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Efficient and Strategy-proof Cardinal Rules on a Restricted Domain

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Efficient and Strategy-proof Cardinal Rules on a Restricted Domain[*]

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Abstract

We establish several impossibility results on the design of resource allocation rules based on the cardinal information of agents' preferences. We focus on the restricted preference domain comprising "ordinally-identical" linear utility functions. That is, we assume that all agents have identical ordinal preferences for goods, while the valuation may differ from agent to agent. Our findings have implications for real-life applications such as (i) divisible goods allocation problems (e.g., international food aid allocation problems) and (ii) stochastic assignment of indivisible goods (e.g., priority design). In the main theorems, we demonstrate that the combination of efficiency and strategy-proofness is incompatible with any one of the following standard fairness notions: equal division lower bound and envy-freeness.

Journal of Economic Literature Classification Numbers: C78, D47, D71.

Keywords: Cardinal rules; Priority design; Strategy-proofness.

1 Introduction

Efficient and strategy-proof resource allocation rules are widely acknowledged to be extremely unfair in many environments, such as voting (Gibbard, 1973; Satterth-

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waite, 1975) and exchange economy (Hurwicz, 1972). [1] This study aims to establish that when designing a resource allocation rule, a *rule* for short, that is based on cardinal information of agents' preferences, the combination of strategy-proofness and efficiency is incompatible with either *equal division lower bound* or *envy-freeness* — standard fairness notions in various resource allocation problems. Equal division lower bound requires that at the selected allocation, each agent should receive an assignment at least as good as the equal division. On the other hand, envy-freeness requires that at the selected allocation, no agent should prefer any other agent's assignment to their own.

A notable feature of the setup we employ is that the domain of a rule is considerably restricted. In this study, we assume that while all agents have identical (strict) preferences for goods, their valuations may differ from agent to agent. Due to this restriction, our setup captures the following practically important problems.

Problem 1 (International food aid allocation). Consider the problem of rationing food aid by international institutions, such as the World Food Programme (WFP), to nations in need of food support. Suppose that the reserved rice crops to be rationed are of three types: harvested this year, last year, and older. While all nations would unanimously prefer newer rice crops, the evaluation of each type may vary across nations.

Problem 2 (Priority design). Priority design involves determining the order of precedence among agents. In many priority-based matching problems including school seat allocation (Abdulkadiroğlu and Sönmez, 2003) and donated organ allocation (Roth, Sönmez, and Ünver, 2004), such an order (or a family of orders) is given as a primitive data of the market under consideration. We focus on the instant at which the priority order is formed. As long as the matching system for the final allocation respects priority, agents are naturally expected to unanimously prefer a

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[1]Throughout this paper, *efficiency* refers to the Paretian efficiency concept, which requires that no agent's assignment be improved without hurting others. *Strategy-proofness* is one of the most important incentive compatibility conditions, which requires that truth-telling be a weakly dominant strategy of each agent in the preference revelation game induced from the resource allocation rule.

higher priority.[2]

In real-life matching markets, each has a conventional way of determining a priority order.[3] There is no doubt of the practical significance of proposing an improvement to the current method while respecting the background of conventions. However, in this study, we develop another research line of priority design at the abstract level without any particular context. It corresponds to the problems of (i) breaking ties contained in the conventional methods and (ii) determining a priority order for agents without any superficial differences except for the valuation on the priority.

Technically, priority design problems can involve two types of formulations. The first is a deterministic model of priority design, in which the combination of strategy-proofness and non-bossiness results in dictatorship (Svensson, 1999).[4] Thus, we focus on the other formulation of the problem: probabilistic priority design.

As long as agents have ordinally-identical preferences for goods, as in Problem 1 and 2, the clearinghouse must elicit additional information about their circumstances. Note that many promising ordinal rules, including random priority rule and probabilistic serial rule (Bogomolnaia and Moulin, 2001), always assign the equal division for Problem 1 and 2. This observation highlights that the heterogeneity of ordinal

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[2]Position auction (Edelman, Ostrovsky, and Schwarz, 2007; Varian, 2007) is also an assignment problem of indivisible objects to agents with identical preferences. A difference is that unlike priority design, position auctions are accompanied with side payments. Zhou and Serizawa (2018) provide an analysis of Walrasian equilibria in position auctions.

Sönmez (2023) elaborates a market design paradigm called *minimalist market design*. Greenberg, Pathak, and Sönmez (2023) provide a priority design for the branching of the US Army from this perspective.

[3]For example, the priority ranking for school choice in the city of Boston is described in Abdulkadiroğlu and Sönmez (2003). Another example is the priority point system for donated kidneys, which is described in the policy book of the Organ Procurement and Transplantation Network (OPTN). See `https://optn.transplant.hrsa.gov/media/eavh5bf3/optn_policies.pdf` for a detailed description.

[4]Non-bossiness is an auxiliary condition according to which misreporting of an agent should not affect any other agents' assignments whenever the assignment of the deviating agent does not change. A comprehensive explanation of this concept is presented in Thomson (2016). Note that all allocations are (ex-post) efficient under the domain consisting of ordinally-identical preferences and the objects equal to the number of agents. Note also that the additional property of *neutrality* in Svensson's characterization is used to extend the serial dictatorship result to a heterogenous domain. Thus, serial dictatorship rules are the only rules that satisfy strategy-proofness and non-bossiness in deterministic priority design problems.

preferences is the source of the welfare gain for these methods. In this paper, we consider the design problem of cardinal rules in which each agent submits a valuation for each good. That is, we assume that agents submit a linear utility function over consumptions to the clearinghouse. On this preference domain, we establish several impossibility results for cardinal rules for Problem 1 and 2: no efficient and strategy-proof cardinal rule satisfies either equal division lower bound or envy-freeness.

1.1 Related Literature

In the context of pure exchange economy, Hurwicz (1972) first points out that efficient and strategy-proof rules have to be dictatorial in the case of two goods and two agents. This result is generalized to the case with m goods and two agents in Zhou (1991). Finally, Serizawa (2002) demonstrates that any efficient and strategy-proof rule is neither individually rational nor symmetric in the case of m goods and n agents.[5] Individual rationality is a standard fairness notion that requires that at the selected allocation, no agent should be worse off than the initial endowment. Symmetry is a weak fairness notion that requires that any two agents who have an identical utility function receive same utility at the selected allocation.

Hylland and Zeckhauser (1979) proposes the market mechanism for stochastic assignment of indivisible objects. Zhou (1990), the most closely related paper to ours, proves that no efficient and strategy-proof rule is symmetric in Hylland and Zeckhauser's model. We also establish impossibility results in a similar form as mentioned in the previous subsection. Technically, there are three differences between Zhou's and our results. First, Zhou's impossibility result is established on the domain of ordinally-heterogeneous linear utility functions with ties, which is broader than the one considered in this study. Due to this difference, his impossibility result is silent

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[5]Additional results for a two-agent economy can be found in Schummer (1997), Ju (2003), Hashimoto (2008), Momi (2013a), and Cho (2014). Serizawa and Weymark (2003) and Momi (2013b, 2017, 2020) extend the n agent result. Moreover, Cho and Thomson (2023) show that no ordinal rule is strategy-proof, efficient and symmetric on the domain of ordinally-heterogenous linear preferences. Due to the difference in the setups, the main result in Cho and Thomson (2023) and those in this study do not imply each other.

in our environment.[6] Second, one of our results establishes an impossibility with the fairness notion of equal division lower bound, which is logically independent of symmetry. Third, one of the fairness notions we employ is envy-freeness, which is logically stronger than symmetry. Consequently, our result does not directly work as an alternative proof for his result. For these reasons, the main result in Zhou (1990) and those in this study do not imply each other.

Since the publication of Bogomolnaia and Moulin (2001)'s work about the probabilistic assignment problems of indivisible objects, many authors have studied lottery rules based on ordinal information.[7] As we focus on the cardinal rules for Problem 1 and 2, we do not consider ordinal rules in this paper.

The remainder of this paper is organized as follows. Section 2 introduces the model and main axioms. In Section 3, we present the main results. Section 4 concludes the paper. All the proofs are provided in Appendix.

2   Model

Let $N := \{1, \ldots, n\}$ be the set of agents with $3 \leq n < +\infty$. Let $M := \{1, \ldots, m\}$ be the set of goods with $2 \leq m < +\infty$. Let $\Omega \in \mathbb{R}^m_{++}$ denote social endowment. For simplicity, we assume that social endowment is standardized as $\Omega = (1, \ldots, 1)$.

In this paper, we focus on two types of consumption sets. The feasibility condition is controlled by the following consumption capacity: Given $\bar{r}, \underline{r} \in \mathbb{R}_+ \cup \{+\infty\}$ with $\underline{r} \leq \bar{r}$, the consumption set is defined as $X(\underline{r}, \bar{r}) := \{x_i \in [0, 1]^m \mid \underline{r} \leq \sum_{p=1}^m x_{ip} \leq \bar{r}\}$. The set of allocations is denoted by $\mathcal{A}(\underline{r}, \bar{r}) := \{a \in X(\underline{r}, \bar{r})^N \mid \sum_{i \in N} a_i = \Omega\}$. For simplicity, we abuse the notations X and $\mathcal{A}$ to denote $X(\underline{r}, \bar{r})$ and $\mathcal{A}(\underline{r}, \bar{r})$, respectively.

Throughout this paper, we assume that each agent has linear utility for feasible consumptions. A utility function $U_i : \mathbb{R}^m_+ \to \mathbb{R}$ is **linear** *if* there exists $u_i \in \mathbb{R}^m \backslash \{0\}$

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[6]Except when the number of agents is $n = 3$. This topic is revisited later.

[7]Among them, Martini (2016) and Nesterov (2017) extend the impossibility result presented in Bogomolnaia and Moulin (2001). Alva and Manjunath (2020) also establish a similar impossibility result in the context of two-sided matching. Recently, Ranjbar and Feizi (2023) establish an impossibility result for ordinal rules on the ordinally-identical, except for indifference, domain.

such that $U_i(x_i) = \sum_{p \in M} u_{ip} x_{ip}$ for all $x_i \in \mathbb{R}^m_+$. Let $\mathcal{U}$ be a restricted domain comprising linear utility functions ordinally-identical each other. Formally, a linear utility function $U_i : \mathbb{R}^m_+ \to \mathbb{R}$ belongs to $\mathcal{U}$ *if and only if* the characteristic $u_i \in \mathbb{R}^m_+ \backslash \{0\}$ of U_i satisfies $1 = u_{i1} > u_{i2} > \ldots > u_{im} \geq 0$. Note that every admissible utility function excludes ties among goods, i.e., for each $k, \ell \in M$ with $k \neq \ell$, $u_{ik} \neq u_{i\ell}$.

A **problem** is a 5-tuple $(N, M, \Omega, \mathcal{A}, U)$, where $U \in \mathcal{U}^N$. As the first four elements of N, M, Ω and $\mathcal{A}$ are fixed in our analysis, a problem is simply represented by a utility profile $U \in \mathcal{U}^N$.

A system of resource allocation is represented by a rule. A **rule** is a function that assigns an allocation for each problem, that is, a function from $\mathcal{U}^N$ to $\mathcal{A}$. Our generic notation for a rule is φ. To introduce axioms for a rule, we define several efficiency and fairness concepts on the allocation level. The following is a standard Paretian efficiency concept. An allocation $a \in \mathcal{A}$ is **efficient (E)** at $U \in \mathcal{U}^N$ *if* there is no allocation $b \in \mathcal{A}$ such that (i) $U_i(b_i) \geq U_i(a_i)$ for all $i \in N$, and (ii) $U_i(b_i) > U_i(a_i)$ for some $i \in N$. In the main theorems, we focus on the following two fairness notions. The first one requires that no agent should be worse off than the equal division. An allocation $a \in \mathcal{A}$ satisfies **equal division lower bound (EDLB)** at $U \in \mathcal{U}^N$ *if* $U_i(a_i) \geq U_i\left(\frac{\Omega}{n}\right)$ for all $i \in N$. The second one requires that no agent should prefer the any other agent's assignment to their own. An allocation $a \in \mathcal{A}$ is **envy-free (EF)** at $U \in \mathcal{U}^N$ *if* $U_i(a_i) \geq U_i(a_j)$ for all $i, j \in N$.

Based on the normative statements in the previous paragraph, we introduce axioms for a rule. A rule φ is **efficient (E)** (Resp. **equal division lower bound (EDLB)**, **envy-free (EF)**) *if* for each $U \in \mathcal{U}^N$, the selected allocation $\varphi(U) \in \mathcal{A}$ is efficient (Resp. equal division lower bound, envy-free) at U.

To implement a rule in an intended manner, we need to prevent strategic behavior of agents. The following incentive compatibility condition requires that no agent can be better off by misreporting their preferences. Formally, a rule φ is **strategy-proof (SP)** *if* for each $U \in \mathcal{U}^N$, each $i \in N$, and each $U_i' \in \mathcal{U}$, $U_i(\varphi_i(U)) \geq U_i(\varphi_i(U_i'; U_{-i}))$, where $(U_i'; U_{-i})$ denotes the profile obtained from U by replacing U_i with U_i'.

3 Results

We establish several impossibility results under one of the following conditions A and B that constrain the feasibility.

Assumption A. $\underline{r} = 0$ *and* $\bar{r} = +\infty$.

Under Assumption A, a problem represents an allocation problem of m types of perfectly divisible goods to n agents. Since we assume that each agent has a greater value for good $k - 1$ than for good k for $k = 2, \ldots, m$, the problem captures the international food aid allocation problem (Problem 1) mentioned in Section 1.

Assumption B. $m = n$ *and* $\underline{r} = \bar{r} = 1$.

Under Assumption B, a problem represents a stochastic assignment of $m(= n)$ indivisible objects among n agents. Under Assumption B, $\mathcal{A}$ denotes the set of $n \times n$ doubly stochastic matrices whose row and column sums coincide with 1. Each stochastic allocation $a \in \mathcal{A}$ is said to be deterministic if each entry of a is 0 or 1. Note that a deterministic allocation represents a matching between agents and objects. According to the Birkhoff-von Neumann theorem, every stochastic allocation $a \in \mathcal{A}$ can be represented by a convex combination of deterministic allocations. Thus, any stochastic allocation can be implemented by a lottery on matchings. Since we assume that each agent prefers object $k - 1$ to object k for $k = 2, \ldots, n$, the problem captures the priority design problem (Problem 2) mentioned in Section 1.

The main results of this paper are as follows.

Theorem 1. *Under any assumption of A or B, no rule satisfies SP, E and EDLB.*

Theorem 2. *Under any assumption of A or B, no rule is SP, E and EF.*

Next, we show the independence of axioms in Theorem 1 and 2. That is, dropping one of the axioms in Theorem 1 and 2 leads to the existence of a rule. We show it by examples.

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Example 1 (*SP* and *E*, but not *EDLB* and *EF* rule)**.** We define a priority rule φ^P depending on the feasibility constraint under consideration. Under Assumption A, for each $U \in \mathcal{U}$, let $\varphi(U) \in \mathcal{A}$ be such that $\varphi_1^P(U) = \Omega$ and $\varphi_i^P(U) = 0$ for all $i \in N \backslash \{1\}$. Under Assumption B, for each $U \in \mathcal{U}$, let $\varphi^P(U) \in \mathcal{A}$ be such that $\varphi_{ii}^P(U) = 1$ for each $i \in N$. The priority rule φ^P is *SP* and *E*. However, φ^P satisfies neither *EDLB* or *EF*.

Example 2 (*E*, *EDLB* and *EF*, but not *SP* rule)**.** Let φ^W be a selection from the Walrasian equilibrium with slack (Mas-Colell, 1992) under the equal division. Then, φ^W satisfies *E*, *EDLB* and *EF*. However, it is not *SP*.

Example 3 (*SP*, *EDLB* and *EF*, but not *E* rule)**.** Let φ^E be the constant rule that always assigns $\left(\frac{\Omega}{n}, \ldots, \frac{\Omega}{n} \right)$. Then, the equal division rule φ^E satisfies *SP*, *EDLB* and *EF*. However, φ^E is not *E*.

Before closing this section, we refer to an unsolved problem related to the main results. In Theorem 1 and 2, whether *EDLB* and *EF* could be replaced by the following weaker fairness notions of *S* and *MULG*, respectively, is open.[8]

Symmetry (S): $\forall i, j \in N, \forall U \in \mathcal{U}^N, [U_i = U_j \Rightarrow U_i(\varphi_i(U)) = U_j(\varphi_j(U))]$.

Minimum utility level guarantee (MULG): $\exists \epsilon > 0$ s.t. $\forall U \in \mathcal{U}^N, \forall i \in N, U_i(\varphi_i(U)) \geq \epsilon$.

We believe that the following weaker result is helpful in promoting this research direction.

Proposition 1. *Under any assumption of A or B, any SP and E rules violate at least one of S and MULG.*

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[8]Zhou (1990) proves that no rule is *SP*, *E* and *S* under Assumption B when $n = 3$. Some further comments on this topic are mentioned at Proposition 2 in Appendix.

4 Conclusion

In this study, we examined the design of efficient and strategy-proof cardinal rules on a restricted domain consisting of ordinally-identical utility functions. In the divisible goods allocation problems and stochastic assignment of indivisible objects captured by Assumption A and B, it is shown that no efficient and strategy-proof rule satisfies any one of the following standard fairness notions: equal division lower bound and envy-freeness.

Appendix: Proofs

A.1 Sketch of the proof

First, we introduce additional notations and a preliminary fact. Given a utility function in $\mathcal{U}$, the associated characteristic vector is denoted by replacing the capital letter U with the small letter u. For example, the associated characteristic vectors of U_i, U_i' and U_i^δ in $\mathcal{U}$ are denoted by u_i, u_i' and u_i^δ, respectively.

Fact 1. *Let $U \in \mathcal{U}^N$ be such that all agents have an identical utility function. Then,*

> *(i). $\forall a \in \mathcal{A}$, a is efficient at U,*
>
> *(ii). $\forall a, b \in \mathcal{A}, \sum_{i \in N} U_i(a_i) = \sum_{i \in N} U_i(b_i)$,*
>
> *(iii). $\forall a \in \mathcal{A}$, $\left[a \text{ satisfies EDLB at } U \Rightarrow \forall i \in N, U_i(a_i) = U_i\left(\frac{\Omega}{n}\right) \right]$, and*
>
> *(iv). $\forall a \in \mathcal{A}$, $\left[a \text{ satisfies } S \text{ at } U \Rightarrow \forall i \in N, U_i(a_i) = U_i\left(\frac{\Omega}{n}\right) \right]$.*

Proof. Obvious. □

Given a profile $U \in \mathcal{U}^N$ and a coalition $S \subseteq N$, let U_S be the subprofile defined as $U_S := (U_i)_{i \in S}$. Moreover, for any $U, U' \in \mathcal{U}^N$ and $S \subseteq N$, let $(U_S'; U_{-S})$ be the profile obtained from U by replacing the subprofile U_S with U_S'.

$$U^{(0)} = (U_0, U_0, U_0, \ldots, U_0) \qquad U^{(1,\delta)} = (\underline{U_0^\delta}, U_0, U_0, \ldots, U_0)$$

$$U^{(1)} = (\underline{U_0'}, U_0, U_0, \ldots, U_0) \qquad U^{(2,\delta)} = (U_0', \underline{U_0^\delta}, U_0, \ldots, U_0)$$

$$U^{(2)} = (U_0', \underline{U_0'}, U_0, \ldots, U_0) \qquad U^{(3,\delta)} = (U_0', U_0', \underline{U_0^\delta}, U_0, \ldots, U_0)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$U^{(n-1)} = (U_0', \ldots, \underline{U_0'}, U_0) \qquad U^{(n,\delta)} = (U_0', U_0', \ldots, U_0', \underline{U_0^\delta})$$

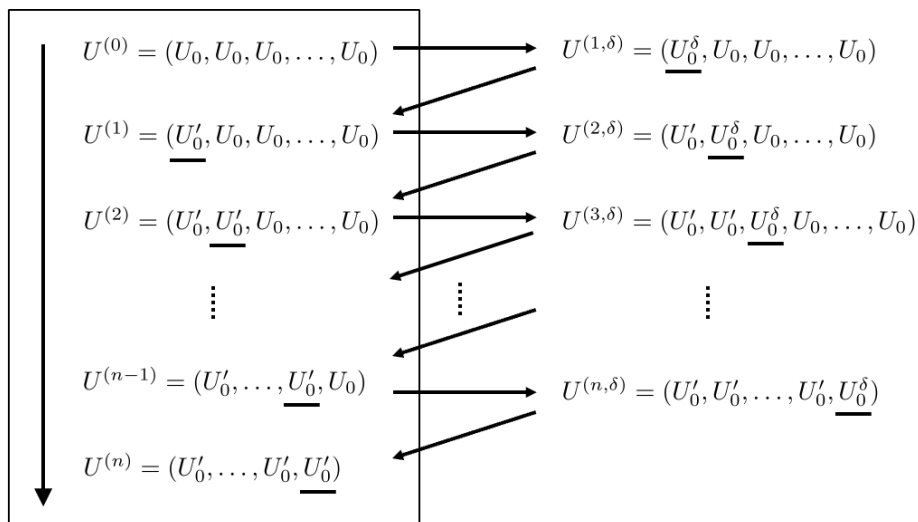$$U^{(n)} = (U_0', \ldots, U_0', \underline{U_0'})$$

Figure 1: Common structure of proofs of Theorems 1 and 2 and Proposition 1.

Note: The difference in a profile relative to the immediate predecessor is highlighted by the underlined part.

In the sequel, we frequently use the following two notations. Given $U_0 \in \mathcal{U}$, let $\mathcal{U}(U_0) := \{U_i \in \mathcal{U} \mid (i) \; \forall p \in M \backslash \{2\}, u_{ip} = u_{0p}, \text{ and } (ii) \; u_{i2} > u_{02}\}$. Given $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$, let $\mathcal{U}(U_0, U_0') := \{U_i^\delta \in \mathcal{U}(U_0) \mid \delta := u_{i2}^\delta \in (u_{02}, u_{02}')\}$.

In the following two subsections, we prove Theorems 1 and 2 and Proposition 1. We establish the impossibility results under Assumption A in Subsection A.2, whereas Subsection A.3 provides proofs for the results under Assumption B. Before commencing the proofs, we begin with their sketch to promote clarity.

All the proofs have a common structure. First, they are proven by contradiction. Therefore, in the first place of the proof, we assume that there exists a rule that satisfies a set of axioms under consideration. Then, a pair of utility functions U_0 and U_0', which are identical, except for the evaluation of good 2 is fixed; that is, $U_0' \in \mathcal{U}(U_0)$. Choosing U_0 and U_0' such that they exhibit very small utility for goods 2 to m is also important.[9] The proof is a replacement process starting from a uniform profile $U^{(0)} := (U_0, \ldots, U_0)$ and moving to another uniform profile $U^{(n)} := (U_0', \ldots, U_0')$. In each step of the process, we replace an agent's preference from U_0 to U_0'. This is the main stream of the proof structure, which is boxed in Figure 1. Note that in

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[9]This condition is denoted as $(*)$ in the proofs.

the first step of the proof, $U^{(0)}$, all agents enjoy the utility level of the equal division with respect to U_0 because they have an identical utility function (Fact 1). However, in the last step, $U^{(n)}$, we show that they do not keep the utility level at the equal division with respect to U'_0, notwithstanding the uniformity of the profile.

To reach the above conclusion, we have to identify the utility levels of the agents in each step of the process $U^{(0)}, U^{(1)}, U^{(2)}, \ldots, U^{(n)}$. The utility levels of all agents at $U^{(1)}$ are completely identified in Lemma 2 for the case under Assumption A and Lemma 4 for the other case under Assumption B.

The arguments on and after $U^{(2)}$ differ depending on the feasibility constraint under consideration. In the easier case under Assumption A, the identification of the utility level at $U^{(2)}, \ldots, U^{(n)}$ is carried out by using an induction argument (Step 2 and 3 in the proof of Proposition 1 under Assumption A). Note that in these profiles, except $U^{(n)}$, there are only two types of agents: U'_0-type and U_0-type. Thus, as long as S or a stronger requirement EF is assumed, the variety of the utility levels in the economy is at most two. Thus, if the utility level of one of U'_0- and U_0-type agents in addition to total utility of the economy is known, the utility level of the other type can be identified.[10][11] To identify the utility level of U'_0-type agents at $U^{(k)}$, we first consider the utility level of agent k at an intermediate profile $U^{(k,\delta)}$ obtained by replacing $U_k^{(k-1)} = U_0$ with $U_0^\delta \in \mathcal{U}(U_0, U'_0)$. The utility level at $U^{(k)}$ is obtained by considering the limit $\delta \to u'_{02}$.

On the other hand, in the harder case under Assumption B, we let agent 3's utility level at $U^{(2)}$ be at an unknown number x.[12] This operation is necessary because we cannot identify the exact utility level at $U^{(2)}$ in this case. However, as partial information, we demonstrate that x is not equal to the level of equal division with respect to U'_0 (Step 2 in the proof of Theorem 2 under Assumption B). Following

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[10]Among our results, S is not assumed only in Theorem 1. As a matter of fact, the proof of Theorem 1 is completed before the induction argument. Refer to the final paragraph in this subsection.

[11]Lemma 1 and Lemma 3 play an essential role in identifying the total utility level at each step of the process. Roughly speaking, they state that at an efficient allocation, the agents with the greatest value for good 2 exhaust good 2 (unless the feasibility constraint binds) whenever they receive a positive amount of good 1.

[12]Here, agent 3 is a representative of U_0-type agents at $U^{(2)}$.

the induction argument in the previous paragraph, we identify the utility levels at $U^{(3)}, \ldots, U^{(n)}$ as functions with a single variable x (Step 3 and 4 in the proof of Theorem 2 under Assumption B). Then, at the last step $U^{(n)}$, we find that x must be equal to the utility level of the equal division with respect to U'_0. Thus, we obtain a contradiction.

Finally, we would like to remark that the full range of the above process is required only for the following three cases: Proposition 1 under Assumption A, Theorem 2 under Assumption B and Proposition 1 under Assumption B. For other cases, the process hits a contradiction at $U^{(2)}$ at the latest.

A.2 Proofs of Theorems 1 and 2 and Proposition 1 under Assumption A

In this subsection, proofs of Theorems 1 and 2 and Proposition 1 under Assumption A are provided. We begin with two lemmas. Lemma 1 shows that in an efficient allocation, agents with the greatest evaluation of good 2 exhaust good 2 whenever at least one of them receives a positive amount of good 1.

Lemma 1. *Suppose that $\underline{r} = 0$ and $\overline{r} = +\infty$. Letting $U_0 \in \mathcal{U}$, suppose that $U \in \mathcal{U}^N$ satisfies that for each $i \in N$ and each $p \in M \backslash \{2\}, u_{ip} = u_{0p}$. Let N_1 be the set of agents who have the greatest evaluation on the good 2, i.e., $N_1 := \{i \in N | u_{i2} = \max\{u_{12}, \ldots, u_{n2}\}\}$. Let $a \in \mathcal{A}$ be such that $\sum\limits_{i \in N_1} a_{i1} > 0$. If a is efficient at U, then $\sum_{i \in N_1} a_{i2} = 1$.*

Proof. Suppose to the contrary that $\sum_{i \in N_1} a_{i2} < 1$. That is, $a_{i2} > 0$ for some $i \in N \backslash N_1$. Since $\sum_{j \in N_1} a_{j1} > 0$, there is $i' \in N_1$ such that $a_{i'1} > 0$. Let $\epsilon > 0$ be sufficiently small so that $a_{i'1} > \epsilon u_{i'2}$ and $a_{i2} > \epsilon$. Note that $\epsilon u_{i'2}$ represents the amount of good 1 "indifferent to ϵ units of good 2" for agent i'. Define $b \in \mathcal{A}$ as

follows:

$$b_{i'p} := \begin{cases} a_{i'p} - \epsilon u_{i'2} & \text{if } p = 1 \\ a_{i'p} + \epsilon & \text{if } p = 2 \\ a_{i'p} & \text{o.w.} \end{cases}, \ b_{ip} := \begin{cases} a_{ip} + \epsilon u_{i'2} & \text{if } p = 1 \\ a_{ip} - \epsilon & \text{if } p = 2 \\ a_{ip} & \text{o.w.} \end{cases}, \text{ and } b_j := a_j \text{ for } j \in N \backslash \{i, i'\}.$$

It is obvious that $U_{i'}(b_{i'}) = U_{i'}(a_{i'})$, $U_i(b_i) > U_i(a_i)$ and $U_j(b_j) = U_j(a_j)$ for all $j \in N \backslash \{i, i'\}$. However, a is *efficient* at U, a contradiction. $\qquad\square$

Lemma 2 demonstrates the effect of single-agent deviation from the uniform utility profile specified in the statement. Notice that it completely characterizes the utility levels of all agents at the resulting profile.

Lemma 2. *Suppose that $\underline{r} = 0$ and $\overline{r} = +\infty$. Suppose also that a SP and E rule φ satisfies S or EDLB. Let $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$ be such that*

$$(*) \quad \frac{1}{2mn} > u_{02}'. \text{[13]}$$

Let $U \in \mathcal{U}^N$ be $U := (U_0, \dots, U_0)$. For each $i \in N$, let $U_i' := U_0'$. Then, $\varphi(U_i'; U_{-i}) \in \mathcal{A}$ is welfare-equivalent with $a^{(i)} \in \mathcal{A}$ defined as follows:

$$a_{ip}^{(i)} := \begin{cases} \frac{1}{n} - \frac{n-1}{n} u_{02} & \text{if } p = 1 \\ 1 & \text{if } p = 2 \\ \frac{1}{n} & \text{o.w.} \end{cases} \text{ and } a_{jp}^{(i)} := \begin{cases} \frac{1}{n} + \frac{1}{n} u_{02} & \text{if } p = 1 \\ 0 & \text{if } p = 2 \\ \frac{1}{n} & \text{o.w.} \end{cases} \text{ for } j \in N \backslash \{i\},$$

That is, (i) $U_i'(\varphi_i(U_i'; U_{-i})) = U_i'(a_i^{(i)})$, and (ii) $U_j(\varphi_j(U_i'; U_{-i})) = U_j(a_j^{(i)}) \left(= U_j\left(\frac{\Omega}{n}\right)\right)$ for all $j \in N \backslash \{i\}$.

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[13] Implication of $(*)$: When agent i with U_0' (or U_0, $U_0^\delta \in \mathcal{U}(U_0, U_0')$) consumes all resources, except for good 1, the agent gets utility $u_{02}' + \sum_{p=3}^m u_{0p} < m u_{02}' < m \frac{1}{2mn} = \frac{1}{2n}$. This implies that if agent i is assigned 0 unit of good 1, the highest level of utility that can be achieved is lower than $\frac{1}{2n}$, which is less than half the utility level under the equal division $(\frac{\Omega}{n}, \dots, \frac{\Omega}{n})$. Thus, under assumption $(*)$, to achieve the utility level $\frac{1}{n}$, the agent needs to consume good 1 greater than $\frac{1}{2n}$ units.

Proof. First, note that the following claims 1 and 2, stating about the effect of deviation of agent i from U, hold. Claim 1 states that agent i receives 1 unit of good 2 at $(U_i^\delta; U_{-i})$ and $(U_i'; U_{-i})$. Claim 2 states that agent i receives the same level of utility from goods in $M \setminus \{2\}$ at $(U_i^\delta; U_{-i})$ and $(U_i'; U_{-i})$.

Claim 1. $\forall U_i^\delta \in \mathcal{U}(U_0, U_0'), \varphi_{i2}(U_i^\delta; U_{-i}) = \varphi_{i2}(U_i'; U_{-i}) = 1.$[14]

Claim 2. $\forall U_i^\delta \in \mathcal{U}(U_0, U_0'), \varphi_{i1}(U_i'; U_{-i}) + \sum_{p=3}^{m} u_{ip}\varphi_{ip}(U_i'; U_{-i}) = \varphi_{i1}(U_i^\delta; U_{-i}) + \sum_{p=3}^{m} u_{ip}\varphi_{ip}(U_i^\delta; U_{-i}).$[15]

To prove (i), note that for each $U_i^\delta \in \mathcal{U}(U_0, U_0')$,

$$U_i^\delta(\varphi_i(U_i^\delta; U_{-i})) \geq U_i^\delta(\varphi_i(U)) \qquad (\because \varphi \text{ is } SP)$$

$$\Leftrightarrow \varphi_{i1}(U_i^\delta; U_{-i}) + \delta + \sum_{p=3}^{m} u_{ip}\varphi_{ip}(U_i^\delta; U_{-i}) \geq U_i^\delta(\varphi_i(U)) \qquad (\because \text{Claim 1})$$

$$\Leftrightarrow \varphi_{i1}(U_i'; U_{-i}) + \delta + \sum_{p=3}^{m} u_{ip}\varphi_{ip}(U_i'; U_{-i}) \geq U_i^\delta(\varphi_i(U)). \qquad (\because \text{Claim 2})$$

In the last inequality, letting $\delta \to u_{i2}$, we obtain $U_i(\varphi_i(U_i'; U_{-i})) \geq U_i(\varphi_i(U))$. As the converse of this inequality is obvious ($\because \varphi$ is SP), $U_i(\varphi_i(U_i'; U_{-i})) = U_i(\varphi_i(U))$. Adding $(u_{i2}' - u_{i2})$ to the left- and right-hand side of this equality, respectively, we obtain

$$U_i(\varphi_i(U_i'; U_{-i})) + (u_{i2}' - u_{i2}) = U_i'(\varphi_i(U_i'; U_{-i})) \ (\because \text{Claim 1})$$

and

$$U_i(\varphi_i(U)) + (u_{i2}' - u_{i2}) = \frac{1}{n} + \frac{1}{n}u_{i2} + \frac{1}{n}\sum_{p=3}^{m} u_{ip} + (u_{i2}' - u_{i2}) = U_i'\left(a_i^{(i)}\right).$$

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[14] *Proof of Claim 1:* First, note that by Fact 1 (iii) and (iv), $U_i(\varphi_i(U)) = \frac{1}{n}\sum_{p=1}^{m} u_{ip}$. By the implication of $(*)$ pointed out in footnote 13, $\varphi_{i1}(U) > \frac{1}{2n}$. Thus, $U_i^\delta(\varphi_i(U)) > \frac{1}{2n}$. Given this observation, we show the first part of Claim 1. Suppose to the contrary that $\varphi_{i2}(U_i^\delta; U_{-i}) < 1$. Then, by Lemma 1, $\varphi_{i1}(U_i^\delta; U_{-i}) = 0$. Thus, by $(*)$, $U_i^\delta\left(\varphi_i(U_i^\delta; U_{-i})\right) < \frac{1}{2n}$. This violates SP of φ.

Similarly, $\varphi_{i2}(U_i'; U_{-i}) = 1$ is proved.

[15] *Proof of Claim 2:* Note that U_i' and U_i^δ are identical except for the evaluation on good 2. Note also that by Claim 1, $\varphi_{i2}(U_i'; U_{-i}) = \varphi_{i2}(U_i^\delta; U_{-i})$. Thus, if Claim 2 is not true, agent i has an incentive to deviate from one of $(U_i'; U_{-i})$ and $(U_i^\delta; U_{-i})$ to the other, a violation of SP of φ.

This completes the proof of (i).

Next, we prove (ii). The total utility level achieved at $(U_i'; U_{-i})$ is $U_i'(\varphi_i(U_i'; U_{-i})) + \sum_{j \in N \setminus \{i\}} U_j(\varphi_j(U_i'; U_{-i})) = 1 + u_{02}' + \sum_{p=3}^{m} u_{0p}$ because $\varphi_{i2}(U_i'; U_{-i}) = 1$ by Claim 1. Thus, by (i), the total utility level, excluding agent i's, is $\sum_{j \in N \setminus \{i\}} U_j(\varphi_j(U_i'; U_{-i})) = \left(1 + u_{02}' + \sum_{p=3}^{m} u_{0p}\right) - \left\{\left(\frac{1}{n} - \frac{n-1}{n} u_{i2}\right) + u_{i2}' + \frac{1}{n} \sum_{p=3}^{m} u_{ip}\right\} = (n-1)U_0\left(\frac{\Omega}{n}\right)$. Since φ satisfies S or $EDLB$, for each $j \in N \setminus \{i\}$, $U_j(\varphi_j(U_i'; U_{-i})) = U_0\left(\frac{\Omega}{n}\right) = U_j\left(\frac{\Omega}{n}\right)$. $\quad\square$

Proof of Theorem 1 under Assumption A. Suppose to the contrary that φ satisfies SP, E and $EDLB$. Let $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$ be such that

$$(*) \quad \frac{1}{2mn} > u_{02}'.$$

Let $U \in \mathcal{U}^N$ be $U := (U_0, \ldots, U_0)$. Let $U_1' := U_0'$. Let $U_2^\delta \in \mathcal{U}(U_0, U_0')$. Let $a^{(2)} \in \mathcal{A}$ be the allocation defined in the statement of Lemma 2. First, note that the following claim, which asserts that the agent 2's assignment of good 2 remains 0 when she changes her reporting to U_2^δ at $(U_1', U_2; U_{-\{1,2\}})$, is true.

Claim. $\varphi_{22}(U_1', U_2^\delta; U_{-\{1,2\}}) = 0.$ [16]

By Claim above and Claim 2 in Lemma 2, $\varphi_{22}(U_1', U_2; U_{-\{1,2\}}) = 0 = \varphi_{22}(U_1', U_2^\delta; U_{-\{1,2\}})$. Because U_2 and U_2^δ are identical except for the evaluation on good 2,

$$
\begin{aligned}
U_2^\delta\left(\varphi_2(U_1', U_2^\delta; U_{-\{1,2\}})\right) &= U_2\left(\varphi_2(U_1', U_2; U_{-\{1,2\}})\right) && (\because \varphi \text{ is } SP.) \\
&= U_2\left(\frac{\Omega}{n}\right) && (\because \text{Lemma } 2(ii)) \\
&< U_2^\delta\left(\frac{\Omega}{n}\right). && (\because u_{02} < \delta)
\end{aligned}
$$

However, this violates the assumption that φ satisfies $EDLB$. $\quad\square$

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[16] *Proof of Claim:* Suppose to the contrary that $\varphi_{22}(U_1', U_2^\delta; U_{-\{1,2\}}) > 0$. By Lemma 1, $\varphi_{11}(U_1', U_2^\delta; U_{-\{1,2\}}) = 0$. Thus, by (*), $\frac{1}{2n} > U_1'\left(\varphi_1(U_1', U_2^\delta; U_{-\{1,2\}})\right)$. On the other hand, note that by Lemma 2 (ii), $U_1(\varphi_1(U_1, U_2^\delta; U_{-\{1,2\}})) = U_1\left(\frac{\Omega}{n}\right) > \frac{1}{2n}$. By Claim 1 in the proof of Lemma 2, $\varphi_{12}(U_1, U_2^\delta; U_{-\{1,2\}}) = 0$. Thus, since U_1 and U_1' are identical except for the evaluation on good 2, $U_1'(\varphi_1(U_1, U_2^\delta; U_{-\{1,2\}})) = U_1(\varphi_1(U_1, U_2^\delta; U_{-\{1,2\}}))$. Thus, $U_1'(\varphi_1(U_1, U_2^\delta; U_{-\{1,2\}})) > \frac{1}{2n}$. Thus, $U_1'(\varphi_1(U_1, U_2^\delta; U_{-\{1,2\}})) > U_1'\left(\varphi_1(U_1', U_2^\delta; U_{-\{1,2\}})\right)$, a violation of SP of φ.

Proof of Theorem 2 under Assumption A. Suppose to the contrary that φ satisfies *SP, E* and *EF*. Let $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$ be such that

$$(*) \quad \frac{1}{2mn} > u_{02}'.$$

Let $U \in \mathcal{U}^N$ be $U := (U_0, \dots, U_0)$. Let $U_1' := U_0', U_2' := U_0'$ and $U_2^\delta \in \mathcal{U}(U_0, U_0')$. First, note that the following three claims, which refer to the effect of agent 2's deviation from $(U_1', U_2; U_{-\{1,2\}})$, hold. Claim 1 states that agent 2's assignment of good 2 remains 0 when the deviation is U_2^δ. Claim 2 states that the utility level of agent 2 remains the same when the deviation is U_2^δ. Finally, Claim 3 states that it also remains the same when the deviation is U_2'.

Claim 1. $\varphi_{22}(U_1', U_2^\delta; U_{-\{1,2\}}) = 0$.[17]

Claim 2. $U_2^\delta \left(\varphi_2(U_1', U_2^\delta; U_{-\{1,2\}}) \right) = U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. [18]

Claim 3. $U_2' \left(\varphi_2(U_1', U_2'; U_{-\{1,2\}}) \right) = U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right) \left(= U_2 \left(\frac{\Omega}{n} \right) \right)$. [19]

Under the allocation $\varphi(U_1', U_2'; U_{-\{1,2\}})$, only agent 1 and 2 consume good 2, i.e.,

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[17] *Proof of Claim 1:* Same as the proof of Claim in the proof of Theorem 1 under Assumption A.

[18] *Proof of Claim 2:* Since φ is *SP*, $U_2 \left(\varphi_2(U_1', U_2^\delta; U_{-\{1,2\}}) \right) \leq U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$ and $U_2^\delta \left(\varphi_2(U_1', U_2^\delta; U_{-\{1,2\}}) \right) \geq U_2^\delta \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. Note that by Claim 1 and Claim 1 of Lemma 2, $\varphi_{22}(U_1', U_2^\delta; U_{-\{1,2\}}) = 0 = \varphi_{22}(U_1', U_2; U_{-\{1,2\}})$. Thus, U_2 and U_2^δ are interchangeable in the two inequalities above because they are identical except for the evaluation on good 2

[19] *Proof of Claim 3:* First, by *SP* of φ, $U_2' \left(\varphi_2(U_1', U_2'; U_{-\{1,2\}}) \right) \geq U_2' \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. Note that U_2 and U_2' are identical except for the evaluation on good 2. Thus, since $\varphi_{22}(U_1', U_2; U_{-\{1,2\}}) = 0$ ($\because$ Claim 1 of Lemma 2), $U_2' \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right) = U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. Thus, we obtain $U_2' \left(\varphi_2(U_1', U_2'; U_{-\{1,2\}}) \right) \geq U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$.

Conversely, by *SP* of φ, $U_2^\delta \left(\varphi_2(U_1', U_2^\delta; U_{-\{1,2\}}) \right) \geq U_2^\delta \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. By Claim 2, $U_2^\delta \left(\varphi_2(U_1', U_2^\delta; U_{-\{1,2\}}) \right) = U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. Thus, we obtain $U_2^\delta \left(\varphi_2(U_1', U_2'; U_{-\{1,2\}}) \right) \leq U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. Letting $\delta \to u_{22}'$, we obtain $U_2' \left(\varphi_2(U_1', U_2'; U_{-\{1,2\}}) \right) \leq U_2 \left(\varphi_2(U_1', U_2; U_{-\{1,2\}}) \right)$. Together with the inequality in the previous paragraph, the desired equation is obtained.

$$\varphi_{12}(U_1', U_2'; U_{-\{1,2\}}) + \varphi_{22}(U_1', U_2'; U_{-\{1,2\}}) = 1.^{20}$$

$$\sum_{i=3}^{n} U_i\Big(\varphi_i(U_1', U_2'; U_{-\{1,2\}})\Big) = \Big(1 + u_{02}' + \sum_{p=3}^{m} u_{0p}\Big) - \Big(\frac{2}{n} + \frac{2}{n}u_{02} + \frac{2}{n}\sum_{p=3}^{m} u_{0p}\Big)$$

$$= \frac{n-2}{n}\Big(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\Big) + (u_{02}' - u_{02}).$$

Since φ is S, for each $h \in N\setminus\{1,2\}, U_h\left(\varphi_h(U_1', U_2'; U_{-\{1,2\}})\right) = \left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}\right) + \frac{1}{n-2}(u_{02}' - u_{02})$. Note that $\varphi_h(U_1', U_2'; U_{-\{1,2\}})$ does not contain good 2, and $U_2' = U_0'$ and $U_h = U_0$ are identical except for the evaluation on good 2. Thus,

$$U_2'\left(\varphi_h(U_1', U_2'; U_{-\{1,2\}})\right) = \left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}\right) + \frac{1}{n-2}(u_{02}' - u_{02})$$

$$> U_0\left(\frac{\Omega}{n}\right) \qquad\qquad (\because\ u_{02}' > u_{02})$$

$$= U_2'\left(\varphi_2(U_1', U_2'; U_{-\{1,2\}})\right) \qquad\qquad (\because\ \text{Claim 3})$$

However, this violates the assumption that φ satisfies EF. $\qquad\square$

Proof of Proposition 1 under Assumption A. Suppose to the contrary that φ satisfies SP, E, S and $MULG$. Let $\epsilon > 0$ be a positive number associated with $MULG$ of φ. Let $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$ be such that

$$(*) \qquad \frac{1}{2m}\min\left\{\frac{1}{n}, \epsilon\right\} > u_{02}'.$$

Let $U, U' \in \mathcal{U}^N$ be $U := (U_0, \ldots, U_0)$ and $U' := (U_0', \ldots, U_0')$. Letting $U^{(0)} := U$, let $U^{(k)} := \left(U_k'; U_{-k}^{(k-1)}\right)$ for each $k \in \{1, \ldots, n\}$. Moreover, for each $k \in \{1, \ldots, n\}$ and each $U_k^\delta \in \mathcal{U}(U_0, U_0')$, let $U^{(k,\delta)} := \left(U_k^\delta; U_{-k}^{(k-1)}\right)$. The proof proceeds in four steps.

Step 1. (i) $\forall k \in \{2, \ldots, n\}, \forall U_k^\delta \in \mathcal{U}(U_0, U_0'), \sum_{i=1}^{k-1} \varphi_{i2}\left(U^{(k,\delta)}\right) = 1$, and (ii) $\forall k \in \{2, \ldots, n\}, \sum_{i=1}^{k} \varphi_{i2}\left(U^{(k)}\right) = 1$.

Proof of Step 1: We only show (i) because (ii) is proved in the same manner. Suppose

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[20]This is proved in the same manner as Claim in proof of Theorem 1 under Assumption A (See Footnote 16).

not. Then, by Lemma 1, $\varphi_{i1}(U^{(k,\delta)}) = 0$ for all $i \in \{1, \ldots, k-1\}$. Then, by $(*)$, these agents $i \in \{1, \ldots, k-1\}$ cannot get the utility level greater than or equal to ϵ, a contradiction. This completes the proof of Step 1.

Step 2. $\forall k \in \{1, \ldots, n-1\}, U'_{k+1}\left(\varphi_{k+1}(U^{(k+1)})\right) = U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right)$.

Proof of Step 2: Let $k \in \{1, \ldots, n-1\}$ be arbitrary. By Step 1, $\sum_{i=1}^{k} \varphi_{i2}(U^{(k+1,\delta)}) = 1$ and $\sum_{i=1}^{k} \varphi_{i2}(U^{(k)}) = 1$. Thus, agent $k+1$ receives 0 unit of good 2 at both allocations, i.e., $\varphi_{k+1,2}(U^{(k+1,\delta)}) = 0 = \varphi_{k+1,2}(U^{(k)})$.[21]

$$U^{\delta}_{k+1}\left(\varphi_{k+1}(U^{(k+1,\delta)})\right) = U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right). \tag{A}$$

Again, by SP, $U^{\delta}_{k+1}\left(\varphi_{k+1}(U^{(k+1,\delta)})\right) \geq U^{\delta}_{k+1}\left(\varphi_{k+1}(U^{(k+1)})\right) = \varphi_{k+1,1}(U^{(k+1)}) + \delta\varphi_{k+1,2}(U^{(k+1)}) + \sum_{p=3}^{m} u_{0p}\varphi_{k+1,p}(U^{(k+1)})$. Thus, combining this inequality with (A),

$$U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right) \geq \varphi_{k+1,1}(U^{(k+1)}) + \delta\varphi_{k+1,2}(U^{(k+1)}) + \sum_{p=3}^{m} u_{0p}\varphi_{k+1,p}(U^{(k+1)}).$$

Letting $\delta \to u'_{02}$, we obtain $U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right) \geq U'_{k+1}\left(\varphi_{k+1}(U^{(k+1)})\right)$.

Conversely, by SP of φ, $U'_{k+1}\left(\varphi_{k+1}(U^{(k+1)})\right) \geq U'_{k+1}\left(\varphi_{k+1}(U^{(k)})\right)$. Note that $\varphi_{k+1,2}(U^{(k)}) = 0$. Thus, $U'_{k+1}\left(\varphi_{k+1}(U^{(k)})\right) = U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right)$ because U'_{k+1} and U_{k+1} are identical except for the evaluation on good 2. Thus, $U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right) \leq U'_{k+1}\left(\varphi_{k+1}(U^{(k+1)})\right)$. This completes the proof of Step 2.

Step 3. For each $k \in \{2, \ldots, n-1\}$,

$$U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right) = \left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}\right) + \left[\frac{1}{n-k} + \sum_{\ell=1}^{k-2}(-1)^{\ell}\frac{\prod_{h=1}^{\ell}\{(k+1)-h\}}{\prod_{h=1}^{\ell+1}\{n-(k+1)+h\}}\right](u'_{02} - u_{02}),$$

where the second term of the coefficient of $u'_{02} - u_{02}$ is well-defined only if $k \geq 3$. We regard it as 0 when $k = 2$.

Proof of Step 3: By induction. First, suppose that $k = 2$. By Step 2, $U'_2\left(\varphi_2(U^{(2)})\right) =$

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[21]To avoid confusion, we add a comma between $k+1$ and 2 in the subscript of φ.

$U_2\left(\varphi_2(U^{(1)})\right) = \frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}$ ($\because$ Lemma 2 (ii)). Thus, since φ is S,
$U_1'\left(\varphi_1(U^{(2)})\right) + U_2'\left(\varphi_2(U^{(2)})\right) = 2\left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}\right)$. Note that the total utility level at $\varphi(U^{(2)})$ is $\sum_{i=1}^{n} U_i^{(2)}\left(\varphi_i(U^{(2)})\right) = 1 + u_{02}' + \sum_{p=3}^{m} u_{0p}$ because $\varphi_{12}(U^{(2)}) + \varphi_{22}(U^{(2)}) = 1$ ($\because$ Step 1). Thus,

$$\sum_{i=3}^{n} U_i^{(2)}\left(\varphi_i(U^{(2)})\right) = \left(1 + u_{02}' + \sum_{p=3}^{m} u_{0p}\right) - 2\left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}\right)$$

$$= \frac{n-2}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + (u_{02}' - u_{02}).$$

Since φ is S,

$$U_3^{(2)}\left(\varphi_3(U^{(2)})\right) = \frac{1}{n-2}\left\{\frac{n-2}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + (u_{02}' - u_{02})\right\}$$

$$= \frac{1}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + \frac{1}{n-2}(u_{02}' - u_{02}) \qquad (B)$$

This is the desired equality for the case $k = 2$.

Now, suppose that $k \geq 3$. The induction hypothesis is given as

$$U_k\left(\varphi_k(U^{(k-1)})\right) = \left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}\right) + \left\{\frac{1}{n-(k-1)} + \sum_{\ell=1}^{k-3}(-1)^\ell \frac{\prod_{h=1}^{\ell}(k-h)}{\prod_{h=1}^{\ell+1}(n-k+h)}\right\}(u_{02}' - u_{02}).$$

By Step 2, $U_k'\left(\varphi_k(U^{(k)})\right) = U_k\left(\varphi_k(U^{(k-1)})\right)$. Thus, since φ is S, $\sum_{i=1}^{k} U_i'\left(\varphi_i(U^{(k)})\right) = k\left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}\right) + k\left\{\frac{1}{n-(k-1)} + \sum_{\ell=1}^{k-3}(-1)^\ell \frac{\prod_{h=1}^{\ell}(k-h)}{\prod_{h=1}^{\ell+1}(n-k+h)}\right\}(u_{02}' - u_{02})$. Note that the total utility level at $\varphi(U^{(k)})$ is $\sum_{i=1}^{n} U_i^{(k)}\left(\varphi_i(U^{(k)})\right) = 1 + u_{02}' + \sum_{p=3}^{m} u_{0p}$

because $\sum_{i=1}^{k} \varphi_{i2}(U^{(k)}) = 1$ ($\because$ Step 1). Thus,

$$\sum_{i=k+1}^{n} U_i^{(k)}\left(\varphi_i(U^{(k)})\right)$$

$$= \left(1 + u'_{02} + \sum_{p=3}^{m} u_{0p}\right) - k\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) - k\left\{\frac{1}{n-(k-1)} + \sum_{\ell=1}^{k-3}(-1)^\ell \frac{\prod_{h=1}^{\ell}(k-h)}{\prod_{h=1}^{\ell+1}(n-k+h)}\right\}(u'_{02} - u_{02})$$

$$= \frac{n-k}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + \left\{1 - \frac{k}{n-(k-1)} - k\sum_{\ell=1}^{k-3}(-1)^\ell \frac{\prod_{h=1}^{\ell}(k-h)}{\prod_{h=1}^{\ell+1}(n-k+h)}\right\}(u'_{02} - u_{02})$$

Since φ is S,

$$U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right)$$

$$= \frac{1}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + \left[\frac{1}{n-k} - \frac{k}{(n-k)\{n-(k-1)\}} - \frac{k}{n-k}\sum_{\ell=1}^{k-3}(-1)^\ell \frac{\prod_{h=1}^{\ell}(k-h)}{\prod_{h=1}^{\ell+1}(n-k+h)}\right](u'_{02} - u_{02})$$

$$= \frac{1}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + \left[\frac{1}{n-k} - \frac{k}{(n-k)\{n-(k-1)\}} + \sum_{\ell=2}^{k-2}(-1)^\ell \frac{\prod_{h=1}^{\ell}(k+1-h)}{\prod_{h=1}^{\ell+1}\{n-(k+1)+h\}}\right](u'_{02} - u_{02})$$

$$= \frac{1}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + \left[\frac{1}{n-k} + \sum_{\ell=1}^{k-2}(-1)^\ell \frac{\prod_{h=1}^{\ell}(k+1-h)}{\prod_{h=1}^{\ell+1}\{n-(k+1)+h\}}\right](u'_{02} - u_{02}).$$

This completes the proof of Step 3.

Step 4. $U'_n(\varphi_n(U^{(n)})) \neq \frac{1}{n} + \frac{1}{n}u'_{02} + \frac{1}{n}\sum_{p=3}^{m} u_{0p}$

Proof of Step 4: For the case with $n = 3$, by Step 2 and (B), $U'_3(\varphi_3(U^{(3)})) = U_3(\varphi_3(U^{(2)})) = \frac{1}{3} + \frac{1}{3}u'_{02} + \frac{1}{3}\sum_{p=3}^{m} u_{0p} + (u'_{02} - u_{02}) > \frac{1}{3} + \frac{1}{3}u'_{02} + \frac{1}{3}\sum_{p=3}^{m} u_{0p}$. In the

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subsequent part, suppose that $n \geq 4$. By Step 2 and 3, we have

$$U_n'(\varphi_n(U^{(n)})) = U_n(\varphi_n(U^{(n-1)}))$$

$$= \frac{1}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + \left[\frac{1}{n-(n-1)} + \sum_{\ell=1}^{n-1-2}(-1)^\ell \frac{\prod_{h=1}^{\ell}(n-1+1-h)}{\ell+1 \prod_{h=1}^{\ell}\{n-(n-1+1)+h\}}\right](u_{02}' - u_{02})$$

$$= \frac{1}{n}\left(1 + u_{02} + \sum_{p=3}^{m} u_{0p}\right) + \left\{1 + \sum_{\ell=1}^{n-3}(-1)^\ell \frac{(n-1!)}{(\ell+1)!(n-\ell-1)!}\right\}(u_{02}' - u_{02}).$$

To complete the proof of Step 4, we show

$$1 + \sum_{\ell=1}^{n-3}(-1)^\ell \frac{(n-1!)}{(\ell+1)!(n-\ell-1)!} \neq \frac{1}{n} \tag{C}$$

in the following two cases separately.

Case 1. n is odd.

Since $n-3$ is even, note that $\sum_{\ell=1}^{n-3}(-1)^\ell \frac{(n-1)!}{(\ell+1)!(n-\ell-1)!}$ contains even number of terms. Observe the following two facts about this summation.

- The k-th term, i.e., $\ell = k$, is $(-1)^k \frac{(n-1)!}{(k+1)!(n-k-1)!}$.

- The k-th term from the last, i.e., $\ell = (n-2)-k$, is $(-1)^{n-2-k} \frac{(n-1)!}{(n-2-k+1)!(n-n+2+k-1)!} = (-1)^{n-2-k} \frac{(n-1)!}{(n-k-1)!(k+1)!}$.

Note that k is odd if and only if $n-2-k$ is even. Thus, $\sum_{\ell=1}^{n-3}(-1)^\ell \frac{(n-1)!}{(\ell+1)!(n-\ell-1)!} = 0$. This completes the proof of (C) when n is odd.

Case 2. n is even.

We prove, by an induction argument, that the left-hand side of (C) is negative for $n = 4, 6, 8, \ldots$. For the case with $n = 4$, $1 + (-1)^1 \frac{(4-1)!}{(1+1)!(4-1-1)!} = -\frac{1}{2}$. Now, letting n be an even number greater than 4, suppose that $1 +$

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$\sum_{\ell=1}^{(n-2)-3} (-1)^\ell \frac{\{(n-2)-1\}!}{(\ell+1)!\{(n-2)-\ell-1)\}!} < 0$. Then,

$$1 + \sum_{\ell=1}^{n-3} (-1)^\ell \frac{(n-1)(n-2)\dots(n-\ell)}{(\ell+1)!}$$

$$= (n-1)(n-2)\left\{1 + \sum_{\ell=1}^{n-5} (-1)^\ell \frac{(n-3)(n-4)\dots(n-\ell)}{(\ell+1)!}\right\} - \{(n-1)(n-2)-1\} + \left\{\frac{(n-1)(n-2)}{3\cdot 2} - \frac{n-1}{2}\right\}.$$

In the right-hand side of the above equation, the first term is negative due to the induction hypothesis. In addition, the second and third term is also negative because $-(n-1)(n-2) + 1 + \frac{(n-1)(n-2)}{3\cdot 2} - \frac{n-1}{2} = -\frac{5}{6}(n-1)(n-2) + \frac{3-n}{2} < 0$. This completes the proof of (C) when n is even.

Since we obtain (C), the proof of Step 4 is completed.

Now, we complete the proof of Proposition 1 under Assumption A. Step 4 violates Fact 1 (iv), a contradiction. $\square$

A.3 Proofs of Theorems 1 and 2 and Proposition 1 under Assumption B

This subsection provides proofs of Theorems 1 and 2 and Proposition 1 under Assumption B. We begin with proofs of the three lemmas. Lemma 3 is similar to, but explicitly different from, Lemma 1 in the previous subsection. A critical difference emerges from the upper bound of the feasibility constraint $\bar{r} = 1$. That is, even if an agent who has the greatest evaluation of good 2 receives a positive amount of good 1, the agent cannot exhaust good 2 at the efficient allocation under consideration. Although the agent does not exhaust good 2, the agent's assignment is filled up with goods 1 and 2 .

Lemma 3. *Suppose that $m = n$ and $\underline{r} = \bar{r} = 1$. Letting $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$, let $U \in \mathcal{U}^N$ and $U' \in \mathcal{U}^N$ be such that $U := (U_0, \dots, U_0)$ and $U' := (U_0', \dots, U_0')$.*

(i). Let $i \in N$. Suppose that $a \in \mathcal{A}$ is E at $(U_i'; U_{-i})$. Suppose also that $a_{i1} > 0$. Then, $a_{i1} + a_{i2} = 1$.

(ii). Let $i, j \in N$ be $i \neq j$. Let $U_i^\delta \in \mathcal{U}(U_0, U_0')$. Suppose that $a \in \mathcal{A}$ is E at $(U_j', U_i^\delta, U_{-\{i,j\}})$. Suppose also that $a_{j1} > 0$ and $a_{i1} > 0$. Then, *(ii-1)* $a_{j1} + a_{j2} = 1$, and *(ii-2)* $a_{j2} + a_{i2} = 1$.

(iii). Let $S \subseteq N$ be $|S| \geq 2$. Suppose that $a \in \mathcal{A}$ is E at (U_S', U_{-S}). Suppose also that $a_{i1} > 0$ for all $i \in S$. Then, $\sum_{i \in S} a_{i2} = 1$.

(iv). Letting $S \subseteq N$ be $|S| \geq 2$, let $i \in N \backslash S$. Let $U_i^\delta \in \mathcal{U}(U_0, U_0')$. Suppose that $a \in \mathcal{A}$ is E at $(U_S', U_i^\delta; U_{-S \cup \{i\}})$. Suppose also that $a_{j1} > 0$ for all $j \in S$. Then, $\sum_{j \in S} a_{j2} = 1$.

Proof. (i) Suppose to the contrary that $a_{i1} + a_{i2} < 1$. By assumption and the hypothesis, we have

- $\exists p \in M \backslash \{1, 2\}$ s.t. $a_{ip} > 0$, and

- $\exists j \in N \backslash \{i\}$ s.t. $a_{j2} > 0$.

Let $\alpha := \min\{a_{i1}, a_{ip}, a_{j2}\}(> 0)$ and $\lambda := \frac{u_{i2}' - u_{ip}}{1 - u_{ip}}$. Note that $\lambda \in (0, 1)$. Define $b \in \mathcal{A}$ as follows:

$$
b_{iq} := \begin{cases} a_{i1} - \lambda\alpha. & \text{if } q = 1 \\ a_{i2} + \alpha & \text{if } q = 2 \\ a_{ip} - (1 - \lambda)\alpha & \text{if } q = p \\ a_{iq} & \text{o.w.} \end{cases}, \quad b_{jq} := \begin{cases} a_{j1} + \lambda\alpha. & \text{if } q = 1 \\ a_{j2} - \alpha & \text{if } q = 2 \\ a_{jp} + (1 - \lambda)\alpha & \text{if } q = p \\ a_{jq} & \text{o.w.} \end{cases}
$$

and $b_{-\{i,j\}} := a_{-\{i,j\}}$. We show that b Pareto-dominates a at $(U_i'; U_{-i})$. First, $U_i'(b_i) = (a_{i1} - \lambda\alpha) + u_{i2}'(a_{i2} + \alpha) + u_{ip}\{a_{ip} - (1 - \lambda)\alpha\} + \sum_{q \in M \backslash \{1, 2, p\}} u_{iq} a_{iq} = U_i'(a_i) - \alpha\{\lambda -$

$u'_{i2} + u_{ip}(1 - \lambda)\} = U'_i(a_i).$[22] Second,

$$U_j(b_j) = (a_{j1} + \lambda\alpha) + u_{j2}(a_{j2} - \alpha) + u_{jp}\{a_{jp} + (1 - \lambda)\alpha\} + \sum_{q \in M\setminus\{1,2,p\}} u_{jq}a_{jq}$$

$$= U_j(a_j) + \alpha\{\lambda - u_{j2} + u_{jp}(1 - \lambda)\}$$
$$> U_j(a_j) + \alpha\{\lambda - u'_{i2} + u_{ip}(1 - \lambda)\} \quad (\because u'_{i2} > u_{i2} = u_{j2})$$
$$= U_j(a_j).$$

Finally, since b and a are identical for agents in $N\setminus\{i, j\}$, b Pareto-dominates a at $(U'_i; U_{-i})$. However, this violates the assumption that a is E at $(U'_i; U_{-i})$.

(ii) First, we show (ii-1). Suppose to the contrary that $a_{j1} + a_{j2} < 1$. By the hypothesis and assumption, we have

- $\exists p \in M\setminus\{1, 2\}$ s.t. $a_{jp} > 0$, and

- $\exists j' \in N\setminus\{j\}$ s.t. $a_{j'2} > 0$.

Thus, following the same argument as the proof of (i), we obtain the conclusion.

Next, we show (ii-2). Suppose to the contrary that $a_{j2} + a_{i2} < 1$. By assumption and the hypothesis, we have

- $\exists p \in M\setminus\{1, 2\}$ s.t. $a_{ip} > 0$,[23] and

- $\exists j' \in N\setminus\{i, j\}$ s.t. $a_{j'2} > 0$.

Thus, following the same argument as the proof of (i), we obtain the conclusion.

(iii) Suppose to the contrary that $\sum_{i \in S} a_{i2} < 1$. Note that $a_{i1} + a_{i2} < 1$ for some $i \in S$.[24] Thus, we have

- $\exists p \in M\setminus\{1, 2\}$ s.t. $a_{ip} > 0$, and

---

[22] By definition of λ, $\lambda - u'_{i2} + u_{ip}(1 - \lambda) = 0$.

[23] *Proof:* Suppose to the contrary that $a_{i1} + a_{i2} = 1$. By an assumption of (ii-2), $a_{j1} + a_{j2} = 1$. As $\Omega_1 = \Omega_2 = 1$, $a_{j2} + a_{i2} = 1$, a contradiction.

[24] Suppose to the contrary that $a_{i1} + a_{i2} = 1$ for all $i \in S$. Since the total amount of good 1 and 2 is 2, $|S| = 2$. Consequently, $\sum_{i \in S} a_{i2} = 1$, a violation of the contradiction hypothesis of the proof of (iii).

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- $\exists j \in N \backslash S$ s.t. $a_{j2} > 0$.

Thus, following the same argument as the proof of (i), we obtain the conclusion.

(iv) Following the same argument as the proof of (iii), we obtain the conclusion. $\square$

Lemma 4 describes the effect of a single-agent deviation from a uniform profile. Item (i) specifies the assignment of the deviating agent. Items (ii) and (iii) show that the utility levels of before and after the deviation is invariant in terms of the original utility functions. Moreover, it shows that the level is identical to that of the equal division.

Lemma 4. *Suppose that $m = n$ and $\underline{r} = \bar{r} = 1$. Let φ be a SP and E rule satisfying S or EDLB. Let $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$ be such that*

$$(*) \quad \frac{1}{2n^2} > u_{02}'.$$

Let $i \in N$ be arbitrary. Let $U \in \mathcal{U}^N$ be such that $U := (U_0, \ldots, U_0)$. Let $U_i' := U_0'$ and $U_i^\delta \in \mathcal{U}(U_0, U_0')$. Then,

- *(i). $\varphi_{i1}(U_i'; U_{-i}) = \frac{1}{n} - \frac{1}{n} \sum_{p \in M \backslash \{1,2\}} \frac{u_{02} - u_{0p}}{1 - u_{02}}$, $\varphi_{i2}(U_i'; U_{-i}) = \frac{1}{n} + \frac{1}{n} \sum_{p \in M \backslash \{1,2\}} \frac{u_{02} - u_{0p}}{1 - u_{02}}$, and $\varphi_{ip}(U_i'; U_{-i}) = 0$ for all $p \in M \backslash \{1, 2\}$.*

- *(ii). $U_i(\varphi_i(U_i'; U_{-i})) = U_i\left(\frac{\Omega}{n}\right)$.*

- *(iii). $U_j(\varphi_j(U_i'; U_{-i})) = U_j\left(\frac{\Omega}{n}\right)$ for all $j \in N \backslash \{i\}$.*

Proof. First, note that the following three claims, each of which refers to assignments and the utility level of agent i under a deviation from U, hold. Claim 1 states that agent i's assignment of good 1 remains positive when she submits U_i^δ or U_i' instead of U_i. Claim 2 states that agent i receives the identical assignment under $(U_i^\delta; U_{-i})$ and $(U_i'; U_{-i})$. Finally, Claim 3 states that the assignment under the deviation U_i' is at least as good as the original assignment, i.e., $\varphi_i(R)$, under the original utility function U_i.

Claim 1. $\varphi_{i1}(U_i^\delta; U_{-i}) > 0$ かつ $\varphi_{i1}(U_i'; U_{-i}) > 0$ [25]

Claim 2. $\varphi_i(U_i^\delta; U_{-i}) = \varphi_i(U_i'; U_{-i})$. [26]

Claim 3. $U_i(\varphi_i(U_i'; U_{-i})) \geq U_i(\varphi_i(U))$. [27]

We prove (ii) first. Since φ is *SP*, $U_i(\varphi_i(U)) \geq U_i(\varphi_i(U_i'; U_{-i}))$. Combining with Claim 3, we obtain $U_i(\varphi_i(U_i'; U_{-i})) = U_i(\varphi_i(U))$. The right-hand side is equal to $U_i\left(\frac{\Omega}{n}\right)$ due to Fact 1 (iii), (iv).

Next, we show (i). Note that

$$
\begin{aligned}
U_i(\varphi_i(U_i'; U_{-i})) &= \varphi_{i1}(U_i'; U_{-i}) + u_{02}\varphi_{i2}(U_i'; U_{-i}) && (\because \text{Lemma 3 (i)}) \\
&= \varphi_{i1}(U_i'; U_{-i}) + u_{02}\left(1 - \varphi_{i1}(U_i'; U_{-i})\right) && (\because \overline{r} = \underline{r} = 1) \\
&= (1 - u_{02})\varphi_{i1}(U_i'; U_{-i}) + u_{02}.
\end{aligned}
$$

Thus, by (ii), we obtain $(1 - u_{02})\varphi_{i1}(U_i'; U_{-i}) + u_{02} = \frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p}$. Thus, $\varphi_{i1}(U_i'; U_{-i}) = \frac{1}{n} - \frac{1}{n}\sum_{p=3}^{n} \frac{u_{02}-u_{0p}}{1-u_{02}}$. Since agent i only consumes good 1 and 2 (Lemma 3 (i)), $\varphi_{i2}(U_i'; U_{-i}) = 1 - \varphi_{i1}(U_i'; U_{-i}) = 1 - \left(\frac{1}{n} - \frac{1}{n}\sum_{p=3}^{n} \frac{u_{02}-u_{0p}}{1-u_{02}}\right) = \frac{n-1}{n} + \frac{1}{n}\sum_{p=3}^{n} \frac{u_{02}-u_{0p}}{1-u_{02}}$. This completes the proof of (i).

Finally, we show (iii). As agents in $N\backslash\{i\}$ collectively consumes all resources not

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[25] *Proof of Claim 1:* We only prove the former as the latter is proved similarly. Suppose to the contrary that $\varphi_{i1}(U_i^\delta; U_{-i}) = 0$. Notice that $U_i^\delta\left(\varphi_i(U_i^\delta; U_{-i})\right) < \frac{1}{2n}$ due to (∗). On the other hand, since $U_i(\varphi_i(U)) > \frac{1}{n}$ ($\because$ Fact 1 (iii) and (iv)), $\varphi_{i1}(U) > \frac{1}{2n}$ due to (∗). Thus, $U_i^\delta(\varphi_i(U)) > \frac{1}{2n} > U_i^\delta(\varphi_i(U_i^\delta; U_{-i}))$. However, this violates *SP* of φ, a contradiction.

[26] *Proof of Claim 2:* Before we begin the proof of Claim 2, note that by Claim 1 and Lemma 3 (i), $\varphi_{i1}(U_i^\delta; U_{-i}) + \varphi_{i2}(U_i^\delta; U_{-i}) = 1$ and $\varphi_{i1}(U_i'; U_{-i}) + \varphi_{i2}(U_i'; U_{-i}) = 1$. Suppose to the contrary that $\varphi_i(U_i^\delta; U_{-i}) \neq \varphi_i(U_i'; U_{-i})$. Then, the ratio of good 1 contained in one of $\varphi_i(U_i^\delta; U_{-i})$ and $\varphi_i(U_i'; U_{-i})$ is greater than that of the other's. Thus, agent i has an incentive to deviate from one of $(U_i^\delta; U_{-i})$ and $(U_i'; U_{-i})$ to the other. This violates the assumption that φ is *SP*.

[27] *Proof of Claim 3:* By *SP* of φ, $U_i^\delta(\varphi_i(U_i^\delta; U_{-i})) \geq U_i^\delta(\varphi_i(U))$. As the left-hand side of this inequality is equal to $\varphi_{i1}(U_i'; U_{-i}) + \delta\varphi_{i2}(U_i'; U_{-i})$ ($\because$ Lemma 3 (i) and Claim 2), $\varphi_{i1}(U_i'; U_{-i}) + \delta\varphi_{i2}(U_i'; U_{-i}) \geq U_i^\delta(\varphi_i(U))$. Letting $\delta \to u_{02}$, we obtain the conclusion.

assigned to agent i, by (i),

$$\sum_{j \in N \setminus \{i\}} \varphi_{j1}(U_i'; U_{-i}) = 1 - \varphi_{i1}(U_i'; U_{-i}) = \frac{n-1}{n} + \frac{1}{n} \sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}}$$

$$\sum_{j \in N \setminus \{i\}} \varphi_{j2}(U_i'; U_{-i}) = 1 - \varphi_{i2}(U_i'; U_{-i}) = \frac{1}{n} - \frac{1}{n} \sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}}$$

$$\sum_{j \in N \setminus \{i\}} \varphi_{jp}(U_i'; U_{-i}) = 1 \quad \text{for all } p \in M \setminus \{1, 2\}.$$

Thus, noting that all agents in $N \setminus \{i\}$ have the identical utility function u_0,

$$\sum_{j \in N \setminus \{i\}} U_j(\varphi_j(U_i'; U_{-i})) = \left(\frac{n-1}{n} + \frac{1}{n} \sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}} \right) + u_{02} \left(\frac{1}{n} - \frac{1}{n} \sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}} \right) + \sum_{p=3}^{n} u_{0p}$$

$$= (n-1) \left(\frac{1}{n} + \frac{1}{n} u_{02} + \frac{1}{n} \sum_{p=3}^{n} u_{0p} \right).$$

Since φ satisfies S or $EDLB$, $U_j(\varphi_j(U_i'; U_{-i})) = \frac{1}{n} + \frac{1}{n} u_{02} + \frac{1}{n} \sum_{p=3}^{n} u_{0p}$ for each $j \in N \setminus \{i\}$. $\qquad \square$

Lemma 5 considers agent i's deviation from U_0 to $U_0^\delta \in \mathcal{U}(U_0, U_0')$ at $(U_j'; U_{-j})$, where $U_j' = U_0'$ and $U_{-j} = (U_0, \ldots, U_0)$. That is, after the first deviation of agent j, another agent i exhibits a relatively "small" deviation. Item (i) shows that the amount of good 1 assigned to agent j coincides with that of good 2 assigned to agent i. Item (ii) shows that the amount of good 1 assigned to agent j does not increase after agent i's deviation (see Lemma 4 (i)).

Lemma 5. *Suppose that $m = n$ and $\underline{r} = \overline{r} = 1$. Let φ be a SP and E rule satisfying S or EDLB. Let $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$ be such that*

$$(*) \quad \frac{1}{2n^2} > u_{02}'.$$

Let $i, j \in N$ be such that $i \neq j$. Let $U \in \mathcal{U}^N$ be $U := (U_0, \ldots, U_0)$. Let $U_j' := U_0'$ and $U_i^\delta \in \mathcal{U}(U_0, U_0')$. Then,

(i). $\varphi_{j1}(U_j', U_i^\delta; U_{-\{i,j\}}) = \varphi_{i2}(U_j', U_i^\delta; U_{-\{i,j\}})$, and

(ii). $\varphi_{j1}(U'_j, U_i^\delta; U_{-\{i,j\}}) \leq \frac{1}{n} - \frac{1}{n}\sum_{p \in M\setminus\{1,2\}} \frac{u_{02}-u_{0p}}{1-u_{02}}.$

Proof. First, note that the following claim holds. It states that agent i's deviation strategy U_i^δ at $(U'_j, U_i; U_{-\{i,j\}})$ does not deprive all probability share of good 1 from agent i or j.

Claim. $\varphi_{j1}(U'_j, U_i^\delta; U_{-\{i,j\}}) > 0$ and $\varphi_{i1}(U'_j, U_i^\delta; U_{-i}) > 0$. [28]

We prove (i). By Claim and Lemma 3 (ii), we have $\varphi_{j1}(U'_j, U_i^\delta; U_{-\{i,j\}}) + \varphi_{j2}(U'_j, U_i^\delta; U_{-\{i,j\}}) = 1$ (i.e., agent j only consumes goods 1 and 2) and $\varphi_{j2}(U'_j, U_i^\delta; U_{-\{i,j\}}) + \varphi_{i2}(U'_j, U_i^\delta; U_{-\{i,j\}}) = 1$ (i.e., agents i and j exhaust good 2). Thus, (i) is shown.

Next, we prove (ii). Suppose to the contrary that $\varphi_{j1}(U'_j, U_i^\delta; U_{-\{i,j\}}) > \frac{1}{n} - \frac{1}{n}\sum_{p \in M\setminus\{1,2\}} \frac{u_{02}-u_{0p}}{1-u_{02}}$. Then,

$U_j\left(\varphi_j(U'_j, U_i^\delta; U_{-\{i,j\}})\right)$

$> U_j\left(\frac{1}{n} - \frac{1}{n}\sum_{p=3}^{n}\frac{u_{02}-u_{0p}}{1-u_{02}}, \ \frac{n-1}{n} + \frac{1}{n}\sum_{p=3}^{n}\frac{u_{02}-u_{0p}}{1-u_{02}}, 0, \ldots, 0\right)$ ($\because$ Contradiction hypothesis and Lemma 3(ii))

$= U_j\left(\varphi_j(U'_j, U_i; U_{-\{i,j\}})\right)$ ($\because$ Lemma 4 (i))

$= U_j\left(\frac{\Omega}{n}\right)$ ($\because$ Lemma 4 (ii))

$= U_j\left(\varphi_j(U_j, U_i^\delta; U_{-\{i,j\}})\right).$ ($\because$ Lemma 4 (iii))

However, this is a violation of *SP* of φ, a contradiction. □

Proof of Theorem 1 under Assumption B. Suppose to the contrary that φ satisfies *SP*, *E* and *EDLB*. Let $U_0 \in \mathcal{U}$ and $U'_0 \in \mathcal{U}(U_0)$ be such that

$$(*) \quad \frac{1}{2n^2} > u'_{02}.$$

Let $U \in \mathcal{U}^N$ be $U := (U_0, \ldots, U_0)$. Let $U'_1 := U'_0$ and $U'_2 := U'_0$. Let $U_2^\delta \in \mathcal{U}(U_0, U'_0)$. As was shown in the proof of Lemma 5, $\varphi_{11}(U'_1, U'_2, U_{-\{1,2\}}) > 0$ and

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[28] *Proof of Claim:* We only prove the first inequality as the latter is proved similarly. Suppose to the contrary that $\varphi_{j1}(U'_j, U_i^\delta; U_{-\{i,j\}}) = 0$. By Lemma 4 (iii), $U_j\left(\varphi_j(U_j, U_i^\delta; U_{-\{i,j\}})\right) = U_j\left(\frac{\Omega}{n}\right)$. Thus, by $(*)$, $\varphi_{j1}(U_j, U_i^\delta; U_{-\{i,j\}}) > \frac{1}{2n}$. Thus, $U'_j\left(\varphi_j(U_j, U_i^\delta; U_{-\{i,j\}})\right) > \frac{1}{2n}$. On the other hand, the combination of the contradiction hypothesis and $(*)$ results in $U'_j\left(\varphi_j(U'_j, U_i^\delta; U_{-\{i,j\}})\right) < \frac{1}{2n}$, a violation of *SP* of φ.

$\varphi_{21}(U_1', U_2', U_{-\{1,2\}}) > 0$.[29] Thus, by Lemma 3 (ii-2), $\varphi_{12}(U_1', U_2', U_{-\{1,2\}}) + \varphi_{22}(U_1', U_2', U_{-\{1,2\}}) = 1$. Thus, supposing, without loss of generality, that $\varphi_{22}(U_1', U_2', U_{-\{1,2\}}) \geq \frac{1}{2}$, let $\epsilon := \varphi_{22}(U_1', U_2', U_{-\{1,2\}}) - \frac{1}{2}$. Moreover, let $b := \varphi(U_1', U_2', U_{-\{1,2\}})$ and $a^\delta := \varphi(U_1', U_2^\delta, U_{-\{1,2\}})$, for simplicity.

The following inequality shows an upper bound of $U_2^\delta(a_2^\delta)$.

Step 1. $\left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p} \right) + (\delta - u_{02})\left(\frac{1}{n} - \frac{1}{n}\sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}} \right) \geq U_2^\delta(a_2^\delta)$.

Proof of Step 1. Since φ is SP, $U_2\left(\varphi_2(U_1', U_2, U_{-\{1,2\}}) \right) \geq U_2(a_2^\delta)$. Thus, $\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p} \geq a_{21}^\delta + u_{02}a_{22}^\delta + \sum_{p=3}^{n} u_{0p}a_{2p}^\delta$. Thus,

$$\left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p} \right) + (\delta - u_{02})a_{22}^\delta \geq \left(a_{21}^\delta + u_{02}a_{22}^\delta + \sum_{p=3}^{n} u_{0p}a_{2p}^\delta \right) + (\delta - u_{02})a_{22}^\delta$$

$$= U_2^\delta(a_2^\delta)$$

By Lemma 5 (i) and (ii), $\frac{1}{n} - \frac{1}{n}\sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}} \geq a_{22}^\delta$. Combining this with the above inequality, we obtain the desired conclusion. This completes the proof of Step 1.

Next, we provide a lower bound of $U_2^\delta(b_2)$.

Step 2. $U_2^\delta(b_2) \geq \left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p} \right) - (u_{02}' - \delta)\left(\frac{1}{2} + \epsilon \right)$.

Proof of Step 2. First, note that $U_2'(b_2) \geq \frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}$ since φ satisfies *EDLB*. Thus, the total utility $U_2'(b_2)$ of b_2 minus $u_{02}'\left(\frac{1}{2} + \epsilon \right)$ (the utility from good 2) is at least $\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p} - u_{02}'\left(\frac{1}{2} + \epsilon \right)$. Since U_2' and U_2^δ are identical except for the evaluation on good 2,

$$U_2^\delta(b_2) - \delta b_{22} \geq \frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p} - u_{02}'\left(\frac{1}{2} + \epsilon \right).$$

This completes the proof of Step 2.

Step 3. Concluding.

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[29]See Footnote 28.

By Step 1 and 2,

$$U_2^\delta(b_2) - U_2^\delta(a_2^\delta)$$

$$\geq \left\{ \left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p} \right) - (u_{02}' - \delta)\left(\frac{1}{2} + \epsilon \right) \right\}$$

$$- \left\{ \left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p} \right) + (\delta - u_{02})\left(\frac{1}{n} - \frac{1}{n}\sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}} \right) \right\}$$

$$= (u_{02}' - \delta)\left(\frac{1}{n} - \frac{1}{2} - \epsilon \right) + (\delta - u_{02})\frac{1}{n}\sum_{p=3}^{n} \frac{u_{02} - u_{0p}}{1 - u_{02}}. \qquad (D)$$

Letting $\delta \to u_{02}'$, (D) converges to $(u_{02}' - u_{02})\frac{1}{n}\sum_{p=3}^{n} \frac{u_{02}-u_{0p}}{1-u_{02}} > 0$. Thus, for $\delta \in (u_{02}, u_{02}')$ sufficiently close to u_{02}', (D) is positive. However, this means that $U_2^\delta(b_2) > U_2^\delta(a_2^\delta)$, a violation of SP of φ. $\qquad \square$

Proof of Theorem 2 under Assumption B. Suppose to the contrary that φ is SP, E and EF. Let $U_0 \in \mathcal{U}$ and $U_0' \in \mathcal{U}(U_0)$ be such that

$$(*) \quad \frac{1}{2n^2} > u_{02}'.$$

Let $U, U' \in \mathcal{U}^N$ be $U := (U_0, \ldots, U_0)$ and $U' := (U_0', \ldots, U_0')$. Letting $U^{(0)} := U$, let $U^{(k)} := \left(U_k'; U_{-k}^{(k-1)} \right)$ for each $k \in \{1, \ldots, n\}$. Moreover, for each $k \in \{1, \ldots, n\}$ and each $U_k^\delta \in \mathcal{U}(U_0, U_0')$, let $U^{(k,\delta)} := \left(U_k^\delta; U_{-k}^{(k-1)} \right)$. The proof proceeds in five steps.

Step 1. (i) $\forall U_2^\delta \in \mathcal{U}(U_0, U_0'), \varphi_{12}\left(U^{(2,\delta)} \right) + \varphi_{22}\left(U^{(2,\delta)} \right) = 1$, (ii) $\forall k \in \{3, \ldots, n\}, \forall U_k^\delta \in \mathcal{U}(U_0, U_0'), \sum_{i=1}^{k-1} \varphi_{i2}\left(U^{(k,\delta)} \right) = 1$, and (iii) $\forall k \in \{2, \ldots, n\}, \sum_{i=1}^{k} \varphi_{i2}\left(U^{(k)} \right) = 1$.

Proof of Step 1. (i) Suppose not. Then, by Lemma 3 (ii), an agent $i \in \{1, 2\}$ receives 0 unit of good 1. On the other hand, there exists $j \in N\backslash\{1,2\}$ who receives at least $\frac{1}{n-1}$ units of good 1 because good 1 is not disposed at all ($\because m = n$ and $\underline{r} = \bar{r} = 1$). Consequently, by $(*)$, agent i prefers j's assignment to her own. However, this violates the assumption that φ is EF.

(ii) and (iii) are proved similarly. This completes the proof of Step 1.

In the following steps, let x be the utility level of agent 3 at $\varphi_3(U^{(2)})$, i.e., $x :=$ $U_3\left(\varphi_3(U^{(2)})\right)$. Step 2 shows that it does not coincide with the utility level of equal division under U_0'.

Step 2. $x \neq \frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}$.

Proof of Step 2. Suppose to the contrary that $x = \frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}$. Let $b := \varphi(U^{(2)})$. Note that by Step 1 (iii), $b_{12} + b_{22} = 1$. Without loss of generality, we assume $b_{22} \geq \frac{1}{2}$. Let $\epsilon := b_{22} - \frac{1}{2}$. Note that $\left(\frac{1}{n} + \frac{1}{n}u_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\right) + (\delta -$ $u_{02})\left(\frac{1}{n} - \frac{1}{n}\sum_{p=3}^{n}\frac{u_{02}-u_{0p}}{1-u_{02}}\right) \geq U_2^\delta(a_2^\delta).$[30] Moreover, the value of $U_2^\delta(b_2)$ is given as $U_2^\delta(b_2) = \left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\right) - (u_{02}' - \delta)\left(\frac{1}{2} + \epsilon\right).$[31] Thus, following the same reasoning as in Step 3 of the proof of Theorem 1 under Assumption B, we obtain $U_2^\delta(b_2) > U_2^\delta(a_2^\delta)$ for some $\delta \in (u_{02}, u_{02}')$ sufficiently close to u_{02}'. However, this is a violation of SP of φ, a contradiction. This completes the proof of Step 2.

Step 3. $\forall k \in \{2, \ldots, n-1\}, U_{k+1}'\left(\varphi_{k+1}(U^{(k+1)})\right) = U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right)$.

Proof of Step 3. Same as Step 2 in the proof of Proposition 1 under Assumption A. This completes the proof of Step 3.

In the sequel, for each $k \in \{3, \ldots, n-1\}$, let $A_k := (n-3)(n-4)\ldots(n-k)$ and $B_k := 3 \cdot 4 \cdot \ldots \cdot k$.

Step 4. For each $k \in \{3, \ldots, n-1\}$,

$$U_{k+1}\left(\varphi_{k+1}(U^{(k)})\right) = \frac{A_k - (-1)^k B_k}{A_k}\left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\right) + (-1)^k \frac{B_k}{A_k}x.$$

Proof of Step 4. We prove it by an induction argument. Suppose that $k = 3$. Since

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[30]Proof is same as Step 1 of the proof of Theorem 1 under Assumption B.

[31]*Proof.* Since φ is S, $U_3^{(2)}(b_3) = \ldots = U_n^{(2)}(b_n) = x$. As $b_{12} + b_{22} = 1$ ($\because$ Step 1 (iii)), the total utility level of b at $U^{(2)}$ is $\sum_{i \in N} U_i^{(2)}(b_i) = 1 + u_{02}' + \sum_{p=3}^{n} u_{0p}$. Thus, since $U_1^{(2)}(b_1) + U_2^{(2)}(b_2) = \left(1 + u_{02}' + \sum_{p=3}^{n} u_{0p}\right) - (n-2)x = 2x$, $U_2^{(2)}(b_2) = x$ ($\because \varphi$ is S). Thus, the utility from b_2 at $U_2^{(2)}$ except for good 2 is $x - u_{02}'\left(\frac{1}{2} + \epsilon\right)$. Since $U_2^{(2)}$ and U_2^δ are identical except for the evaluation on good 2, $U_2^\delta(b_2) - \delta\left(\frac{1}{2} + \epsilon\right) = x - u_{02}'\left(\frac{1}{2} + \epsilon\right)$. Thus, the desired equation is obtained.

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$U_3'\big(\varphi_3(U^{(3)})\big) = U_3\big(\varphi_3(U^{(2)})\big) = x \ (\because \text{Step 3})$, $\sum_{i=1}^{3} U_i^{(3)}\big(\varphi_i(U^{(3)})\big) = 3x \ (\because \varphi \text{ is } S)$.
Note that by Step 1 (iii), $\sum_{i=1}^{3} \varphi_{i2}(U^{(3)}) = 1$. Thus, the total utility at $\varphi(U^{(3)})$ is given as $\sum_{i=1}^{n} U_i^{(3)}\big(\varphi_i(U^{(3)})\big) = 1 + u_{02}' + \sum_{p=3}^{n} u_{0p}$. Thus, $\sum_{i=4}^{n} U_i^{(3)}\big(\varphi_i(U^{(3)})\big) = \big(1 + u_{02}' + \sum_{p=3}^{n} u_{0p}\big) - 3x$. Since φ is S, $U_4^{(3)}\big(\varphi_4(U^{(3)})\big) = \frac{1}{n-3}\big\{\big(1 + u_{02}' + \sum_{p=3}^{n} u_{0p}\big) - 3x\big\} = \frac{(n-3)-(-1)^3 \cdot 3}{n-3}\big(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\big) + (-1)^3 \frac{3}{n-3}x$.

Let $k \geq 4$. We assume that

$$U_k\big(\varphi_k(U^{(k-1)})\big) = \frac{A_{k-1} - (-1)^{k-1}B_{k-1}}{A_{k-1}}\left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\right) + (-1)^{k-1}\frac{B_{k-1}}{A_{k-1}}x.$$

Since $U_k'\big(\varphi_k(U^{(k)})\big) = U_k\big(\varphi_k(U^{(k-1)})\big) \ (\because \text{Step 3})$, $\sum_{i=1}^{k} U_i^{(k)}\big(\varphi_i(U^{(k)})\big) = kU_k\big(\varphi_k(U^{(k-1)})\big)$ $(\because \varphi \text{ is } S)$. Note that by Step 1 (iii), $\sum_{i=1}^{k} \varphi_{i2}(U^{(k)}) = 1$. Thus, the total utility at $\varphi(U^{(k)})$ is given as $\sum_{i=1}^{n} U_i^{(k)}\big(\varphi_i(U^{(k)})\big) = 1 + u_{02}' + \sum_{p=3}^{n} u_{0p}$. Thus, $\sum_{i=k+1}^{n} U_i^{(k)}\big(\varphi_i(U^{(k)})\big) = \big(1 + u_{02}' + \sum_{p=3}^{n} u_{0p}\big) - kU_k\big(\varphi_k(U^{(k-1)})\big)$. Since φ is S, by the induction hypothesis,

$$U_{k+1}^{(k)}\big(\varphi_{k+1}(U^{(k)})\big)$$

$$= \frac{1}{n-k}\left[\left(1 + u_{02}' + \sum_{p=3}^{n} u_{0p}\right) - k\left\{\frac{A_{k-1} - (-1)^{k-1}B_{k-1}}{A_{k-1}}\left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\right) + (-1)^{k-1}\frac{B_{k-1}}{A_{k-1}}x\right\}\right]$$

$$= \frac{A_k - (-1)^k B_k}{A_k}\left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\right) + (-1)^k \frac{B_k}{A_k}x.$$

This completes the proof of Step 4.

Step 5. Concluding

By Step 3 and 4,

$$U_n'\big(\varphi_n(U^{(n)})\big) = U_n\big(\varphi_n(U^{(n-1)})\big)$$

$$= \frac{A_{n-1} - (-1)^{n-1}B_{n-1}}{A_{n-1}}\left(\frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}\right) + (-1)^{n-1}\frac{B_{n-1}}{A_{n-1}}x.$$

Regarding the last line as a real-valued function with a single variable x, it is linear. Thus,

$$U_n'\big(\varphi_n(U^{(n)})\big) = \frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p} \quad \Leftrightarrow \quad x = \frac{1}{n} + \frac{1}{n}u_{02}' + \frac{1}{n}\sum_{p=3}^{n} u_{0p}.$$

Since $x \neq \frac{1}{n} + \frac{1}{n}u'_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p}$ ($\because$ Step 2), $U'_n\left(\varphi_n(U^{(n)})\right) \neq \frac{1}{n} + \frac{1}{n}u'_{02} + \frac{1}{n}\sum_{p=3}^{n} u_{0p}$. However, this contradicts Fact 1 (iv). $\qquad\square$

In the case with $n = 3$, note that Step 1 of the preceding proof can be completed even if EF is replaced with a weaker fairness notion S. Since the proofs of Steps 2 to 5 have no opportunity to directly use EF instead of S, the preceding proof works as an alternative proof of the following result. [32]

Proposition 2 (Theorem 2 in Zhou (1990)). *Suppose that $n = 3$. Under Assumption B, no rule is SP, E and S.*[33]

Proof of Proposition 1 under Assumption B. Suppose to the contrary that φ satisfies SP, E, S and $MULG$. Let $\epsilon > 0$ be the utility level guaranteed by φ ($\because$ $MULG$). Let $U_0 \in \mathcal{U}$ and $U'_0 \in \mathcal{U}(U_0)$ be such that

$$(*) \quad \frac{1}{2n} \min\left\{\frac{1}{n}, \epsilon\right\} > u'_{02}.$$

Define $U, U', U^{(0)}, U^{(1)}, \ldots, U^{(n)}, U^\delta_k, U^{(1,\delta)}, \ldots, U^{(n,\delta)}$ in the same manner as in the proof of Theorem 2 under Assumption B. The following claim asserts the same as Step 1 in the proof of Theorem 2 under Assumption B. Here, we prove it by utilizing $MULG$ instead of EF.

Claim. (i) $\forall U^\delta_2 \in \mathcal{U}(U_0, U'_0), \varphi_{12}\left(U^{(2,\delta)}\right) + \varphi_{22}\left(U^{(2,\delta)}\right) = 1$, (ii) $\forall k \in \{3, \ldots, n\}, \forall U^\delta_k \in \mathcal{U}(U_0, U'_0), \sum_{i=1}^{k-1} \varphi_{i2}\left(U^{(k,\delta)}\right) = 1$, and (iii) $\forall k \in \{2, \ldots, n\}, \sum_{i=1}^{k} \varphi_{i2}\left(U^{(k)}\right) = 1$.

Proof of Claim. (i) Suppose not. Then, by Lemma 3 (ii), an agent $i \in \{1, 2\}$ receives 0 unit of good 1. Consequently, by $(*)$, agent i's utility level at $\varphi_i\left(U^{(2,\delta)}\right)$ is smaller than ϵ. This is a violation of $MULG$, a contradiction.

(ii) and (iii) are proved similarly. This completes the proof of Claim.

Note that there is no opportunity to directly apply EF instead of S through Step

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[32] Actually, the proof is completed at the end of Step 2 when $n = 3$.

[33] The proof of Theorem 2 in Zhou (1990) consists of two parts. The first half precisely corresponds to a proof of Proposition 2. In the latter half, he expands this result to the case with many agents by utilizing heterogeneity of ordinal preferences.

2 to 5 in the proof of Theorem 2 under Assumption B. Thus, the same argument
leads to a contradiction. □

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