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Note on Gale's conjecture in one-sided matching problems *

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Abstract

We consider the problem of stochastically allocating n indivisible objects to n agents when each agent assigns cardinal utility values to the objects. In this context, Zhou (1990) demonstrates Gale's conjecture in a stronger form: No rule is strategy-proof, ex ante efficient, and symmetric. We further strengthen this impossibility theorem by relaxing the requirement of symmetry. Consequently, we indicate that every strategy-proof and ex ante efficient rule satisfies neither symmetry nor the equal division lower bound.

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1 Introduction

We consider the problem of allocating indivisible objects without monetary transfers, known as the house allocation problem (Hylland and Zeckhauser, 1979; Svensson, 1999). This class of problems, where each agent has unit demand, is not only important in the sense of possessing direct applications such as on-campus housing (Abdulkadiroğlu and Sönmez, 1999), but also as a special case or at least a building block of more complex and practically important problems, e.g., school choice

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(Abdulkadiroğlu and Sönmez, 2003). Hylland and Zeckhauser (1979) propose using lotteries to solve the house allocation problem. Under a setup in which each agent is an expected utility maximizer, they show that the pseudo-market rule can achieve *ex ante efficient* and *envy-free* allocation.[1] However, their proposed method may allow agents to be better off by misreporting their preferences. Specifically, it fails to satisfy the representative incentive compatibility concept of *strategy-proofness*. Consequently, an important question remained unanswered: Is there a strategy-proof, ex ante efficient, and fair allocation rule? Gale (1987) offered a concrete formulation of this problem and Zhou (1990) later proved a stronger form of Gale's conjecture. That is, he proved that no rule is strategy-proof, ex ante efficient, and symmetric.[2] We further strengthen this impossibility theorem by replacing symmetry with a weaker condition. Consequently, we show that every strategy-proof and ex ante efficient rule satisfies neither symmetry nor the equal division lower bound. In this context, the equal division lower bound requires that a rule always selects a probabilistic allocation that is at least as good as the perfectly fair lottery (uniform distribution) for all agents. Therefore, one of the implications of our theorem (Corollary 2) is that if a social planner wishes to design a rule better than the perfectly fair lottery, at least one of strategy-proofness or ex ante efficiency must be abandoned.

Since the publication of Bogomolnaia and Moulin's (2001) work, many authors have studied lottery rules based on ordinal preference information. Among them, Martini (2016) and Nesterov (2017) extend the impossibility results presented in Bogomolnaia and Moulin (2001). Ranjbar and Feizi (2023) also study the same topic on a restricted preference domain. As we focus on cardinal rules, we do not consider ordinal rules in this study. Alva and Manjunath (2020) establish a similar impossibility result in the context of both one- and two-sided matching with outside options. They demonstrate that no rule is strategy-proof, ex post efficient, and ex ante individually rational.[3] Alongside Zhou (1990), Anno (2023) also studies the

[1] Generally, a system of resource allocation is called a rule, which is formalized as a function that assigns an allocation for each resource allocation problem.

A probabilistic allocation is ex ante efficient if it is impossible to improve any agent without making another agent worse off in terms of expected utility. A probabilistic allocation is envy-free if no agent prefers the probabilistic assignment of other agents to the agent's own in terms of expected utility.

[2] A rule is symmetric if two agents have an identical utility function, then the rule selects a probabilistic allocation in which these two agents enjoy the same expected utility.

[3] Ex post efficiency, which is weaker than ex ante efficiency, requires that each deterministic allocation in the support of the selected probabilistic allocation has no Pareto-improving deterministic

tension between efficiency, fairness, and incentives for cardinal rules.

The remainder of this paper is organized as follows. Section 2 introduces the model and axioms. Section 3 presents the main results, and Section 4 concludes the paper.

2 Model

2.1 Basics

Let $N := \{1, \ldots, n\}$ be a set of **agents**. We assume that at least three agents exist. Let $\mathcal{O} := \{o_1, \ldots, o_m\}$ be a set of **objects**. Each agent consumes exactly one object. We assume that $|\mathcal{O}| = |N|$, i.e., $m = n$, and that there is no outside option.

An assignment for agent i specifies the object agent i receives. Formally, an m-dimensional vector $P_i = (P_{io})_{o\in\mathcal{O}} \in \{0, 1\}^m$ is called a deterministic assignment *if* $\sum_{o\in\mathcal{O}} P_{io} = 1$. Note that $P_{io} = 1$ indicates that agent i receives object o. We sometimes interpret a deterministic assignment as an $m \times 1$ matrix. A list of deterministic assignments $P = (P_1, \ldots, P_n)$ is called a deterministic allocation, or matching, *if* the sum of each row of P is equal to one. Let $\mathcal{M}$ be the set of deterministic allocations.

The purpose of this study is to investigate the house allocation problem when randomization is introduced. Therefore, we extend the concept of assignment to include randomization. A **probabilistic assignment**, or **assignment**, for agent i is a probability distribution over the objects. That is, we call an m-dimensional vector $P_i = (P_{io})_{o\in\mathcal{O}} \in [0, 1]^m$ a probabilistic assignment *if* $\sum_{o\in\mathcal{O}} P_{io} = 1$, where P_{io} denotes the probability for agent i to receive object o. Let Δ be the set of probabilistic assignments, i.e., $\Delta := \left\{P_i = (P_{io})_{o\in\mathcal{O}} \in [0, 1]^m \middle| \sum_{o\in\mathcal{O}} P_{io} = 1\right\}$. A list of probabilistic assignments $P = (P_1, \ldots, P_n) \in \Delta^N$ is called a **probabilistic allocation**, or **allocation**, *if* the sum of each row of P is equal to one. Let $\mathcal{P}$ be the set of probabilistic allocations. Note that $\mathcal{M} \subseteq \mathcal{P}$.

A probability distribution over $\mathcal{M}$ is called a lottery. Let $\mathcal{L}$ be the set of lotteries.

Remark 1. Note that every lottery $\ell \in \mathcal{L}$ induces a probabilistic allocation $\sum_{M\in\mathcal{M}} \ell_M M$. Conversely, the following theorem, known as Birkhoff-von Neumann theorem, holds:

allocation.

Ex ante individual rationality requires that each agent finds that the selected probabilistic allocation is at least as good as the agent's own initial endowment (outside option). This concept is well-defined only if the model under consideration includes the outside option.

for any $P \in \mathcal{P}$, there exists $\ell \in \mathcal{L}$ such that $P = \sum_{M \in \mathcal{M}} \ell_M M$. Thanks to the Birkhoff-von Neumann theorem, every probabilistic allocation can be implemented by a lottery.

We assume that the preferences of each agent are embodied by a utility function. Let $\mathcal{U} := \{u_i \in \mathbb{R}^{\mathcal{O}} | \max_{o \in \mathcal{O}} u_i(o) = 1 \text{ and } \min_{o \in \mathcal{O}} u_i(o) = 0\}$ be the set of admissible utility functions over objects. Note that each utility function is standardized to attain a utility level of 1 at the maximizers and 0 at the minimizers.[4] The utility function, which is characterized by the following two features, plays an important role in the proof of our main theorem: (i) a unique maximizer exists, and (ii) other objects are indifferent. Formally, a utility function $u_i \in \mathcal{U}$ is **single-minded** if there exists $o \in \mathcal{O}$ such that $u_i(\mathcal{O} \setminus \{o\}) = \{0\}$. For each $o \in \mathcal{O}$, let $\mathcal{U}(o)$ be the set of single-minded utility functions whose utility-maximizer is o.[5]

In this study, we assume that each agent is an expected utility maximizer. For each $u_i \in \mathcal{U}$ and each $P_i \in \Delta$, let $Eu_i(P_i)$ be the expected utility of agent i with the utility function u_i at a probabilistic assignment P_i, i.e., $Eu_i(P_i) := \sum_{o \in \mathcal{O}} P_{io} u_i(o)$.

A **house allocation problem**, or **problem** for short, is a list $(N, \mathcal{O}, u)$, where $u \in \mathcal{U}^N$. As we fix N and $\mathcal{O}$ throughout this study, a problem is represented simply by a utility function profile. Thus, $\mathcal{U}^N$ denotes the set of problems.

A resource allocation system is represented by a **rule** that assigns a probabilistic allocation for each problem. That is, a rule is a function from $\mathcal{U}^N$ to $\mathcal{P}$. Our generic notation for a rule is φ.

2.2 Axioms

First, we introduce several properties of the probabilistic allocation. The efficiency property we employ is an adaptation of the standard Paretian efficiency concept for our setup. An allocation $P \in \mathcal{P}$ is **ex ante efficient (EAE)** at $u \in \mathcal{U}^N$ *if* there exists no allocation $Q \in \mathcal{P}$ such that $Eu_i(Q_i) \geq Eu_i(P_i)$ for all $i \in N$, and $Eu_i(Q_i) > Eu_i(P_i)$ for some $i \in N$. In this note, we consider three types of fairness properties. An allocation $P \in \mathcal{P}$ is **symmetric (S)** at $u \in \mathcal{U}^N$ *if* for all $i, j \in N$ with $u_i = u_j$, $Eu_i(P_i) = Eu_i(P_j)$. An allocation $P \in \mathcal{P}$ satisfies the **equal division**

[4] We employ the preference domain adopted by Zhou (1990). As the main result of this note is an impossibility theorem, it is preserved on any broader domains.

[5] The set $\mathcal{U}(o)$ is a singleton. This notation is useful when we describe an axiom in the next subsection.

lower bound (EDLB) at $u \in \mathcal{U}^N$ *if* for all $i \in N$, $Eu_i(P_i) \geq Eu_i\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. The following weak fairness property says that every member of a group of agents with a common single-minded preference should get at least $\frac{1}{n}$ utility if it is physically possible. Formally, an allocation $P \in \mathcal{P}$ satisfies the **equal division lower bound for single-minded agents (EDLB-SMA)** at $u \in \mathcal{U}^N$ *if* for any $S \in 2^N \backslash \{\emptyset\}$,

$$\left[\exists o \in \mathcal{O} \text{ s.t. } (\forall i \in S, u_i \in \mathcal{U}(o)) \text{ and } 1 - \sum_{i \in N \backslash S} P_{io} \geq \frac{|S|}{n}\right] \Rightarrow \forall i \in S, Eu_i(P_i) \geq \frac{1}{n}.$$

Note that each of S and EDLB implies EDLB-SMA.

Next, we introduce the properties of a rule. A rule φ is **ex ante efficient (EAE)** (resp. **symmetric (S)**, **equal division lower bound (EDLB)**, **equal division lower bound for single-minded agents (EDLB-SMA)**) *if* for each $u \in \mathcal{U}^N$, the selected allocation $\varphi(u)$ is ex ante efficient (resp. symmetric, equal division lower bound, equal division lower bound for single-minded agents) at u. Finally, we introduce an incentive property of a rule. A rule φ is **strategy-proof (SP)** *if* for each $u \in \mathcal{U}^N$, each $i \in N$, and each $u_i' \in \mathcal{U}$, $Eu_i(\varphi_i(u)) \geq Eu_i(\varphi_i(u_i', u_{-i}))$, where (u_i', u_{-i}) denotes the profile obtained from u by replacing u_i with u_i'.

3 Result

The following is the main result of this study, which highlights the tripartite tension between efficiency, fairness, and incentives. A proof is presented after providing the three lemmas.

Theorem 1. *No rule is SP, EAE and EDLB-SMA.*

Lemma 1 states an intuitively obvious fact regarding EAE allocation. Given $u \in \mathcal{U}^N$, let i be an agent such that $u_i(o) > 0$. Suppose that under an EAE allocation $\varphi(u)$, each agent $j \in N \backslash \{i\}$ evaluates object o as the worst, or $\varphi_j(u) = 0$. Then, the assignment $\varphi_i(u)$ does not contain a positive probability share of objects worse than o at u_i. Additionally, if there is no object indifferent to o at u_i, $\varphi_{io}(u) = 1 - \sum_{\substack{o' \in \mathcal{O} \\ u_i(o') > u_i(o)}} \varphi_{io'}(u)$.

Lemma 1. *Suppose that φ is EAE. Let $o \in \mathcal{O}, i \in N$ and $u \in \mathcal{U}^N$ be such that*

(Lem 1-i) $u_i(o) > 0$ *and* $\nexists o' \in \mathcal{O}\backslash\{o\}$ *s.t.* $u_i(o') = u_i(o)$, *and*

(Lem 1-ii) $\forall j \in N\backslash\{i\}, [u_j(o) > 0 \Rightarrow \varphi_{jo}(u) = 0]$.

Then, $\varphi_{io}(u) = 1 - \sum_{\substack{o' \in \mathcal{O} \\ u_i(o') > u_i(o)}} \varphi_{io'}(u)$.

Proof. Suppose to the contrary that $\varphi_{io}(u) < 1 - \sum_{\substack{o' \in \mathcal{O} \\ u_i(o') > u_i(o)}} \varphi_{io'}(u)$. Note that there exists $o' \in \mathcal{O}$ such that $\varphi_{io'}(u) > 0$ and $u_i(o) > u_i(o')$ ($\because$ Lem 1-i). Note also that there exists $j \in N\backslash\{i\}$ such that $\varphi_{jo}(u) > 0$ ($\because m = n$). Letting $\varepsilon := \min\{\varphi_{io'}(u), \varphi_{jo}(u)\}$, consider the following exchange between agents i and j: (i) ε units out of $\varphi_{jo}(u)$ is transferred to agent i, and (ii) ε units out of $\varphi_{io'}(u)$ is transferred to agent j. The resulting allocation Pareto-dominates $\varphi(u)$ at u. However, this contradicts that φ is EAE. □

Before we proceed to Lemma 2, we introduce the following notation. Given $o, o' \in \mathcal{O}$ with $o \neq o'$, let $\mathcal{U}(o, o') := \{u_i \in \mathcal{U} | u_i(o) = 1, u_i(o') \in (0,1)$ and $u_i(\mathcal{O}\backslash\{o, o'\}) = \{0\}\}$.

Lemma 2 illustrates the effect of the deviation from a single-minded preference. Suppose that agent i has $u_i \in \mathcal{U}(o)$. Suppose also that an object $o'(\neq o)$ is a common worst object for all agents. Then, under the SP and EAE rules, when only agent i's valuation of object o' solely increases, the probability share of object o assigned to agent i is invariant.

Lemma 2. *Suppose that* φ *is SP and EAE. Let* $o, o' \in \mathcal{O}$ *with* $o \neq o'$, $i \in N$, $u \in \mathcal{U}^N$ *and* $u'_i \in \mathcal{U}$ *be such that*

(Lem 2-i) $u_i \in \mathcal{U}(o)$ *and* $u'_i \in \mathcal{U}(o, o')$, *and*

(Lem 2-ii) $\forall j \in N\backslash\{i\}, u_j(o') = 0$.

Then, $\varphi_{io}(u'_i, u_{-i}) = \varphi_{io}(u)$.

Proof. Suppose to the contrary that $\varphi_{io}(u'_i, u_{-i}) \neq \varphi_{io}(u)$. If $\varphi_{io}(u'_i, u_{-i}) > \varphi_{io}(u)$, agent i has the incentive to misreport u'_i at u ($\because u_i \in \mathcal{U}(o)$). However, this contradicts that φ is SP. Thus, in the sequel, we assume $\varphi_{io}(u'_i, u_{-i}) < \varphi_{io}(u)$. Note that the following claim holds.

Claim. $\forall u''_i \in \mathcal{U}(o, o'), \varphi_{io}(u''_i, u_{-i}) = \varphi_{io}(u'_i, u_{-i})$. [6]

[6] *Proof of Claim:* Suppose the contrary. Assume, without loss of generality, that $\varphi_{io}(u''_i, u_{-i}) < \varphi_{io}(u'_i, u_{-i})$. By Lemma 1, $\varphi_{io}(u'_i, u_{-i}) + \varphi_{io'}(u'_i, u_{-i}) = 1$ and $\varphi_{io}(u''_i, u_{-i}) + \varphi_{io'}(u''_i, u_{-i}) = 1$. Thus, $Eu''_i(\varphi_i(u'_i, u_{-i})) > Eu''_i(\varphi_i(u''_i, u_{-i}))$, a violation of SP of φ.

Let $a \in (0,1)$ be such that $a < \frac{\varphi_{io}(u)-\varphi_{io}(u'_i,u_{-i})}{1-\varphi_{io}(u'_i,u_{-i})}$.[7] Let $u''_i \in \mathcal{U}(o,o')$ be such that $u''_i(o') = a$. Then,

$$\begin{aligned} Eu''_i\left(\varphi_i(u)\right) &= \varphi_{io}(u) + a\varphi_{io'}(u) \\ &\geq \varphi_{io}(u) \\ &> \varphi_{io}(u'_i,u_{-i}) + a(1-\varphi_{io}(u'_i,u_{-i})) \quad (\because a < \tfrac{\varphi_{io}(u)-\varphi_{io}(u'_i,u_{-i})}{1-\varphi_{io}(u'_i,u_{-i})}) \\ &= \varphi_{io}(u''_i,u_{-i}) + a(1-\varphi_{io}(u''_i,u_{-i})) \quad (\because \text{Claim}) \\ &= \varphi_{io}(u''_i,u_{-i}) + a\varphi_{io'}(u''_i,u_{-i}) \quad (\because \text{Lemma 1}) \\ &= Eu''_i\left(\varphi_i(u''_i,u_{-i})\right), \end{aligned}$$

a violation of SP of φ. □

In Lemma 3, we consider a situation where there is an object o which is the unique maximizer for all agents. Suppose that agents i and j have a common object $o'(\neq o)$ that is their second-most preferred, with i having a higher valuation on o'. Suppose also that object o' is the worst object for the agents other than i and j. In this case, if agent i receives a positive probability share of object o, then (i) the probability share of object o' will be fully assigned to agents i and j, and (ii) agent i's assignment will consist only of the probability share of objects o and o'.

Lemma 3. *Suppose that φ is EAE. Let $o, o' \in \mathcal{O}$ with $o \neq o'$, $u \in \mathcal{U}^N$ and $i, j \in N$ with $i \neq j$ be such that*

(Lem 3-i) $u_i, u_j \in \mathcal{U}(o,o')$ *and* $u_i(o') > u_j(o')$,

(Lem 3-ii) $\varphi_{io}(u) > 0$, *and*

(Lem 3-iii) $\forall k \in N\backslash\{i,j\}, [u_k(o) = 1, u_k(o') = 0$ *and* $\{\forall o'' \in \mathcal{O}\backslash\{o\}, u_k(o'') < 1\}]$.

Then, (i) $\varphi_{io'}(u) + \varphi_{jo'}(u) = 1$, *and (ii)* $\varphi_{io}(u) + \varphi_{io'}(u) = 1$.

Proof. (i) Suppose to the contrary that $\varphi_{io'}(u) + \varphi_{jo'}(u) < 1$. Note that there exists $o'' \in \mathcal{O}\backslash\{o,o'\}$ such that $\varphi_{io''}(u) > 0$ or $\varphi_{jo''}(u) > 0$. Without loss of generality, assume that $\varphi_{io''}(u) > 0$. Note also that there exists $k \in N\backslash\{i,j\}$ such that $\varphi_{ko'}(u) > 0$ ($\because m = n$). Then, an allocation that Pareto-dominates $\varphi(u)$ is constructed through the transfer between agents i and k similar to the one in the proof of Lemma 1, a contradiction.

(ii) For notational simplicity, let $a := u_i(o')$ and $b := u_j(o')$. Note that $\varphi_{io'}(u) < 1$ ($\because$ Lem 3-ii). Thus, thanks to (i), $\varphi_{jo'}(u) > 0$.

[7] The right-hand side of the inequality is positive because $\varphi_{io}(u'_i,u_{-i}) < \varphi_{io}(u) \leq 1$.

To prove (ii), suppose to the contrary that $\varphi_{io}(u)+\varphi_{io'}(u) < 1$. In this case, there exists $o'' \in \mathcal{O}\backslash\{o, o'\}$ such that $\varphi_{io''}(u) > 0$. Choose sufficiently small $\varepsilon > 0$ such that $\varepsilon < \varphi_{io}(u)$, $\frac{\varepsilon}{b} < \varphi_{jo'}(u)$ and $\frac{\varepsilon}{b}(1-b) < \varphi_{io''}(u)$. Define $P \in \Delta^N$ as follows:

$$P_{i\tilde{o}} := \begin{cases} \varphi_{io}(u) - \varepsilon & \text{if } \tilde{o} = o, \\ \varphi_{io'}(u) + \frac{\varepsilon}{b} & \text{if } \tilde{o} = o', \\ \varphi_{io''}(u) - \frac{\varepsilon}{b}(1-b) & \text{if } \tilde{o} = o'', \\ \varphi_{i\tilde{o}}(u) & \text{o.w.} \end{cases} \quad P_{j\tilde{o}} := \begin{cases} \varphi_{jo}(u) + \varepsilon & \text{if } \tilde{o} = o, \\ \varphi_{jo'}(u) - \frac{\varepsilon}{b} & \text{if } \tilde{o} = o', \\ \varphi_{jo''}(u) + \frac{\varepsilon}{b}(1-b) & \text{if } \tilde{o} = o'', \\ \varphi_{j\tilde{o}}(u) & \text{o.w.} \end{cases} \quad \text{and}$$

$$P_k := \varphi_k(u) \text{ for } k \in N\backslash\{i, j\}.$$

Obviously, P is feasible, i.e., $P \in \mathcal{P}$. Moreover,

$$\begin{aligned} Eu_i(P_i) &= (\varphi_{io}(u) - \varepsilon) + a\left(\varphi_{io'}(u) + \frac{\varepsilon}{b}\right) \\ &> \varphi_{io}(u) + a\varphi_{io'}(u) \qquad (\because a > b) \\ &= Eu_i(\varphi_i(u)), \end{aligned}$$

and the assignments at P and $\varphi(u)$ are indifferent for other agents. Thus, P Pareto-dominates $\varphi(u)$ at u, a contradiction. □

A direct consequence of Lemma 3 is that under the assumptions of Lemma 3, the amount of the probability share of object o assigned to agent i is the same as that of o' assigned to agent j. This observation is effectively used in the proof of Theorem 1.

Now, we provide a proof of Theorem 1.

Proof of Theorem 1. Suppose to the contrary that φ is SP, EAE and EDLB-SMA. Let $u^0 \in \mathcal{U}^N$ be such that $u_i^0 \in \mathcal{U}(o_1)$ for all $i \in N$. Let $u \in \mathcal{U}^N$ be such that $u_1, u_2 \in \mathcal{U}(o_1, o_2)$ and $u_i \in \mathcal{U}(o_1, o_i)$ for each $i \in N\backslash\{1, 2\}$. The proof proceeds in three steps.

Step 1. $\forall S \in 2^N, \left[\{1, 2\} \not\subseteq S \Rightarrow \forall i \in N, \varphi_{io_1}(u_S, u_{-S}^0) = \frac{1}{n}\right]$.[8]
Proof of Step 1: We prove this by the induction on $|S|$. For $|S| = 0$, $\varphi_{io_1}(u^0) = \frac{1}{n}$ for each $i \in N$ because φ is EDLB-SMA. Let $S \in 2^N$ be such that $|S| \geq 1$. Suppose

[8]The notation (u_S, u_{-S}^0) denotes the profile in which each agent in S has the preference u_i while each agent in $N\backslash S$ has the preference u_i^0.

that for any $S' \in 2^N$ with $|S'| = |S| - 1$, if $\{1,2\} \not\subseteq S'$, then $\varphi_{io_1}(u_{S'}, u^0_{-S'}) = \frac{1}{n}$ for all $i \in N$.

Let $i \in S$ be arbitrary. For $S' := S \backslash \{i\}$, $\varphi_{io_1}(u_{S'}, u^0_{-S'}) = \frac{1}{n}$ because of the induction hypothesis. Note that by Lemma 2, $\varphi_{io_1}(u_S, u^0_{-S}) = \varphi_{io_1}(u_{S'}, u^0_{-S'})$. Thus, $\varphi_{io_1}(u_S, u^0_{-S}) = \frac{1}{n}$. Since $i \in S$ is arbitrary, $1 - \sum_{i \in S} \varphi_{io_1}(u_S, u^0_{-S}) = \frac{|N \backslash S|}{n}$. Thanks to EDLB-SMA, $\varphi_{jo_1}(u_S, u^0_{-S}) = \frac{1}{n}$ for each $j \in N \backslash S$. This completes the proof of Step 1.

Step 2. Let $\{i,j\} := \{1,2\}$. Suppose that $u_i(o_2) > u_j(o_2)$. Then, (i) $\varphi_{io_1}(u) = \frac{1}{n}$ and $\varphi_{io_2}(u) = \frac{n-1}{n}$, and (ii) $\varphi_{jo_1}(u) = \frac{1}{n}$ and $\varphi_{jo_2}(u) = \frac{1}{n}$.

Proof of Step 2: Applying Step 1 to $S = N \backslash \{i\}$, $\varphi_{ko_1}(u^0_i, u_{-i}) = \frac{1}{n}$ for each $k \in N$. Moreover, by Lemma 1, $\varphi_{jo_2}(u^0_i, u_{-i}) = \frac{n-1}{n}$, and $\varphi_{ko_k}(u^0_i, u_{-i}) = \frac{n-1}{n}$ for each $k \in N \backslash \{1,2\}$. Thus, $\varphi_i(u^0_i, u_{-i}) = \begin{pmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix}$ ($\because m = n$). Summing up, we obtain

$$\varphi(u^0_i, u_{-i}) = \begin{array}{c} \\ o_1 \\ o_2 \\ o_3 \\ \vdots \\ o_n \end{array} \begin{array}{c} \begin{array}{ccccc} i & j & 3 & \cdots & n \end{array} \\ \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{n-1}{n} & 0 & \cdots & 0 \\ \frac{1}{n} & 0 & \frac{n-1}{n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & 0 & 0 & \cdots & \frac{n-1}{n} \end{pmatrix} \end{array}. \tag{1}$$

Similarly, applying Step 1 to $S = N \backslash \{j\}$, we obtain

$$\varphi(u^0_j, u_{-j}) = \begin{array}{c} \\ o_1 \\ o_2 \\ o_3 \\ \vdots \\ o_n \end{array} \begin{array}{c} \begin{array}{ccccc} i & j & 3 & \cdots & n \end{array} \\ \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{n-1}{n} & \frac{1}{n} & 0 & \cdots & 0 \\ 0 & \frac{1}{n} & \frac{n-1}{n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{n} & 0 & \cdots & \frac{n-1}{n} \end{pmatrix} \end{array}. \tag{2}$$

The following claim is an immediate consequence of (1) and (2).

Claim. $\varphi_{io_1}(u) \leq \frac{1}{n}$ and $\varphi_{jo_1}(u) \leq \frac{1}{n}$. [9]

Now, we complete the proof of Step 2. As φ is SP, $Eu_j(\varphi_j(u)) \geq Eu_j(\varphi_j(u_j^0, u_{-j}))$. Thus, by (2), we have

$$\varphi_{jo_1}(u) + u_j(o_2)\varphi_{jo_2}(u) \geq \frac{1}{n} + \frac{1}{n}u_j(o_2). \tag{3}$$

This inequality implies that $\varphi_{jo_2}(u) \geq \frac{1}{n}$ ($\because$ The latter part of Claim). As an implication of Lemma 3 is $\varphi_{jo_2}(u) = \varphi_{io_1}(u)$, $\varphi_{io_1}(u) \geq \frac{1}{n}$. Therefore, combined with the first part of Claim, we obtain $\varphi_{io_1}(u) = \frac{1}{n}$. Thus, by (ii) of Lemma 3, $\varphi_{io_2}(u) = \frac{n-1}{n}$. The proof of (i) is completed.

Finally, we show (ii). As $\varphi_{io_1}(u) = \frac{1}{n}$, by the implication of Lemma 3 again, $\varphi_{jo_2}(u) = \frac{1}{n}$. Substituting this to (3), we obtain $\varphi_{jo_1}(u) \geq \frac{1}{n}$. Thus, combined with the latter part of Claim, we obtain $\varphi_{jo_1}(u) = \frac{1}{n}$. This completes the proof of Step 2.

Step 3. Concluding.

We assume that $u_2(o_2) > u_1(o_2)$. Let $u_1' \in \mathcal{U}(o_1, o_2)$ be such that $u_1'(o_2) > u_2(o_2)$. Then, by Step 2, we have (i) $\varphi_{1o_1}(u_1, u_{-1}) = \frac{1}{n}$ and $\varphi_{1o_2}(u_1, u_{-1}) = \frac{1}{n}$, and (ii) $\varphi_{1o_1}(u_1', u_{-1}) = \frac{1}{n}$ and $\varphi_{1o_2}(u_1', u_{-1}) = \frac{n-1}{n}$. As we assume that $n \geq 3$, $Eu_1(\varphi_1(u_1', u_{-1})) > Eu_1(\varphi_1(u_1, u_{-1}))$, a contradiction. □

Note that the three axioms in Theorem 1 are independent. That is, dropping one of these axioms leads to the existence of a rule. For this, refer to examples of rules provided by Zhou (1990).

The following two impossibility results are immediate corollaries of Theorem 1.

Corollary 1 (Zhou, 1990; Nesterov, 2017)**.** *No rule satisfies SP, EAE and S.*

Corollary 2 (Anno, 2023)**.** *No rule satisfies SP, EAE and EDLB.*

Remark 2. In this study, Theorem 1 is established under the assumption that $|\mathcal{O}| = |N|$. This assumption is utilized three times within the series of proofs (Lemmas 1, 3, and Step 2 of Theorem 1). Among these, this assumption is critical only for the proof of Step 2 of Theorem 1. To extend Theorem 1 to general cases, a proof technique for this part must be developed.

[9] *Proof of Claim:* To prove the first part, suppose to the contrary that $\varphi_{io_1}(u) > \frac{1}{n}$. Note that by (1), $\varphi_{io_1}(u_i^0, u_{-i}) = \frac{1}{n}$. Thus, agent i has the incentive to misreport u_i at (u_i^0, u_{-i}). However, this violates the SP of φ.

Similarly, the latter is proven using (2). This completes the proof of Claim.

4 Conclusion

We investigated the problem of stochastically allocating indivisible objects when each agent is an expected utility maximizer. In this environment, we demonstrated that the combination of strategy-proofness and ex ante efficiency is incompatible with the weak fairness notion called the equal division lower bound for single-minded agents.

In the context of pure exchange economies, resource allocation rules satisfying both strategy-proofness and Pareto efficiency have been shown to violate various fairness notions (Hurwicz, 1972; Zhou, 1991; Serizawa, 2002). An extreme result in this research line is given in Momi (2017). He demonstrated that every strategy-proof and Pareto efficient rule is an alternately dictatorial rule in which each economy has an individual who consumes all resources. In the context of the stochastic allocation of indivisible objects, such a rule can be interpreted as one in which each problem has an agent who receives 1 unit of the probability share of the most preferred object. At this stage, we do not know the exact degree of unfairness provoked by the combination of strategy-proofness and ex ante efficiency. We believe that the analysis in this study is useful in this research line.[10]

References

Abdulkadiroğlu, A., and T. Sönmez (1999): "House Allocation with Existing Tenants," *Journal of Economic Theory*, 88, 233–260.

——— (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Alva, S., and V. Manjunath (2020): "The Impossibility of Strategy-proof, Pareto efficient, and Individually Rational Rules for Fractional Matching," *Games and Economic Behavior*, 119, 15–29.

Anno, H. (2023): "Efficient and strategy-proof cardinal rules on a restricted domain," Discussion paper NO. 1387, University of Tsukuba.

[10]When the preference domain is restricted to the one with ordinally-identical utility functions, a rule satisfying strategy-proofness and ex ante efficiency may not be alternately dictatorial (Anno, 2023).

Bogomolnaia, A., and H. Moulin (2001): "A New Solution to the Random Assignment Problem," *Journal of Economic Theory*, 100, 295–328.

Gale, D. (1987): "College Course Assignments and Optimal Lotteries," University of California at Berkeley.

Hurwicz, L. (1972): "On Informationally Decentralized Systems," in *Decision and Organization*, ed. by G. McGuire, and R. Radner. North Holland, Amsterdam.

Hylland, A., and R. Zeckhauser (1979): "The Efficient Allocation of Individuals to Positions," *Journal of Political Economy*, 87, 293–314.

Martini, G. (2016): "Strategy-proof and Fair Assignment is Wasteful," *Games and Economic Behavior*, 98, 172–179.

Momi, T. (2017): "Efficient and Strategy-proof Allocation Mechanisms in Economies with Many Goods," *Theoretical Economics*, 12, 1267–1306.

Nesterov, A. S. (2017): "Fairness and Efficiency in Strategy-proof Object Allocation Mechanisms," *Journal of Economic Theory*, 170, 145–168.

Ranjbar, H. H., and M. Feizi (2023): "Random Assignments with Uniform Preferences: An Impossibility Result," *Operations Research Letters*, 51(3), 304–307.

Serizawa, S. (2002): "Inefficiency of Strategy-Proof Rules for Pure Exchange Economies," *Journal of Economic Theory*, 106, 219–241.

Svensson, L.-G. (1999): "Strategy-proof Allocation of Indivisible Goods," *Social Choice and Welfare*, 16, 557–567.

Zhou, L. (1990): "On a Conjecture by Gale about One-Sided Matching Problems," *Journal of Economic Theory*, 52, 123–135.

——— (1991): "Inefficiency of Strategy-Proof Allocation Mechanisms in Pure Exchange Economies," *Social Choice and Welfare*, 8, 247–257.