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Fair priority-completion in assignment problems*

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Abstract

We consider the problem of assigning indivisible objects to agents prioritized within their affiliated institutions. An example is the assignment of student exhange programs to students prioritized only in their own departments. As students from different departments are incomparable, the problem is formalized as a priority-based indivisible goods allocation problem with incomplete priority. We show that each weak core allocation is attained by a priority rule associated with a priority-completion, and vice versa. Moreover, we advocate a class of completions satisfying two fairness notions: interpersonal and interinstitutional fairness.

Journal of Economic Literature Classification Numbers : C78, D47, D71.

Keywords: Market design; Incomplete priority; Exclusion core; Priority rule

1 Introduction

Suppose that the administration office of a university faces the problem of assigning indivisible objects, such as scholarships and student exchange programs, to students prioritized only in their own departments. For example, assume that students e_1, e_2 belonging to the department of economics and l_1, l_2, l_3, l_4 of the department of law

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Table 1: Example of completions

(i)

Econ.	Priority	Law	Priority
e_1	1	l_1	3
		l_2	4
e_2	2	l_3	5
		l_4	6

(ii)

Econ.	Priority	Law	Priority
e_1	1	l_1	2
		l_2	3
e_2	6	l_3	4
		l_4	5

(iii)

Econ.	Priority	Law	Priority
e_1	5	l_1	1
		l_2	2
e_2	6	l_3	3
		l_4	4

are searching for opportunities to study abroad. Assume also that each department gives a higher priority to students with smaller indices. This situation is similar to the so-called "school choice" problem, but exhibits the following two features:

Feature 1. Priority is **incomplete** because students of different departments are incomparable.

Feature 2. Priority is **common**, that is, each exchange program gives priority to students in the same manner. This is because the administration office refers to the priority order submitted by departments.

As long as the administration office is able to bear the cost of having an additional interview or examination, the incomparability of students could be resolved by creating new data. However, the cost of extracting such additional data could become enormous as the number of applicants increases. In this study, we pursue market design without such a cost. The second feature is typically observed in the problems that the administration office assigns indivisible objects to students, e.g., dorm room assignment (Abdulkadiroğlu and Sönmez, 1999).

We formalize the problem with the above features as a priority-based indivisible goods allocation problem with an incomplete priority (Balbuzanov and Kotowski, 2019). Based on the relationship between several core concepts (Theorem 1 and Proposition 1), we prove that the weak core is characterized by the range of priority rule, also known as the serial-dictatorship rule, with completions of priority (Theorem 2), where completion is a complete priority order consistent with the given incomplete priority. As shown in Table 1, the class of completions contains a wide range of possibilities in terms of fairness. For example, the completions shown in (i) and (iii) are completely favorable to one of the departments, whereas the middle completion (ii) shows a balanced distribution of the opportunity. Among the various completions, we advocate a method for selecting a plausible class based on two fairness notions: interpersonal and interinstitutional fairness (Theorem 3).

1.1 Related literature

In one-to-one matching problems, a model with social endowments is called the house allocation problem (Hylland and Zeckhauser, 1979), one with private ownership is called a housing market (Shapley and Scarf, 1974), and one with mixed ownership is termed a house allocation problem with existing tenants (Abdulkadiroğlu and Sönmez, 1999). Balbuzanov and Kotowski (2019) generalize these models to an indivisible goods allocation problem with incomplete priority structure. To attain an efficient allocation, the Gale's top-trading cycles algorithm (Shapley and Scarf, 1974), or its variant, is applied to these problems. For a special class of problems with homogeneous priority, a priority rule is used to attain efficiency (Svensson, 1999).

School choice is a house allocation problem with multiple copies of objects (Abdulkadiroğlu and Sönmez, 2003). The vast literature on school choice contains research papers on priority with ties (Erdil and Ergin, 2008; Ehlers and Erdil, 2010), which look similar, but are different from the incomplete priority case. A difference is that ties are exogenously given in the former case, but not in the latter. This makes the set of completions far richer than that of tie-breakers. For example, in Table 1, since e_2 is not comparable with the law department students, e_2 could be prioritized to l_1 in some completion (Table 1(i)). However, on another completion (Table 1(iii)), l_1 can be prioritized to e_1, which is explicitly prioritized to e_2 in the given priority order. This reversal never occurs under a complete priority with ties. Thus, an appropriate research direction to find a plausible completion may start with the formation of ties under a given incomplete priority (Hatakeyama and Kurino, 2022). Apportionment is related to this line (Balinski and Young, 2001). As is discussed in detail in Section 4, our approach takes a different direction. We transform the problem of finding a completion into the allocation problem of quantified opportunity (point) to establish a system under which agents who receive greater points, receive a higher priority. This transformation enables us to define two fairness concepts, interpersonal and interinstitutional fairness, which in turn are helpful in establishing an intuitive system to allocate objects.

Another related topic is the reduction of interview burden in the application process for fellowship (Melcher, Ashlagi and Wapnir, 2018, 2019; Ashlagi et al., 2020). Our result, which helps to eliminate any additional interview burden on constructing a complete priority order, critically depends on a given priority order, although the information it delivers is partial. Thus, our result is silent in this setting without any transcendental information on priority.

The contributions of the current study can be summarized as follows: Under the assumption of common priority (Feature 2), we show the following.

- Extending the Balbuzanov and Kotowski's priority-based formalization of property right to the many-to-one setting, we show the relationship among several core concepts in terms of set-theoretic inclusion (Theorem 1 and Proposition 1).
- The weak core is characterized by the range of priority rule with priority-completions (Theorem 2).
- We formalize the concept of interpersonal and interinstitutional fair point allocations to pick a class of conpletions. Moreover, we advocate a concrete method, the (generalized) midpoint rule, to implement it. We provide a characterization of the generalized midpoint rule based mainly on the two fairness notions (Theorem 3).

The remainder of this paper is organized as follows. Section 2 formalizes the priority-based indivisible good allocation problem with an incomplete priority structure. In Section 3, we generalize the concept of property rights formalized in Balbuzanov and Kotowski (2019) and define the several core concepts. We then provide two characterization results for the weak core (Theorems 1 and 2). In Section 4, we first consider the refinement of the weak core in terms of fairness notions in the ex-post sense. Then, we turn to the design of the completion selection rule through point allocations. We propose a rule called the midpoint rule, and characterize a class of rules including it. In Section 5, we discuss the related topics. The proofs are presented in Appendix B.

2 Model

We introduce an indivisible goods allocation problem with an incomplete priority. It is the one in Balbuzanov and Kotowski (2019) with multiple copies of objects.[1] Let $N = \{1, 2, \ldots, n\}$ be a set of agents and $\mathcal{O} = \{o_1, o_2, \ldots, o_m\}$ be a set of real objects. Each real object $o \in \mathcal{O}$ has $q_o \in \mathbb{N}$ copies. Let $q := (q_o)_{o\in\mathcal{O}} \in \mathbb{N}^m$ be a quota vector. We assume the existence of a null object $o_0 \notin \mathcal{O}$ that represents no consumption. For each $o \in \mathcal{O}$, a binary relation $\succeq_o$ on N is provided, which represents the priority of consuming o. We assume that $\succeq_o$ is reflexive, transitive, and anti-symmetric, but not necessarily complete.[2] Letting $\succeq := (\succeq_o)_{o\in\mathcal{O}}$, we call $\succeq$ the priority structure. Each agent $i \in N$ has a preference represented by a complete, transitive and anti-symmetric binary relation R_i on $\mathcal{O} \cup \{o_0\}$. Let $\mathcal{R}$ be the set of preference relations. For each $R_i \in \mathcal{R}$, let P_i be the anti-symmetric part of R_i, that is, $o\ P_i\ o'$ if and only if $o\ R_i\ o'$ and not $o'\ R_i\ o$. Let $\mathcal{R}^N$ be the set of preference profiles.

A **problem** is a 5-tuple $(N, \mathcal{O} \cup \{o_0\}, \succeq, q, R)$, where $R \in \mathcal{R}^N$. Throughout this paper, we fix $N, \mathcal{O} \cup \{o_0\}, \succeq$ and q. Thus, each problem is simply denoted by a preference profile R. An allocation is a function $a : N \to \mathcal{O} \cup \{o_0\}$ such that for each $o \in \mathcal{O}$, $|a^{-1}(o)| \leq q_o$. Let $\mathcal{A}$ be the set of allocations.

In this study, we concentrate on the class of problems that satisfy the following condition.

Assumption 1 (Common priority)**.** *The priority structure $\succeq$ is **common** if*

$$\forall o, o' \in \mathcal{O}, \succeq_o = \succeq_{o'} .$$

Hereafter, if there is no confusion, the symbol $\succeq$ not only represents a profile of priority relations, but also the individual priority relation $\succeq_o$ that all objects share.

Balbuzanov and Kotowski (2019) investigate another class of problems with acyclic priority structure. The priority structure $\succeq$ is **acyclic** *if* for each $o \in \mathcal{O}$, and each $\{i, j, h\} \subseteq N$, if $i \succ_o j$ and $i \not\succeq_o h$, then $h \succ_{o'} j$ for all $o' \in \mathcal{O}\setminus\{o\}$. As shown in

[1] Balbuzanov and Kotowski (2019) deal with two types of matching problems: simple economies with the initial endowment structure, and relational economies with the priority structure. The model we borrow is the latter.

[2] A binary relation on X, denoted as $\geq$, is reflexive if for each $x \in X$, $x \geq x$. A binary relation $\geq$ on X is complete if for each $\{x, y\} \subseteq X$, $x \geq y$ or $y \geq x$. A binary relation $\geq$ on X is transitive if for each $\{x, y, z\} \subseteq X$, $x \geq y$ and $y \geq z$ imply $x \geq z$. A binary relation $\geq$ on X is anti-symmetric if for each $\{x, y\} \subseteq X$, $x \geq y$ and $y \geq x$ imply $x = y$.

the following example, when $|N| \geq 3$ and $|\mathcal{O}| \geq 2$, acyclicity and Assumption 1 are independent, that is, one of the two conditions does not imply the other. Thus, the results of Balbuzanov and Kotowski (2019) and ours are independent.

Example 1 (Common priority structure may not be acyclic.)**.** Suppose that $|N| \geq 3$ and $|\mathcal{O}| \geq 2$. Let $i, j, h \in N$. Suppose that for each $o \in \mathcal{O}$, $i \succ_o j$ and h is not comparable with i and j under $\succeq_o$. Then, for any $o' \in \mathcal{O}$, $h \not\succ_{o'} j$. Therefore, the common priority structure described above is not acyclic. ◇

3 Result I: Core under the common priority

In this section, we investigate the three types of cores under a common priority structure. Roughly speaking, an allocation belongs to the core if it is impossible for any coalition $C \subseteq N$ to be better off by reallocating the objects "owned" by C. Thus, to define a concept of core, we need to establish a notion of property rights.

The concept of property rights we adopt is based on primitive data of the priority structure. The formulation is proposed in Balbuzanov and Kotowski (2019): given an allocation $a \in \mathcal{A}$, a coalition C conditionally owns i) objects assigned to agents subordinate to at least one member in C, and ii) objects disposed under a.[3]

Example 2 (Conditional endowment system in Balbuzanov and Kotowski (2019))**.** Let $N = \{1, 2, \ldots, 8\}$. Suppose that the priority structure $\succeq$ orders the agents, as depicted in Figure 1. Given an allocation $a \in \mathcal{A}$, the coalition of boxed numbers conditionally owns objects assigned to agents in the dotted area. The left figure shows that coalition $\{1\}$ owns $\{a(1), a(2), a(3)\}$ (and objects disposed at a, if any). In the right figure, coalition $\{2, 4\}$ owns $\{a(2), a(3)\} \cup \{a(4), a(5), a(6)\}$ (and objects disposed at a, if any). ◇

Note that in the example above, ownership does not count the overlapping of assignments. To extend the idea to our setup with multiple copies of real objects, we need to introduce the following extension of an object and allocation. The set of **extended objects** is defined as $\bar{\mathcal{O}} := \{o_{k\ell} \mid 1 \leq k \leq m \text{ and } 1 \leq \ell \leq q_{o_k}\}$. The set of **extended allocations** is defined as $\bar{\mathcal{A}} := \{\bar{a} : N \to \bar{\mathcal{O}} \cup \{o_0\} \mid \forall o \in \bar{\mathcal{O}}, |\bar{a}^{-1}(o)| \leq 1\}$.

[3]Here, we use the word "conditional" because the ownership depends on the given allocation a.

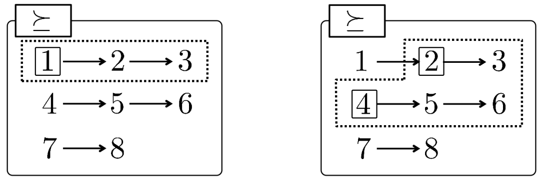

Figure 1: Conditional endowment system

Note: Each arrow represents that the agent on the root is prioritized to the one on the head.

Given an allocation $a \in \mathcal{A}$, the corresponding extended allocation $\bar{a} \in \bar{\mathcal{A}}$ is defined as follows: i) if $a(i) \in \mathcal{O}$(say o_k), let $\bar{a}(i) = o_{k\ell}$, where $\ell = |\{j \in N \mid j < i \text{ and } a(j) = o_k\}| + 1$; and ii) if $a(i) = o_0$, let $\bar{a}(i) = o_0$.

Now, we extend the concept of ownership in Balbuzanov and Kotowski (2019) to a many-to-one setting. Given an allocation $a \in \mathcal{A}$, the **conditional endowment system** is the function $\omega_a : 2^N \to 2^{\bar{\mathcal{O}}}$ such that for each $C \in 2^N$, ω_a assigns the set of extended objects owned by C.[4] Formally,

$$\omega_a(C) := \big\{o_{k\ell} \in \bar{\mathcal{O}} \mid \exists i \in C \text{ s.t. } i \succeq_{o_k} \bar{a}^{-1}(o_{k\ell})\big\} \cup \big(\bar{\mathcal{O}} \backslash \bar{a}(N)\big).$$

In words, the first part of the right-hand side indicates the set of extended objects assigned to agents subordinate to at least one member of C, and the second part is the set of extended objects disposed at $\bar{a}$.

Given an allocation $a \in \mathcal{A}$ and a coalition $C \subseteq N$, we say that $a|_C$ is **achievable** on $\bar{X} \subseteq \bar{\mathcal{O}}$ if the number of agents in C who receive $o_k \in \mathcal{O}$ at a does not exceed the number of o_k's copies in $\bar{X}$. Formally, for each $k \in \{1, 2, \ldots, m\}$, $\big|\{i \in C \mid a(i) = o_k\}\big| \leq \big|\{x \in \bar{X} \mid \exists \ell \in \{1, 2, \ldots, q_{o_k}\} \text{ s.t. } x = o_{k\ell}\}\big|$. To introduce the three types of cores, we first define the concept of blocking in three ways.

Definition 1 (Weak block)**.** *An allocation $a \in \mathcal{A}$ is weakly blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$ if*

(i) $\forall i \in C, b(i)\ R_i\ a(i)$,

(ii) $\exists i \in C$ s.t. $b(i)\ P_i\ a(i)$, and

(iii) $b|_C$ is achievable on $\omega_a(C)$.

Definition 2 (Strong block)**.** *An allocation $a \in \mathcal{A}$ is strongly blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$ if*

[4] Balbuzanov and Kotowski (2019) call ω_a the weak conditional endowment system.

(i) $\forall i \in C, b(i)\ P_i\ a(i)$, and

(ii) $b|_C$ is achievable on $\omega_a(C)$.

Definition 3 (Exclusion block)**.** An allocation $a \in \mathcal{A}$ is exclusion blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$ *if*

(i) $\forall i \in C, b(i)\ P_i\ a(i)$, and

(ii) $\forall i \in N, [a(i)\ P_i\ b(i) \Rightarrow \bar{a}(i) \in \omega_a(C)]$.

An allocation $a \in \mathcal{A}$ belongs to the **strong core** (Resp. **weak core**, **exclusion core**) at $R \in \mathcal{R}^N$ *if* no coalition $C \in 2^N \backslash \{\emptyset\}$ weakly (Resp. strongly, exclusion) blocks a through any allocation at R. Let $\mathcal{SC}(R)$, $\mathcal{WC}(R)$ and $\mathcal{EC}(R)$ be the strong, weak, and exclusion cores at R, respectively.[5] The first result summarizes the relationships between the three core concepts.

Theorem 1. *For each* $R \in \mathcal{R}^N, \mathcal{SC}(R) \subseteq \mathcal{EC}(R) = \mathcal{WC}(R)$.

Next, we consider the existence of the core. The following example demonstrates that the existence of the strong core is not guaranteed.

Example 3 (Strong core may be empty)**.** Suppose that $N = \{1, 2, 3, 4\}$, $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$ and $q = (1, 1, 1, 1)$. Suppose also that the priority structure is given as follows: $1 \succ 2$ and $3 \succ 4$. Let $R \in \mathcal{R}^N$ be such that for each $i \in N$, $o_1\ R_i\ o_2\ R_i\ o_3\ R_i\ o_4\ R_i\ o_0$. In Table 2, each allocation in the left column is weakly blocked by the corresponding coalition in the right column. Thus, $\mathcal{SC}(R) = \emptyset$. ◇

In contrast to the strong core, the weak core (= exclusion core) is always non-empty. Moreover, they are achieved by adjusting a quite simple algorithm, the priority rule. First, we introduce the notation to define the priority rule. A binary relation $\trianglerighteq$ on N is a **completion** of $\succeq$ *if* i) $\trianglerighteq$ is complete, transitive, and anti-symmetric, and ii) for each $\{i, j\} \subseteq N$, if $i \succeq j$, then $i \trianglerighteq j$. Let $\Gamma(\succeq)$ be the set of completions of $\succeq$.

Given a completion $\trianglerighteq \in \Gamma(\succeq)$, the **priority rule associated with** $\trianglerighteq$, $\varphi^{\trianglerighteq}$, is the function from $\mathcal{R}^N$ to $\mathcal{A}$ defined as follows: Suppose that $i_1 \trianglerighteq i_2 \trianglerighteq \ldots \trianglerighteq i_n$. Let

[5] Balbuzanov and Kotowski (2019) call $\mathcal{EC}$ the direct exclusion core in the context of simple economy. In addition to $\mathcal{EC}$, they introduce another exclusion core concept based on an extension of the conditional endowment system. However, under Assumption 1, the two exclusion core concepts coincide. See Appendix A for a more detailed description.

Table 2: Strong core may be empty.

Allocation	Blocking coalition	Allocation	Blocking coalition	Allocation	Blocking coalition
$o_1o_2o_3o_4$	$\{1,3\}$	$o_1o_3o_2o_4$	$\{1,4\}$	$o_1o_4o_2o_3$	$\{2,3\}$
$o_1o_2o_4o_3$	$\{3\}$	$o_1o_3o_4o_2$	$\{3\}$	$o_1o_4o_3o_2$	$\{3\}$
$o_2o_1o_3o_4$	$\{1\}$	$o_3o_1o_2o_4$	$\{1\}$	$o_4o_1o_2o_3$	$\{1\}$
$o_2o_1o_4o_3$	$\{1\}$	$o_3o_1o_4o_2$	$\{1\}$	$o_4o_1o_3o_2$	$\{1\}$

Note: The sequence $o_io_jo_ko_\ell$ indicates the allocation $a \in \mathcal{A}$ such that $a(1) = o_i$, $a(2) = o_j$, $a(3) = o_k$ and $a(4) = o_\ell$. We omit allocations in which some agents receive the null object because they are weakly blocked by a single agent. The table exhausts the allocations in which $\{1, 2\}$ consumes $\{o_1, o_2\}$, $\{o_1, o_3\}$ or $\{o_1, o_4\}$. Since $\{1, 2\}$ and $\{3, 4\}$ are symmetric, the table exhausts all allocations.

$R \in \mathcal{R}^N$. In step 1, agent i_1 selects the most favorite object from $\mathcal{O} \cup \{o_0\}$. If it is a real object, the quota vector is updated by subtracting 1 from the quota of the object. Letting $\varphi^{\trianglerighteq}_{i_1}(R)$ be the selected object, we proceed to the next step. In step 2, agent i_2 selects the most favorite object from the remaining real objects under the updated quota and the null object. If it is a real object, the quota vector is updated by subtracting 1 from the latest quota of the object. Letting $\varphi^{\trianglerighteq}_{i_2}(R)$ be the selected object, we proceed to the next step. By repeating this process n times, we obtain $\varphi^{\trianglerighteq}(R) := \left(\varphi^{\trianglerighteq}_{i_k}(R)\right)_{k=1}^{n} \in \mathcal{A}$.

The range of the priority rule is denoted by $\Phi(R)$, that is, for each $R \in \mathcal{R}^N$, $\Phi(R) := \{\varphi^{\trianglerighteq}(R) | \trianglerighteq \in \Gamma(\succeq)\}$. Based on this, we provide a complete characterization of the weak core as follows.

Theorem 2. *For each $R \in \mathcal{R}^N, \mathcal{WC}(R) = \Phi(R)$.*

4 Result II: Priority-completion on the basis of interpersonal and interinstitutional fairness

In this section, we focus on the refinement of the weak core in terms of fairness. Fairness is as important as efficiency in evaluating an allocation or a rule in a wide range of allocation problems.[6] In the sequel, we return to the original problem that the administration office of a university assigns indivisible objects to students prioritized only in their own department. To capture this problem, we adopt the following

[6]As is shown in Lemma 3 in Appendix B, each weak core allocation is Pareto efficient.

assumption, in addition to Assumption 1. Note that the symbol $\succeq$ in the statement represents the priority order that all objects have in common.

Assumption 2. *Let $\mathcal{I}$ be a partition of N such that*

- *$\forall I \in \mathcal{I}$, the priority $\succeq$ is complete on I, and*
- *$\forall I, I' \in \mathcal{I}$ with $I \neq I'$, $\forall i \in I, \forall i' \in I'$, $i \not\succeq i'$.*

We call each $I \in \mathcal{I}$ an institution. To avoid trivial cases, we assume $|\mathcal{I}| \geq 2$.

4.1 Refinement of the core on the basis of ex-post fairness notions

In this subsection, we define the interpersonal and interinstitutional fairness notions in the ex-post sense. The following is a standard notion of fairness in the matching literature. An allocation $a \in \mathcal{A}$ is interpersonally-fair at $R \in \mathcal{R}^N$ *if* no agent has justified envy, that is, there exists no pair of agents $(i, j) \in N \times N$ such that i) $a(j)\ P_i\ a(i)$, and ii) $i \succ j$. Note that this property is achieved by the priority rule under any completion, that is, every weak core allocation is interpersonally-fair (Theorem 2). Thus, this fairness notion alone is not helpful for refining the weak core.

To define an interinstitutional fairness notion, we need a tool to evaluate an allocation from an institutional perspective. The following notation $\rho_k^I(a, R)$ denotes the ratio of agents in I that receive the k-th best object at a. Formally, the rank distribution of $a \in \mathcal{A}$ for $I \in \mathcal{I}$ at $R \in \mathcal{R}^N$ is $\rho^I(a, R) \in \mathbb{R}_+^{m+1}$ defined as follows. For each $k \in \{1, 2, \ldots, m+1\}$, $\rho_k^I(a, R) := \frac{|\{i \in I | a(i) \text{ is the } k\text{-th favorite object at } R_i.\}|}{|I|}$. An allocation $a \in \mathcal{A}$ is interinstitutionally-fair in the ex-post sense at $R \in \mathcal{R}^N$ *if* there exists no pair of institutions $(I, I') \in \mathcal{I} \times \mathcal{I}$ such that $\rho^I(a, R)$ stochastically dominates $\rho^{I'}(a, R)$, i.e., $\sum_{\ell=1}^k \rho_k^I(a, R) \geq \sum_{\ell=1}^k \rho_k^{I'}(a, R)$ for all $k \in \{1, 2, \ldots, m+1\}$, and the inequality is strict for some $k \in \{1, 2, \ldots, m+1\}$.

Although the interinstitutional fairness notion above seems reasonable, it is too strong to practice, as shown in the following example.

Example 4 (A problem in which no weak core allocation is interinstitutionally-fair in the ex-post sense)**.** Suppose that $|N| \geq 2$, $|\mathcal{O}| \geq 2$ and $q = (1, 1, \ldots, 1)$. Assume, without loss of generality, that for $I, I' \in \mathcal{I}$ with $I \neq I'$, $1 = \max_{\succeq} I$ and $2 = \max_{\succeq} I'$.

Table 3: Example of completions

(i) $\trianglerighteq_1$

I	Priority	I'	Priority
i_1	1	i'_1	2
		i'_2	3
i_2	6	i'_3	4
		i'_4	5

(ii) $\trianglerighteq_2$

I	Priority	I'	Priority
i_1	2	i'_1	1
		i'_2	3
i_2	5	i'_3	4
		i'_4	6

(iii) $\trianglerighteq_3$

I	Priority	I'	Priority
i_1	3	i'_1	1
		i'_2	2
i_2	4	i'_3	5
		i'_4	6

Let $R \in \mathcal{R}^N$ be such that i) $o_1 \ R_i \ o_2 \ R_i \ o_0 \ R_i \ \ldots$ for $i \in \{1, 2\}$, and ii) $o_0 \ R_i \ \ldots$ for $i \geq 3$. Then, for any completion $\trianglerighteq \in \Gamma(\succeq)$, under the allocation $\varphi^{\trianglerighteq}(R)$, all members in one of I and I' receive the best object, whereas the other institution contains one member who receives the second-best object. Thus, $\varphi^{\trianglerighteq}(R)$ is not interinstitutionally-fair in the ex-post sense at R. Because $\trianglerighteq$ is arbitrary, all allocations in $\Phi(R)\,(= \mathcal{WC}(R))$ are not interinstitutionally-fair in the ex-post sense at R. $\diamond$

Hereafter, we do not pursue the selection of fair allocations in the ex-post sense. Instead, we turn to the problem of selecting fair priority-completions to establish an allocation system that attains interpersonal and interinstitutional fairness in the ex-ante sense.

4.2 Point allocation approach to the priority-completion

To clarify the difficulty in selecting a fair completion, let us consider the following example. Suppose that there are two institutions $I = \{i_1, i_2\}$ and $I' = \{i'_1, \ldots, i'_4\}$ such that $i_1 \succ i_2$ and $i'_1 \succ i'_2 \succ i'_3 \succ i'_4$. Table 3 shows three examples of completion that may be regarded as "interinstitutionally-fair" in the following two senses: (1) the balance of the size of tiers, and (2) the distribution of favorable treatment between tiers.

Note that the three completions share identical upper and lower tiers, both consisting of one member of I and two members of I'. Completion $\trianglerighteq_1$ is favorable to I in the upper tier and I' in the lower tier. In completion $\trianglerighteq_3$, the favorable treatment in the upper and lower tiers is reversed compared with $\trianglerighteq_1$. Completion $\trianglerighteq_2$ is favorable to either I or I' in either tier. Thus, using one of these completions, we may attain an interinstitutionally fair state in terms of (1) and (2). Difficulties arise when we generalize the criteria (1) and (2) to the cases with many institutions containing various numbers of members (imagine $\mathcal{I} = \{I_3, I_5, I_7, I_{11}, I_{13}, I_{17}, I_{19}\}$ with $|I_k| = k$). Let us call the scenario of choosing a completion by generalizing (1) and (2) direct selection (DS).

In this study, we pursue another scenario, which we call indirect selection (IS), to choose a completion. We generate a completion through an alloocation of quantified opportunity: point allocation. A **point allocation** is a function $\alpha : N \to [0,1]$ such that

$$\forall i, j \in N, i \succ j \Rightarrow \alpha(i) > \alpha(j).$$

Let $\mathcal{A}^P$ be the set of point allocations. IS is a procedure for selecting a point allocation $\alpha \in \mathcal{A}^P$ to choose a completion $\trianglerighteq \in \Gamma(\succeq)$ such that

$$\forall i, j \in N, \alpha(i) > \alpha(j) \Rightarrow i \triangleright j.$$[7]

We say that such completion $\trianglerighteq$ is consistent with the point allocation α. Then, using the selected completion $\trianglerighteq$, we utilize the priority rule $\varphi^{\trianglerighteq}$ to determine an object allocation.[8] We point out that IS has the following two advantages over DS.

- First, IS is simpler. While DS accompanies with the difficulties pertaining to discrete resource allocation problems such as the generalization of (1) and (2), it is easier to define fairness notions under IS.

- Second, IS is more flexible. As discussed below, the system equipped with the quantified opportunity makes it easier for institutions to express their preferences on completions. For example, if institution I' in Table 3 would like to be more supportive to the students in the upper tier, it may prefer $\trianglerighteq_3$ to the

[7] Ties are broken arbitrarily.

[8] The idea of point allocation itself is not new to generate a priority. The most important example is the priority point system for the allocation of medical resources during the public health emergency (Vincent et al., 1996; White et al., 2009; Piscitello et al., 2020). In Japan, point allocation is used to generate a priority order for patients waiting for kidney transplantation through donation by a deceased donor. The recipient-selection rule of the Japan Society for Transplantation is in `https://www.jotnw.or.jp/files/page/medical/manual/doc/rec-kidney.pdf`, accessed on January 2, 2023. Another example is the allocation of nursery school seats in Osaka, Japan. The priority point system of Osaka city is described in `https://www.city.osaka.lg.jp/kodomo/cmsfiles/contents/0000574/574018/14riyoutyouseikijyunn.pdf`, accessed on January 2, 2023. The translation of these documents is available upon request.

others. This policy could be realized by a point allocation with extra points for i'_1 and i'_2.

Based on the idea of point allocation, it is possible to formalize an interinstitutional fairness notion in a natural way. The following axiom requires that the assignment of the quantified opportunity to institutions be proportional to the number of members each institution has.

Definition 4. A point allocation $\alpha \in \mathcal{A}^P$ is **interinstitutionally-fair (IIF)** *if*

$$\forall I, I' \in \mathcal{I}, \frac{1}{|I|}\sum_{i \in I} \alpha(i) = \frac{1}{|I'|}\sum_{i \in I'} \alpha(i).$$

Next, we establish the notion of interpersonal fairness. To this end, we introduce a notation that expresses the relative position of an agent in an affiliated institution. For each $I \in \mathcal{I}$ and each $i \in I$, let r_i^I be the reverse rank of agent i in the institution I, that is, $r_i^I := |I| - |\{j \in I | j \succ i\}|$. Note that if i is the k-th highest-priority agent in I, r_i^I is $|I| - (k-1)$.

As long as each institution has an equal right to the objects in the market, each agent may regard a point allocation as fair if the points they obtain are similar to those of the agents with similar positions in other institutions. For example, supposing that agent i is the 4-th priority agent in an institution with 40 members, she may accept a point assignment similar to that of the 10-th priority agent in an institution with 100 members, but may complain to the one similar to the points for the 10-th priority agent in an institution with 10 members. This is because the allocation ignores the fact that agent i's relative position (top 10 %) is far better than that of the 10-th agent among 10 members. The following axiom reflects this criterion straightforwardly. A point allocation $\alpha \in \mathcal{A}^P$ is **strongly interpersonally-fair (SIPF)** *if* there exists no pair of institutions $(I, I') \in \mathcal{I} \times \mathcal{I}$ such that for some $(i, i') \in I \times I'$, (1) $\alpha(i') > \alpha(i)$ and (2) $\frac{r_i^I - 1}{|I|} > \frac{r_{i'}^{I'} - 1}{|I'|}$.[9] Although this axiom, which actually singles out a completion except for ties, seems reasonable, it could be too restrictive in other aspects. For example, SIPF point allocations always assign the greatest points to the top agent in the institution with the largest population. That is, there is no chance for the top agents in small institutions to be at the top of the

[9]The condition (2) is equivalent with $\frac{(|I|+1) - r_i^I}{|I|} < \frac{(|I'|+1) - r_{i'}^{I'}}{|I'|}$.

completion under any SIPF point allocation. The following axiom overcomes the extreme inflexibility of SIPF, respecting the relative position of agents to a certain extent.

Definition 5. A point allocation $\alpha \in \mathcal{A}^P$ is **weakly interpersonally-fair (WIPF)** *if* there exists no pair of institutions $(I, I') \in \mathcal{I} \times \mathcal{I}$ such that for some $(i, i') \in I \times I'$,

(1) $\alpha(i') \geq \alpha(i)$,

(2) $\frac{r_i^I - 1}{|I|} \geq \frac{r_{i'}^{I'}}{|I'|}$, and

(3) at least one of inequalities (1) and (2) is strict.

For an interpretation of the inequality (2) of WIPF, we need to examine the concept of relative position more closely. Let us consider the following example. Let i be a member of an institution I with $|I| = 100$, and suppose that i is given the 10-th highest priority in I. Then, the "relative position" of i might be characterized by two information: "i is a top 10 % agent" and "i is not a top 9 % agent." Note that the endpoints of the interval $\left[\frac{r_i^I - 1}{|I|}, \frac{r_i^I}{|I|}\right]$ clearly represent the information.[10] Thus, the interval could be understood as a set of potentially possible evaluations of agent i's relative positions. Based on this understanding, inequality (2) indicates that agent i is an agent whose worst-evaluated relative position is not lower than agent i''s best-evaluated relative position.

4.3 Midpoint rule

To select an IIF and WIPF point allocation, it is noteworthy that the following is a sufficient condition for a point allocation $\alpha \in \mathcal{A}^P$ to be WIPF.

$$\forall I \in \mathcal{I}, \forall i \in I, \alpha(i) \in \left[\frac{r_i^I - 1}{|I|}, \frac{r_i^I}{|I|}\right].^{11}$$

Specifically, the midpoint of the interval for each agent results in an IIF and WIPF allocation. To see this, let $\alpha^M \in \mathcal{A}^P$ be the point allocation defined as follows. For each $I \in \mathcal{I}$ and each $i \in I$, let

$$\alpha^M(i) := \frac{2r_i^I - 1}{2|I|} \left(= \frac{1}{2} \cdot \frac{r_i^I - 1}{|I|} + \frac{1}{2} \cdot \frac{r_i^I}{|I|}\right).$$

[10] This is the r_i^I-th interval from the left when we divide $[0, 1]$ into $|I|$ equal parts.

[11] Later, it is shown that this is not a necessary condition.

Then, for each $I \in \mathcal{I}$, $\sum_{i \in I} \alpha^M(i) = \frac{|I|}{2}$. Thus, α^M satisfies IIF in addition to WIPF.

A natural question that arises is, whether there are other point allocations that satisfy IIF and WIPF? The answer is yes. Further, infinitely many IIF and WIPF point allocations exist. For example, fixing an institution $I \in \mathcal{I}$ with even members, let ϵ be such that $\frac{1}{2|I|} > \epsilon > 0$. Let $\beta^+, \beta^- \in \mathcal{A}^P$ be defined as

$$\beta^+(i) = \begin{cases} \alpha^M(i) + \epsilon & \text{if } i \in I \text{ and } r_i^I > \frac{|I|}{2}, \\ \alpha^M(i) - \epsilon & \text{if } i \in I \text{ and } r_i^I \leq \frac{|I|}{2}, \\ \alpha^M(i) & \text{o.w.} \end{cases} \quad \beta^-(i) = \begin{cases} \alpha^M(i) - \epsilon & \text{if } i \in I \text{ and } r_i^I > \frac{|I|}{2}, \\ \alpha^M(i) + \epsilon & \text{if } i \in I \text{ and } r_i^I \leq \frac{|I|}{2}, \\ \alpha^M(i) & \text{o.w.} \end{cases}$$

That is, β^+ (Resp. β^-) is a point allocation, under which I is supportive to the upper-half (Resp. lower-half) members relative to the canonical allocation $\alpha^M \in \mathcal{A}^P$. Obviously, β^+ and β^- satisfy the sufficient condition for WIPF above. Moreover, $\sum_{i \in I} \beta^+(i) = \sum_{i \in I} \beta^-(i) = \sum_{i \in I} \alpha^M(i) = \frac{|I|}{2}$. Thus, β^+ and β^- are IIF and WIPF. Thus, there are infinitely many IIF and WIPF point allocations including β^+ and β^-.

Now, we would like to reconsider our direction of market design, especially the degree of refinement of point allocations in terms of axioms. We have proposed two desirable properties to single out some point allocations. However, we should be careful about further selection, especially regarding the selection that might be appropriate to depend on the local information or the policy of institutions, for example, the selection from β^+ and β^-. In the following, we propose a system in which the clearing house selects a set of point allocations (candidates for the final selection). Among the suggested point allocations, the system selects one that reflects on the evaluation policy submitted by the institutions. Formally,

The midpoint rule (abbreviated as MR).

Step 1. Each institution $I \in \mathcal{I}$ submits a weight vector $w^I \in [-1, 1]^I$ with $\sum_{i \in I} w^I(i) = 0$ to the clearing house.[12] Set $w := (w^I)_{I \in \mathcal{I}}$.

Step 2. The clearing house picks the point allocation $\alpha^w \in \mathcal{A}^P$ defined as follows: for each $I \in \mathcal{I}$ and each $i \in I$,

$$\alpha^w(i) := \alpha^M(i) + \frac{w_i^I}{2|I|}.$$

[12] We assume that for each $i, j \in I$ with $r_i^I = r_j^I + 1$, $w^I(j) = 1$ implies $w^I(i) \neq -1$.

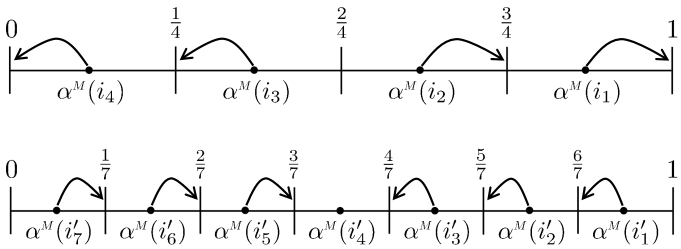

Figure 2: The point allocation in Example 5.

Step 3. The clearing house picks a completion $\trianglerighteq \in \Gamma(\succeq)$ consistent with α^w.[13]

In the MR, the set of suggested point allocations by the clearing house is $\{\alpha^w \in \mathcal{A}^P \mid w \in [-1,1]^N \text{ and } \forall I \in \mathcal{I}, \sum_{i\in I} w_i = 0\}$. Note that $\alpha^w(i)$ belongs to the interval $\left[\frac{r_i^I-1}{|I|}, \frac{r_i^I}{|I|}\right]$ because the second term $\frac{w_i^I}{2|I|}$ of $\alpha^w(i)$ represents a small gap, at most half the length of the interval, from the midpoint of the interval. The weight vector w can be understood as the profile of the institutional evaluation policy.

Example 5 (The midpoint rule)**.** Suppose that there are two institutions $I = \{i_1, \ldots, i_4\}$ and $I' = \{i'_1, \ldots, i'_7\}$. Priority is given to agents with smaller indices at each institution, that is, $i_1 \succ i_2 \succ \ldots \succ i_4$ and $i'_1 \succ i'_2 \succ \ldots \succ i'_7$. Assume that I is supportive of higher-priority agents i_1 and i_2 whereas I' is supportive of lower-priority agents i'_5, i'_6 and i'_7. To realize the scoring policy, assume that the institutions submit the following weight vectors: $w^I = (1,1,-1,-1)$ and $w^{I'} = (-1,-1,-1,0,1,1,1)$. Then, the midpoint rule selects the point allocation $\alpha^w = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i'_1 & i'_2 & i'_3 & i'_4 & i'_5 & i'_6 & i'_7 \\ 1 & \frac{3}{4} & \frac{1}{4} & 0 & \frac{6}{7} & \frac{5}{7} & \frac{4}{7} & \frac{1}{2} & \frac{3}{7} & \frac{2}{7} & \frac{1}{7} \end{pmatrix}$, as described in Figure 2. Thus, the completion $\trianglerighteq \in \Gamma(\succeq)$ consistent with α^w is $i_1 \triangleright i'_1 \triangleright i_2 \triangleright i'_2 \triangleright i'_3 \triangleright i'_4 \triangleright i'_5 \triangleright i'_6 \triangleright i_3 \triangleright i'_7 \triangleright i_4$. ◇

We characterize a class of sets of point allocations, including the range of the MR, with the following axioms. A set of point allocations $S \subseteq \mathcal{A}^P$ satisfies

Interinstitutional fairness (IIF): $\forall \alpha \in S, \alpha$ is IIF.

Weak interpersonal fairness (WIPF): $\forall \alpha \in S, \alpha$ is WIPF.

Independence (IND): $\forall \alpha, \beta \in S, \forall I \in \mathcal{I}, \exists \gamma \in S$ s.t. $\gamma|_I = \alpha|_I$ and $\gamma|_{N\setminus I} = \beta|_{N\setminus I}$.

Continuity (CON): $\forall I \in \mathcal{I}, \forall i, j \in I$ with $r_i^I = r_j^I + 1, \sup_{\alpha\in S} \alpha(j) = \inf_{\alpha \in S} \alpha(i)$.

Existence of the standard evaluation (ESE): $\forall I \in \mathcal{I}, \exists \alpha \in S, \forall i \in I, \alpha(i) = \frac{\sup_{\beta\in S}\beta(i) + \inf_{\beta\in S}\beta(i)}{2}$.

[13]Ties in α^w are broken arbitrarily.

Closedness (CLO): **(i).** $\forall i \in N, \exists \overline{\alpha} \in S$ s.t. $\overline{\alpha}(i) = \sup_{\alpha \in S} \alpha(i)$. In particular, for each $I \in \mathcal{I}$ with $i = \max_{\succeq} I$, there exists $\overline{\alpha} \in S$ such that $\overline{\alpha}(i) = 1$. **(ii).** $\forall i \in N, \exists \underline{\alpha} \in S$ s.t. $\underline{\alpha}(i) = \inf_{\alpha \in S} \alpha(i)$. In particular, for each $I \in \mathcal{I}$ with $i = \min_{\succeq} I$, there exists $\underline{\alpha} \in S$ such that $\underline{\alpha}(i) = 0$.

IIF and WIPF are the main axioms in this section. Other auxiliary axioms are interpreted as follows. IND requires that the point allocation within an institution be unaffected by the allocations in other institutions. CON requires that the supremum of the potentially possible assignment for a lower-priority agent should not exceed that the infimum of that for a higher-priority agent, that is, $\sup_{\alpha \in S} \alpha(j) \leq \inf_{\alpha \in S} \alpha(i)$. Moreover, it requires that there be no gap between them. ESE embodies a kind of convexity of the potentially possible evaluations so that the convex combination of the supremum and the infimum is available as an evaluation. CLO is a technical condition that requires the existence of the greatest and smallest of the potentially possible evaluations. These six axioms almost characterize the range of the midpoint rule. To be precise, let us introduce the following class of procedures, which consists of a variant of the MR.

The generalized midpoint rule (abbreviated as GMR).
Let $f : [0, 1] \to [0, 1]$ be such that

- GMR-1. f is a continuous strictly increasing function with $f(0) = 0$ and $f(1) = 1$,
- GMR-2. $\forall I, I' \in \mathcal{I}, \frac{1}{|I|}\left[\sum_{i \in I} f\left(\frac{r_i^I}{|I|}\right) - \frac{1}{2}\right] = \frac{1}{|I'|}\left[\sum_{i \in I'} f\left(\frac{r_i^{I'}}{|I'|}\right) - \frac{1}{2}\right]$, and
- GMR-3. $\forall I \in \mathcal{I}, \forall i \in I, f\left(\frac{r_i^I}{|I|}\right) - f\left(\frac{r_i^I - 1}{|I|}\right) \leq \frac{1}{2}$.

For each $I \in \mathcal{I}$ and each $i \in I$, let $T_i^f := f\left(\frac{r_i^I}{|I|}\right) - f\left(\frac{r_i^I - 1}{|I|}\right)$.

- Step 1. Each institution $I \in \mathcal{I}$ submits a weight vector $w^I \in [-1, 1]^I$ with $\sum_{i \in I} w^I(i) T_i^f = 0$ to the clearing house.[14] Set $w := (w^I)_{I \in \mathcal{I}}$.
- Step 2. The clearing house picks the point allocation $\alpha^w \in \mathcal{A}^P$ defined as follows: for each $I \in \mathcal{I}$ and each $i \in I$,

$$\alpha^w(i) := \frac{1}{2}\left[f\left(\frac{r_i^I - 1}{|I|}\right) + f\left(\frac{r_i^I}{|I|}\right)\right] + \frac{w^I(i) T_i^f}{2}.$$

[14] We assume that for each $i, j \in I$ with $r_i^I = r_j^I + 1$, $w^I(j) = 1$ implies $w^I(i) \neq -1$.

Step 3. The clearing house picks a completion $\trianglerighteq \in \Gamma(\succeq)$ consistent with α^w.

Let $\mathcal{A}_f^P$ be the range of the generalized midpoint rule associated with f, i.e., $\mathcal{A}_f^P := \{\alpha^w \in \mathcal{A}^P \mid w \in [-1,1]^N \text{ and } \sum_{i \in I} w(i) T_i^f = 0 \text{ for all } I \in \mathcal{I}\}$.

The class of GMRs consists of a variant of the MR parameterized by transformations f satisfying GMR-1, GMR-2 and GMR-3. Under a GMR with a transformation f, the interval that represents the potentially possible relative positions of an agent $i \in I$ is distorted into $\left[f\left(\frac{r_i^I - 1}{|I|}\right), f\left(\frac{r_i^I}{|I|}\right)\right]$. Obviously, the GMR with the identity mapping is the MR. GMR-2 requires that the interinstitutional proportion of the sum of the midpoints be preserved under the transformation f. GMR-3 requires that each distorted interval not exceed the half of the entire interval $[0,1]$.[15] The main theorem of this section states that a set of point allocations S has (one of) the widest variety of options among the sets satisfying the six axioms if and only if S is the range of a GMR.

Theorem 3. *Let $S \subseteq \mathcal{A}^P$. The set of point allocations S is a maximal element of $\mathcal{S} := \{S' \subseteq \mathcal{A}^P \mid S'$ satisfies IIF, WIPF, IND, CON, ESE and CLO$\}$ with respect to $\supseteq$ if and only if there exists $f : [0,1] \to [0,1]$ satisfying GMR-1, GMR-2 and GMR-3 such that $S = \mathcal{A}_f^P$.*

Unless there is an unambiguous advantage of adopting a particular distortion f, we advocate the simplest rule in the class of GMRs—the MR—for the selection of a completion.

As a final remark, we point out that $\supseteq$-maximality in Theorem 3 is indispensable. The following example shows that there exists $S \in \mathcal{S}$ such that S does not coincide with the range of any generalized midpoint rule.

Example 6. Let $S := \{\alpha^w \in \mathcal{A}_{id}^P \mid w \in \{-1,0,1\}^N \text{ and } \sum_{i \in I} w(i) = 0 \text{ for all } I \in \mathcal{I}\}$, where $id : [0,1] \to [0,1]$ denotes the identity mapping. Obviously, $S \in \mathcal{S}$. However, $S \subsetneq \mathcal{A}_{id}^P$. By Lemma 5 in Appendix B, S does not coincide with the range of any generalized midpoint rule.

[15] If there is such $i \in I$, it is impossible to give her the point assignment that corresponds to the right endpoint of the new interval, keeping the points for I constant. Note that at least an extra $\frac{1}{4}$ point is needed to give her the right endpoint of the interval under the GMR. However, the maximum points exploited from $I \setminus \{i\}$ cannot exceed $\frac{1}{4}$ because the total length of their new intervals is smaller than $\frac{1}{2}$.

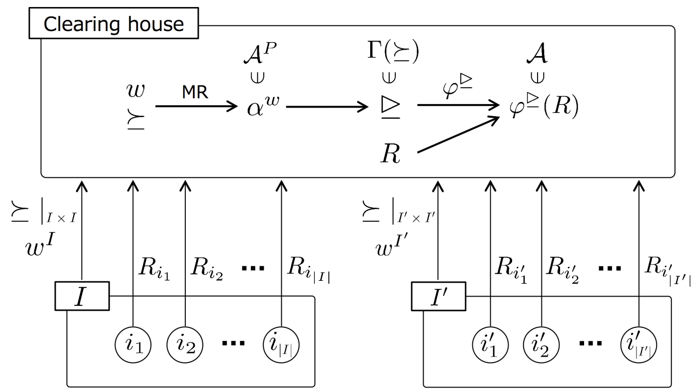

Figure 3: Implementation of priority rule under the midpoint rule .

Note: Each institution submits the priority order on the own members (i.e., $\succeq|_{I\times I}$ and $\succeq|_{I'\times I'}$) and the weight vector (i.e., w^I and $w^{I'}$). At the same time, each agent $i \in N$ submits own preference R_i to the clearing house. Then, the clearing house processes the collected data: Based on the incomplete priority $\succeq$ and the list of weight vectors $w = (w^I)_{I\in\mathcal{I}}$, the midpoint rule first selects the point allocation α^w, which in turn is transformed to a complete priority order $\trianglerighteq$ consistent with α^w. Finally, the priority rule associated with $\trianglerighteq$ selects the weak core allocation $\varphi^{\trianglerighteq}(R)$.

5 Discussion

Strategic issues

In our environment, for any completion $\trianglerighteq \in \Gamma(\succeq)$, the priority rule associated with $\trianglerighteq$, $\varphi^{\trianglerighteq}$, is **strategy-proof**. That is, for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i^{\trianglerighteq}(R)\ R_i\ \varphi_i^{\trianglerighteq}(R'_i; R_{-i})$, where $(R'_i; R_{-i})$ is the preference profile obtained from R by replacing R_i with R'_i. Thus, no individual can manipulate the allocation system through the strategic reporting.

To prevent strategic manipulation of institutions, we should pay attention to the subtlety of the system. Note that the admissible action of institutions is to submit a priority order on their own members and a weight vector (Figure 3). Under these circumstances, it is important for agents to submit their preferences directly to the clearing house. Alternatively, if agents submit their preferences through their own institutions, institutions may have an incentive to misreport the priority order and the weight vector. To see this, suppose that a student i belonging to department I wishes to get a student-exchange program $o \in \mathcal{O}$, which is not popular among students for some reason (for example, o is in a country with a disputed region, or a region hit hard by infectious diseases, etc). In this case, because it is anticipated that the exchange program o is less competitive, the department I may have the incentive to give a lower priority to i to enhance the welfare of other students in I without harming i.[16]

Although this is not the case for our fixed-population model, strategic candidacy should also be kept under surveillance. Under the midpoint rule, a department can increase the total points assigned to it by hiring students who actually do not wish to study abroad. To prevent this manipulation, the administration office should keep annual records of the withdrawal rate of each department to punish those with an extraordinarily high rate.

Unequal institutions

In this study, we investigate the assignment problem of heterogeneous objects among institutions with equal rights to the objects[17]. However, the Covid-19 pandemic has proved the importance of an assignment rule for homogeneous objects (e.g., ventilators, ICU beds, and vaccines, etc.) to unequal groups of agents. Let I_1 and I_2 be the sets of health care workers and other patients, respectively. Under the pandemic, in rationing scarce medical resources, the unequal treatment between I_1 and I_2 could be justified for some ethical values, including the principles of saving the most lives and promoting instrumental value.[18]

Let o_1 be the medical resource under consideration. Suppose that x units (out of q_{o_1}) are reserved for agents in I_1. In this situation, Sönmez et al. (2021) introduce two rationing methods below:[19]

- First, $q_{o_1} - x$ units of o_1 are fairly assigned to I_1 and I_2 (The population of two equal groups is in the ratio of $|I_1| : |I_2|$). Then, x units of o_1 are assigned to the unassigned agents in I_1.

- First, x units of o_1 are assigned to agents in I_1. Then, $q_{o_1} - x$ units of o_1 are fairly assigned to the unassigned agents in I_1 and I_2 (The population of two equal groups is in the ratio of $|I_1| - x : |I_2|$).

[16]This is a manipulation behavior similar to the one observed in multiple assignment problems (Pápai, 2001; Klaus and Miyagawa, 2001; Hatfield, 2009; Coreno and Balbuzanov, 2022).

[17]The individual assignments to agents in an institution collectively forms an assignment to the institution.

[18]See Recommendation 2 in Emanuel et al. (2020).

[19]The description of the methods here is a practically adjusted version of the cutoff equilibrium proposed in Pathak et al. (2021), which endogenously determines i) the reserved units for each category and ii) who receives a unit from respective categories.

Note that both methods include an instance that treats the two groups equally in rationing unreserved units. The characterization of the midpoint rule justifies a class of completions in that instance.

6 Conclusion

This study examines the problem of assigning indivisible objects to agents prioritized within their affiliated institutions. To this end, we formalized the problem as a priority-based indivisible goods allocation problem with incomplete priorities. Extending the concept of conditional property rights (Balbuzanov and Kotowski, 2019) to a many-to-one setting, we investigated the relationships between several core concepts. Unlike the case with an acyclic priority structure, under the assumption of common priority, the weak core coincides with the exclusion core (Theorem 1) and is characterized by the range of the priority rule with priority-completion (Theorem 2).

To implement the priority rule, we approached the problem of selecting a priority-completion within the framework of point allocation. We advocate a rule called the midpoint rule, which is sufficiently flexible for institutions to express their policy for point allocation. In the main theorem (Theorem 3), we give a characterization of the ranges of generalized midpoint rules based mainly on two fairness notions: interpersonal and interinstitutional fairness.

Appendix A: Indirect exclusion core under the common priority

In addition to the conditional endowment system in Section 3, Balbuzanov and Kotowski (2019) introduce an extension of the concept. Given an allocation $a \in \mathcal{A}$, **the extended conditional endowment system** $\Omega_a : 2^N \to 2^{\bar{\mathcal{O}}}$ is defined as follows: for each $C \subseteq N$,

$$\Omega_a(C) := \omega_a \left(\cup_{p=0}^{\infty} C_p \right),$$

where $C_0 = C$ and $C_p = C_{p-1} \cup (\bar{a}^{-1} \circ \omega_a)(C_{p-1})$.

The exclusion core in the main text, based on the conditional endowment system ω_a, is called the **direct exclusion core** in Balbuzanov and Kotowski (2019).[20] On

[20]Precisely speaking, they use the terminology only for the setting with initial endowment, that

the other hand, the exclusion core defined by replacing ω_a with Ω_a is called the **indirect exclusion core**.[21] Under the assumption of common priority, these two core concepts coincide as shown in the following proposition.

Proposition 1. *Under Assumption 1, for each $a \in \mathcal{A}$ and each $C \in 2^N$, $\omega_a(C) = \Omega_a(C)$.*

Proof. First, we show the following two claims.

Claim 1. $\omega_a(C) = w_a(C_1)$

Proof of Claim 1: By definition, $C \subseteq C_1$. As ω_a is monotonic, $\omega_a(C) \subseteq \omega_a(C_1)$. In the following, we show $\omega_a(C) \supseteq \omega_a(C_1)$ by contradiction. Suppose to the contrary that there exists $o_{k\ell} \in \omega_a(C_1)$ such that $o_{k\ell} \notin \omega_a(C)$. As $o_{k\ell} \notin \bar{\mathcal{O}} \backslash \bar{a}(N)$, $\bar{a}(i) = o_{k\ell}$ for some $i \in N$. Let $j \in C_1$ be $\succeq_{o_k}$-maximal in C_1 such that $j \succeq_{o_k} i$.

<u>Case 1. $j \notin C$</u>: As $j \in C_1$ and $j \notin C$, $j \in (\bar{a}^{-1} \circ \omega_a)(C)$. Thus, there exists a real object $o_{k'\ell'}$ such that $\bar{a}(j) = o_{k'\ell'}$. As $o_{k'\ell'} \in \omega_a(C) \backslash \left(\bar{\mathcal{O}} \backslash \bar{a}(N)\right)$, there exists $j' \in C$ such that $j' \succ_{o_{k'}} j$. By Assumption 1, $\succeq_{o_k} = \succeq_{o_{k'}}$. Thus, $j' \succ_{o_k} j$. Since $j' \in C \subseteq C_1$, this contradicts the choice of j.

<u>Case 2. $j \in C$</u>: As $j \succeq_{o_k} i$, $o_{k\ell} \in \omega_a(\{j\})$. As $\{j\} \subseteq C$, the monotonicity of ω_a implies $o_{k\ell} \in \omega_a(C)$, a contradiction.

Since all cases result in a contradiction, we conclude that $\omega_a(C) \supseteq \omega_a(C_1)$. This completes the proof of Claim 1.

Claim 2. $\forall p \in \mathbb{N}, C_1 = C_{p+1}$

Proof of Claim 2: The induction argument with Claim 1 brings the conclusion. This completes the proof of Claim 2.

is, simple economy. Here, we borrow it for the case with priority structure.

[21] Along with the terminology in Balbuzanov and Kotowski (2019), we should call it the **strong exclusion core** or the **indirect exclusion core based on** ω_a because they define several indirect exclusion core concepts by replacing ω_a with other basic property rights formulations. Here, we choose a simpler terminology.

Now, we return to the proof of $\omega_a(C) = \Omega_a(C)$.

$$\begin{aligned}
\Omega_a(C) &= \omega_a\left(\cup_{p=0}^{\infty} C_p\right) && \\
&= \omega_a\left(C_0 \cup C_1\right) && (\because \text{ Claim 2}) \\
&= \omega_a(C_1) && (\because C_0 \subseteq C_1) \\
&= \omega_a(C) && (\because \text{ Claim 1})
\end{aligned}$$

□

By Proposition 1, the direct and indirect exclusion cores coincide under Assumption 1. For this reason, we introduced only the direct exclusion core in the main text, simply calling it the exclusion core.

Appendix B: Proofs

Lemma 1. *Suppose that an allocation $a \in \mathcal{A}$ is exclusion blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at $R \in \mathcal{R}^N$. Then, for $C' := \{i \in N \mid b(i)\ P_i\ a(i)\}$,*

$$\forall i \in N, \left[a(i)\ P_i\ b(i) \Rightarrow \bar{a}(i) \in \omega_a(C')\right].$$

Proof. Note that ω_a is monotonic. Thus, since $C \subseteq C'$, $\bar{a}(i) \in \omega_a(C) \subseteq \omega_a(C')$. □

Before we prove Lemma 2, we introduce a standard notion of efficiency. An allocation $a \in \mathcal{A}$ is Pareto efficient at $R \in \mathcal{R}^N$ *if* there exists no allocation $b \in \mathcal{A}$ such that i) $b(i)\ R_i\ a(i)$ for all $i \in N$, and ii) $b(i)\ P_i\ a(i)$ for some $i \in N$.

Lemma 2. *Suppose that $a \in \mathcal{A}$ is Pareto efficient at $R \in \mathcal{R}^N$. Then,*

$$\forall b \in \mathcal{A}, \forall i \in N, \left[b(i)\ P_i\ a(i) \Rightarrow \exists j \in N \text{ s.t. } \bar{a}(j) = \bar{b}(i)\right].$$

Proof. Suppose to the contrary that for some $b \in \mathcal{A}$ and $i \in N$, $b(i)\ P_i\ a(i)$ and

$$\nexists j \in N \text{ s.t. } \bar{a}(j) = \bar{b}(i).$$

Then, $\bar{b}(i) \in \left(\bar{\mathcal{O}} \backslash \bar{a}(N)\right) \cup \{o_0\}$. Let $c \in \mathcal{A}$ be such that $c(i) = b(i)$ and $c|_{N\backslash\{i\}} = a|_{N\backslash\{i\}}$. Then, a is not Pareto efficient at R, a contradiction. □

Lemma 3. *Let $R \in \mathcal{R}^N$ and $a \in \mathcal{WC}(R)$. Then, a is Pareto efficient at R.*

Proof. Suppose to the contrary that for some $b \in \mathcal{A}$, i) $b(i)\ R_i\ a(i)$ for all $i \in N$, and ii) $b(i)\ P_i\ a(i)$ for some $i \in N$. Let $C := \{i \in N \mid b(i)\ P_i\ a(i)\}$. Since $a \in \mathcal{WC}(R)$, $b(i) \neq o_0$ for all $i \in C$. Moreover, by the definition of C, $b(i) = a(i)$ for $i \notin C$. Thus, $b|_C$ could be constructed by reallocating

- the real objects disposed at a and
- the real objects assigned to members of C.

Note that $\big(\bar{\mathcal{O}}\backslash\bar{a}(N)\big) \cup \bar{a}(C) \subseteq \omega_a(C)$. Thus, $b|_C$ is achievable on $\omega_a(C)$. This contradicts $a \in \mathcal{WC}(R)$. □

Proof of Theorem 1. Let $R \in \mathcal{R}^N$. Obviously, $\mathcal{SC}(R) \subseteq \mathcal{WC}(R)$. Thus, we only prove $\mathcal{EC}(R) = \mathcal{WC}(R)$.

($\mathcal{EC}(R) \supseteq \mathcal{WC}(R)$) Suppose to the contrary that there exists $a \in \mathcal{WC}(R)$ such that a coalition $C \in 2^N\backslash\{\emptyset\}$ exclusion blocks through an allocation $b \in \mathcal{A}$ at R. By Lemma 1, we may assume, without loss of generality, that (i) $i \in C$ if and only if $b(i)\ P_i\ a(i)$, and (ii) $a(i)\ P_i\ b(i)$ implies $\bar{a}(i) \in \omega_a(C)$. We show that $b|_C$ is achievable on $\omega_a(C)$.

To this end, we first show that $\omega_a(C)$ contains at least one copy of $b(i)$ for each $i \in C$. The following inductive argument brings a chain of agents $\{i_1, \dots, i_p\}$ that finally hits an agent, whose assignment under $\bar{a}$ is a copy of $b(i)$ belongin to $\omega_a(C)$. Note that, by Lemma 2 and 3, there exists $i_1 \in N$ such that $\bar{a}(i_1) = \bar{b}(i)$. For each $p \geq 1$, the search for a chain stops if one of the following three conditions holds.

Stopping Rule 1 (SR 1). $i_p \in C$.

Stopping Rule 2 (SR 2). $i_p \notin C$ and $a(i_p)\ P_{i_p}\ b(i_p)$.

Stopping Rule 3 (SR 3). $i_p \notin C$, $a(i_p) = b(i_p)$ and $\bar{b}(i_p) \notin \bar{a}(N)$.

If one of SR 1, 2, or 3 holds, then $\{i_1, \dots, i_p\}$ is the desired chain with the length p. If none of the three rules hold, i.e., $i_p \notin C$, $a(i_p) = b(i_p)$ and $\bar{b}(i_p) \in \bar{a}(N)$, go to the next step. Note that in the latter case, since $\bar{b}(i_p) \in \bar{a}(N)$, there exists $i_{p+1} \in N$ such that $\bar{a}(i_{p+1}) = \bar{b}(i_p)$. The following two are important features of the chain (at step p in the construction).

$$[\forall q \in \{1, \dots, p-1\}, a(i_q) = b(i_q)] \text{ and } [\forall q \in \{1, \dots, p-1\}, \bar{a}(i_{q+1}) = \bar{b}(i_q)]$$

Note that for any $p \geq 2$, $i_p \notin \{i_1, \ldots, i_{p-1}\}$.[22] Since N is finite, one of the stopping rules holds for some i_p with $p \leq n$. Then, under SR 1, the copy of $b(i)$ is $\bar{a}(i_p) \in \omega_a(\{i_p\}) \subseteq \omega_a(C)$. Under SR 2, it is $\bar{a}(i_p) \in \omega_a(C)$ ($\because (ii)$). Under SR 3, it is $\bar{b}(i_p) \in \omega_a(C)$ $\left(\because \bar{\mathcal{O}} \backslash \bar{a}(N) \subseteq \omega_a(C)\right)$.

Next, we show that, for any $i, j \in C (i \neq j)$, the copies of $b(i)$ and $b(j)$ in $\omega_a(C)$ found by the above argument are different. Let $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_{p'}\}$ be the chains of agents to find copies of $b(i)$ and $b(j)$, respectively. We claim that $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_{p'}\} = \emptyset$.[23] Thus, $i_p \neq j_{p'}$. Note that the following three cases exhaust all possible combinations of i_p and $j_{p'}$. In any case, the copies of $b(i)$ and $b(j)$ found by the corresponding chains are different.

Case 1. Both i_p and $j_{p'}$ satisfy one of SR 1 or 2: For this case, the copies of $b(i)$ and $b(j)$ are $\bar{a}(i_p)$ and $\bar{a}(j_{p'})$, respectively. If they are identical, then $i_p = j_{p'}$, a contradiction.

Case 2. Both i_p and $j_{p'}$ satisfy SR 3: For this case, the copies of $b(i)$ and $b(j)$ are $\bar{b}(i_p)$ and $\bar{b}(j_{p'})$, respectively. If they are identical, then $i_p = j_{p'}$, a contradiction.

Case 3. i_p satisfies one of SR 1 or 2 while $j_{p'}$ satisfies SR 3: For this case, the copies of $b(i)$ and $b(j)$ are $\bar{a}(i_p)$ and $\bar{b}(j_{p'})$, respectively. If they are identical, then $\bar{a}(i_p) = \bar{b}(j_{p'})$. This implies $\bar{b}(j_{p'}) \in \bar{a}(N)$, a contradiction to SR 3.

Summing up, $b|_C$ is achievable on $\omega_a(C)$, a contradiction to $a \in \mathcal{WC}(R)$.

($\mathcal{EC}(R) \subseteq \mathcal{WC}(R)$) Suppose to the contrary that there exists $a \in \mathcal{EC}(R)$ such that a is strongly blocked by a coalition $C \in 2^N \backslash \{\emptyset\}$ through an allocation $b \in \mathcal{A}$ at R, i.e., (i) $b(i)\ P_i\ a(i)$ for all $i \in C$, and (ii) $b|_C$ is achievable on $\omega_a(C)$. Note that , by (ii), there exists an injective function $T : C \to \omega_a(C)$ such that for each $i \in C$,

[22] *Proof.* Suppose to the contrary that $i_p \in \{i_1, i_2, \ldots, i_{p-1}\}$. Assume, without loss of generality, that p is the smallest among such indices. Let $p' \in \{1, \ldots, p-1\}$ be $i_p = i_{p'}$. Then, $\bar{b}(i_{p'-1}) = \bar{a}(i_{p'}) = \bar{a}(i_p) = \bar{b}(i_{p-1})$, where $i_0 = i$. Thus, $i_{p'-1} = i_{p-1}$, a contradiction to the choice of p.

[23] *Proof.* Suppose to the contrary that for $q \in \{1, \ldots, p\}$ and $q' \in \{1, \ldots, p'\}$, $i_q = j_{q'}$. Assume, without loss of generality, that q is the smallest first coordinate among the pairs of such indices. Then, $\bar{b}(i_{q-1}) = \bar{a}(i_q) = \bar{a}(j_{q'}) = \bar{b}(j_{q'-1})$, where $i_0 = i$ and $j_0 = j$. Thus, $i_{q-1} = j_{q'-1}$, a contradiction to the choice of q.

$T(i) \in \{o_{k1}, \ldots, o_{kq_k}\}$, where k satisfies $b(i) = o_k$. Let $b' \in \mathcal{A}$ be such that

$$b'(i) = \begin{cases} b(i) & \text{if } i \in C, \\ o_0 & \text{if } i \notin C \text{ and } \bar{a}(i) \in T(C), \\ a(i) & \text{o.w.} \end{cases}$$

Note that for any $i \in N$, $a(i)\ P_i\ b'(i)$ only if $i \notin C$ and $\bar{a}(i) \in T(C)$. Thus, $\bar{a}(i) \in \omega_a(C)$. Summing up, the coalition C exclusion blocks a through b' at R, a contradiction. □

Theorem 4 in Balbuzanov and Kotowski (2019) shows that any indirect exclusion core allocation can be reached by the generalized top-trading cycles (GTTC) algorithm which coincides with the priority rule under the assumption of common priority.[24] By Proposition 1 in Appendix A and Theorem 1, $\mathcal{WC}(R)$ coincides with the indirect exclusion core in the current setting. Thus, to prove $\mathcal{WC}(R) \subseteq \Phi(R)$, the argument for the proof of Theorem 4 in Balbuzanov and Kotowski (2019) works apart from the fact that the current setting is accompanied by multiple copies of objects. We provide a proof for completeness.

Proof of Theorem 2. Let $R \in \mathcal{R}^N$.

($\mathcal{WC}(R) \supseteq \Phi(R)$) Suppose to the contrary that there exists $\trianglerighteq \in \Gamma(\succeq)$ such that $a := \varphi^{\trianglerighteq}(R) \notin \mathcal{WC}(R)$. Let $C \in 2^N \backslash \{\emptyset\}$ be a coalition that strongly blocks a through an allocation $b \in \mathcal{A}$ at R, i.e., (i) $b(i)\ P_i\ a(i)$ for all $i \in C$, and (ii) $b|_C$ is achievable on $\omega_a(C)$. Let $i_0 \in C$ be the highest-priority agent in C with respect to $\trianglerighteq$. First, we show the following claim.

Claim. $\forall i \in N, [a(i) = b(i_0) \Rightarrow i \triangleright i_0]$.

Proof of Claim: Suppose to the contrary that for some $i \in N$ with $a(i) = b(i_0)$, $i_0 \triangleright i$. This implies that under the priority rule associated with $\trianglerighteq$, at least one unit of $b(i_0)(= a(i))$ remains at the step in which $\varphi^{\trianglerighteq}_{i_0}(R)(= a(i_0))$ is determined. Thus, $\varphi^{\trianglerighteq}_{i_0}(R)\ R_{i_o}\ b(i_0)\ P_{i_o}\ a(i_0)$, a contradiction. This completes the proof of Claim.

[24]See Appendix A for the definition of the indirect exclusion core.

By (ii), $\omega_a(C)$ contains at least one unit of $b(i_0)$. Let $o_{k\ell} \in \omega_a(C)$ be a copy of $b(i_0)$. Note that $o_{k\ell} \in \bar{a}(N)$.[25] By the definition of $\omega_a(C)$,

$$\exists i \in C \text{ s.t. } i \succeq_{o_k} \bar{a}^{-1}(o_{k\ell}).$$

Let $j_0 := \bar{a}^{-1}(o_{k\ell})$. Since $\trianglerighteq$ is a completion of $\succeq_{o_k}$, $i \trianglerighteq j_0$. Because i_0 is the $\trianglerighteq$-greatest in C, $i_0 \trianglerighteq i$. Thus, $i_0 \trianglerighteq j_0$. However, $a(j_0) = o_k = b(i_0)$, a contradiction to Claim.

($\mathcal{WC}(R) \subseteq \Phi(R)$) By Theorem 1, $\mathcal{EC}(R) = \mathcal{WC}(R)$. Thus, we show $\mathcal{EC}(R) \subseteq \Phi(R)$. Let $a \in \mathcal{EC}(R)$ be arbitrary. Let $G_1 := \{i \in N \mid i \text{ is } \succeq\text{-maximal}\}$. We first prove

$$(*) \exists i \in G_1 \text{ s.t. } \bar{a}(i) \text{ is a copy of } i\text{'s most favorite object with respect to } R_i$$

by contradiction. Suppose the contrary. We show that there exists a cycle of agents $(i_1, j_1, \ldots, i_p, j_p, i_{p+1})$ such that

(i) $\{i_1, \ldots, i_p\} \subseteq G_1$,

(ii) $\forall p' \in \{1, \ldots, p\}$, $\bar{a}(j_{p'})$ is a copy of $i_{p'}$'s most favorite object with respect to $R_{i_{p'}}$,

(iii) $\forall p' \in \{1, \ldots, p\}, i_{p'+1} \succeq j_{p'}$,

(iv) $|\{i_1, \ldots, i_p\}| = p = |\{j_1, \ldots, j_p\}|$, and

(v) $i_{p+1} = i_1$.

The following inductive procedure, by definition, finds a cycle satisfying (i)-(v).

Step 1. Fix $i_1 \in G_1$ arbitrarily. Let $\alpha_{i_1} := \max_{R_{i_1}} \mathcal{O}$. By the contradiction hypothesis, $a(i_1) \neq \alpha_{i_1}$. As a is Pareto efficient at R ($\because$ Lemma 3), $\bar{\mathcal{O}} \backslash \bar{a}(N)$ does not contain a copy of α_{i_1}. Thus, there exists $j_1 \in N$ such that $a(j_1) = \alpha_{i_1}$. By the definition of G_1, there exists $i_2 \in G_1$ such that $i_2 \succeq j_1$. If $i_2 \in \{i_1\}$, (i_1, j_1, i_2) is the desired cycle. Otherwise, i.e., $i_2 \notin \{i_1\}$, go to Step 2.

Induction hypothesis. Let $p \geq 2$. Suppose that a sequence $(i_1, j_1, \ldots, i_{p-1}, j_{p-1}, i_p)$ satisfies the conditions (i) - (iv). Suppose also that $i_p \notin \{i_1, \ldots, i_{p-1}\}$.

[25] *Proof.* Suppose to the contrary that $o_{k\ell} \in \bar{\mathcal{O}} \backslash \bar{a}(N)$. Then, at least one unit of $o_k (= b(i_0))$ remains at the step in which $\varphi^{\trianglerighteq}_{i_0}(R)$ is determined. Thus, $\varphi^{\trianglerighteq}_{i_0}(R) \; R_{i_o} \; b(i_0) \; P_{i_o} \; a(i_0)$, a contradiction.

Step p. Let $\alpha_{i_p} := \max_{R_{i_p}} \mathcal{O}$. By the contradiction hypothesis, $a(i_p) \neq \alpha_{i_p}$. As a is Pareto efficient at R ($\because$ Lemma 3), $\bar{\mathcal{O}} \backslash \bar{a}(N)$ does not contain a copy of α_{i_p}. Thus, there exists $j_p \in N$ such that $a(j_p) = \alpha_{i_p}$. If $j_p \in \{j_1, \ldots, j_{p-1}\}$, $(i_{p'}, j_{p'}, \ldots, i_p, j_p, i_{p+1})$ is the desired cycle, where $j_{p'} = j_p$ and $i_{p+1} := i_{p'}$. Otherwise, i.e., $j_p \notin \{j_1, \ldots, j_{p-1}\}$, then by the definition of G_1, there exists $i_{p+1} \in G_1$ such that $i_{p+1} \succeq j_p$. If $i_{p+1} \in \{i_1, \ldots, i_p\}$, $(i_{p'}, j_{p'}, \ldots, i_p, j_p, i_{p+1})$ is the desired cycle, where $i_{p'} = i_{p+1}$. Otherwise, i.e., $i_{p+1} \notin \{i_1, \ldots, i_p\}$, go to Step $p+1$.

As N is finite, the procedure stops in a finite steps. This completes the description of a procedure that finds a cycle $(i_1, j_1, \ldots, i_p, j_p, i_{p+1})$ satisfying the properties (i)-(v).

Now, we return to the proof of $(*)$. Let $C := \{i_1, \ldots, i_p\}$. We define $b \in \mathcal{A}$ as follows.

$$b(i) = \begin{cases} \alpha_{i_{p'}} & \text{if } i \in C \text{ and } i = i_{p'}, \\ o_0 & \text{if } i \in \{j_1, \ldots, j_p\} \backslash C, \\ a(i) & \text{o.w.} \end{cases}$$

Obviously, $b(i)\ P_i\ a(i)$ for all $i \in C$. By definition, for any $i \in N$ with $a(i)\ P_i\ b(i)$, $i \in \{j_1, \ldots, j_p\} \backslash C$. By (iii), for such i, $\bar{a}(i) \in \omega_a(C)$. However, this contradicts $a \in \mathcal{EC}(R)$. This completes the proof of $(*)$.

Finally, we construct $\trianglerighteq \in \Gamma(\succeq)$ such that $a = \varphi^{\trianglerighteq}(R)$. By $(*)$,

$$\exists i'_1 \in G_1 \text{ s.t. } \bar{a}(i'_1) \text{ is a copy of the } i'_1\text{'s most favorite object with respect to } R_{i'_1}.$$

Define a subproblem by removing i'_1 with one unit of $a(i'_1)$ if $a(i'_1)$ is a real object. Remove only i'_1 if $a(i'_1) = o_0$. Let G_2 be the set of $\succeq$-maximal agents in the subproblem. By $(*)$,

$$\exists i'_2 \in G_2 \text{ s.t. } \bar{a}(i'_2) \text{ is a copy of the } i'_2\text{'s most favorite object with respect to } R_{i'_2}.$$ [26]

By repeating this procedure n times, we obtain a sequence of agents $\{i'_p\}_{p=1}^n$. Let $\trianglerighteq$ be $i'_1 \trianglerighteq i'_2 \trianglerighteq \ldots \trianglerighteq i'_n$. Obviously, $\trianglerighteq \in \Gamma(\succeq)$ and $a = \varphi^{\trianglerighteq}(R)$. □

Lemmas 4 and 5 below are used in the proof of Theorem 3. In the sequel, $\mathcal{S}$

[26] Here, "the most favorite object" indicates the one in the subproblem.

denotes the set of subsets of $\mathcal{A}^P$ satisfying IIF, WIPF, IND, CON, ESE, and CLO, as defined in the statement of Theorem 3,

Lemma 4. *Let $S \in \mathcal{S}$. Then, there exists $f : [0,1] \to [0,1]$ satisfying GMR-1, GMR-2 and GMR-3 such that $S \subseteq \mathcal{A}^P_f$.*

Proof. For each $I \in \mathcal{I}$ and each $i \in I$, let $\overline{x}^I_i := \sup_{\alpha \in S} \alpha(i)$ and $\underline{x}^I_i := \inf_{\alpha \in S} \alpha(i)$. We first prove the following claims.

Claim 1. $\forall I, I' \in \mathcal{I}, \forall i \in I, \forall i' \in I', \frac{r^{I'}_{i'}}{|I'|} \leq \frac{r^I_i}{|I|} \Rightarrow \overline{x}^{I'}_{i'} \leq \overline{x}^I_i$.

Proof of Claim 1: Suppose to the contrary that $\frac{r^{I'}_{i'}}{|I'|} \leq \frac{r^I_i}{|I|}$ and $\overline{x}^{I'}_{i'} > \overline{x}^I_i$. Note that $r^I_i < |I|$.[27] Thus, we have $j \in I$ such that $r^I_j = r^I_i + 1$. By CON, $\overline{x}^I_i = \inf_{\alpha \in S} \alpha(j)$. Thus, by CLO(ii), there exists $\alpha \in S$ such that $\alpha(j) = \overline{x}^I_i$. Moreover, by CLO(i), there exists $\beta \in S$ such that $\beta(i') = \overline{x}^{I'}_{i'}$. Note that $I \neq I'$.[28] By IND, there exists $\gamma \in S$ such that $\gamma|_I = \alpha|_I$ and $\gamma|_{N \setminus I} = \beta|_{N \setminus I}$. Since $\gamma(j) < \gamma(i')$ and $\frac{r^{I'}_{i'}}{|I'|} \leq \frac{r^I_i}{|I|} = \frac{r^I_j - 1}{|I|}$, γ is not WIPF, a contradiction. This completes the proof of Claim 1.

Claim 2. $\forall I, I' \in \mathcal{I}, \forall i \in I, \forall i' \in I', \frac{r^{I'}_{i'}}{|I'|} < \frac{r^I_i}{|I|} \Rightarrow \overline{x}^{I'}_{i'} < \overline{x}^I_i$.

Proof of Claim 2: Suppose to the contrary that $\frac{r^{I'}_{i'}}{|I'|} < \frac{r^I_i}{|I|}$ and $\overline{x}^{I'}_{i'} \geq \overline{x}^I_i$. By Claim 1, $\overline{x}^{I'}_{i'} = \overline{x}^I_i$. Note that $r^I_i < |I|$.[29] Thus, there exists $j \in I$ such that $r^I_j = r^I_i + 1$. By CON, $\inf_{\alpha \in S} \alpha(j) = \overline{x}^I_i \left(= \overline{x}^{I'}_{i'}\right)$. By CLO(ii), there exists $\alpha \in S$ such that $\alpha(j) = \overline{x}^I_i$. Moreover, by CLO(i), there exists $\beta \in S$ such that $\beta(i') = \overline{x}^{I'}_{i'}$. Note that $I \neq I'$.[30] By IND, there exists $\gamma \in S$ such that $\gamma|_I = \alpha|_I$ and $\gamma|_{N \setminus I} = \beta|_{N \setminus I}$. Thus, $\gamma(j) = \gamma(i')$ and $\frac{r^{I'}_{i'}}{|I'|} < \frac{r^I_i}{|I|} = \frac{r^I_j - 1}{|I|}$. Thus, γ is not WIPF, a contradiction. This completes the proof of Claim 2.

[27] *Proof of $r^I_i < |I|$:* If $r^I_i = |I|$, $1 = \overline{x}^I_i < \overline{x}^{I'}_{i'}$ by CLO(ii). This contradicts the definition of $\overline{x}^{I'}_{i'}$.

[28] *Proof of $I \neq I'$:* If $I = I'$, $\overline{x}^{I'}_{i'} > \overline{x}^I_i$. Thus, by the definition of $\mathcal{A}^P$, $r^I_{i'} > r^I_i$, a contradiction.

[29] *Proof of $r^I_i < |I|$:* Suppose to the contrary that $r^I_i = |I|$. Then, by CLO(i), $1 = \overline{x}^I_i = \overline{x}^{I'}_{i'}$. Since $\frac{r^{I'}_{i'}}{|I'|} < 1$, there exists $j' \in I'$ such that $r^{I'}_{j'} = r^{I'}_{i'} + 1$. By CLO(i), there exists $\delta \in S$ such that $\delta(i') = \overline{x}^{I'}_{i'} = 1$. For this δ, $\delta(j') > \delta(i') = 1$. However, this contradicts the definition of $\mathcal{A}^P$.

[30] *Proof of $I \neq I'$:* Suppose to the contrary that $I = I'$. By CLO(i), there exists $\delta \in S$ such that $\delta(i') = \overline{x}^{I'}_{i'}$. Since $\frac{r^{I'}_{i'}}{|I'|} < \frac{r^I_i}{|I|}$, $r^{I'}_{i'} < r^I_i$. Thus, $\delta(i') < \delta(i)$ ($\because$ the definition of $\mathcal{A}^P$). This contradicts $\overline{x}^I_i = \overline{x}^{I'}_{i'}$.

By Claims 1 and 2, there exists a function $f : [0,1] \to [0,1]$ satisfying GMR-1 such that

$$\forall I \in \mathcal{I}, \forall i \in I, f\left(\frac{r_i^I}{|I|}\right) = \overline{x}_i^I.$$

In the sequel, we show that f satisfies GMR-2, GMR-3 and $S \subseteq \mathcal{A}_f^P$.

(f satisfies GMR-2): By ESE, there exists $\alpha_f^M \in S$ such that $\alpha_f^M(i) = \frac{\overline{x}_i^I + \underline{x}_i^I}{2}$ for each $I \in \mathcal{I}$ and each $i \in I$. For each $I \in \mathcal{I}$ with $I = \{i_1, \ldots, i_K\}$ and $i_K \succ \ldots \succ i_1$,

$$\begin{aligned}
\sum_{i\in I} \alpha_f^M(i) &= \frac{\overline{x}_{i_1}^I + \underline{x}_{i_1}^I}{2} + \ldots + \frac{\overline{x}_{i_K}^I + \underline{x}_{i_K}^I}{2} \\
&= \frac{0 + \overline{x}_{i_1}^I}{2} + \frac{\overline{x}_{i_1}^I + \overline{x}_{i_2}^I}{2} + \ldots + \frac{\overline{x}_{i_{K-1}}^I + 1}{2} \ (\because \text{ CON and CLO }) \\
&= \sum_{k=1}^{K-1} \overline{x}_{i_k}^I + \frac{1}{2} \\
&= \sum_{i\in I} \overline{x}_i^I - \frac{1}{2}. \qquad \text{(A)}
\end{aligned}$$

Thus, combined with IIF, f satisfies GMR-2.

(f satisfies GMR-3): Suppose to the contrary that there exist $I \in \mathcal{I}$ and $i \in I$ such that $\overline{x}_i^I - \underline{x}_i^I > \frac{1}{2}$. Note that $\overline{x}_i^I - \underline{x}_i^I = f\left(\frac{r_i^I}{|I|}\right) - f\left(\frac{r_i^I - 1}{|I|}\right)$ ($\because$ CON and CLO(ii)). By CLO, there exist $\alpha, \beta \in S$ such that $\alpha(i) = \overline{x}_i^I$ and $\beta(i) = \underline{x}_i^I$. Since S satisfies IIF and IND, the point assignment to institution $I \in \mathcal{I}$ under any point allocation in S is equal to (A). Thus, $\overline{x}_i^I + \sum_{j\in I\setminus\{i\}} \alpha(j) = \underline{x}_i^I + \sum_{j\in I\setminus\{i\}} \beta(j) < \left(\overline{x}_i^I - \frac{1}{2}\right) + \sum_{j\in I\setminus\{i\}} \beta(j)$. Thus, $\sum_{j\in I\setminus\{i\}} (\beta(j) - \alpha(j)) > \frac{1}{2}$. Note that for each $j \in I\setminus\{i\}, \overline{x}_j^I - \underline{x}_j^I \geq \beta(j) - \alpha(j)$ because $\alpha(j), \beta(j) \in [\underline{x}_j^I, \overline{x}_j^I]$. Thus, $\sum_{j\in I\setminus\{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right) > \frac{1}{2}$. Summing up,

$$\left(\overline{x}_i^I - \underline{x}_i^I\right) + \sum_{j\in I\setminus\{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right) > \frac{1}{2} + \frac{1}{2} = 1.$$

However, the left-hand side of the inequality is 1, a contradiction.

($S \subseteq \mathcal{A}_f^P$): First, we introduce a notation. For each $I \in \mathcal{I}$ and each $i \in I$, let $T_i^f := \overline{x}_i^I - \underline{x}_i^I$. Let $\alpha \in S$ be arbitrary. For each $I \in \mathcal{I}$ and each $i \in I$, there exists $w_i \in [-1,1]$ such that $\alpha(i) = \frac{\underline{x}_i^I + \overline{x}_i^I}{2} + w_i \frac{T_i^f}{2}$ ($\because \alpha(i) \in [\underline{x}_i^I, \overline{x}_i^I]$). Since S satisfies IIF and IND, the point assignment to institution $I \in \mathcal{I}$ under any point allocation in S

is equal to (A), i.e., $\sum_{i\in I}\overline{x}_i^I - \frac{1}{2} = \sum_{i\in I}\alpha(i)$. Thus,

$$\begin{aligned}\sum_{i\in I}\overline{x}_i^I - \frac{1}{2} &= \sum_{i\in I}\left(\frac{\underline{x}_i^I + \overline{x}_i^I}{2} + w_i\frac{T_i^f}{2}\right)\\ &= \sum_{i\in I}\alpha_f^M(i) + \sum_{i\in I} w_i\frac{T_i^f}{2}\\ &= \left(\sum_{i\in I}\overline{x}_i^I - \frac{1}{2}\right) + \sum_{i\in I} w_i\frac{T_i^f}{2}.\end{aligned}$$

Thus, $\sum_{i\in I} w_i T_i^f = 0$. Thus, $\alpha \in \mathcal{A}_f^P$. □

Lemma 5. *Let $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be such that*

- *both f and g satisfy GMR-1, GMR-2 and GMR-3, and*
- *$\exists I \in \mathcal{I}, \exists i \in I$ s.t. $f\left(\frac{r_i^I}{|I|}\right) \neq g\left(\frac{r_i^I}{|I|}\right)$.*

Then, $\mathcal{A}_f^P \not\subseteq \mathcal{A}_g^P$.

Proof. We show the conclusion in the following two cases separately.

Case 1. $f\left(\frac{r_i^I}{|I|}\right) < g\left(\frac{r_i^I}{|I|}\right)$.

Since f and g satisfy GMR-1, $f(1) = 1 = g(1)$. Thus, $r_i^I < |I|$. Thus, there exists $j \in I$ such that $r_j^I = r_i^I + 1$. Thus, by the definition of GMR, there exists $\alpha \in \mathcal{A}_f^P$ such that $\alpha(j) = f\left(\frac{r_i^I}{|I|}\right)$. Because $\min_{\beta\in\mathcal{A}_g^P}\beta(j) = g\left(\frac{r_i^I}{|I|}\right)$, $\alpha \notin \mathcal{A}_g^P$.

Case 2. $f\left(\frac{r_i^I}{|I|}\right) > g\left(\frac{r_i^I}{|I|}\right)$.

By the definition of GMR, there exists $\alpha \in \mathcal{A}_f^P$ such that $\alpha(i) = f\left(\frac{r_i^I}{|I|}\right)$. Because $\max_{\beta\in\mathcal{A}_g^P}\beta(i) = g\left(\frac{r_i^I}{|I|}\right)$, $\alpha \notin \mathcal{A}_g^P$. □

Proof of Theorem 3. ($\Leftarrow$) For each $I \in \mathcal{I}$ and each $i \in I$, let $\overline{x}_i^I := f\left(\frac{r_i^I}{|I|}\right)$, $\underline{x}_i^I := f\left(\frac{r_i^I - 1}{|I|}\right)$ and $T_i^f := \overline{x}_i^I - \underline{x}_i^I$.

($\mathcal{A}_f^P$ satisfies WIPF): Let $\alpha \in \mathcal{A}_f^P$ be arbitrary. Suppose that $\alpha(i') \geq \alpha(i)$ and $\frac{r_{i'}^{I'}}{|I'|} \leq \frac{r_i^I - 1}{|I|}$ for $I, I' \in \mathcal{I}, i \in I$ and $i' \in I'$.

First, we show that $\frac{r^{I'}_{i'}}{|I'|} = \frac{r^I_i - 1}{|I|}$. Suppose to the contrary that $\frac{r^{I'}_{i'}}{|I'|} < \frac{r^I_i - 1}{|I|}$. Since f is strictly increasing ($\because$ GMR-1), $f\left(\frac{r^{I'}_{i'}}{|I'|}\right) < f\left(\frac{r^I_i - 1}{|I|}\right)$. By the definition of GMR,

$$\alpha(i') \leq f\left(\frac{r^{I'}_{i'}}{|I'|}\right) < f\left(\frac{r^I_i - 1}{|I|}\right) \leq \alpha(i),$$

a contradiction. Thus, $\frac{r^{I'}_{i'}}{|I'|} = \frac{r^I_i - 1}{|I|}$.

Next, we show that $\alpha(i') = \alpha(i)$. Since $\frac{r^{I'}_{i'}}{|I'|} = \frac{r^I_i - 1}{|I|}$, by the definition of GMR, $\alpha(i') \leq f\left(\frac{r^{I'}_{i'}}{|I'|}\right) = f\left(\frac{r^I_i - 1}{|I|}\right) \leq \alpha(i)$. Thus, $\alpha(i') = \alpha(i)$.

Summing up the previous two paragraphs, α satisfies WIPF. Thus, $\mathcal{A}^P_f$ satisfies WIPF.

($\mathcal{A}^P_f$ satisfies IIF): Let $\alpha \in \mathcal{A}^P_f$ be arbitrary. Suppose that for $w \in [-1, 1]^N$, $\alpha(i) = \frac{\underline{x}^I_i + \overline{x}^I_i}{2} + w_i \frac{T^f_i}{2}$. Let $I, I' \in \mathcal{I}$ be arbitrary. By GMR-2, $\frac{1}{|I|}\left(\sum_{i \in I} \overline{x}^I_i - \frac{1}{2}\right) = \frac{1}{|I'|}\left(\sum_{i' \in I'} \overline{x}^{I'}_{i'} - \frac{1}{2}\right)$. Thus,

$$\begin{aligned}
\sum_{i \in I} \alpha(i) &= \sum_{i \in I}\left(\frac{\underline{x}^I_i + \overline{x}^I_i}{2} + w_i \frac{T^f_i}{2}\right) \\
&= \sum_{i \in I} \frac{\underline{x}^I_i + \overline{x}^I_i}{2} \qquad \left(\because \sum_{i \in I} w_i T^f_i = 0\right) \\
&= \sum_{i \in I} \overline{x}^I_i - \frac{1}{2}.
\end{aligned}$$

The same calculation yields that $\sum_{i' \in I'} \alpha(i') = \sum_{i' \in I'} \overline{x}^{I'}_{i'} - \frac{1}{2}$. Thus, $\frac{1}{|I|}\sum_{i \in I} \alpha(i) = \frac{1}{|I'|}\sum_{i' \in I'} \alpha(i')$. Thus, α satisfies IIF. Thus, $\mathcal{A}^P_f$ satisfies IIF.

($\mathcal{A}^P_f$ satisfies IND): Obvious.

($\mathcal{A}^P_f$ satisfies CLO): We only show CLO(i) since the proof of CLO(ii) is similar. Let $I \in \mathcal{I}$ and $i \in I$ be arbitrary. We construct a point allocation in $\mathcal{A}^P_f$ under which the point assignment for i is $\overline{x}^I_i$ $\left(= \frac{\overline{x}^I_i + \underline{x}^I_i}{2} + \frac{\overline{x}^I_i - \underline{x}^I_i}{2}\right)$. By GMR-3, $\sum_{j \in I \setminus \{i\}}\left(\overline{x}^I_j - \underline{x}^I_j\right) \geq \overline{x}^I_i - \underline{x}^I_i$. Thus, $\sum_{j \in I \setminus \{i\}} \frac{\overline{x}^I_j - \underline{x}^I_j}{2} \geq \frac{\overline{x}^I_i - \underline{x}^I_i}{2}$. Note that the right-hand side of the inequality respresents the extra points needed for i's assignment to be $\overline{x}^I_i$ beyond the midpoint of $[\underline{x}^I_i, \overline{x}^I_i]$. On the other hand, the left-hand side of the inequality represents the sum of the points maximally removed from other agents $j \in I\backslash\{i\}$ beyond the midpoint of $[\underline{x}_j^I, \overline{x}_j^I]$. Let $w \in [-1,1]^N$ be

$$w_k := \begin{cases} 1 & \text{if } k = i, \\ -\frac{\overline{x}_i^I - \underline{x}_i^I}{\sum_{j \in I\backslash\{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right)} & \text{if } k \in I\backslash\{i\}, \\ 0 & \text{if } k \in N\backslash I. \end{cases}$$

Because $\sum_{k \in I} w_k T_k^f = (\overline{x}_i^I - \underline{x}_i^I) + \sum_{k \in I\backslash\{i\}} \left(-\frac{\overline{x}_i^I - \underline{x}_i^I}{\sum_{j \in I\backslash\{i\}} \left(\overline{x}_j^I - \underline{x}_j^I\right)}\right)(\overline{x}_k^I - \underline{x}_k^I) = 0$, $\alpha^w \in \mathcal{A}_f^P$. Note that $\alpha^w(i) = \overline{x}_i^I$. Thus, $\mathcal{A}_f^P$ satisfies CLO(i).

($\mathcal{A}_f^P$ satisfies ESE): Since $\mathcal{A}_f^P$ satisfies CLO, by the definition of GMR, for each $I \in \mathcal{I}$ and each $i \in I$, $\inf_{\alpha \in \mathcal{A}_f^P} \alpha(i) = f\left(\frac{r_i^I - 1}{|I|}\right)$ and $\sup_{\alpha \in \mathcal{A}_f^P} \alpha(i) = f\left(\frac{r_i^I}{|I|}\right)$. Let $w = (0, \dots, 0) \in [-1,1]^N$. Obviously, $\alpha^w \in \mathcal{A}_f^P$ satisfies $\alpha^w(i) = \frac{\inf_{\alpha \in \mathcal{A}_f^P} \alpha(i) + \sup_{\alpha \in \mathcal{A}_f^P} \alpha(i)}{2}$ for all $i \in I$.

($\mathcal{A}_f^P$ satisfies CON): Let $I \in \mathcal{I}$. Suppose that $i, j \in I$ are such that $r_i^I = r_j^I + 1$. Since $\mathcal{A}_f^P$ satisfies CLO, by the definition of GMR, $\sup_{\alpha \in \mathcal{A}_f^P} \alpha(j) = f\left(\frac{r_j^I}{|I|}\right) = f\left(\frac{r_i^I - 1}{|I|}\right) = \inf_{\alpha \in \mathcal{A}_f^P} \alpha(i)$.

($\mathcal{A}_f^P$ is $\supseteq$-maximal in $\mathcal{S}$): Suppose to the contrary that there exists $T \in \mathcal{S}$ such that $\mathcal{A}_f^P \subsetneq T$. By Lemma 4, there exists $g : [0,1] \to [0,1]$ satisfying GMR-1, GMR-2 and GMR-3 such that $T \subseteq \mathcal{A}_g^P$. Thus, $\mathcal{A}_f^P \subsetneq \mathcal{A}_g^P$. However, this contradicts Lemma 5.

($\Rightarrow$) By Lemma 4, there exists $f : [0,1] \to [0,1]$ satisfying GMR-1, GMR-2 and GMR-3 such that $S \subseteq \mathcal{A}_f^P$. Note that $\mathcal{A}_f^P \in \mathcal{S}$ as shown in the sufficiency part of the proof of Theorem 3. Since S is a $\supseteq$-maximal element in $\mathcal{S}$, $S = \mathcal{A}_f^P$. □

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