x_k^i)$, which implies that there is an $i(g) \in M_k^*$ such that $a_i(g) > p_k$ and $x^{i(g)} = 0$. Hence we have $a_i(g) + I_{i(g)} = u_{i(g)}(a^{i(g)}) + I_{i(g)} > p_k + I_{i(g)} = u_{i(g)}(x^{i(g)}) + m^{i(g)}$, which is a contradiction. Q.E.D.

Theorem 12. There exists a competitive equilibrium in the generalized assignment market .

Proof. Theorem 10 ensures the existence of a competitive equilibrium in the agent-assignment market. Hence it follows from Theorem 11 that there exists a competitive equilibrium in the generalized assignment market corresponding to the competitive equilibrium in the agent-assignment market. Q.E.D.

Theorem 13. The core of the generalized assignment market is nonempty .

Proof. It can be shown in the usual way that the competitive allocations are included by the core . Hence Theorem 12 implies that the core is nonempty . Q.E.D.

In the assignment market, the core always coincides with the set of all competitive allocations. It is, however, not necessarily true in the generalized assignment market, but we can give a weak condition for the equivalence .

In Kaneko [1976a] , the following theorems were given as more general versions in the generalized assignment market with the transferable utility assumption. As the proofs of them are the almost same as Lemma 3 and Theorem II in Kaneko [1976 a] , we do not give the proofs in this paper .

If two sllers i_1 and i_2 in M have the same preference orderings and the same initial holdings, the sellers are called the <u>same type</u>. We consider the generlaized assignment market in the following theorems:

<u>Theorem 14</u>. If for each $i \in M_k$, there is at least one seller $i' \in M_k$ ($i \neq i'$) who is the same type with i, then in any allocation $(x^{M \cup N}, m^{M \cup N})$ in the core, the k-th indivisible commodities are exchanged at a common prices, i.e., there is a p_k such that

$$(3.11) \quad m^i = I_i + p_k(w_i - x^i_k) \quad \text{for } \forall i \in M_k$$

$$(3.12) \quad m^i = I_i - p_k \quad \text{for } \forall i \in N \text{ with } x^i = e^k .$$

<u>Theorem 15</u>. If for each $i \in M$, there is at least one seller i' ($i' \neq i$) with i, then the core coincides with the set of all competitive allocations .

In Kaneko [1976 a] , the core was considered in the case where the supposition of Theorem 14 is not true, i.e., a seller becomes a monopolist in a certain sense .[10] It says that the

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10). See Theorem III of Kaneko [1976a] .

core permits price discrimination. This result is also true in the generalized assignment market without the transferable utility assumption, but as it can be gained in the almost same way, we give an explanation only in diagram .

Let $s = 1$. Then the supply curve and the demand curve are drawn in Figure 1. Let us consider the case where the supposition of Theorem 14 is not true . Let $a_M = \min \{ a_i(g) | i \in M - \{1\}, g = 1, \ldots, w_i \}$. If a_M is greater than the intersection of the supply and demand curves ,e.g., a_M in Figure 1, then the core permits price discrimination, in any allocation in the core, seller 1 trades the commodity at different prices not more than a_M with different buyers . See Figure 1 . Seller 1 is considered to be a monopolist in this sense . If a_M is not greater than the intersection, the core coincides with the set of all competitive allocations . Of course, the commodity is traded at a common price in the intersection . This is a more precise sufficient condition than Theorem 15 for the equivalence of the core and the set of all competitive allocations, which is corresponding to the supposition of Theorem III of Kaneko [1976 a] .

When $s \geqq 2$, the similar price discrimination occures in the case where the supposition of Theorem 14 is not true . But as the indivisible commodities are not substantially different, any seller must regard the sellers owning the other commodities beside the ones owning the same commodity as competitors of him.

This fact makes the price discrimination in the core narrow .

Clearly the sufficient condition for the equivalence given by Theorem 15 is very weak. Furthermore even if the equivalence does not hold, the price discrimination in the core is not too large . Thus we get a conclusion that in most class of generalized assignment markets , the competitive equilibria may be representatives of the core . By this reason we can concentrate our consideration on the competitive equilibria in the succeeding sections.

Lastly we note that permissible coalitions can be constrained to a subclass of that of all the coalitions in the generalized assignment market similarly to Corollary 6 , which was shown in the case with the transferable utility assumption in Lemma 2 of Kaneko [1976 a] .

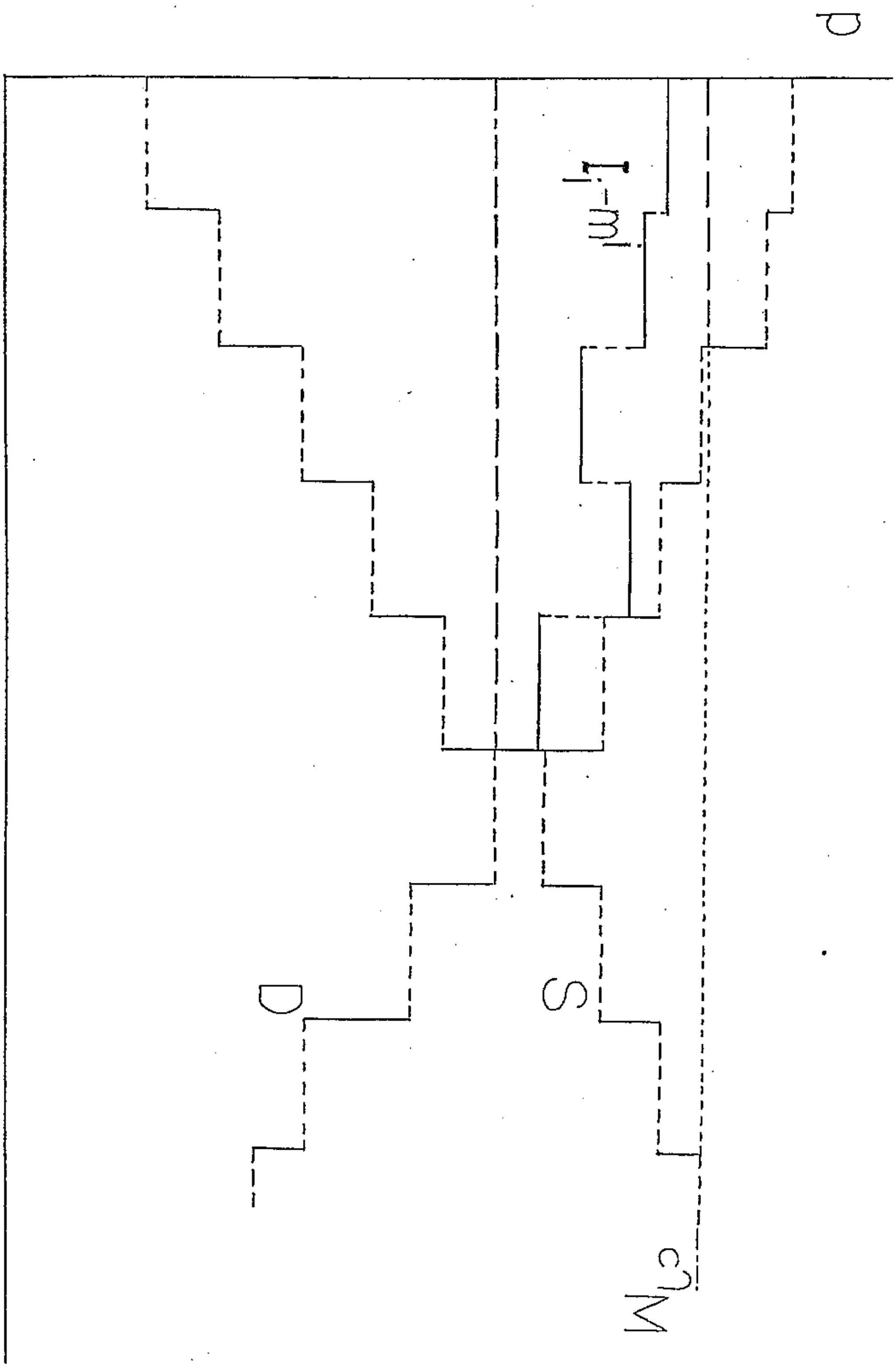

Figure 1.

4. Application to a Housing Market

In the present and next sections we consider an application of the theory of the generalized assignment market given in the previous section to a housing market. We give a simple model of a housing market which is depicted as a special case of the generalized assignment market, and consider a competitive rental values of houses. We should note that the conclusion of the previous section is that the competitive equilibria can be represntatives of the core in most class of generalized assignment markets. Hence we shall consider only the competitive equilibria.

We consider a one-period model of housing market (M,N) in which there are s kinds of houses for rent, i.e., services yielded by houses in the period are exchanged for money. In this section we call a member i in M a house-owner and a member j in N a household. Each house-owner $i \in M_k$ ($k = 1,\ldots,s$) owns w_i units of the k-th house to be leased at the initial point of the period. No household in N owns any house nor has rented any house at the initial point of the period. Hence i's initial endowment of houses and money (a^i, I_i) is $(w_i e^k, I_i)$ if $i \in M_k$ and $(0, I_i)$ if $i \in N$. The housing market (M,N) is assumed to be a generalized assignment market. It is noted that

$$(4.1) \quad \sum_{i \in M_k} w_i > 0 \quad \text{for } \forall k = 1,\ldots,s .$$

We assume the following conditions:

(<u>F'</u>)(<u>Constant Marginal Utility of the Indivisible Commodities</u>):
$a_i(g) = a_k$ for $\forall i \in M_k$ ($k = 1,\ldots,s$) and $g = 1,\ldots,w_i$,

(<u>G</u>): all the households $i \in N$ have the same preference ordering R .

Assumption (F') implies that house-owner i's ($i \in M_k$) utility function $u_i(x) + m$ satisfies

$$(4.2) \quad u_i(x) = \begin{cases} a_k x_k & \text{if } x_k \leqq w_i \\ a_k w_i & \text{if } x_k \geqq w_i \end{cases} .$$

We note that a_k is independent of house-owners in M_k , and that x is the amount of remaining houses, i.e., $w_i e^k - x$ is the amount of houses which house-owner i leases in the period. We call a_k an <u>evaluation value</u> of the k-th house. We also note that $a_k >$ 0 for $\forall k = 1,\ldots,s$ by ($\overline{B}$').

Assumptions (F') and (G) impose homogeneities on the house-owners' and households' preference orderings. a_k is the least value measured in terms of money at which house-owners in M_k can lease one unit of the k-th house in the period without any decrease of their utilities less than those of not leasing.

We reorder the households 1,...,n as follows:

$$(4.3) \qquad I_1 \geqq I_2 \geqq \cdots \geqq I_n \quad .$$

Next we assume that for $\forall i \in N$

(H)(Decreasing Marginal Utility of Money): if $(x,m_1)Q(y,m_2)$, $m_1 < m_2$ and $\delta > 0$, then $(x,m_1+\delta)P(y,m_2+\delta)$.

Assumption (H) means that each household's marginal uility of money is decreasing. As it is assumed by (E) that each house-owner's marginal utility of money is constant, assumption (H) seems to be inconsistent with (E). But the plausibility of assuming them is discussed in the paragraph after (3.1).

Example 16. Let $h_1,\ldots,h_s$ be positive real numbers. We define a function $f(x)$ on I_+^s by

$$(4.4) \qquad f(ge^k) = \begin{cases} h_k & \text{if } g \geqq 1 \\ 0 & \text{if } g = 0 \end{cases} ,$$

$$f(x) = \max \left\{ h_k \mid x_k \geqq 1 \right\} .$$

Let g(m) be a strictly concave, continuous and increasing function on E_+ with $\lim_{m \to \infty} g(m) = \infty$. Then the function $U(x,m) = f(x) + g(m)$ gives a preference ordering R on X which satisfies assumptions (A),(B''),(C) and (H). Of course, any monotonically increasing transformation of this U(x,m) is also a utility function representing the preference ordering R. For R given by U(x,m) to satisfy assumption (D), it must hold that

$$(4.5) \quad g(I_n) > \max_k h_k \quad .$$

<u>Lemma</u> <u>17</u>.(<u>i</u>). If $(x,m_1)\Omega(y,m_2)$ and $0 < \delta \leqq m_2 < m_1$, then $(x,m_1-\delta)P(y,m_2-\delta)$.

(<u>ii</u>). If $(e^k,0)P(e^f,0)$ and $(e^k,m_1)\Omega(e^f,m_2)$, then $m_1 < m_2$.

<u>Proof</u>. (<u>i</u>). Suppose that $(y,m_2-\delta)R(x,m_1-\delta)$. By (A) and (C) there is a $b \geqq 0$ such that $(y,m_2-\delta)\Omega(x,m_1-\delta+b)$. Since $m_1-\delta+b > m_2-\delta$, we have $(y,m_2)P(x,m_1+b)$ by (H). But by (A) we have $(x,m_1+b)R(x,m_1)\Omega(y,m_2)$, i.e., $(x,m_1+b)R(y,m_2)$, which is a contradiction.

(<u>ii</u>). By (A) and (C) there is a $b > 0$ such that $(e^k,0)\Omega(e^f,b)$.

By (H) we have $(e^k, m_1)P(e^f, m_1+b)$. Hence we have $(e^f, m_2)P(e^f, m_1+b)$. This implies $m_2 > m_1+b > m_1$ by (A). Q.E.D.

We assume for simplicity that when households can consume no amount of money, there is no indifference between any pair of houses, i.e., for any k and f ($k \neq f$) $(e^k,0)P(e^f,0)$ or $(e^f,0)P(e^k,0)$. Furthermore we reorder $1,\ldots,s$ such that

$$(\underline{I}):\ (e^1,0)P(e^2,0)P\ \ldots\ P(e^s,0)\ .$$

Assumption (I) is equivalent to the following condition (4.6), which gives us a clear meaning of assumption (I):

$$(4.6)\quad (e^1,m)P(e^2,m)P\ \ldots\ P(e^s,m)\quad \text{for } \forall m \geqq 0\ .$$

That is, if the amount of money after paying the rental values are the same, households have the same preference ordering as that given by (I). We show that assumption (I) implies (4.6). If $(e^k,m)Q(e^{k+1},m_1)$, then we have $m < m_1$ by Lemma 17.(ii), which implies $(e^k,m)P(e^{k+1},m)$ by (A) .

We define G(k) ($k = 1,\ldots,s$) by

$$(4.7)\quad G(k) = \sum_{t=1}^{k} \sum_{i \in M_t} w_i\ .$$

We call G(k) the <u>k-th marginal household</u> if $G(k) \leqq n$, i.e., $G(k) \in N$. We call the f-th house the <u>marginal house</u> if

$$(4.8) \quad \sum_{k=1}^{f-1} \overline{\sum_{i \in M_k}} w_i < n \leqq \sum_{k=1}^{f} \overline{\sum_{i \in M_k}} w_i$$

$$\text{or } f = s \quad \text{if } n \geqq \sum_{k=1}^{s} \overline{\sum_{i \in M_k}} w_i \quad .$$

These terminologies will play important role in the following .

We assume:

$$(\underline{J}): \ (e^1, I_n - a_1) P (e^2, I_n - a_2) P \ \ldots \ P(e^s, I_n - a_s) P(0, I_n) \ .$$ [11)]

Assumption (J) means that when house-owners lease houses at the evaluation values $a_1, \ldots, a_s$, the poorest household n prefers to rent a house in the same order as that of (I). Then household n wants to rent a house rather than not to rent any house . Assumption (J) imposes a kind of homogeneity on sizes of houses. When the first kind of house is very large so that $I_n - a_1$ is very small, but when there is a kind k of house such that $I_n - a_k$ is not

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11). Of course, it is assumed that $I_n \geqq a_k$ for $\forall k \leqq s$.

small, household n would prefer $(e^k, I_n - a_k)$ to $(e^1, I_n - a_1)$, though he prefers $(e^1, 0)$ to $(e^k, 0)$. Assumption (J) excepts such cases. We can give an extreme example such that the sizes of all the houses are the same and they are distinct with respect to only the distances from the center of a city. But the following example shows that the assumption permits differentiations of a_k ($k = 1, \ldots, s$) to a certain extent .

Example 18. Let $s = 7$. Let $I_n = 80000$ Yen , $U(x,m) = f(x) + g(m)$ be given by Example 16, and $g(m) = \sqrt{m}$. Let h_k and a_k ($k = 1, 2, \ldots, 7$) be given as Table 1. Then $U(e^k, I_n - a_k)$ is given as Table 1.

k	1	2	3	4	5	6	7
h_k	80	75	70	65	60	55	50
a_k	22000 Yen	20000	18000	16000	14000	12000	10000
$U(e^k, I_n - a_k)$	320.8	319.9	319.0	318.0	316.9	315.8	314.6

Table 1 .

In fact, assumption (J) can be replaced by the following weaker condition (4.9) in the following essential argument;

(4.9) $(e^k, I_{G(k)}-a_k)P(e^{k+1}, I_{G(k)}-a_{k+1})$ for $\forall k$ such that $G(k) \leqq n$, and $(e^f, I_n-a_f)P(e^{f+1}, I_n-a_{f+1})P \quad \ldots \quad P(e^s, I_n-a_s)P$ $(0, I_n)$, where f is the marginal house ,

which is equivalent to

(4,10) $(e^f, I_i-a_f)P(e^{f+1}, I_i-a_{f+1})P \quad \ldots \quad P(e^s, I_i-a_s)P(0, I_i)$
for $\forall i \in N$ with $I_i \geqq I_n$,

(4.11) $(e^k, I_i-a_k)P(e^{k+1}, I_i-a_{k+1})P \quad \ldots \quad P(e^s, I_i-a_s)P(0, I_i)$
for $\forall i \in N$ with $I_i \geqq I_{G(k)}$.

It should be noted that there will be propositions in the following which will not be true under (4.9). But this assumption makes the following argument complex and does our intuition unclear. Hence we do not replace assumption (J) by (4.9) in this paper .

<u>Lemma 19</u>. $(e^1, I_i-a_1)P(e^2, I_i-a_2)P \quad \ldots \quad P(e^s, I_i-a_s)P(0, I_i)$ for $\forall i \in N$.

<u>Proof</u>. By (A) and (C) there is a $b > 0$ such that (e^{k-1}, I_n-a_{k-1}) $Q(e^k, I_n-a_k+b)$. By (I) and Lemma 17.(ii) we have $I_n-a_{k-1} <$

$I_n - a_k + b$. By (H) we have $(e^{k-1}, I_i - a_{k-1})R(e^k, I_i - a_k + b)$. Hence we have $(e^{k-1}, I_i - a_{k-1})P(e^k, I_i - a_k)$ by (A) .

By (A) and (C) there is a $b > 0$ such that $(e^s, I_n - a_s)Q(0, I_n + b)$. Since $I_n - a_s < I_n + b$, we have $(e^s, I_i - a_s)R(0, I_i + b)P(0, I_i)$ by (A) and (H). Q.E.D.

In the present and next sections we call a competitive price vector $p = (p_1, \dots, p_s)$ a <u>competitive rent vector</u> and each p_k ($k = 1, \dots, s$) a <u>competitive rental value</u>.

We showed the existence of a competitive equilibrium in the generalized assignment market in the previous section, but now we can construct a competitive equilibrium.

<u>Theorem 20</u>. The rent vector $p = (p_1, \dots, p_s)$ which is given in (i),(ii) or (iii) is a competitive rent vector:

(<u>i</u>): If $G(f-1) < n < G(f)$, then p is defined by

$$(4.12) \quad p_k = a_k \quad \text{for } \forall k \geqq f$$

$$(e^k, I_{G(k)} - p_k)Q(e^{k+1}, I_{G(k)} - p_{k+1}) \quad \text{for } \forall k = 1, \dots, f-1 .$$

(ii): If $G(f) = n$, then p is defined by

$$(4.13)\quad p_k = a_k \quad \text{for } \forall k \geqq f+1 \quad \text{and } a_f \leqq p_f \leqq p_f^*$$

$$(e^k, I_{G(k)} - p_k)\Omega(e^{k+1}, I_{G(k)} - p_{k+1}) \quad \text{for } \forall k = 1,\ldots,f-1 \ ,$$

where p_f^* is defined by $(e^f, I_{G(f)} - p_f^*)\Omega(e^{f+1}, I_{G(f)} - a_{f+1})$ if $f < s$ and $(e^f, I_{G(f)} - p_f^*)\Omega(0, I_{G(f)})$ if $f = s$.

(iii): If $G(s) > n$, then p is defined by

$$(4.14)\quad (e^k, I_{G(k)} - p_k)\Omega(e^{k+1}, I_{G(k)} - p_{k+1}) \quad \text{for } \forall k = 1,\ldots,s-1$$

$$p_s^1 \leqq p_s \leqq p_s^2 \quad ,$$

where $(e^s, I_{G(s)+1} - p_s^1)\Omega(0, I_{G(s)+1})$ and $(e^s, I_{G(s)} - p_s^2)\Omega(0, I_{G(s)})$.

Proof. Initially we prove in case (i) that there is a rent vector satisfying (4.12) and that the rent vector is competitive. As case (ii) can be proved similarly, we omit the proof .

By Lemma 19 and (D), it holds that $(e^f, I_{G(f-1)} - a_f)P(0, I_{G(f-1)})$ $P(e^{f-1}, 0)$. Hence there is a $b_{f-1} > 0$ by (C) such that $(e^f, I_{G(f-1)} - a_f)\Omega(e^{f-1}, b_{f-1})$. It holds by Lemma 19 that $(e^{f-1}, I_{G(f-1)} - a_{f-1})$ $P(e^f, I_{G(f-1)} - a_f)\Omega(e^{f-1}, b_{f-1})$, which implies $I_{G(f-1)} - a_{f-1} > b_{f-1}$

by (A). Hence there is a p_{f-1} such that $b_{f-1} = I_{G(f-1)}-p_{f-1}$, i.e., $(e^{f-1}, I_{G(f-1)}-p_{f-1})\Omega(e^{f}, I_{G(f-1)}-a_{f})$ and $I_{G(f-1)} > p_{f-1} > a_{f-1}$.

By (I) and Lemma 17.(ii) we have $I_{G(f-1)}-p_{f-1} < I_{G(f-1)}-a_{f}$. Hence we have $(e^{f-1}, I_{G(f-2)}-p_{f-1})R(e^{f}, I_{G(f-2)}-a_{f})$ by (H). Since $p_{f-1} > a_{f-1}$, we have $(e^{f-1}, I_{G(f-2)}-a_{f-1})P(e^{f-1}, I_{G(f-2)}-p_{f-1})$. Hence we get $(e^{f-2}, I_{G(f-2)}-a_{f-2})P(e^{f-1}, I_{G(f-2)}-a_{f-1})P(e^{f-1}, I_{G(f-2)}-p_{f-1})R(e^{f}, I_{G(f-2)}-a_{f})P(0, I_{G(f-2)})P(e^{f-2}, 0)$. This implies that there is a b_{f-2} such that $(e^{f-1}, I_{G(f-2)}-p_{f-1})\Omega(e^{f-2}, b_{f-2})$ and $I_{G(f-2)}-a_{f-2} > b_{f-2} > 0$. Let $b_{f-2} = I_{G(f-2)}-p_{f-2}$. This p_{f-2} satisfies $(e^{f-2}, I_{G(f-2)}-p_{f-2})\Omega(e^{f-1}, I_{G(f-2)}-p_{f-1})$ and $I_{G(f-2)} > p_{f-2} > a_{f-2}$.

By the same argument we get $p_{f-3}, \ldots, p_1$. These $p_1, \ldots, p_{f-1}$ satisfies (4.12). Furthermore it is noted that these satisfy

$$(4.15) \qquad I_{G(k)} > p_k > a_k \quad \text{for } \forall k = 1, \ldots, f-1 .$$

We define $(x^{M \cup N}, m^{M \cup N})$ as follows:

$$(4.16) \qquad x^i = \begin{cases} 0 & \text{if } i \in M_k \text{ and } k < f \\ w_i e^k & \text{if } i \in M_k \text{ and } k > f , \end{cases}$$

$$\sum_{i \in M_f} x^i + \sum_{i \in N} x_f^j e^f = \sum_{i \in M_f} w_i e^f \ , \ 0 \leqq x_f^i \leqq w_i \quad \text{for } \forall i \in M_f \ ,$$

(4.17) $x^j = e^k$ if $G(k-1) < j \leqq G(k)$ and $j \in N$,

and

(4.18) $m^i = I_i + p(w_i e^k - x^i)$ for $\forall i \in M_k$ ($k = 1, \ldots, s$)

$m^j = I_j - px^j$ for $\forall j \in N$.

We shall prove that $(p,(x^{M \cup N}, m^{M \cup N}))$ is a competitive equilibrium. It is easily verified that $(x^{M \cup N}, m^{M \cup N})$ defined above is an allocation. We need to show that $(p,(x^{M \cup N}, m^{M \cup N}))$ satisfies (2.30) .

Let j be a household such that $G(k-1)+1 \leqq j \leqq G(k)$ and $k < f$. Since $(e^k, I_{G(k)}-p_k)\Omega(e^{k+1}, I_{G(k)}-p_{k+1})$ and $I_j \geqq I_{G(k)}$, we have $(e^k, I_j-p_k)R(e^{k+1}, I_j-p_{k+1})$ by (H), because $I_{G(k)}-p_k < I_{G(k)}-p_{k+1}$ by (I) and Lemma 17.(ii) . Similarly we can show that $(e^{k+1}, I_j-p_{k+1})R(e^{k+2}, I_j-p_{k+2})R \ \ldots \ R(e^{f-1}, I_j-p_{f-1})R(e^f, I_j-p_f)$. By Lemma 19 we have $(e^f, I_j-p_f)P(e^{f+1}, I_j-p_{f+1})P \ \ldots \ P(e^s, I_j-p_s)P(0, I_j)$. Since $(e^{k-1}, I_{G(k-1)}-p_{k-1})\Omega(e^k, I_{G(k-1)}-p_k)$ and $I_j \leqq$

$I_{G(k-1)}$, we have $(e^k, I_j - p_k) R (e^{k-1}, I_j - p_{k-1})$ by Lemma 17.(i) , or $I_j < p_{k-1}$, because $I_{G(k-1)} - p_{k-1} < I_{G(k-1)} - p_k$ by (I) and Lemma 17.(ii). Similarly we can prove that if I_j is not smaller than $p_{k-2}, \ldots, p_1$, then $(e^{k-1}, I_j - p_{k-1}) R (e^{k-2}, I_j - p_{k-2}) R \ \ldots \ R \ (e^1, I_j - p_1)$. When k = f , the first relation is not necessary to be shown, but the others have been shown without the condition $k < f$. Hence we have shown that for $\forall j \in N$ ($G(k-1)+1 \leqq j \leqq G(k)$), $(e^k, I_j - p_k) R (e^g, I_j - p_g)$ for $\forall g$ ($g \neq k$ and $I_j \geqq p_g$) and $(e^k, I_j - p_k) P (0, I_j)$. For any $(x, I_j - px)$ with $I_j \geqq px$, there is a g such that $(e^g, I_j - p_k) R (x, I_j - px)$ by (A) and (B") . Hence we have shown that $(e^k, I_j - p_k) R (x, m)$ for $\forall (x, m) \in X$ with $px + m \leqq I_j$.

Since $p_k > a_k$ for $\forall k = 1, \ldots, f-1$ by (4.15) and $p_k = a_k$ for $\forall k = f, \ldots, s$ by (4.12), it holds that

$$u_i(x^i) + I_i + p(w_i e^k - x) = \max_{z \in I_+^s} (u_i(z) + I_i + p(w_i e^k - z))$$

$$\text{for } \forall i \in M_k \ (k = 1, \ldots, s) .$$

Let us consider case (iii). We define $(x^{M \cup N}, m^{M \cup N})$ as follows:

(4.19) $\quad x^i = 0 \quad$ for $\forall i \in M$,

$$(4.20) \qquad x^j = \begin{cases} e^k & \text{if } G(k-1)+1 \leqq j \leqq G(k) \text{ and } k \leqq s \\ 0 & \text{if } j > G(s) \quad , \end{cases}$$

and

$$(4.21) \qquad m^i = I_i + p(w_i e^k - x^i) \qquad \text{for } \forall i \in M_k \ (k = 1,\dots,s)$$

$$m^j = I_j - px^j \qquad \text{for } \forall j \in N \ .$$

Similarly to the proof of (i), we can prove that $(p,(x^{M \cup N},m^{M \cup N}))$ is a competitive equilibrium. Q.E.D.

Example 21. Let us consider Example 18 . Let $G(5) < n < G(6)$, and let $I_{G(k)}$ ($k = 1,\dots,5$) be given as Table 2 . Condition (4.12) is written as follows:

$$(4.22) \qquad p_6 = a_6 \quad \text{and} \quad p_7 = a_7$$

$$h_k + \sqrt{I_{G(k)} - p_k} = h_{k+1} + \sqrt{I_{G(k)} - p_{k+1}}$$

$$\text{for } \forall k = 1,\dots,5 \ .$$

By solving this equation we get the competitive rent vector p given as Table 2 .

k	1	2	3	4	5	6	7
$I_{G(k)}$	130000 Yen	120000	110000	100000	90000		
p_k	27036 Yen	23803	20676	17662	14768	12000	10000

Table 2 .

In Example 21 we got a monotonically decreasing slope of a competitive rental values, but in fact, this property holds generally .

Theorem 22. Let $(p,(x^{M\cup N},m^{M\cup N}))$ be an arbitrary competitive equilibrium and let f be the marginal house in (M,N) . In $(p, (x^{M\cup N},m^{M\cup N}))$, the following propositions (i),(ii),(iii) and (iv) hold:

(i): Each household $i \in N$ rents at most one unit of house, i.e., $x^i = e^k$ for some $k \leqq s$ or $x^i = 0$. When $n > \sum_{k=1}^{s} \sum_{i \in M_k} w_i$, all the house-owners $i \in M$ lease all the units of houses, i.e., $x^i = 0$. When $n \leqq \sum_{k=1}^{s} \sum_{i \in M_k} w_i$, each household $i \in N$ rents exactly one unit of the kinds 1,...,s of houses, i.e., $x^i = e^k$ for some $k \leqq s$.

(ii): If $I_i > I_{i_o}$ ($i, i_o \in N$), $x^i = e^k$ and $x^{i_o} = e^{k_o}$, then $k \leqq k_o$.

(iii): $p_k > a_k$ for $\forall k = 1, \ldots, f-1$ and $p_k \leqq a_k$ for $\forall k = f+1, \ldots, s$.

(iv): $p_1 > p_2 > \cdots > p_f$.

Proof. If $p_k < a_k$, then the house-owners $i \in M_k$ never lease any houses . Hence if $\sum_{j \in N} x_k^j > 0$, then $p_k \geqq a_k$, because $(x^{M \cup N}, m^{M \cup N})$ is an allocation . Suppose that $\sum_{k=1}^{s} x_k^j \geqq 2$ for some $j \in N$, i.e., household j rents more than one unit of house . Then, by (A) and (B''), there is a k such that $x_k^j \geqq 1$ and $(e^k, I_j - p_k)P(x^j, I_j - px)$, because $p_t \geqq a_t > 0$ for $\forall t$ with $x_t^j \geqq 1$. This is a contradiction . We have shown that each household rents at most one unit of house .

Suppose that $n > \sum_{k=1}^{s} \sum_{i \in M_k} w_i$. If there is a k such that $p_k \leqq a_k$, then $(e^k, I_j - p_k)P(0, I_j)$ for $\forall j \in N$ by Lemma 19 and (A) . This implies that the total demand is n, which is over the potential total supply $\sum_{k=1}^{s} \sum_{i \in M_k} w_i$. This is a contradiction . Hence we get $p_k > a_k$ for $\forall k \geqq s$. This means that all the house-owners leases all their houses .

Suppose that $n \leqq \sum_{k=1}^{s} \sum_{i \in M_k} w_i$. If there is a household $j \in N$ who does not rent any house, then it must hold that $(0,I_j)R(e^k,I_j-p_k)$ for $\forall k \leqq s$. This implies that $I_j-a_k > I_j-p_k$, i.e., $p_k > a_k$ for $\forall k \leqq s$, because $(e^k,I_j-a_k)P(0,I_j)$ for $\forall k \leqq s$ by Lemma 19 . It follows that the total supply is $\sum_{k=1}^{s} \sum_{i \in M_k} w_i$. Since the household j does not rent any house, the total demand is smaller than n . Hence the total supply is over the total demand , which is a contradiction . We have shown that every household rents exactly one unit of house .

Suppose that $x^j = e^k$ ($j \in N$) . Since $(p,(x^{M \cup N},m^{M \cup N}))$ is a competitive equilibrium, we have $(e^k,I_j-p_k)R(e^t,I_j-p_t)$ for $\forall t \neq k$ and $p_k \geqq a_k$. If $p_t \leqq a_t$ for some $t < k$, then we have $(e^k,I_j-a_k)R(e^k,I_j-p_k)R(e^t,I_j-p_t)R(e^t,I_j-a_t)$ by (A) . This contradicts Lemma 19 . Hence we have shown that if $x^j = e^k$ for some $j \in N$ and k , then $p_t > a_t$ for $\forall t < k$. If there is a $j \in N$ such that $x^j = e^k$ for some $k > f$, then $p_t > a_t$ for $\forall t < k$. This means that all the houses of the kinds 1,...,f are leased , i.e., they are rented by households , because $(p,(x^{M \cup N},m^{M \cup N}))$ is a competitive equilibrium . Since the household j rents one unit of the k-th house ($k > f$) , the total demand is over

$\sum_{t=1}^{f} \sum_{i \in M_t} w_i$, which is not smaller than n . That is, the total demand is over n , which is a contradiction . Thus we have shown that each household rents exactly one unit of house of the kinds of 1,...,f , and that $p_k > a_k$ for $\forall k = 1,\dots,f-1$ and $p_k \leqq a_k$ for $\forall k = f+1,\dots,s$. We complete the proofs of (i) and (iii) .

Next we show (ii) . Suppose $k_o < k$. By (I) we have $(e^{k_o},0)P(e^k,0)$. Since $(p,(x^{M\cup N},m^{M\cup N}))$ is a competitive equilibrium , we have $(e^{k_o},I_{i_o}-p_{k_o})R(e^k,I_{i_o}-p_k)$. By (A) and (C) there is a $b \geqq 0$ such that $(e^{k_o},I_{i_o}-p_{k_o})Q(e^k,I_{i_o}-p_k+b)$. By Lemma 17.(ii) we have $I_{i_o}-p_{k_o} < I_{i_o}-p_k+b$. Then we have $(e^{k_o},I_i-p_{k_o})P(e^k,I_i-p_k+b)R(e^k,I_i-p_k)$ by (A) and (H). This contradicts the supposition that $(p,(x^{M\cup N},m^{M\cup N}))$ is a competitive equilibrium .

We show (iv) . In the above we showed that there is a $j \in N$ for $\forall k \leqq f$ such that $x^j = e^k$. Since $(p,(x^{M\cup N},m^{M\cup N}))$ is a competitive equilibrium, we have $(e^k,I_j-p_k)R(e^{k-1},I_j-p_{k-1})$. Since $(e^{k-1},0)P(e^k,0)$, we get $I_j-p_k > I_j-p_{k-1}$ by Lemma 17.(ii) and (A) . This implies $p_k < p_{k-1}$. Q.E.D.

When the number of households is not greater than that of houses owned by the house-owners, each household rents exactly one unit of the k-th house ($1 \leqq k \leqq f$). If otherwise, the house-owners lease all the houses, but there is a household who can not rent any house . The proposition (ii) means that a household having a greater income rents a more preferred house, and that the income levels of households renting houses are greater than those of households not renting any houses if they exists . The proposition (iv) means that the competitive rental values have a slope decreasing from the most preferred house 1 to the marginal house f . We can illustrate it . See Figure 2 .

In the following we assume for simplicity:

$$(\underline{K}): \quad \sum_{k=1}^{f-1} \sum_{i \in M_k} w_i + 1 \leqq n < \sum_{k=1}^{f} \sum_{i \in M_k} w_i \ .$$

This assumption makes us to concentrate our consideration on the case (i) of Theorem 20 . The following arguments results will be gained in the cases (ii) and (iii) of Theorem 20 without any essential change.

Furthermore we choose a competitive rent vector from the set of all competitive rent vectors and concentrate our consideration upon it . We call $p = (p_1, \dots, p_s)$ the <u>maximal</u> <u>competitive</u> <u>rent</u> <u>vector</u> if

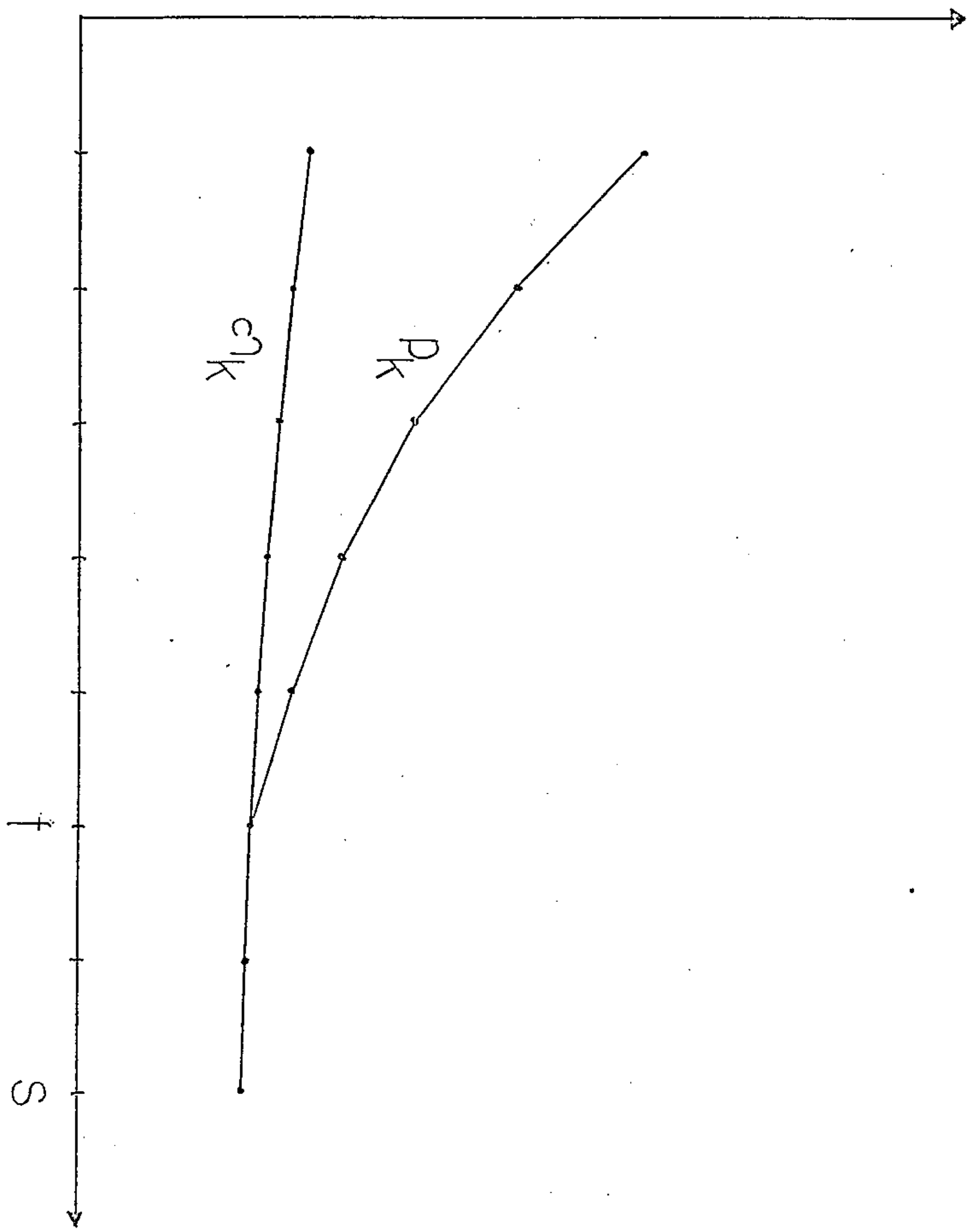

Figure 2.

(4.23) p is a competitive rent vector,

(4.24) for every competitive rent vector $p' = (p_1', \dots, p_s')$,
$p_k \geqq p_k'$ for $\forall k \leqq s$.

It should be noted that if the maximal competitive rent vector exists, then it is unique .

Lemma 23. The maximal competitive rent vector p exists, and is given by (4.12) of Theorem 20 .

Proof. It is sufficient to show that p given by (4.12) of Theorem 20 is maximal .

Let p' be an arbitrary competitive rent vector. In Theorem 22 we got $p_k' \leqq a_k$ for $\forall k \geqq f+1$. If $p_f' > a_f$, then the total supply is at least $\sum_{k=1}^{f} \sum_{i \in M_k} w_i$, which is over n by (K). This is a contradiction . Hence we have $p_k' \leqq a_k$ for $\forall k = f+1, \dots, s$.

By Theorem 22.(ii) we can assume without loss of generality that in an arbitrary competitive equilibrium $(p, (x^{M \cup N}, m^{M \cup N}))$, each k-th marginal household G(k) ($k \leqq f-1$) rents one unit of the k-th house, i.e., $x^{G(k)} = e^k$. Since p' is a competitive rent vector, we have $(e^{f-1}, I_{G(f-1)} - p_{f-1}') R (e^f, I_{G(f-1)} - a_f)$. But since $(e^{f-1}, I_{G(f-1)} - p_{f-1}) Q (e^f, I_{G(f-1)} - a_f)$ by (4.12), we have

$I_{G(f-1)}-p'_{f-1} \geqq I_{G(f-1)}-p_{f-1}$ by (A), i.e., $p'_{f-1} \leqq p_{f-1}$. Then we have $(e^{f-2},I_{G(f-2)}-p'_{f-2})R(e^{f-1},I_{G(f-2)}-p'_{f-1})R(e^{f-1},I_{G(f-2)}-p_{f-1})Q(e^{f-2},I_{G(f-2)}-p_{f-2})$ by (A),(4.12) and the supposition that p' is a competitive rent vector . Hence we have $(e^{f-2},I_{G(f-2)}-p'_{f-2})R(e^{f-2},I_{G(f-2)}-p_{f-2})$. This implies $p'_{f-2} \leqq p_{f-2}$ by (A). Similarly we get $p_k \geqq p_{k'}$ for $\forall k \leqq f$. Q.E.D.

In the cases (ii) or (iii) of Theorem 20 , p defined by (4.13) and p^*_f or (4.14) and p^2_s can be shown to be a maximal competitive rent vector respectively .

5. Comparative Statics

In this section we consider effects of some changes of certain parameters of the housing market (M,N) upon the competitive rental values. Let (M*,N*) be a housing market which is yielded by changes of certain parameters from the housing market (M,N), and which is also assumed to be a generalized assignment market satisfying all the assumptions in Section 4. $M^* = \{1^{*\prime},\ldots,m^{*\prime}\}$ is the set of all house-owners, and $N^* = \{1^*, \ldots, n^*\}$ is the set of all households. Of course, members in M* or N* may be different from those in M or N respectively. Because we think that economic agents are also parameters in our economy. The house-owners in M* have utility functions in the form given by (4.2) with the evaluation values $a_1^*,\ldots,a_s^*$. The households in M* have the same preference ordering R as that of the households in N. Similarly to the previous section we define the k-th marginal household G*(k) in (M*,N*) by

$$(5.1)\quad G^*(k) = \Big(\sum_{t=1}^{k} \sum_{i\in M_t^*} w_i\Big)^* \text{ and } G^*(k) \lesseqgtr n^* .$$

Let f and f* be the marginal house in (M,N) and in (M*,N*) respectively, i.e., $G(f-1) < n < G(f)$ and $G^*(f^*-1) < n^* < G^*(f^*)$.

Here we should give a brief interpretation to our comparative statics. The two markets (M,N) and (M^*,N^*) are considered to be ones held at different two times. Let (M,N) and (M^*,N^*) be markets held at times t and t^* respectively ($t < t^*$). Every house appearing in (M,N) and (M^*,N^*) has the term of a contract, which may be different. Suppose that a house owned by a house-owner in M is engaged on T year contract with a household at time t. If $t^* < t+T$, then the house does not appear in the market (M^*,N^*) unless the contract has been cancelled at time t^*. Hence if the house appears in (M^*,N^*), then $t^* \geqq t+T$ or the contract is cancelled before time t^* and it has not been engaged between $t+T$ (or time when the contract is cancelled) and time t^*. Of course, houses appearing in (M^*,N^*) may not appear in (M,N), that is, they are newly-built ones or ones the contracts of which have expired at time t^* or have been cancelled. Thus we can interpret the relationship between (M,N) and (M^*,N^*) . Precise parts of the picture of the interpretation would be drawn without difficulty.

Initially we consider an effect of a change of income levels upon the maximal competitive rental values.

<u>Theorem 24</u>. Assume that the marginal houses in (M,N) and in (M^*,N^*) are the same, i.e., $f = f^*$ and that $a_f = a_f^*$. Assume that

$$(5.2) \quad I_{G^*(1)} - I_{G(1)} \geqq \quad \cdots \quad \geqq I_{G^*(f-1)} - I_{G(f-1)} > 0 .$$

Let p and p* be the maximal competitive rent vectors in (M,N) and in (M*,N*) respectively. Then it holds that

$$(5.3) \quad p_1^* - p_1 > \cdots > p_{f-1}^* - p_{f-1} > 0 .$$

Proof. By (A), Lemma 23 and (5.2), we have $(e^{f-1}, I_{G^*(f-1)}-p_{f-1}^*)$ $Q(e^f, I_{G^*(f-1)}-a_f)P(e^f, I_{G(f-1)}-a_f)Q(e^{f-1}, I_{G(f-1)}-p_{f-1})$, which implies $(e^{f-1}, I_{G^*(f-1)}-p_{f-1}^*)P(e^{f-1}, I_{G(f-1)}-p_{f-1})$. Then we have $I_{G^*(f-1)}-p_{f-1}^* > I_{G(f-1)}-p_{f-1}$ by (A), i.e., $I_{G^*(f-1)}-I_{G(f-1)} > p_{f-1}^*-p_{f-1}$. By (5.2) we have $I_{G^*(f-2)}-I_{G(f-2)} > p_{f-1}^*-p_{f-1}$, i.e., $I_{G^*(f-2)}-p_{f-1}^* > I_{G(f-2)}-p_{f-1}$. By this inequality, (A) and Lemma 23 we get $(e^{f-2}, I_{G^*(f-2)}-p_{f-2}^*)Q(e^{f-1}, I_{G^*(f-2)}-p_{f-1}^*)P(e^{f-1}, I_{G(f-2)}-p_{f-1})Q(e^{f-2}, I_{G(f-2)}-p_{f-2})$, which implies $I_{G^*(f-2)}-p_{f-2}^* > I_{G(f-2)}-p_{f-2}$ by (A). Similarly we get $I_{G^*(k)}-p_k^* > I_{G(k)}-p_k$ for $\forall k \leqq f-1$.

Since $(e^{f-1}, I_{G(f-1)}-p_{f-1})Q(e^f, I_{G(f-1)}-a_f)$ and $p_{f-1} > a_f$ by Lemma 23 and Theorem 22.(iv), we get $(e^{f-1}, I_{G^*(f-1)}-p_{f-1})P$ $(e^f, I_{G^*(f-1)}-a_f)$ by (H). By Lemma 23 we have $(e^{f-1}, I_{G^*(f-1)}-$

$p_{f-1})P(e^f, I_{G*(f-1)}-a_f)Q(e^{f-1}, I_{G*(f-1)}-p^*_{f-1})$, which implies $I_{G*(f-1)} -p_{f-1} > I_{G*(f-1)}-p^*_{f-1}$ by (A) , i.e., $p^*_{f-1} - p_{f-1} > 0$.

Let $b_k = (I_{G*(k)}-p^*_k) - (I_{G(k)}-p_k)$ ($k \leqq f-2$). Since $(e^k, I_{G(k)}-p_k)Q(e^{k+1}, I_{G(k)}-p_{k+1})$ and $p_k > p_{k+1}$, we have $(e^k, I_{G(k)} -p_k+b_k)P(e^{k+1}, I_{G(k)}-p_{k+1}+b_k)$ by (H). But since $(e^{k+1}, I_{G*(k)}-p^*_{k+1}) Q(e^k, I_{G(k)}-p_k+b_k)P(e^{k+1}, I_{G(k)}-p_{k+1}+b_k)$, we get $I_{G*(k)}-p^*_{k+1} > I_{G(k)}-p_{k+1}+b_k$, i.e., $p^*_k - p_k > p^*_{k+1} - p_{k+1}$. Q.E.D.

This theorem says as follows: Assume that the marginal houses and their evaluation values a_f and a^*_{f*} of house-owners are the same, and that the income level of the k-th marginal household $G(k)$ ($k < f$) increases more than that of the t-th marginal household $G(t)$ if $t > k$. Then the increment of the maximal competitive rental value of a more preferred house is larger than that of a less preferred house. In the special case where the income levels increase at a proportion (1+a), i.e., $I_{G*(k)} = (1 + a)I_{G(k)}$ for $\forall k \leqq f-1$, the increments of the maximal competitive rental values have the same property. This case is considered in the following example. We can illustrate it. See Figure 3 .

Example 25. Let us consider the housing market (M,N) given in

Examples 18 and 21 . Let $I_{G*(k)} = (1+0.1)I_{G(k)}$ for $\forall k \leqq 5$, which is written as Table 3 . Let the other parameters be fixed. Then the maximal competitive rental values are given as Table 3.

k	1	2	3	4	5	6	7
$I_{G*(k)}$	143000 Yen	132000	121000	110000	99000		
p_k^*	27888 Yen	24471	21167	17982	14924	12000	10000
$p_k^*-p_k$	852 Yen	668	491	320	156	0	0
p_k^*/p_k	1.0315	1.0281	1.0237	1.0181	1.0106	1	1

Table 3.

Though the income levels increase uniformly at 10% , the rates of the increments of the rental values are not equal. The rate of the most preferred house 1 is 3.15 % but that of the 5-th house is 1.06 % . In this example we get the interesting result that the maximal competitive rental value of a more preferred house increases more than that of a less in both senses of the absolute magnitude and the proportion. But the proposition with respect to proportion is not generally true .

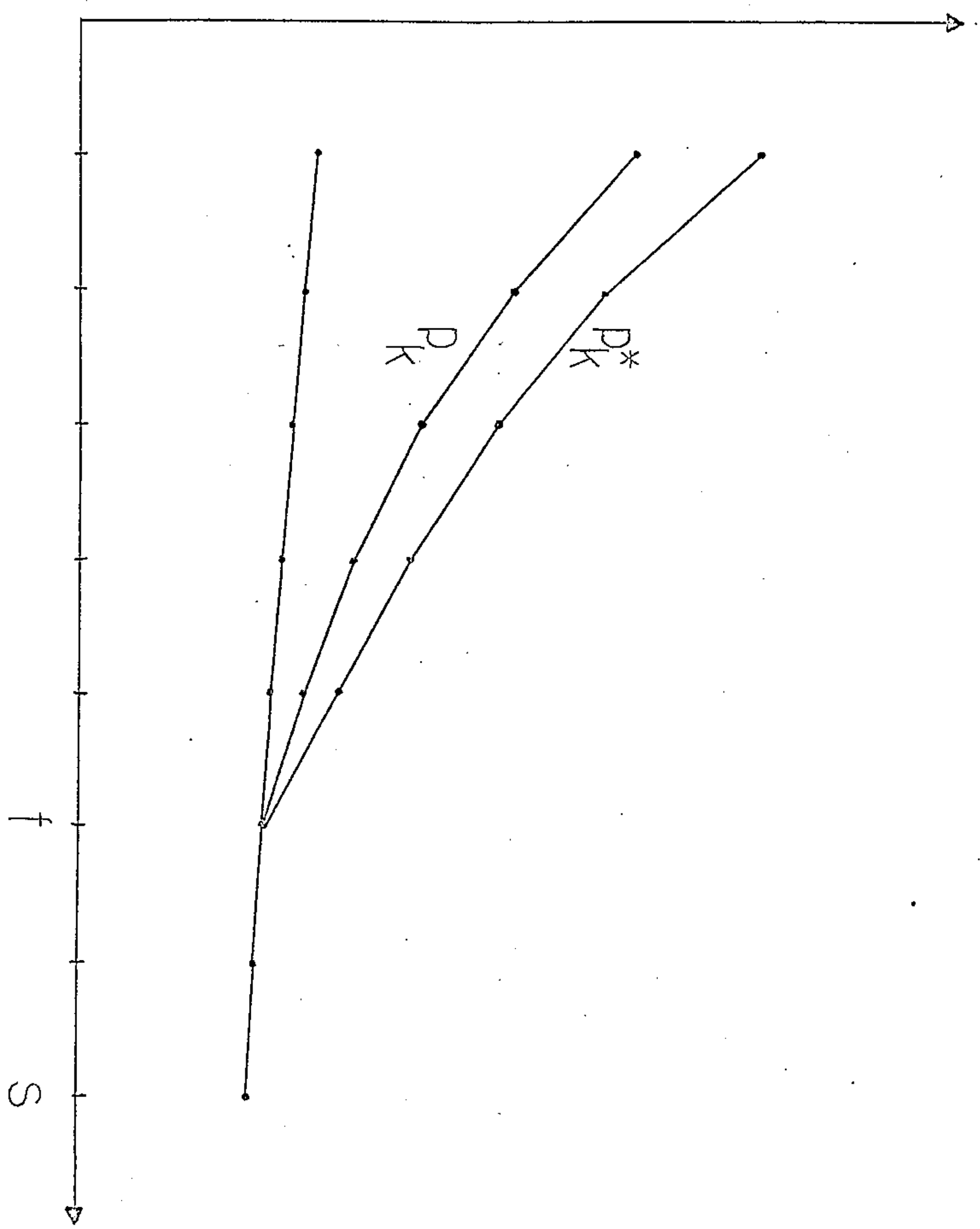

Figure 3.

The phenomenon given in Theorem 24 can be explained intuitively as follows: When households $j \in N$ get larger incomes I'_j than I_j, they must decide how they distribute the increments $I'_j - I_j$ among general consumptions and renting houses. If the rental values are the same, they want to rent more preferred houses than those rented now. These actions of households make the demands of more preferred houses larger than the supplies. Hence the rental values of more preferred houses increase more than those of less.

<u>Theorem 26</u>. Assume that $f = f^*$, and that $I_{G^*(k)} = I_{G(k)}$ for $\forall k = 1,\ldots,f-1$. And assume that $a^*_f > a_f$. Let p and p* be the maximal competitive rent vectors in (M,N) and in (M*,N*) respectively. Then it holds that

$$(5.4) \quad 0 < p^*_1 - p_1 < \cdots < p^*_{f-1} - p_{f-1} < a^*_f - a_f$$

$$(5.5) \quad 1 < p^*_1/p_1 < \cdots < p^*_{f-1}/p_{f-1} < a^*_f/a_f \ .$$

<u>Proof</u>. Since $(e^{f-1}, I_{G(f-1)} - p^*_{f-1}) \Omega (e^f, I_{G(f-1)} - a^*_f)$ by Lemma 23 and $p^*_{f-1} > a^*_f$ by Theorem 22.(iv), we have $(e^{f-1}, I_{G(f-1)} - p^*_{f-1} + (a^*_f - a_f)) P (e^f, I_{G(f-1)} - a^*_f + (a^*_f - a_f)) = (e^f, I_{G(f-1)} - a_f)$ by (H). Since

$(e^{f-1}, I_{G(f-1)}-p_{f-1})Q(e^f, I_{G(f-1)}-a_f)$, we have $I_{G(f-1)}-p^*_{f-1} + (a^*_f - a_f) > I_{G(f-1)}-p_{f-1}$ by (A), i.e., $a^*_f-a_f > p^*_{f-1}-p_{f-1}$. Since $(e^{f-1}, I_{G(f-1)}-p_{f-1})Q(e^f, I_{G(f-1)}-a_f)P(e^f, I_{G(f-1)}-a^*_f)Q(e^{f-1}, I_{G(f-1)}-p^*_{f-1})$ by Lemma 23 and (A), we have $I_{G(f-1)}-p_{f-1} > I_{G(f-1)}-p^*_{f-1}$, i.e., $p_{f-1} < p^*_{f-1}$. By repeating the same argument, we get $p^*_{k+1}-p_{k+1} > p^*_k-p_k > 0$ for $\forall k = 1,\ldots,f-1$.

Since $p_1 > p_2 > \cdots\ p_{f-1} > a_f$ by Theorem 22.(iv), we get (5.5) by (5.4). Q.E.D.

When the marginal house and the income levels of the marginal households do not change, but when the evaluation value a_f of the marginal house increases, the maximal competitive rental values also increase. But in this case the shape of the increments of the rental values has a different tendency from that of Theorem 24 but rather a converse one, that is, the increment of the rental value of a less preferred house is larger that of a more. See Figure 4. As the evaluation value a_f reflects the cost of building one unit of the marginal house f, we can think that the increment of the evaluation value a_f is that of the cost of building the house. Hence the rises in the rental values given in Theorem 26 may be caused by an increment of the cost of building the marginal house.

<u>Example 27</u>. Let us consider the housing market (M,N) in Examples

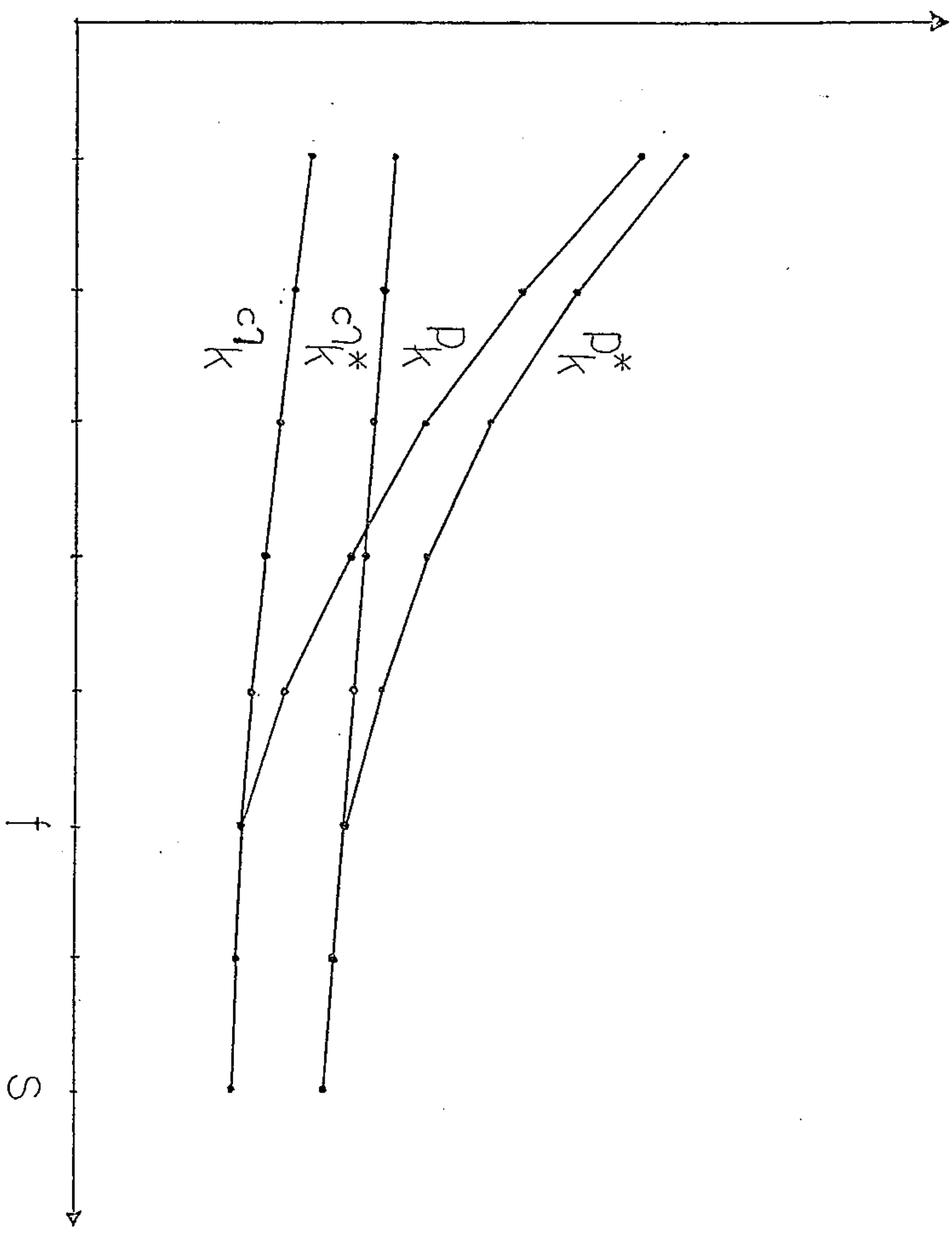

Figure 4.

18 and 21 . Let $a_k^* = (1+0.1)a_k$ for $\forall k = 1,\ldots,7$, i.e., a_k^* are given as Table 4 . Let the other parameters be fixed. Then the maximal competitive rent vector p* is given as Table 4 .

k	1	2	3	4	5	6	7
a_k^*	24200 Yen	22000	19800	17600	15400	13200	11000
p_k^*	28140 Yen	24924	21815	18820	15946	13200	11000
$p_k^*-p_k$	1104 Yen	1121	1139	1158	1178	1200	1000
p_k^*/p_k	1.0409	1.0471	1.0551	1.0656	1.0798	1	1

Table 4 .

We should note that even though a_k^* ($k = 1,\ldots,7$) was increased, assumption (J) is true yet, which is verified by computing $U(e^k, 80000-a_k^*)$ similarly to Example 18 .

<u>Theorem</u> <u>28</u>. Assume that $a^*_{f^*} = a_{f^*}$, and that $I_{G(k)} = I_{G^*(k)}$ for $\forall k \leqq f-1$. Assume that $f < f^*$. Let p and p* be the maximal competitive rent vectors in (M,N) and in (M*,N*) respectively. Then it holds that

$$(5.6) \quad 0 < p_1^*-p_1 < \cdots < p_f^*-p_f$$

$$(5.7) \quad 1 < p_1^*/p_1 < \cdots < p_f^*/p_f \quad .$$

Proof. If $p_f^* \leqq a_f$, then we have $(e^f, I_{G^*(f^*-1)}-p_f^*)R(e^f, I_{G^*(f^*-1)}-a_f)P(e^{f^*}, I_{G^*(f^*-1)}-a_{f^*}^*)Q(e^{f^*-1}, I_{G^*(f^*-1)}-p_{f^*-1}^*)$ by (A), Lemmata 19 and 23, which contradicts that p^* is a competitive rent vector. Hence we have $p_f^* > a_f$. We get $(e^{f-1}, I_{G(f-1)}-p_{f-1})Q(e^f, I_{G(f-1)}-a_f)P(e^f, I_{G(f-1)}-p_f^*)Q(e^{f-1}, I_{G(f-1)}-p_{f-1}^*)$ by Lemma 23 and (A), which implies $I_{G(f-1)}-p_{f-1} > I_{G(f-1)}-p_{f-1}^*$, i.e., $p_{f-1}^* > p_{f-1}$. Similarly we get $p_k^* > p_k$ for $\forall k \leqq f$.

Let $k \leqq f-1$ and $b_k = p_k^*-p_k$. Since $(e^k, I_{G(k)}-p_k^*)Q(e^{k+1}, I_{G(k)}-p_{k+1}^*)$ and $p_k^* > p_{k+1}^*$ by Theorem 22.(iv), we have $(e^k, I_{G(k)}-p_k^*+b_k)P(e^{k+1}, I_{G(k)}-p_{k+1}^*+b_k)$ by (H). But since $(e^k, I_{G(k)}-p^*+b_k) = (e^k, I_{G(k)}-p_k)Q(e^{k+1}, I_{G(k)}-p_{k+1})$ by Lemma 23, we get $I_{G(k)}-p_{k+1} > I_{G(k)}-p_{k+1}^*+b_k$ by (I) and Lemma 17.(ii), i.e., $p_{k+1}^*-p_{k+1} > p_k^*-p_k$.

Since $p_1 > p_2 > \cdots > p_f$, we get (5.7) by (5.6). Q.E.D.

When the income levels of the marginal households and the evaluation values of house-owners do not change, but when the marginal house stirs to a worse house, the shape of the increments of the maximal competitive rental values has the same tendency as that of Theorem 26. That is, the rental value of a less preferred house increases more than that of a more. See Figure 5. This

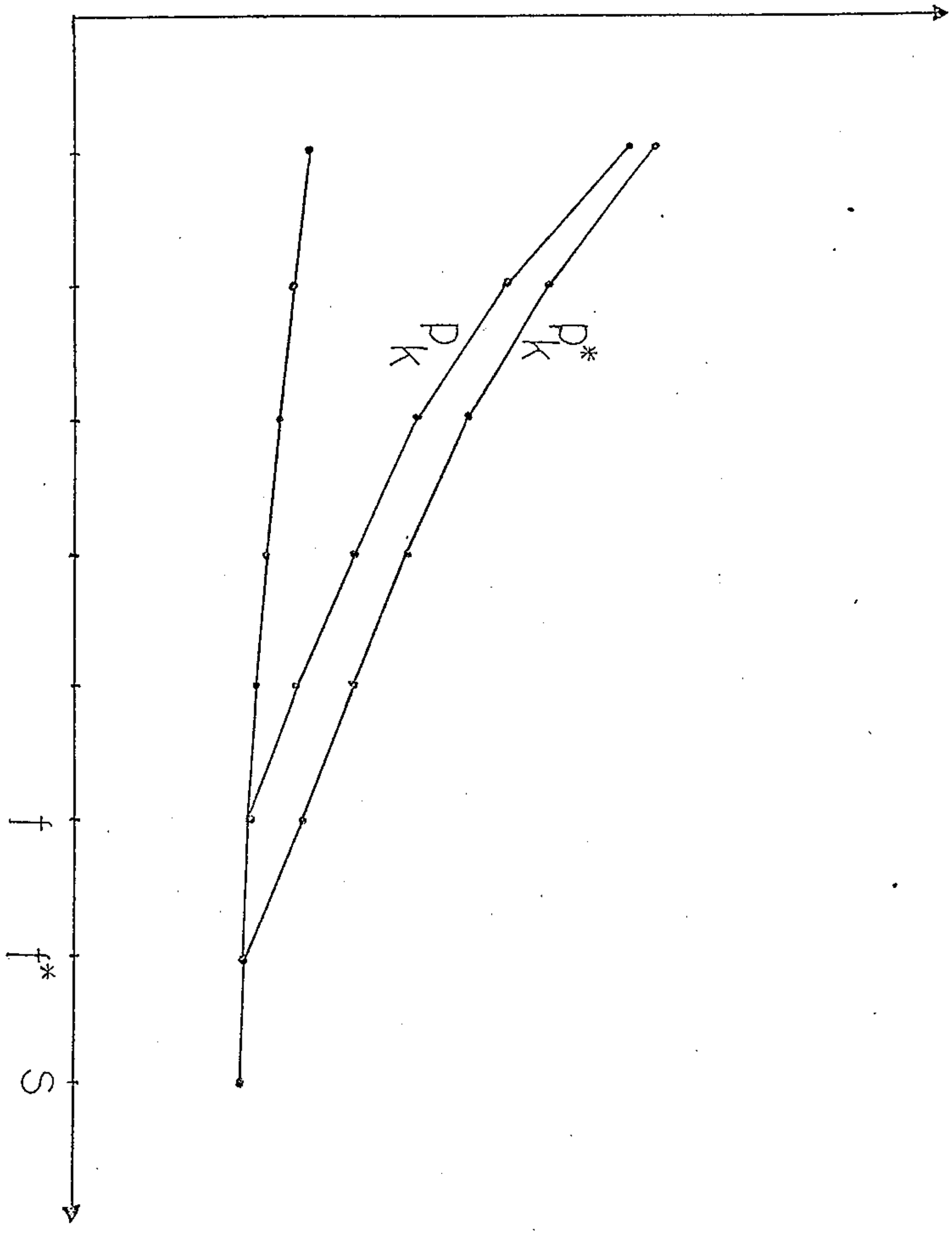

Figure 5.

case may occur when the population of the urban area participating in the housing market increases, i.e., the demands of houses increase more than the supplies. The crucial assumption of this theorem is that $I_{G*(k)} = I_{G(k)}$ for $\forall k < f$, i.e., though the population of the area increases, the increment of population is the group of households having lower income levels. Hence by this theorem we can say nothing in the case where the population of the area increases uniformly on each group of households having a income level. If we want to think of such a case, then we must take the effect of changes of the income levels of the marginal households like Theorem 24 into account. But as they give converse effects, we can not gain any general tendency .

Example 29. Let us consder the market (M,N) given in Examples 17 and 21. Let $f^* = 6$ and $I_{G*(6)} =$ 80000 Yen . Let the other parameters be fixed. Then the maximal competitive rental values p_k^* (k = 1,...,7) are given as Table 5.

k	1	2	3	4	5	6	7
p_k^*	27607 Yen	24382	21265	18261	15377	12621	10000
$p_k^*-p_k$	571 Yen	579	589	599	609	621	0
p_k^*/p_k	1.0211	1.0243	1.0285	1.0339	1.0412	1.0518	1

Table 5.

Finally we consider effects of simultaneous changes of the parameters upon the maximal competitive rental values. The following lemma is necessary to consider them.

<u>Lemma 30</u>. Assume that $f = f^*$ and $a_f = a_f^*$. Assume that

(5.8) $\quad I_{G^*(k)} \geqq I_{G(k)} \quad$ for $\forall k = 1, \ldots, \bar{f}-1$.

Let p and p* be the maximal competitive rent vectors in (M,N) and in (M*,N*) respectively. If there is a $t \leqq \bar{f}-1$ such that $I_{G^*(t)} > I_{G(t)}$ and $I_{G^*(k)} = I_{G(k)}$ for $\forall k > t$, then it holds that

(5.9) $\quad p_k^* > p_k \quad$ for $\forall k \leqq t$ and $p_k^* = p_k$ for $\forall k$ ($t < k \leqq f$).

<u>Proof</u>. Suppose that $I_{G^*(k)} = I_{G(k)}$ for $\forall k$ ($t < k \leqq \bar{f}-1$). Then clearly $p_k^* = p_k$ for $\forall k \geqq t+1$. Since $(e^t, I_{G(t)}-p_t)\Omega(e^{t+1}, I_{G(t)}-p_{t+1})$ by Lemma 23 and $p_t > p_{t+1}$ by Theorem 22.(iv), we have $(e^t, I_{G^*(t)}-p_t)P(e^{t+1}, I_{G^*(t)}-p_{t+1})\Omega(e^t, I_{G^*(t)}-p_t^*)$ by (H) and Lemma 23. We have $p_t < p_t^*$ by (A). Since $(e^{t-1}, I_{G(t-1)}-p_{t-1})\Omega(e^t, I_{G(t)}-p_t)$ by Lemma 23 and $p_{t-1} > p_t$ by Theorem 22.(iv), we have $(e^{t-1}, I_{G^*(t-1)}-p_{t-1})R(e^t, I_{G^*(t-1)}-p_t)P(e^t, I_{G^*(t-1)}-p_t^*)\Omega(e^{t-1}, I_{G^*(t-2)}-p_{t-1}^*)$ by (H),(A) and Lemma 23. This implies $p_{t-1}^* > p_{t-1}$.

Similarly we get $p_k^* > p_k$ for $\forall k \leqq t$. Q.E.D.

Let $\{(M^1,N^1),\ldots,(M^g,N^g)\}$ be a set of housing markets which are generalized assignment markets and which satisfy all the assumptions of Section 4 and (5.10):

$$(5.10) \quad I_G{}^{t+1}{}_{(k)} \geqq I_G{}^{t}{}_{(k)} \quad \text{for } \forall t = 1,\ldots,g-1 \text{ and } k \leqq f^t-2 \text{ .}$$

Here $I_G{}^{t}{}_{(k)}$ and f^t are the income of the k-th marginal household and the marginal house in market (M^t,N^t) respectively. We call $\{(M^1,N^1),\ldots,(M^g,N^g)\}$ a <u>chain</u> between (M^1,N^1) and (M^g,N^g) if (5.10) and the following three conditions hold for each pair (M^t,N^t) and (M^{t+1},N^{t+1}) ($t = 1,\ldots,g-1$):

$$(5.11) \quad f^{t+1} \geqq f^t$$

$$(5.12) \quad I_G{}^{t+1}{}_{(f^t-1)} \geqq I_G{}^{t}{}_{(f^t-1)}$$

$$(5.13) \quad a^{t+1}_{f^{t+1}} \geqq a^{t}_{f^{t+1}} \text{ .}$$

Here a_k^t is the evaluation value of the k-th house in (M^t,N^t) . We call a chain $\{(M^1,N^1),\ldots,(M^g,N^g)\}$ <u>strict</u> if exact one of the inequalities (5.11),(5.12) and (5.13) holds strictly . Let p^t be the maximal competitive rent vector in (M^t,N^t) ($t = 1,\ldots,g$). It follows from Theorems 26, 28 and Lemma 30 that if $\{(M^1,N^1),\ldots,(M^g,N^g)\}$ is a strict chain, then

$$(5.14) \quad p_k^t < p_k^{t+1} < \cdots < p_k^g \qquad \text{for } \forall k \leqq f^t-1 \ .$$

<u>Corollary 31</u>. Assume that $I_{G^*(k)} \geqq I_{G(k)}$ for $\forall k = 1,\ldots,f-2$. Let p and p* be the maximal competitive rent vectors in (M,N) and in (M*,N*) respectively. Assume the following three conditions

$$(5.15) \quad f^* \geqq f$$

$$(5.16) \quad I_{G^*(f-1)} \geqq I_{G(f-1)}$$

$$(5.17) \quad a^*_{f^*} \geqq a_{f^*} \ .$$

If at least one of (5.15),(5.16) and (5.17) holds strictly, then it holds that

$$(5.18) \quad p_k^* > p_k \qquad \text{for } \forall k = 1,\ldots,f-1 \ .$$

Proof. It is easy to construct a strict chain between (M,N) and (M*,N*). Hence we get (5.18) by (5.14). Q.E.D.

Corollary 32. Assume that $I_{G^*(k)} = I_{G(k)}$ for $\forall k \leq \bar{f}-1$. Assume that $f < f^*$ and $a_{f^*} < a^*_{f^*}$. Then it holds that

$$(5.19) \quad 0 < p^*_1 - p_1 < \cdots < p^*_{\bar{f}} - p_f$$

$$(5.20) \quad 0 < p^*_1 / p_1 < \cdots < p^*_f / p_f \quad .$$

Proof. Let (M^1, N^1) be a housing market in which $f^1 = f^*$, $a^1_{f^*} = a_{f^*}$ and $I_{G^1(k)} = I_{G(k)}$ for $\forall k \leq \bar{f}-1$. Then clearly $\{(M,N),(M^1, N^1),(M^*,N^*)\}$ is a strict chain between (M,N) and (M*,N*). Let p^1 be the maximal competitive rent vector in (M^1,N^1). Then by Theorems 26 and 28, we have

$$0 < p^1_1 - p_1 < \cdots < p^1_f - p_f$$

$$0 < p^*_1 - p^1_1 < \cdots < p^*_f - p^1_f \quad .$$

Hence we have (5.19), which implies (5.20). Q.E.D.

6. Remarks

(1). The game V_o was led by the assignment market and was used to prove the nonemptiness of the core of the assignment market. Even if V_o is given as an abstract characteristic function on the collection of the coalitions having not more than three members and if it is defined by (2.10), then the proof of the nonemptiness of the core of V_o is true yet. This game V_o is considered to be a generalization of the assignment game of Shapley and Shubik [1972] to the case without side payments.

(2). The nonemptiness of the core of the assignment market was proved using Scarf's theorem. The theorem gives an algorithm for finding a point of the core. Hence we can use the algorithm to find a point of the core or a competitive equilibrium of the assignment market. The core or the competitive equilibria (of the generalized assignment market) can be obtained by considering the core of the agent-assignment market. Hence the algorithm can be used to find a competitive equilibrium of the generalized assignment market .

(3). We have considered the housing market only in abstract form. It is necessary to specify the utility functions of the households or the evaluation values of the house-owners when we want to apply our theory to urban economics . These studies are not only necessary to develop our theory but also will become main part of it . But in that case, some assumptions of Section 4 (e.g., (F'),(G),(I) or (J)) would not hold . Hence a competitive rent vector or the maximal one would not be given as a solution of a simple system of equations like (4.12) of Theorem 20. (i) . The algorithm mentioned above can be used for this purpose, but I do not think that it is efficient. An important open question is to make an efficient algorithm for finding a competitive rent vector in a general case .

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