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by

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Riemannian Interior Point Methods for Constrained Optimization on Manifolds*

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Abstract

We extend the classical primal-dual interior point method from the Euclidean setting to the Riemannian one. Our method, named the Riemannian interior point method (RIPM), is for solving Riemannian constrained optimization problems. We establish its locally superlinear and quadratic convergence under the standard assumptions. Moreover, we show its global convergence when it is combined with a classical line search. This method is a generalization of the classical framework of primal-dual interior point methods for nonlinear programming proposed by El-Bakry et al. in 1996. Numerical experiments show the stability and efficiency of our method.

Keywords: Riemannian manifold, Constrained optimization problem, Interior point method, Riemannian Newton method, Barrier method.

1 Introduction

We consider the following problem,

$$\begin{array}{ll} \min\limits_{x\in\mathbb{M}} & f(x) \\ \text{s.t.} & h(x)=0, \text{ and } g(x)\leqslant 0, \end{array} \tag{RCOP}$$

where $\mathbb{M}$ is a d-dimensional Riemannian manifold and $f:\mathbb{M}\to\mathbb{R}$, $h:\mathbb{M}\to\mathbb{R}^l$, $l<d$, and $g:\mathbb{M}\to\mathbb{R}^m$ are C^2 (twice continuously differentiable) on the manifold. This problem is called the nonlinear programming problem (NLP) on a Riemannian manifold, or the *Riemannian constrained optimization problem* (RCOP). It appears in many applications, for instance, matrix approximation with nonnegative constraints on a fixed-rank manifold [32] and orthogonal nonnegative matrix factorization on the Stiefel manifold [20]; for more applications, see [23, 27].

The body of knowledge on the Riemannian unconstrained optimization problem (i.e., $h,g=0$), often called simply *Riemannian optimization*, has grown considerably in the last 20 years. In particular, well-known methods in the Euclidean setting, such as steepest descent, Newton, conjugate gradient and trust region, have been extended to the Riemannian setting [1, 17, 5, 29]. By contrast, research on the Riemannian constrained optimization problem is still in its infancy. The earliest studies go back to the optimal conditions in the Riemannian case. Yang et al. [36] extended the Karush Kuhn Tucker (KKT) conditions and the second-order necessary and sufficient conditions to (RCOP). Bergmann and Herzog [2] considered more constraint qualifications (CQs) on manifolds. Yamakawa and Sato [34] proposed sequential optimality conditions, called approximate KKT conditions in the Riemannian case. Liu and Boumal [23] were the first to develop practical algorithms. They extended the augmented Lagrangian method and exact penalty method to (RCOP). Yamakawa and Sato [34] improved the augmented Lagrangian method in order to obtain a solution without CQs. Schiela and Ortiz [30] and Obara et al. [27] proposed the Riemannian sequential quadratic programming method. However, to our knowledge, interior point methods have yet to be considered for (RCOP).

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The advent of interior point methods in the 1980s greatly advanced the field of optimization [33, 37, 16]. By the early 1990s, the success of these methods in linear and quadratic programming ignited interest in using them on nonlinear cases [10, 35]. From the 1990s to the first decade of the 21st century, a large number of interior point methods for nonlinear programming emerged. They proved to be as successful as the linear ones [26, Chapter 19]. A subclass known as *primal-dual* interior point methods is the most efficient practical approach. As described in [24], the primal-dual approach to linear programming was introduced in [25]: it was first developed as an algorithm in [21] and eventually became standard for the nonlinear case as well [10, 35]. Since it seems to be an application of the Newton method for solving the KKT conditions, it has been called the Newton interior point method in some of the literature.

In this paper, we extend the primal-dual interior point algorithms from the Euclidean setting, i.e., $\mathbb{M} \equiv \mathbb{R}^d$ in (RCOP), to the Riemannian setting. We call this extension the *Riemannian interior point method* (RIPM). Under meaningful assumptions in the Riemannian setting, we establish the locally superlinear and quadratic convergence. We also show global convergence with a classical line search. Our method is a generalization of classical local and global convergence theory of interior point methods for nonlinear programming first proposed by El-Bakry et al. [10]. To our knowledge, this paper is the first study to apply the primal-dual interior point method to optimization on Riemannian manifolds.

In Section 2, we give an interpretation of the Riemannian interior point methods; in particular, we define the concept of a *KKT vector field* and give the formulation of its covariant derivative. The implication of the standard assumptions motivates RIPM. We end the section by describing a prototype algorithm 2. In Section 3, we describe the notation, preliminaries, and auxiliary results that we will need later. In Section 4, we prove locally superlinear and quadratic convergence of RIPM. In Section 5, we describe a globally convergent RIPM with the classical line search and prove its convergence in Section 6. Section 7 is a collection of numerical experiments. As a beneficial addition, we give an intuitive example of a barrier method on a sphere manifold.

2 Interpretation of Riemannian Interior Point Methods

We will use three symbols to denote the various manifolds: $\mathbb{M}$ is the manifold appearing in (RCOP); $\mathscr{M} := \mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$ is the product manifold consisting of $\mathbb{M}$ and three Euclidean spaces, and $\mathcal{M}$ refers to a general manifold. Following common usage in the interior-point literature, big letters denote the associated diagonal matrix by $Z = \operatorname{diag}(z_1, \ldots, z_n)$ for $z \in \mathbb{R}^n$, and e the vector of all ones. In the following overview of the Riemannian interior point method, many of the symbols appearing in Riemannian optimization will be used without giving their definitions. The readers may refer to Section 3.1 for the detailed definitions.

2.1 KKT Vector Field

The Lagrangian function of (RCOP) is

$$\mathcal{L}(x, y, z) = f(x) + y^T h(x) + z^T g(x),$$

where $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^m$ are Lagrange multipliers. With respect to the variable x, $\mathcal{L}(\cdot, y, z)$ is a real function on $\mathbb{M}$, and its Riemannian gradient is

$$\operatorname{grad}_x \mathcal{L}(x, y, z) = \operatorname{grad} f(x) + \textstyle\sum_{i=1}^{l} y_i \operatorname{grad} h_i(x) + \sum_{i=1}^{m} z_i \operatorname{grad} g_i(x),$$

where $\operatorname{grad} f(x), \{\operatorname{grad} h_i(x)\}_{i=1}^{l}, \{\operatorname{grad} g_i(x)\}_{i=1}^{m}$ are the Riemannian gradients for the component functions of f, h, g. The active set $A(x) = \{i : g_i(x) = 0, i = 1, \ldots, m\}$ consists of the indices of the active constraints at $x \in \mathbb{M}$. 0_x denotes the zero element of the tangent space $T_x\mathbb{M}$. The Riemannian versions of the KKT conditions [23, Definition 2.3] for (RCOP) are given by

$$\operatorname{grad}_x \mathcal{L}(x, y, z) = 0_x;\ h(x) = 0, g(x) \leqslant 0, z \geqslant 0;\ Zg(x) = 0. \tag{1}$$

With slack variables $s := -g(x)$, the above KKT conditions can be written as

$$F(w) := \begin{pmatrix} F_x := \operatorname{grad}_x \mathcal{L}(x, y, z) \\ F_y := h(x) \\ F_z := g(x) + s \\ F_s := ZSe \end{pmatrix} = 0_w \equiv \begin{pmatrix} 0_x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{2}$$

and $(z, s) \geqslant 0$, where $w := (x, y, z, s) \in \mathscr{M} := \mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$. Note that for the Riemannian KKT conditions, we generate a vector field F on the Riemannian product manifold $\mathscr{M}$, i.e., $F : \mathscr{M} \to T\mathscr{M} \cong T\mathbb{M} \times T\mathbb{R}^l \times T\mathbb{R}^m \times T\mathbb{R}^m$, where $T\mathscr{M}$ denotes the tangent bundle of $\mathscr{M}$. At a point $w = (x, y, z, s) \in \mathscr{M}$, the tangent space appears as $T_w\mathscr{M} \cong T_x\mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$ under the canonical identification $T_v\mathcal{E} \cong \mathcal{E}$ for any vector space $\mathcal{E}$ and any $v \in \mathcal{E}$. In particular, for any $w = (x, y, z, s)$ with $x \in \mathbb{M}$, the first component of $F(w)$ is in $T_x\mathbb{M}$; the multipliers y and z and slack variables s, in turn, are treated as they usually are.

Definition 2.1. The vector field F on $\mathscr{M}$ defined in (2) is called the *KKT vector field* of (RCOP).

Thus, the KKT conditions for (RCOP) can be interpreted as ones for finding a singularity of a vector field on a product manifold with partial nonnegative requirements, namely, $F(w) = 0_w$ and $(z, s) \geqslant 0$.

2.2 Implication of Standard Assumptions

The Newton method is a powerful tool for finding the zeros of nonlinear functions in the Euclidean setting. The generalized Newton method has been studied in the Riemannian setting; it aims to find a singularity of a vector field $F \in \mathfrak{X}(\mathcal{M})$, specifically, a point $p \in \mathcal{M}$ such that,

$$F(p) = 0_p \in T_p\mathcal{M}. \tag{3}$$

Let ∇ be the Riemannian connection on $\mathcal{M}$. The covariant derivative ∇F assigns each point $p \in \mathcal{M}$ a linear operator $\nabla F(p)$ from and to $T_p\mathcal{M}$. The Riemannian Newton iteration for (3) can be performed as follows.

Algorithm 1 (Riemannian Newton Method).

(Step 1) Compute $\xi_k \in T_{p_k}\mathcal{M}$ by solving the equation $\nabla F(p_k)\xi_k = -F(p_k)$.
(Step 2) Compute $p_{k+1} := R_{p_k}(\xi_k)$, where R denotes a retraction on $\mathcal{M}$. Return to Step 1.

We know that, if p^* is a solution of (3) and the operator $\nabla F(p^*)$ is nonsingular, then local superlinear and quadratic convergence hold [12] under certain mild conditions on the map, $p \mapsto \nabla F(p)$. If the Riemannian Newton method is to be applied to (2), we must formulate the covariant derivative of the KKT vector field F at an arbitrary $w \in \mathscr{M}$. Let $\operatorname{Hess}_x \mathcal{L}(w)$ be the Riemannian Hessian of a real function $\mathcal{L}(\cdot, y, z)$. It is a linear operator on $T_x\mathbb{M}$ such that, for any $\Delta x \in T_x\mathbb{M}$,

$$\operatorname{Hess}_x \mathcal{L}(w)\Delta x = \left(\operatorname{Hess} f(x) + \textstyle\sum_{i=1}^{l} y_i \operatorname{Hess} h_i(x) + \sum_{i=1}^{m} z_i \operatorname{Hess} g_i(x)\right) \Delta x, \tag{4}$$

where $\operatorname{Hess} f(x), \{\operatorname{Hess} h_i(x)\}_{i=1}^{l}, and \{\operatorname{Hess} g_i(x)\}_{i=1}^{m}$ are Riemannian Hessian operators for the component functions. Using the notation $\operatorname{Hess}_x \mathcal{L}(w)$, we can prove the following.

Lemma 2.2 (covariant derivative of KKT vector field)**.** *For any $w \in \mathscr{M}$, the linear operator $\nabla F(w) : T_w\mathscr{M} \to T_w\mathscr{M}$ for the KKT vector field F defined in (2) is given by*

$$\nabla F(w)\Delta w = \begin{pmatrix} \operatorname{Hess}_x \mathcal{L}(w)\Delta x + \sum_{i=1}^{l} \Delta y_i \operatorname{grad} h_i(x) + \sum_{i=1}^{m} \Delta z_i \operatorname{grad} g_i(x) \\ \langle \operatorname{grad} h_i(x), \Delta x \rangle_x, \text{ for } i = 1, 2, \ldots, l \\ \langle \operatorname{grad} g_i(x), \Delta x \rangle_x + \Delta s_i, \text{ for } i = 1, 2, \ldots, m \\ Z\Delta s + S\Delta z \end{pmatrix} \tag{5}$$

where $\Delta w = (\Delta x, \Delta y, \Delta z, \Delta s) \in T_x\mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$.

Proof. See Appendix A for a rigorous proof. A compact form is given in (12) later. □

A prior study on Riemannian optimal conditions [36, 2] showed that the following assumptions for (RCOP) are meaningful, see Example 7.1. Their Euclidean versions can be found in [10, Section 4].

Assumption 1 (standard Riemannian assumptions of (RCOP))**.**

(A1) Existence. There exists (x^*, y^*, z^*) satisfying the KKT conditions (1).
(A2) Smoothness of C^2. The functions f, g and h are C^2 on $\mathbb{M}$.
(A3) LICQ. $\{\operatorname{grad} h_i(x^*)\}_{i=1}^{l} \cup \{\operatorname{grad} g_i(x^*) : i \in A(x^*)\}$ is linearly independent in $T_{x^*}\mathbb{M}$.
(A4) Strict complementarity. $(z^*)_i > 0$ if $g_i(x^*) = 0$ for all $i = 1, \cdots, m$.
(A5) Second-order sufficiency. $\langle \operatorname{Hess}_x \mathcal{L}(w^*)\xi, \xi \rangle > 0$ for all nonzero $\xi \in T_{x^*}\mathbb{M}$ satisfying $\langle \xi, \operatorname{grad} h_i(x^*) \rangle = 0$ for $i = 1, \cdots, l$, and $\langle \xi, \operatorname{grad} g_i(x^*) \rangle = 0$ for $i \in A(x^*)$.

The following result motivates the use of the Newton method for solving (2). See Appendix B for the proof, which is a generalization of [10, Proposition 4.1].

Proposition 2.3. *If (A1)-(A5) hold at w^*, then the operator $\nabla F(w^*)$ in (5) is nonsingular.*

2.3 Prototype Algorithm

If we directly apply the Newton method to the KKT vector field, the Newton equation for (2) is $\nabla F(w)\Delta w + F(w) = 0$. As in the usual Euclidean setting, once the iterates reach the boundary of the feasible region, they are forced to stick to it. To keep the iterates sufficiently far from the boundary, we introduce a perturbed complementary equation for some number $\mu > 0$ and define

$$F(w;\mu) := F(w) - \mu\hat{e}, \text{ and } \hat{e} := \hat{e}(w) := (0_x, 0, 0, e). \tag{6}$$

Notice that the perturbation term $\hat{e}$, indeed, is a special vector field on $\mathscr{M}$, not a constant, because 0_x is essentially dependent on w and/or x. We call $F(w;\mu)$ in (6) the *perturbed KKT vector field*.

Note that the covariant derivative of the perturbed KKT vector field is the same as that of the original. From the linearity of the connection ∇, we have at any point $w \in \mathscr{M}$,

$$\nabla F(w;\mu) = \nabla F(w) - \mu\nabla\hat{e}(w) = \nabla F(w), \tag{7}$$

since $\nabla\hat{e}(w)\Delta w = (0_x, 0, 0, 0)$ for all $\Delta w \in T_w\mathscr{M}$. Applying the Newton method to $F(w;\mu) = 0$ yields the perturbed Newton equation, $\nabla F(w;\mu)\Delta w + F(w;\mu) = 0$. From (6) and (7), this equation is equivalent to $\nabla F(w)\Delta w + F(w) = \mu\hat{e}$, which reduces to the ordinary Newton equation when $\mu = 0$.

On the other hand, the next lemma gives a homotopy (or, continuation) derivation that is similar to the case of the Euclidean interior point method [26, Chapter 19]. Note that, very recently, Séguin and Kressner [31] developed continuation methods for Riemannian optimization, which are closely related to our Riemannian interior point method.

Lemma 2.4. *Under the standard assumptions (A1)-(A5) at w^*, there exist a sufficiently small $\bar{\mu} > 0$ and a smooth curve $w : [0, \bar{\mu}) \to \mathscr{M}$ such that $w(0) = w^*$ and $F(w(\mu);\mu) = 0, \forall\mu \in [0, \bar{\mu})$.*

Proof. By Proposition 2.3, we have that $F(w^*;0) = 0$ and $\nabla F(w^*;0)$ is nonsingular. The proof uses the same technique as in [31, Theorem 3.1]. Roughly speaking, it applies the implicit function theorem to the local coordinate representations of the vector field F and its full-rank Jacobian matrix at the solution. □

This smooth curve $\mu \mapsto w(\mu)$ is called the *central path*, whose endpoint $w(0) = w^*$ is a solution of (RCOP). μ is customarily called the *barrier parameter* because $F(w(\mu);\mu) = 0$ can be interpreted as the Riemannian KKT conditions of the following barrier problem:

$$\begin{array}{ll} \min_{(x,s)\in\mathbb{M}\times\mathbb{R}^m} & f(x) - \mu\sum_{i=1}^m \log s_i \\ \text{s.t.} & h(x) = 0,\ g(x) + s = 0. \end{array} \tag{8}$$

See section 7.1 for an illustrative example. Now, let us describe the Riemannian interior point method.

Algorithm 2 (Prototype Algorithm of RIPM).

(Step 0) Let R be a retraction on $\mathbb{M}$. Set $w_0 = (x_0, y_0, z_0, s_0) \in \mathscr{M}$ with $(z_0, s_0) > 0$; $\mu_0 > 0, k \leftarrow 0$.
(Step 1) Check whether w_k satisfies a stopping test for (RCOP).
(Step 2) Solve the following linear system for $\Delta w_k = (\Delta x_k, \Delta y_k, \Delta s_k, \Delta z_k)$,

$$\nabla F(w_k)\Delta w_k = -F(w_k) + \mu_k\hat{e}. \tag{9}$$

(Step 3) Choose γ_k where $0 < \hat{\gamma} \leqslant \gamma_k \leqslant 1$ for some constant $\hat{\gamma}$ and compute the step size,

$$\alpha_k := \min\left\{1, \gamma_k \min_i\left\{-\tfrac{(s_k)_i}{(\Delta s_k)_i} \mid (\Delta s_k)_i < 0\right\}, \gamma_k \min_i\left\{-\tfrac{(z_k)_i}{(\Delta z_k)_i} \mid (\Delta z_k)_i < 0\right\}\right\}. \tag{10}$$

(Step 4) Update: $w_{k+1} = \bar{R}_{w_k}(\alpha_k\Delta w_k)$, i.e., $(x_{k+1}, y_{k+1}, s_{k+1}, z_{k+1}) = (R_{x_k}(\alpha_k\Delta x_k), y_k + \alpha_k\Delta y_k, s_k + \alpha_k\Delta s_k, z_k + \alpha_k\Delta z_k)$. Choose $\mu_{k+1} < \mu_k$; $k \leftarrow k + 1$. Return to Step 1.

2.4 Condensed Form of Perturbed Newton Equation

We will focus our attention to the solution of the linear system (9). To make the formulation of $\nabla F(w)$ in (5) look simpler, we will introduce the following symbols. For $x \in \mathbb{M}$, we define two maps $\mathcal{H}_x : \mathbb{R}^l \to T_x\mathbb{M}$ and $\mathcal{G}_x : \mathbb{R}^m \to T_x\mathbb{M}$ by

$$\mathcal{H}_x v := \textstyle\sum_{i=1}^l v_i \operatorname{grad} h_i(x), \quad \mathcal{G}_x v := \sum_{i=1}^m v_i \operatorname{grad} g_i(x), \tag{11}$$

respectively. We can see that for every $v \in \mathbb{R}^l$ and $\xi \in T_x\mathbb{M}$, $\langle \mathcal{H}_x v, \xi \rangle_x = \sum_{i=1}^l v_i \langle \operatorname{grad} h_i(x), \xi \rangle_x$. Hence, the adjoint operator of $\mathcal{H}_x$ is given by $\mathcal{H}_x^* : T_x\mathbb{M} \to \mathbb{R}^l$,

$$\mathcal{H}_x^* \xi = \left[\langle \operatorname{grad} h_1(x), \xi \rangle_x , \cdots , \langle \operatorname{grad} h_l(x), \xi \rangle_x \right]^T ,$$

Similarly, we have $\mathcal{G}_x^* : T_x\mathbb{M} \to \mathbb{R}^m$ and $\mathcal{G}_x^* \xi = \left[\langle \operatorname{grad} g_1(x), \xi \rangle_x , \cdots , \langle \operatorname{grad} g_m(x), \xi \rangle_x \right]^T$. By using those symbols, we can obtain a compact form of (5) as follows:

$$\nabla F(w) \Delta w = \begin{pmatrix} \operatorname{Hess}_x \mathcal{L}(w) \Delta x + \mathcal{H}_x \Delta y + \mathcal{G}_x \Delta z \\ \mathcal{H}_x^* \Delta x \\ \mathcal{G}_x^* \Delta x + \Delta s \\ Z \Delta s + S \Delta z \end{pmatrix}. \tag{12}$$

If $\mathbb{M} \equiv \mathbb{R}^d$, then $\mathcal{H}_x, \mathcal{G}_x$ are expressed as the Jacobian matrices of h, g, and $\mathcal{H}_x^*, \mathcal{G}_x^*$ are their transposes.

Remark 2.5 (Implementation Details)**.** If $\mathbb{M}$ is a Riemannian submanifold of $\mathbb{R}^n$ equipped with the inherited metric $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$, we can express $\mathcal{H}_x$ and $\mathcal{H}_x^*$ by using only Euclidean gradients. Let Proj_x be the orthogonal projector from $\mathbb{R}^n$ onto $T_x\mathbb{M} \subseteq \mathbb{R}^n$, and $\operatorname{egrad} h_i(x)$ be the Euclidean gradients of h_i at x. It follows from the linearity of Proj_x and $\operatorname{grad} h_i(x) = \operatorname{Proj}_x \operatorname{egrad} h_i(x)$ [5, Proposition 3.61] that $\mathcal{H}_x v = \operatorname{Proj}_x (\sum_{i=1}^l v_i \operatorname{egrad} h_i(x)) = \operatorname{Proj}_x (\operatorname{egrad}_x \langle v, h(x) \rangle)$. On the other hand, since Proj_x is self-adjoint and $\operatorname{Proj}_x \xi = \xi$ for every $\xi \in T_x\mathbb{M}$, we have $\langle \operatorname{grad} h_i(x), \xi \rangle_x = \langle \operatorname{egrad} h_i(x), \operatorname{Proj}_x \xi \rangle = \langle \operatorname{egrad} h_i(x), \xi \rangle$. Thus, $\mathcal{H}_x^* \xi = \left[\langle \operatorname{egrad} h_1(x), \xi \rangle , \cdots , \langle \operatorname{egrad} h_l(x), \xi \rangle \right]^T$. The above can be applied verbatim to $\mathcal{G}_x, \mathcal{G}_x^*$.

Suppose that $(z, s) > 0$. We will solve the next system by using the compact form (12) and the notation in (2):

$$\begin{pmatrix} \operatorname{Hess}_x \mathcal{L}(w) \Delta x + \mathcal{H}_x \Delta y + \mathcal{G}_x \Delta z \\ \mathcal{H}_x^* \Delta x \\ \mathcal{G}_x^* \Delta x + \Delta s \\ Z \Delta s + S \Delta z \end{pmatrix} = \begin{pmatrix} -F_x \\ -F_y \\ -F_z \\ -F_s + \mu e \end{pmatrix}. \tag{13}$$

From the fourth line we can deduce that $\Delta s = Z^{-1} (\mu e - F_s - S \Delta z)$. Substituting it into the third line, one has $\mathcal{G}_x^* \Delta x - Z^{-1} S \Delta z = -Z^{-1} \mu e - g(x)$. A further substitution from the third line $\Delta z = S^{-1} \left[Z \left(\mathcal{G}_x^* \Delta x + F_z \right) + \mu e - F_s \right]$ into the first line yields (13) in a condensed form on $T_x\mathbb{M} \times \mathbb{R}^l$:

$$\mathcal{T}(\Delta x, \Delta y) := \begin{pmatrix} \mathcal{A}_w \Delta x + \mathcal{H}_x \Delta y \\ \mathcal{H}_x^* \Delta x \end{pmatrix} = \begin{pmatrix} c \\ q \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} &\Theta := \mathcal{G}_x S^{-1} Z \mathcal{G}_x^*, \ \mathcal{A}_w := \operatorname{Hess}_x \mathcal{L}(w) + \Theta, \\ &c := -F_x - \mathcal{G}_x S^{-1} \left(Z F_z + \mu e - F_s \right), \ q := -F_y. \end{aligned} \tag{15}$$

Accordingly, the question of how to solve the linear system (14) becomes critical. Like the interior point methods in the Euclidean setting, (14) is essentially a symmetric indefinite linear operator equation. Note that Θ and $\mathcal{A}_w$ are self-adjoint operators from and to $T_x\mathbb{M}$. Hence, $\mathcal{T}$ is self-adjoint on the product space $T_x\mathbb{M} \times \mathbb{R}^l$ with inner product $\langle (\Delta x, \Delta y), (\Delta x', \Delta y') \rangle := \langle \Delta x, \Delta x' \rangle_x + \sum_{i=1}^l \Delta y_i^T \Delta y_i'$. Indeed, (14) is a saddle point problem defined on Hilbert spaces. From the above discussion, if $(z, s) > 0$ holds, then the operator $\nabla F(w)$ in (5) is nonsingular if and only if $\mathcal{T}$ in (14) is nonsingular. The following theorem states two sufficient conditions to ensure that $\mathcal{T}$ is nonsingular. They are a direct extension of the classical results. Note that $\mathcal{H}_x$ in (11) is injective if and only if the set $\{\operatorname{grad} h_i(x)\}_{i=1}^l$ is linearly independent in $T_x\mathbb{M}$.

Corollary 2.6. *In (14), the operator $\mathcal{T}$ is nonsingular if: (i) $\mathcal{A}_w$ and $\mathcal{H}_x^* \mathcal{A}_w^{-1} \mathcal{H}_x$ are nonsingular; alternatively, (ii) $\mathcal{H}_x$ is injective and $\mathcal{A}_w$ is positive definite on the null space of $\mathcal{H}_x^*$.*

3 Preliminaries

Here, we describe some of the preliminary and auxiliary results necessary for understanding Riemannian optimization on the basis of the literature [5, 19, 18]. All symbols are defined in Section 3.1.

3.1 Notation

Let $\mathcal{E}$ and $\mathcal{E}'$ be two finite-dimensional normed vector spaces, on which norms are induced by inner products $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{E}'}$, respectively. Let $T : \mathcal{E} \to \mathcal{E}'$ be a linear operator. The adjoint of T is the linear operator $T^* : \mathcal{E}' \to \mathcal{E}$ defined by the property: for all $v \in \mathcal{E}, u \in \mathcal{E}', \langle Tv, u \rangle_{\mathcal{E}'} = \langle v, T^* u \rangle_{\mathcal{E}}$. The norm of T is $\|T\| := \sup \{\|Tv\|_{\mathcal{E}'} : v \in \mathcal{E}, \|v\|_{\mathcal{E}} = 1, \text{ or, } \|v\|_{\mathcal{E}} \leqslant 1\}$. T is self-adjoint if $T = T^*$. $\mathrm{Sym}(d)$ denotes the set of symmetric matrices of order d. Given two nonnegative sequences $\{u_k\}$, $\{v_k\}$, we write $u_k = O(v_k)$ if there is a positive constant M such that $u_k \leqslant M v_k$ for all sufficiently large k. We write $u_k = o(v_k)$ if $v_k > 0$ and the sequence of ratios $\{u_k / v_k\}$ approaches zero.

Riemannian metrics and gradients Let $\mathcal{M}$ be a finite-dimensional manifold. The tangent space $T_x\mathcal{M}$ contains all the possible directions of curves on $\mathcal{M}$ passing through $x \in \mathcal{M}$. The elements of $T_x\mathcal{M}$ are called tangent vectors and $T\mathcal{M} := \{(x, \xi) : x \in \mathcal{M}, \xi \in T_x\mathcal{M}\}$ is called the tangent bundle. A vector field on a manifold $\mathcal{M}$ is a map $V : \mathcal{M} \to T\mathcal{M}$ with $V(x) \in T_x\mathcal{M}$. The set of all smooth vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$. The inner product $\langle \cdot, \cdot \rangle_x$ on $T_x\mathcal{M}$ is a positive-definite symmetric bilinear form. It induces the norm $\|\xi\|_x := \sqrt{\langle \xi, \xi \rangle_x}$. A Riemannian metric on $\mathcal{M}$ is a choice of inner product $\langle \cdot, \cdot \rangle_x$ for each $x \in \mathcal{M}$ and for all smooth vector fields V, W on $\mathcal{M}$, the function,

$$x \mapsto \langle V(x), W(x) \rangle_x, \tag{16}$$

is smooth from $\mathcal{M}$ to $\mathbb{R}$. A manifold with a Riemannian metric is called a Riemannian manifold. Let $\mathfrak{F}(\mathcal{M})$ denote the set of all smooth *scalar fields* (i.e., real-value function) $f : \mathcal{M} \to \mathbb{R}$. The Riemannian gradient of f is the vector field, $\mathrm{grad}\, f$, uniquely defined by the identities: $\mathcal{D}f(x)[\xi] = \langle \xi, \mathrm{grad}\, f(x) \rangle$ for all $(x, \xi) \in T\mathcal{M}$, where $\mathcal{D}f(x) : T_x\mathcal{M} \to \mathbb{R}$ is the differential of f at x. For $f \in \mathfrak{F}(\mathcal{M})$, $\mathrm{grad}\, f$ is a smooth vector field, hence $x \mapsto \|\mathrm{grad}\, f(x)\|_x$ is a smooth scalar field by (16).

Covariant derivatives and Hessians A Levi-Civita or Riemannian connection on $\mathcal{M}$ is the unique operator $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) : (U, V) \mapsto \nabla_U V$ which has the following properties: $\mathfrak{F}(\mathcal{M})$-linearity in U, $\mathbb{R}$-linearity in V, the Leibniz rule, symmetry and compatibility with the metric. $\nabla_U V$ at x depends on U only through $U(x)$. Thus, we can write $\nabla_\xi V$ to mean $(\nabla_U V)(x)$ for arbitrary $U \in \mathfrak{X}(\mathcal{M})$ such that $U(x) = \xi$. The covariant derivative of $F \in \mathfrak{X}(\mathcal{M})$ determined by ∇ defines a linear operator at each $x \in \mathcal{M}$: $\nabla F(x) : T_x\mathcal{M} \to T_x\mathcal{M}$, by $\nabla F(x)\xi := \nabla_\xi F$. Particularly, the Riemannian Hessian of $f \in \mathfrak{F}(\mathcal{M})$ at x is a self-adjoint operator on $T_x\mathcal{M}$ that is defined as $\mathrm{Hess}\, f(x)\xi := \nabla_\xi \mathrm{grad}\, f$.

Retraction and vector transport A retraction $R : T\mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$ is a smooth map having the following properties: $R_x(0_x) = x$, where R_x denotes the restriction of R to $T_x\mathcal{M}$ and 0_x denotes the zero element of $T_x\mathcal{M}$; with the canonical identification $T_{0_x}(T_x\mathcal{M}) \cong T_x\mathcal{M}$, $\mathcal{D}R_x(0_x) = \mathrm{id}_{T_x\mathcal{M}}$. A vector transport $\mathcal{T}$ on $\mathcal{M}$ is a smooth map $T\mathcal{M} \oplus T\mathcal{M} \to T\mathcal{M} : (\eta, \xi) \mapsto \mathcal{T}_\eta(\xi) \in T\mathcal{M}$ having the following properties, where $\oplus$ denotes the Whitney sum: there exists an associated retraction R such that $\mathcal{T}_{\eta_x}(\xi_x) \in T_{R_x(\eta_x)}\mathcal{M}$; $\mathcal{T}_{0_x}(\xi_x) = \xi_x$; $\mathcal{T}_{\eta_x}(a\xi_x + b\zeta_x) = a\mathcal{T}_{\eta_x}(\xi_x) + b\mathcal{T}_{\eta_x}(\zeta_x)$ for all $a, b \in \mathbb{R}$ and $\eta_x, \xi_x, \zeta_x \in T_x\mathcal{M}$.

Distance, geodesics and completeness Given a piecewise smooth curve $\gamma : [a, b] \to \mathcal{M}$, we define the length of γ as $L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt$. The Riemannian distance $d(p, q)$ is defined to be the infimum of $L(\gamma)$ over all piecewise smooth curves γ joining p and q. It induces the original topology on $\mathcal{M}$; namely, $(\mathcal{M}, d)$ is a metric space. The open ball of radius r centered at p is $B_r(p) := \{q \in \mathcal{M} : d(p, q) < r\}$. A vector field V along a smooth curve γ on $\mathcal{M}$ is said to be parallel if $\nabla_{\gamma'} V = 0$. When γ' itself is parallel we call γ a geodesic. A geodesic joining p and q is minimal if its length equals $d(p, q)$. $\mathcal{M}$ is complete if the geodesics are well-defined on $\mathbb{R}$. Hopf-Rinow's theorem asserts that every pair of points in a complete Riemannian manifold $\mathcal{M}$ can be joined by a minimal geodesic segment. If a unique geodesic joining p to q exists, we denote it by γ_{pq}. Throughout this paper, $\mathcal{M}$ is a complete Riemannian manifold.

Exponential mapping and parallel transport For every $\xi \in T_x\mathcal{M}$, there exists an interval I around zero and a unique geodesic $\gamma(t) : I \to \mathcal{M}$ such that $\gamma(0) = x$ and $\gamma'(0) = \xi$. The map $\xi \mapsto \exp_x \xi := \gamma(1)$ is called the exponential map, which is a retraction. The domain of $\exp_x$ is the whole $T_x\mathcal{M}$ for all $x \in \mathcal{M}$ if and only if $\mathcal{M}$ is complete. Given a smooth curve γ on $\mathcal{M}$, parallel transport of the tangent space at $\gamma(t_0)$ to the tangent space at $\gamma(t_1)$ along γ, $\mathcal{P}_\gamma^{t_1 \leftarrow t_0} : T_{\gamma(t_0)}\mathcal{M} \to T_{\gamma(t_1)}\mathcal{M}$ is defined by $\mathcal{P}_\gamma^{t_1 \leftarrow t_0}(u) = Z(t_1)$, where Z is a unique parallel vector field such that $Z(t_0) = u$. In particular, parallel transport is an isometry. Let R be a retraction on a manifold $\mathcal{M}$; then $\mathcal{T}_{\eta_x}(\xi_x) := \mathcal{P}_\gamma^{1 \leftarrow 0} \xi_x$ is a vector transport with an associated retraction R, where $\mathcal{P}_\gamma$ denotes parallel transport along the curve $t \mapsto \gamma(t) = R_x(t\eta_x)$.

Riemannian product manifold At a point $w = (x, y, z, s)$ in the Riemannian product manifold $\mathscr{M} = \mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$, we have $T_w\mathscr{M} \cong T_x\mathbb{M} \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m$. For $\xi = (\xi_x, \xi_y, \xi_z, \xi_s)$ and $\zeta = (\zeta_x, \zeta_y, \zeta_z, \zeta_s)$ in $T_w\mathscr{M}$, the Riemannian product metric is defined as $\langle \xi, \zeta \rangle_w := \langle \xi_x, \zeta_x \rangle_x + \xi_y^T \zeta_y + \xi_s^T \zeta_z + \xi_z^T \zeta_s$. The induced norm $\|\xi\|_w := \sqrt{\langle \xi, \xi \rangle_w}$. The product distance on $\mathscr{M}$ is defined as $d(w_1, w_2) := \sqrt{d^2(x_1, x_2) + \|y_1 - y_2\|^2 + \|z_1 - z_2\|^2 + \|s_1 - s_2\|^2}$, where $\|\cdot\|$ denotes the l_2 norm. For all $(w, \xi) \in T\mathscr{M}$, $\bar{R}_w(\xi) := (R_x(\xi_x), y + \xi_y, s + \xi_s, z + \xi_z)$ defines a retraction on $\mathscr{M}$. Let $\mathbb{R}^l$ and $\mathbb{R}^m$ equipped with the canonical Euclidean connection; then, the connection ∇ on $\mathbb{M}$ determines a connection on $\mathscr{M}$ (see Lemma A.1). We will conflate the notations ∇ and d on $\mathscr{M}$ and $\mathbb{M}$ since they are clear from context.

3.2 Preliminaries and Auxiliary Results

From $\mathcal{D}R_x(0_x) = \mathrm{id}_{T_x\mathcal{M}}$ and the inverse function theorem, there exists a neighborhood V of 0_x in $T_x\mathcal{M}$ such that R_x is a diffeomorphism on V, i.e., $R_x^{-1}(y)$ is well defined for all $y \in \mathcal{M}$ sufficiently close to x. In this case, $R_x(V) \subseteq \mathcal{M}$ is called a retractive neighborhood of x. Furthermore, the existence of a *totally* retractive neighborhood [38, Theorem 2] shows that for any $\bar{x} \in \mathcal{M}$ there is a neighborhood W of $\bar{x}$ such that $R_x^{-1}(y)$ is well defined for all $x, y \in W$. In what follows, we will suppose that an appropriate neighborhood has been chosen by default for the well-definedness of $R_x^{-1}(y)$.

Multiple Riemannian versions of Lipschitz continuity have been defined [5, Section 10.4]. Here, we consider Lipschitz continuity with respect to general vector transport and its associated retraction. Let $\mathcal{M}$ be a Riemannian manifold endowed with a vector transport $\mathcal{T}$ and an associated retraction R.

Definition 3.1 ([18, Definition 5.2.1])**.** A function $f : \mathcal{M} \to \mathbb{R}$ is *Lipschitz continuously differentiable with respect to* $\mathcal{T}$ in $\mathcal{U} \subseteq \mathcal{M}$ if it is differentiable and if there exists a number $\kappa > 0$ such that, for all $x, y \in \mathcal{U}$, $\|\operatorname{grad} f(y) - \mathcal{T}_\xi \operatorname{grad} f(x)\| \leqslant \kappa \|\xi\|$, where $\xi = R_x^{-1} y$.

Lemma 3.2 ([18, Lemma 5.2.1])**.** *If a function $f : \mathcal{M} \to \mathbb{R}$ is C^2, then, for any $\bar{x} \in \mathcal{M}$, there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ such that f is Lipschitz continuously differentiable with respect to $\mathcal{T}$ in $\mathcal{U}$.*

We get the following result when the gradient is replaced by a general vector field.

Definition 3.3. A vector field F is *Lipschitz continuous with respect to* $\mathcal{T}$ in $\mathcal{U} \subseteq \mathcal{M}$ if there exists a number $\kappa > 0$ such that, for all $x, y \in \mathcal{U}$, $\|F(y) - \mathcal{T}_\xi F(x)\| \leqslant \kappa \|\xi\|$, where $\xi = R_x^{-1} y$.

Lemma 3.4. *If F is a C^1 vector field, then, for any $\bar{x} \in \mathcal{M}$, there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ such that F is Lipschitz continuous with respect to $\mathcal{T}$ in $\mathcal{U}$.*

Proof. The proof of this lemma is similar to that of [18, Lemma 5.2.1]. □

Going one degree higher, let us now discuss the Lipschitz-continuous Hessian of f. Recall that the Hessian of f associates to each x a linear operator $\operatorname{Hess} f(x)$ from and to $T_x\mathcal{M}$.

Definition 3.5 ([19, Assumption 3])**.** A function $f : \mathcal{M} \to \mathbb{R}$ is *twice Lipschitz continuously differentiable with respect to* $\mathcal{T}$ in $\mathcal{U} \subseteq \mathcal{M}$ if it is twice differentiable and if a number $\kappa > 0$ exists such that, for all $x, y \in \mathcal{U}$, $\|\operatorname{Hess} f(y) - \mathcal{T}_\xi \operatorname{Hess} f(x) \mathcal{T}_\xi^{-1}\| \leqslant \kappa d(x, y)$, where $\xi = R_x^{-1} y$.

Lemma 3.6 ([19, Lemma 4])**.** *If $f : \mathcal{M} \to \mathbb{R}$ is C^3, then, for any $\bar{x} \in \mathcal{M}$, there exists a neighborhood $\mathcal{U}$ of $\bar{x}$, such that f is twice Lipschitz continuously differentiable with respect to $\mathcal{T}$ in $\mathcal{U}$.*

Similarly, if Hessian is replaced by a general covariant derivative, we get the following result.

Definition 3.7. The covariant derivative ∇F is *Lipschitz continuous with respect to* $\mathcal{T}$ in $\mathcal{U} \subseteq \mathcal{M}$ if there exists a number $\kappa > 0$ such that, for all $x, y \in \mathcal{U}$, $\|\nabla F(y) - \mathcal{T}_\xi \nabla F(x) \mathcal{T}_\xi^{-1}\| \leqslant \kappa d(x, y)$, where $\xi = R_x^{-1} y$.

Lemma 3.8. *If F is a C^2 vector field, then, for any $\bar{x} \in \mathcal{M}$, there exists a neighborhood $\mathcal{U}$ of x such that the covariant derivative ∇F is Lipschitz continuous with respect to $\mathcal{T}$ in $\mathcal{U}$.*

Proof. This lemma can be proven in a similar way as [19, Lemma 4]. □

Lemma 3.9. *Let $\mathcal{M}$ be a Riemannian manifold endowed with a (smooth) retraction R and let $\bar{x} \in \mathcal{M}$. Then,*

(i) there exist $a_0 > 0, a_1 > 0$, and $\delta_{a_0,a_1} > 0$ such that for all x in a sufficiently small neighborhood of $\bar{x}$ and all $\xi, \eta \in T_x\mathcal{M}$ with $\|\xi\| \leqslant \delta_{a_0,a_1}$ and $\|\eta\| \leqslant \delta_{a_0,a_1}$, $a_0\|\xi - \eta\| \leqslant d(R_x(\eta), R_x(\xi)) \leqslant a_1\|\xi - \eta\|$. In particular, $a_0\|\xi\| \leqslant d(x, R_x(\xi)) \leqslant a_1\|\xi\|$ when we take $\eta = 0$.

(ii) there exist $a_0 > 0$, and $a_1 > 0$ such that for all x in a sufficiently small neighborhood of $\bar{x}$, $a_0\|\xi\| \leqslant d(x, \bar{x}) \leqslant a_1\|\xi\|$ where $\xi = R_{\bar{x}}^{-1}(x)$.

Proof. (i) comes directly from [19, Lemma 2]. To prove (ii), we take $x = \bar{x}$ in (i) so that $a_0\|\xi\| \leqslant d(\bar{x}, R_{\bar{x}}(\xi)) \leqslant a_1\|\xi\|$ for all $\xi \in T_{\bar{x}}\mathcal{M}$ with $\|\xi\| \leqslant \delta_{a_0,a_1}$. Since $R_{\bar{x}}$ is a local diffeomorphism at $0_{\bar{x}} \in T_{\bar{x}}\mathcal{M}$, for all x in a sufficiently small neighborhood of $\bar{x}$, we can take $\xi = R_{\bar{x}}^{-1}(x)$ such that $\|\xi\| \leqslant \delta_{a_0,a_1}$. We complete the proof of (ii) by substituting $R_{\bar{x}}(\xi) = x$. □

Lemma 3.10 ([19, Lemma 8]). *Let F be a C^1 vector field and $\bar{x} \in \mathcal{M}$. Then a neighborhood $\mathcal{U}$ of $\bar{x}$ and a constant c_1 exist such that, for all $x, y \in \mathcal{U}$, $\|\mathcal{P}_\gamma^{0\leftarrow 1}F(y) - F(x) - \int_0^1 \mathcal{P}_\gamma^{0\leftarrow t}\nabla F(\gamma(t))\mathcal{P}_\gamma^{t\leftarrow 0}\eta dt\| \leqslant c_1\|\eta\|^2$, where $\eta = R_x^{-1}(y)$ and $\mathcal{P}_\gamma$ is a parallel transport along the curve $\gamma(t) = R_x(t\eta)$. Particularly, $c_1 = 0$ if $R = \exp$. (e.g., see [13, equality (2.4)].)*

Lemma 3.11. *Let F be a C^2 vector field and $\bar{x} \in \mathcal{M}$. Then a neighborhood $\mathcal{U}$ of $\bar{x}$ and a constant c_2 exist such that, for all $x \in \mathcal{U}$, $\|\mathcal{P}_\gamma^{0\leftarrow 1}F(x) - F(\bar{x}) - \nabla F(\bar{x})\eta\| \leqslant c_2 d^2(\bar{x}, x)$, where $\eta = R_{\bar{x}}^{-1}x$ and $\mathcal{P}_\gamma$ is a parallel transport along the curve $\gamma(t) = R_{\bar{x}}(t\eta)$.*

Proof. Let $LHS := \|\mathcal{P}_\gamma^{0\leftarrow 1}F(x) - F(\bar{x}) - \nabla F(\bar{x})\eta\|$. It follows that

$$LHS \leqslant \left\|\mathcal{P}_\gamma^{0\leftarrow 1}F(x) - F(\bar{x}) - \int_0^1 \mathcal{P}_\gamma^{0\leftarrow t}\nabla F(\gamma(t))\mathcal{P}_\gamma^{t\leftarrow 0}\eta dt\right\| + \left\|\int_0^1 \mathcal{P}_\gamma^{0\leftarrow t}\nabla F(\gamma(t))\mathcal{P}_\gamma^{t\leftarrow 0}\eta dt - \nabla F(\bar{x})\eta\right\|$$

$$\leqslant c_1\|\eta\|^2 + \left\|\int_0^1 \left[\mathcal{P}_\gamma^{0\leftarrow t}\nabla F(\gamma(t))\mathcal{P}_\gamma^{t\leftarrow 0} - \nabla F(\bar{x})\right]\eta dt\right\|. \text{ (by Lemma 3.10)}$$

Let $\theta := \int_0^1 \left[\mathcal{P}_\gamma^{0\leftarrow t}\nabla F(\gamma(t))\mathcal{P}_\gamma^{t\leftarrow 0} - \nabla F(\bar{x})\right]\eta dt$. Note that

$$\|\theta\| \leqslant \int_0^1 \left\|\mathcal{P}_\gamma^{0\leftarrow t}\nabla F(\gamma(t))\mathcal{P}_\gamma^{t\leftarrow 0} - \nabla F(\bar{x})\right\|\|\eta\| dt \leqslant \int_0^1 c_0 d(\bar{x}, R_{\bar{x}}(t\eta))\|\eta\| dt \text{ (by Lemma 3.8)}$$

$$\leqslant \int_0^1 c_0 a_1 t\|\eta\|\|\eta\| dt = \frac{1}{2}c_0 a_1\|\eta\|^2. \text{ (by (i) of Lemma 3.9)}$$

Combining the above results yields $LHS \leqslant (c_1 + \frac{1}{2}c_0 a_1)\|\eta\|^2 \leqslant (c_1 + \frac{1}{2}c_0 a_1)/a_0^2 d^2(\bar{x}, x)$, where the last inequality comes from (ii) of Lemma 3.9. Letting $c_2 := (c_1 + \frac{1}{2}c_0 a_1)/a_0^2$ completes the proof. □

We end this section with the following lemmas.

Lemma 3.12 ([11, Lemma 3.2]). *If ∇F is continuous at p^* and $\nabla F(p^*)$ is nonsingular, then a neighborhood $\mathcal{U}$ of p^* and a positive constant Ξ exist such that, for all $p \in \mathcal{U}$, $\nabla F(p)$ is nonsingular and $\|\nabla F(p)^{-1}\| \leqslant \Xi$.*

Lemma 3.13 ([7, Lemma 3.5]). *Let $(p, v) \in T\mathcal{M}$ such that $\exp_p(v)$ exists and $w \in T_p\mathcal{M} \cong T_v(T_p\mathcal{M})$. Then, $\langle \mathcal{D}\exp_p(v)[v], \mathcal{D}\exp_p(v)[w]\rangle = \langle v, w\rangle_p$. In particular, $\|\mathcal{D}\exp_p(\lambda v)[v]\| = \|v\|$ holds for $\lambda \geqslant 0$.*

4 Local Convergence of RIPM

The local convergence of RIPM results from an application of the *perturbed damped Newton method* and will be described later. Here, we consider again problem (3). The standard assumptions are as follows:

Assumption 2 (Riemannian Newton assumptions of problem (3) [4, Theorem 2]).

(B1) There exists $p^* \in \mathcal{M}$ such that $F(p^*) = 0_{p^*}$.
(B2) The operator $\nabla F(p^*)$ is nonsingular.
(B3) The covariant derivative ∇F is Lipschitz continuous on a neighborhood of p^*.

Then, the perturbed damped Newton method for (3) is as follows: let $0 < \alpha_k \leqslant 1, \mu_k > 0$.

Algorithm 3 (Perturbed Damped Newton Method).

(Step 1) Compute $\xi_k \in T_{p_k}\mathcal{M}$ by solving the perturbed equation $\nabla F(p_k)\xi_k + F(p_k) = \mu_k\hat{e}$.
(Step 2) Compute $p_{k+1} := R_{p_k}(\alpha_k\xi_k)$ by using a step size α_k. Return to Step 1.

Proposition 4.1. *Consider the perturbed damped Newton method for problem (3). Let (B1)-(B3) hold at p^*. Choose parameters μ_k, α_k as below; then there exists a positive constant δ such that for all $d(p_0, p^*) < \delta$, the sequence $\{p_k\}$ is well defined. Furthermore,*
(i) if $\mu_k = o(\|F(p_k)\|)$ and $\alpha_k \to 1$, then $\{p_k\}$ converges to p^* superlinearly;
(ii) if $\mu_k = O(\|F(p_k)\|^2)$ and $1 - \alpha_k = O(\|F(p_k)\|)$, then $\{p_k\}$ converges to p^* quadratically.

Proof. By Lemma 3.12, we can let p_k be sufficiently close to p^* such that $\nabla F(p_k)$ is nonsingular, and $\|\nabla F(p_k)^{-1}\| \leqslant \Xi$. Then, the next iterate, $p_{k+1} := R_{p_k}[\alpha_k \nabla F(p_k)^{-1}(-F(p_k) + \mu_k \hat{e})]$, is well defined, and it follows from $p^* = R_{p_k}(\eta)$ with $\eta := R_{p_k}^{-1} p^*$ and (i) of Lemma 3.9 that

$$d(p_{k+1}, p^*) \leqslant a_1 \|\eta + \alpha_k \nabla F(p_k)^{-1}(F(p_k) - \mu_k \hat{e})\|.$$

Let $r_k := \eta + \alpha_k \nabla F(p_k)^{-1}(F(p_k) - \mu_k \hat{e})$. Algebraic manipulations show that $r_k = (1-\alpha_k)\eta + \alpha_k \nabla F(p_k)^{-1}[\nabla F(p_k)\eta + F(p_k) - \mathcal{P}_\gamma^{0\leftarrow 1} F(p^*) - \mu_k \hat{e}]$, where $\mathcal{P}_\gamma$ is a parallel transport along the curve $\gamma(t) = R_{p_k}(t\eta)$. Thus, using $\|\eta\| \leqslant \frac{1}{a_0} d(p_k, p^*)$ from (ii) of Lemma 3.9, and Lemma 3.11, we have

$$\begin{aligned}
\|r_k\| &\leqslant (1-\alpha_k)\|\eta\| + \alpha_k \|\nabla F(p_k)^{-1}\| \|\mathcal{P}_\gamma^{0\leftarrow 1} F(p^*) - F(p_k) - \nabla F(p_k)\eta\| + \alpha_k \|\nabla F(p_k)^{-1}\| \|\hat{e}\| \mu_k \\
&\leqslant \frac{1}{a_0}(1-\alpha_k) d(p_k, p^*) + \alpha_k \|\nabla F(p_k)^{-1}\| c_2 d^2(p_k, p^*) + \alpha_k \|\nabla F(p_k)^{-1}\| \|\hat{e}\| \mu_k \\
&\leqslant \frac{1}{a_0}(1-\alpha_k) d(p_k, p^*) + \Xi c_2 d^2(p_k, p^*) + \Xi \|\hat{e}\| \mu_k. \text{ (since } 0 < \alpha_k \leqslant 1)
\end{aligned}$$

Combining the above, we conclude that

$$d(p_{k+1}, p^*) \leqslant \kappa_1 (1-\alpha_k) d(p_k, p^*) + \kappa_2 d^2(p_k, p^*) + \kappa_3 \mu_k \tag{17}$$

for some positive constants $\kappa_1, \kappa_2, \kappa_3$. On the other hand, by [15, Lemma 14.5], we have

$$\|F(p_k)\| = O(d(p_k, p^*)). \tag{18}$$

In what follows, we prove assertions (i) and (ii).

(i) Suppose that $\alpha_k \to 1$ and $\mu_k = o(\|F(p_k)\|)$, which together imply $\mu_k = o(d(p_k, p^*))$. By (17), we have

$$\frac{d(p_{k+1}, p^*)}{d(p_k, p^*)} \leqslant \kappa_1 (1-\alpha_k) + \kappa_2 d(p_k, p^*) + \kappa_3 \frac{\mu_k}{d(p_k, p^*)}, \tag{19}$$

and we can take δ sufficiently small and k sufficiently large, if necessary, to conclude that $d(p_{k+1}, p^*) < \frac{1}{2} d(p_k, p^*) < \delta$. Thus, $p_{k+1} \in B_\delta(p^*)$. By mathematical induction, it is easy to show that the sequence $\{p_k\}$ is well defined and converges to p^*. Taking the limit of both sides of (19) proves superlinear convergence.

(ii) Again, we start from (17) and rewrite it as:

$$d(p_{k+1}, p^*) = (1-\alpha_k) O(d(p_k, p^*)) + O(d^2(p_k, p^*)) + O(\mu_k). \tag{20}$$

Suppose that $1-\alpha_k = O(\|F(p_k)\|)$ and $\mu_k = O(\|F(p_k)\|^2)$. Using (18), the above reduces to $d(p_{k+1}, p^*) = O(d^2(p_k, p^*))$. This implies that there exists a positive constant ν such that $d(p_{k+1}, p^*) \leqslant \nu d^2(p_k, p^*)$, and hence, $d(p_{k+1}, p^*) \leqslant \nu d^2(p_k, p^*) \leqslant \nu \delta^2 < \delta$, if δ is sufficiently small. Again, by mathematical induction, the sequence $\{p_k\}$ converges to p^* quadratically. □

The next lemma shows the relationship between the parameter γ_k and step size α_k.

Lemma 4.2. *Let (A1), (A4) hold. Suppose that the step size α_k is as in Algorithm 2, i.e., (10). Define a constant, $\Pi := 2\max\left\{\max_i \left\{\frac{1}{(s^*)_i} \mid (s^*)_i > 0\right\}, \max_i \left\{\frac{1}{(z^*)_i} \mid (z^*)_i > 0\right\}\right\}$. For $\gamma_k \in (0,1)$, if $\Pi \|\Delta w_k\| \leqslant \gamma_k$, then $0 \leqslant 1 - \alpha_k \leqslant (1-\gamma_k) + \Pi \|\Delta w_k\|$.*

Proof. Note that the fourth line of (13) yields $S_k^{-1}\Delta s_k + Z_k^{-1}\Delta z_k = \mu_k (S_k Z_k)^{-1} e - e$, which is exactly the same as in the usual interior point method in the Euclidean setting. Thus, the proof entails directly applying [35, Lemma 3 and 4] for the Euclidean case to the Riemannian case. □

Now, let us establish the local convergence of RIPM in a way that almost replicates the perturbed damped Newton method except for taking care of γ_k. We consider next stronger assumption:

(A2′) Smoothness of C^3. The functions f, g, and h are C^3 on $\mathbb{M}$.

From Lemma 3.6, (A2′) implies that the covariant derivative ∇F is locally Lipschitz continuous at w^*, where F is the associated KKT vector field. Hence, from Proposition 2.3, (A1)-(A5) and (A2′) together show that F satisfies (B1)-(B3).

Theorem 4.3 (Locally quadratic convergence of RIPM). *Consider Algorithm 2 for solving problem (RCOP). Let (A1)-(A5) and (A2′) hold at w^*. Choose $\mu_k = O(\|F(w_k)\|^2)$ and $1-\gamma_k = O(\|F(w_k)\|)$. Then, there exists a positive constant δ such that, for all $d(w_0, w^*) < \delta, w_0 \in \mathscr{M}$, the sequence $\{w_k\}$ is well defined and converges quadratically to w^*.*

Proof. Suppose that $d(w_k, w^*) < \delta$ for sufficiently small δ. Since F satisfies assumptions (B1)-(B3), from the proof of Proposition 4.1 and equation (20), we also have $d(w_{k+1}, w^*) = (1-\alpha_k)O(d(w_k, w^*)) + O(d^2(w_k, w^*)) + O(\mu_k)$. Since $\mu_k = O(\|F(w_k)\|^2)$, and $\|F(w_k)\| = O(d(w_k, w^*))$ by equation (18), we obtain $\mu_k = O(d^2(w_k, w^*))$. Thus, we have

$$\begin{aligned}\|\Delta w_k\| &= \left\|\nabla F(w_k)^{-1}(-F(w_k) + \mu_k \hat{e})\right\| \leqslant \Xi(\|F(w_k)\| + \mu_k\|\hat{e}\|) \text{ (by Lemma 3.12)} \\ &\leqslant O(\|F(w_k)\|) + O(\mu_k) = O(d(w_k, w^*)) + O(d^2(w_k, w^*)) = O(d(w_k, w^*)).\end{aligned}$$

Since δ is sufficiently small, from the above inequalities, the conditions of Lemma 4.2 are satisfied. Notice that $\hat{\gamma}$ is bounded below by some constant $\hat{\gamma}$ in Algorithm 2. Hence, we have $0 \leqslant 1-\alpha_k \leqslant (1-\gamma_k) + \Pi\|\Delta w_k\| = (1-\gamma_k) + O(d(w_k, w^*))$. Finally, we have $d(w_{k+1}, w^*) = (1-\alpha_k)O(d(w_k, w^*)) + O(d^2(w_k, w^*)) + O(\mu_k) \leqslant [(1-\gamma_k) + O(d(w_k, w^*)]O(d(w_k, w^*)) + O(d^2(w_k, w^*)) + O(d^2(w_k, w^*)) = O(d^2(w_k, w^*))$. □

The next theorem can be proven similarly.

Theorem 4.4 (Locally superlinear convergence of RIPM). *Let the assumptions of Theorem 4.3 hold. Choose $\mu_k = o(\|F(w_k)\|)$ and $\gamma_k \to 1$. Then, there exists a positive constant δ such that, for all $d(w_0, w^*) < \delta, w_0 \in \mathscr{M}$, the sequence $\{w_k\}$ is well defined and converges superlinearly to w^*.*

5 Global-Line-Search RIPM Algorithm

In this section, we describe a globally convergent Riemannian interior point algorithm with the classical line search [10] and merit function $\varphi(w) = \|F(w)\|^2$.

At the current point $w = (x, y, z, s)$ and direction $\Delta w = (\Delta x, \Delta y, \Delta z, \Delta s)$, the subsequent iterate is calculated along a curve on $\mathscr{M}$, i.e., $w(\alpha) := \bar{R}_w(\alpha \Delta w)$, for some step length $\alpha > 0$. By introducing $w(\alpha) = (x(\alpha), y(\alpha), z(\alpha), s(\alpha))$, we have $x(\alpha) = R_x(\alpha \Delta x)$, $y(\alpha) = y + \alpha \Delta y$, $z(\alpha) = z + \alpha \Delta z$, and $s(\alpha) = s + \alpha \Delta s$. For a given starting point $w_0 = (x_0, y_0, z_0, s_0)$ with $x_0 \in \mathbb{M}, (z_0, s_0) > 0$, let

$$\tau_1 := \min(Z_0 S_0 e)/(z_0^T s_0/m), \tau_2 := z_0^T s_0/\|F(w_0)\| .$$

Define $f^I(\alpha) := \min(Z(\alpha)S(\alpha)e) - \gamma\tau_1 z(\alpha)^T s(\alpha)/m$, $f^{II}(\alpha) := z(\alpha)^T s(\alpha) - \gamma\tau_2\|F(w(\alpha))\|$, where $\gamma \in (0,1)$ is a constant. Notice that the functions $f^i(\alpha), i = I, II$, depend on the iteration count k, though for simplicity we have omitted the subscript k. For $i = I, II$, define

$$\alpha_k^i := \max_{\alpha \in (0,1]} \{\alpha : f^i(t) \geqslant 0, \text{ for all } t \in (0, \alpha]\}; \tag{21}$$

i.e., α_k^i are either one or the smallest positive root for the functions $f^i(\alpha)$ in $(0, 1]$.

Define a merit function $\varphi : \mathscr{M} \to \mathbb{R}$ by $\varphi(w) := \|F(w)\|_w^2$; then, $\operatorname{grad}\varphi(w) = 2\nabla F(w)^* F(w)$. Note that $\|F(w)\|_w^2 = \|\operatorname{grad}_x L(w)\|_x^2 + \|h(x)\|_2^2 + \|g(x) + s\|_2^2 + \|ZSe\|_2^2$. Moreover, for any nonnegative $z, s \in \mathbb{R}^m$, one has $\|ZSe\|_2 \leqslant z^T s = \|ZSe\|_1 \leqslant \sqrt{m}\|ZSe\|_2$. Hence,

$$\|ZSe\|_2/\sqrt{m} \leqslant \frac{z^T s}{\sqrt{m}} \leqslant \|ZSe\|_2 \leqslant \|F(w)\|_w. \tag{22}$$

Now, we describe the globally convergent Riemannian interior point method.

Algorithm 4 (Global RIPM Algorithm).

(Step 0) Set $w_0 = (x_0, y_0, z_0, s_0)$ with $(z_0, s_0) > 0$; and $\theta \in (0,1), \beta \in (0, 1/2], \gamma_{k-1} \in (1/2, 1), k \leftarrow 0$.
(Step 1) Check whether w_k satisfies a stopping test for (RCOP).
(Step 2) Choose $\sigma_k \in (0,1)$; for w_k, compute the Newton direction Δw_k by solving

$$\nabla F(w_k)\Delta w_k = -F(w_k) + \sigma_k \rho_k \hat{e}, \tag{23}$$

where $z_k^T s_k/m \leqslant \rho_k \leqslant \|F(w_k)\|/\sqrt{m}$.

(Step 3) Step length selection.

(3a) *Centrality conditions*: Let $0.5 < \gamma_k < \gamma_{k-1}$; compute $\bar{\alpha}_k = \min\{\alpha_k^I, \alpha_k^{II}\}$ from (21);

(3b) *Sufficient decrease*: Let $\alpha_k = \theta^t \bar{\alpha}_k$, where t is the smallest nonnegative integer such that

$$\varphi(\bar{R}_{w_k}(\alpha_k \Delta w_k)) - \varphi(w_k) \leqslant \alpha_k \beta \langle \operatorname{grad} \varphi_k, \Delta w_k \rangle . \tag{24}$$

(Step 4) Let $w_{k+1} = \bar{R}_{w_k}(\alpha_k \Delta w_k)$. $k \leftarrow k+1$. Return to Step 1.

With a slight abuse of notation, at the current point w and direction Δw, we define a real-to-real function $\alpha \mapsto \varphi(\alpha)$ by $\varphi(\alpha) := \varphi(\bar{R}_w(\alpha \Delta w))$; then, it follows from the definition of a retraction on a manifold and the chain rule that the derivative of $\varphi(\alpha)$ at $\alpha = 0$ is $\varphi'(0) = \mathcal{D}\varphi(\bar{R}_w(0)) \left[\mathcal{D}\bar{R}_w(0)[\Delta w]\right] = \mathcal{D}\varphi(w)[\Delta w] = \langle \operatorname{grad} \varphi(w), \Delta w \rangle$. Hence, at the k-th point w_k, we have $\varphi_k'(0) = \langle \operatorname{grad} \varphi(w_k), \Delta w_k \rangle$, and condition (24) is to say $\varphi_k(\alpha_k) - \varphi_k(0) \leqslant \alpha_k \beta \varphi_k'(0)$. When $\varphi_k'(0) < 0$, this implies that the backtracking loop of Step (3b) is well defined [26, Lemma 3.1]. The next lemma shows the condition under which the Newton step Δw generated by (23) ensures the descent of the merit function.

Lemma 5.1. *If the direction Δw is the solution of equation (23), then*

$$\langle \operatorname{grad} \varphi(w), \Delta w \rangle = 2(-\|F(w)\|^2 + \sigma \rho z^T s). \tag{25}$$

In this case, Δw is a descent direction for φ at w if and only if $\rho < \|F(w)\|^2 / \sigma z^T s$.

Proof. Let Δw be the solution of (23) and note that $\langle F(w), \hat{e} \rangle = \langle ZSe, e \rangle = z^T s$. Then, $\langle \operatorname{grad} \varphi(w), \Delta w \rangle = \langle 2 \nabla F(w)^* F(w), \Delta w \rangle = 2 \langle F(w), \nabla F(w) \Delta w \rangle = 2 \langle F(w), -F(w) + \sigma \rho \hat{e} \rangle = 2(-\langle F(w), F(w) \rangle + \sigma \rho \langle F(w), \hat{e} \rangle) = 2(-\|F(w)\|^2 + \sigma \rho z^T s)$. □

Proposition 5.2. *If $\|F(w_k)\| \neq 0$, then the direction Δw_k generated by Algorithm 4 is a descent direction for the merit function φ at w_k. Moreover, if the Armijo condition (24) is satisfied, then*

$$\varphi_k(\alpha_k) \leqslant [1 - 2\alpha_k \beta (1 - \sigma_k)] \, \varphi_k(0). \tag{26}$$

Thus, the sequence $\{\varphi_k\}$ is monotonically decreasing.

Proof. Note that the iteration index k is omitted in what follows. Suppose that $\rho \leqslant \|F(w)\| / \sqrt{m}$ and Δw is given by (23), we have (23), we have

$$\begin{aligned} &\varphi'(0) = \langle \operatorname{grad} \varphi(w), \Delta w \rangle = 2(-\varphi(w) + \sigma \rho z^T s) \text{ (by Lemma (5.1))} \\ &\leqslant 2(-\varphi(w) + \sigma \|F(w_k)\| \, z^T s / \sqrt{m}) \leqslant 2(-\varphi(w) + \sigma \|F(w_k)\|^2) = -2(1-\sigma)\varphi(w) < 0. \end{aligned} \tag{27}$$

Alternatively, by Lemma 5.1 it is sufficient to show that $\|F(w)\| / \sqrt{m} < \|F(w)\|^2 / \sigma z^T s$. Note that $\sigma z^T s < z^T s \leqslant \sqrt{m} \|F(w)\|$; then, $1/\sqrt{m} < \|F(w)\| / \sigma z^T s$. Multiplying both sides by $\|F(w)\|$ leads to the desired result. Therefore, Δw_k is a descent direction for the merit function φ at w_k.

Moreover, if condition (24) is satisfied, then by (27), we have $\varphi(\alpha) \leqslant \varphi(0) + \alpha \beta \langle \operatorname{grad} \varphi(w), \Delta w \rangle \leqslant \varphi(0) + \alpha \beta (-2(1-\sigma)\varphi(0)) = [1 - 2\alpha\beta(1-\sigma)]\varphi(0)$. □

Lemma 5.3. *Let $\{w_k\}$ be generated by Algorithm 4. Then, $\chi \varphi(w_k) \leqslant (z_k^T s_k)^2 \leqslant m \varphi(w_k)$, with $\chi := \tau_2^2 / 4$.*

Proof. From the definition of f^{II}, we have $z^T s \geqslant \gamma \tau_2 \|F(w)\| \geqslant \tau_2 / 2 \|F(w)\|$; then, $(z^T s)^2 \geqslant (\tau_2/2)^2 \varphi(w)$. The second inequality follows from (22). □

6 Global Convergence of RIPM

We will show global convergence by following the proof procedure of the *Euclidean* interior point method (EIPM) in [10]. In what follows, we will omit similar content because of space limitations and focus on the difficulties encountered when adapting the proof of EIPM to RIPM. In particular, we will make these difficulties as tractable as in EIPM by proving a series of propositions. Given $\epsilon \geqslant 0$, let us define the set,

$$\Omega(\epsilon) := \left\{ w \in \mathscr{M} : \epsilon \leqslant \varphi(w) \leqslant \varphi_0, \min(ZSe)/(z^T s/m) \geqslant \tau_1/2, z^T s / \|F(w)\| \geqslant \tau_2/2 \right\}.$$

Note that $\Omega(\epsilon)$ is a closed subset of $\mathscr{M}$; $\Omega(\epsilon) \subseteq \Omega(0)$ for any $\epsilon > 0$. We will establish the global convergence of Algorithm 4 under the following assumptions.

Assumption 3.

(C1) In the set $\Omega(0)$, the functions f, h, and g are smooth; the gradients of the equality constraints are linearly independent vector fields, i.e., $\{\operatorname{grad} h_i(x)\}_{i=1}^{l}$ is linearly independent in $T_x\mathbb{M}$ for all x; and $w \mapsto \nabla F(w)$ is Lipschitz continuous;
(C2) The sequences $\{x_k\}$ and $\{z_k\}$ are bounded [9, 3];
(C3) In any compact subset of $\Omega(0)$ where s is bounded away from zero, $\nabla F(w)$ is nonsingular.

6.1 Continuity of Some Special Scalar Fields

To show the boundedness of the sequences generated by Algorithm 4, we need to prove the continuity of some special functions. The claims of section 6.1 are trivial if $\mathbb{M} \equiv \mathbb{R}^d$, but they need to be treated carefully for general $\mathbb{M}$.

If we assign a linear operator $A_x : T_x\mathcal{M} \to T_x\mathcal{M}$ to each point $x \in \mathcal{M}$, then the function $x \mapsto \|A_x\|$ is a scalar field on $\mathcal{M}$, and we should be aware that the operator norm $\|\cdot\|$ depends on x. Lemma 6.1 shows that $x \mapsto \|\hat{A}_x\|_2$ and $x \mapsto \|\hat{A}_x\|_F$ are also well-defined scalar fields, where $\hat{A}_x$ denotes the representing matrix of A_x with respect to an arbitrary orthonormal basis of $T_x\mathcal{M}$.

Lemma 6.1. *Let $\mathcal{M}$ be an n-dimensional Riemannian manifold. Let $x \in \mathcal{M}$ and A_x be a linear operator on $T_x\mathcal{M}$. Choose a basis of $T_x\mathcal{M}$ that is orthonormal with respect to the inner product $\langle\cdot,\cdot\rangle_x$, and let $\hat{A}_x$ denote the representing matrix of A_x under the basis. Then, the values $\|\hat{A}_x\|_2$ and $\|\hat{A}_x\|_F$ are invariant under a change of orthonormal basis; moreover, $\|A_x\| = \|\hat{A}_x\|_2 \leqslant \|\hat{A}_x\|_F$.*

Proof. Suppose that there are two orthonormal bases $\{E_i\}_{i=1}^n, \{E_i'\}_{i=1}^n$ on $T_x\mathcal{M}$. With respect to them, let $P \in \mathbb{R}^{n\times n}$ denote the change-of-basis matrix, i.e., $[P]_{kj} := \langle E_j', E_k\rangle_x$, for $1 \leqslant k, j \leqslant n$, and hence, P is orthogonal. Let $\hat{A}_x, \hat{A}_x' \in \mathbb{R}^{n\times n}$ denote the representing matrices of A_x under the two bases, respectively. We have $\hat{A}_x' = P^{-1}\hat{A}_x P$. Then, $\|\hat{A}_x'\| = \|P^{-1}\hat{A}_x P\| = \|\hat{A}_x\|$ holds for the Frobenius norm or the spectral norm. Therefore, the values $\|\hat{A}_x\|_2$ and $\|\hat{A}_x\|_F$ are invariant under a change of orthonormal basis.

Now, consider an orthonormal basis $\{E_i\}_{i=1}^n$ on $T_x\mathcal{M}$. For $y \in T_x\mathcal{M}$, its representing vector $\hat{y} \in \mathbb{R}^n$ is defined by $y = \sum_{i=1}^n \hat{y}_i E_i$. Accordingly, we have $\widehat{A_x y} = \hat{A}_x\hat{y}$, i.e., $A_x y = \sum_{i=1}^n (\hat{A}_x\hat{y})_i E_i$, and from the orthonormal property of the basis, we have $\|A_x y\|_x^2 = \|\sum_{i=1}^n (\hat{A}_x\hat{y})_i E_i\|_x^2 = \sum_{i=1}^n (\hat{A}_x\hat{y})_i^2 = \|\hat{A}_x\hat{y}\|_2^2$. Thus, $\|A_x y\|_x = \|\hat{A}_x\hat{y}\|_2$ for any y. Finally, $\|A_x\| = \sup_{y\in T_x\mathcal{M}, \|y\|_x=1} \|A_x y\|_x = \sup_{\hat{y}\in\mathbb{R}^n, \|\hat{y}\|_2=1} \|\hat{A}_x\hat{y}\|_2 = \|\hat{A}_x\|_2$. □

Yet, we have not reached clarity on the continuity of the scalar field $x \mapsto \|A_x\|$. The following proposition proves the continuity of an important case of A_x that appears in our problem (RCOP).

Proposition 6.2. *Consider (RCOP). Let $\widehat{\mathrm{H}} f(x) \in \mathrm{Sym}(d)$ denote the representing matrix of $\operatorname{Hess} f(x)$ with respect to an arbitrary orthonormal basis of $T_x\mathbb{M}$. Then, $x \mapsto \|\widehat{\mathrm{H}} f(x)\|$ is a* continuous *scalar field on $\mathbb{M}$, for the Frobenius norm or the spectral norm. Moreover, $x \mapsto \|\operatorname{Hess} f(x)\|$ is a* continuous *scalar field on $\mathbb{M}$.*

Proof. Lemma 6.1 shows that the scalar field $x \mapsto \|\widehat{\mathrm{H}} f(x)\|$ is well defined, so it suffices to prove its continuity. From [22, Corollary 13.8], for each $\bar{x} \in \mathbb{M}$ there is a smooth, orthonormal local frame $\{E_i\}_{i=1}^d$ on a open neighborhood $\mathcal{U}$ of $\bar{x}$; namely, $\{E_1(x), \ldots, E_d(x)\}$ forms an orthonormal basis on $T_x\mathbb{M}$ for all $x \in \mathcal{U}$. Choose such a local frame $\{E_i\}_{i=1}^d$ around $\bar{x}$; then, the representing matrix of $\operatorname{Hess} f(x)$ is given by,

$$[\widehat{\mathrm{H}} f(x)]_{kj} := \langle \operatorname{Hess} f(x)[E_j(x)], E_k(x)\rangle_x = \big\langle (\nabla_{E_j} \operatorname{grad} f)(x), E_k(x)\big\rangle_x \text{ for } 1 \leqslant k, j \leqslant d.$$

The last equality comes from the definition of $\operatorname{Hess} f(x)$. Now, from the smoothness of Riemannian metric, see (16), it follows that $x \mapsto \widehat{\mathrm{H}} f(x)$ is a continuous function from $\mathcal{U} \subseteq \mathbb{M}$ to $\mathrm{Sym}(d)$. Since any matrix norm is continuous, $\|\widehat{\mathrm{H}} f(x)\|$ is continuous on $\mathcal{U}$ containing $\bar{x}$. This argument holds for any $\bar{x} \in \mathbb{M}$. □

The above result can be applied verbatim to the Hessian of $\{h_i\}_{i=1}^l, \{g_i\}_{i=1}^m$ in (RCOP).

Proposition 6.3. *For a given $w = (x, y, z, s) \in \mathscr{M}$, consider the operator $\nabla F(w)$ in (5). Let $\{E_i\}_{i=1}^d$ be an orthonormal basis of $T_x\mathbb{M}$ and $\{e_i\}_{i=1}^l$, $\{\dot{e}_i\}_{i=1}^m$ be the natural bases of $\mathbb{R}^l, \mathbb{R}^m$, respectively. If we choose an orthonormal basis of $T_w\mathscr{M}$ of the form,*

$$\{(E_i, 0, 0, 0)\}_{i=1}^d \cup \{(0_x, e_i, 0, 0)\}_{i=1}^l \cup \{(0_x, 0, \dot{e}_i, 0)\}_{i=1}^m \cup \{(0_x, 0, 0, \dot{e}_i)\}_{i=1}^m, \tag{28}$$

then, the representing matrix of $\nabla F(w)$ is of order $(d+l+2m)$ as below:

$$\hat{\nabla} F(w) = \begin{pmatrix} Q & B & C & 0 \\ B^T & 0 & 0 & 0 \\ C^T & 0 & 0 & I \\ 0 & 0 & S & Z \end{pmatrix},$$

where $Q := Q(w) \in \mathrm{Sym}(d)$ is given by, $[Q]_{kj} := \langle \mathrm{Hess}_x L(w) E_j, E_k \rangle_x$, for $1 \leqslant k, j \leqslant d$, and $B := B(x) = [\widehat{\mathrm{grad}} h_1(x), \cdots, \widehat{\mathrm{grad}} h_l(x)] \in \mathbb{R}^{d\times l}, C := C(x) = [\widehat{\mathrm{grad}} g_1(x), \cdots, \widehat{\mathrm{grad}} g_m(x)] \in \mathbb{R}^{d\times m}$, and the "hat" symbol means the corresponding representing vector under the basis $\{E_i\}_{i=1}^d$.

Moreover, there is a continuous scalar field $T(w)$ on $\mathscr{M}$ such that for any w, $\|Q(w)\|_F \leqslant T(w)$. $x \mapsto \|B(x)\|_F$ and $x \mapsto \|C(x)\|_F$ are continuous scalar fields.

Proof. The matrix $\hat{\nabla} F(w)$ under the basis (28) is obtained by a trivial process, so we will omit its description. By (4), we have $\mathrm{Hess}_x \mathcal{L}(w) = \mathrm{Hess}\, f(x) + \sum_{i=1}^l y_i \,\mathrm{Hess}\, h_i(x) + \sum_{i=1}^m z_i \,\mathrm{Hess}\, g_i(x)$. By the linearity of the representing matrix, $Q(w) := \hat{\mathrm{H}}_x \mathcal{L}(w) = \hat{\mathrm{H}} f(x) + \sum_{i=1}^l y_i \hat{\mathrm{H}} h_i(x) + \sum_{i=1}^m z_i \hat{\mathrm{H}} g_i(x)$, under the basis $\{E_i\}_{i=1}^d$. Thus,

$$\|Q(w)\|_F \leqslant \|\hat{\mathrm{H}} f(x)\|_F + \sum_{i=1}^l |y_i| \|\hat{\mathrm{H}} h_i(x)\|_F + \sum_{i=1}^m |z_i| \|\hat{\mathrm{H}} g_i(x)\|_F =: T(w).$$

From Proposition 6.2, $\|\hat{\mathrm{H}} f(x)\|_F$, $\|\hat{\mathrm{H}} h_i(x)\|_F$, and $\|\hat{\mathrm{H}} g_i(x)\|_F$ are all continuous functions of x. It follows from the definition of $T(w)$ that T is continuous. As for $B(x)$, since the basis $\{E_i\}_{i=1}^d$ is orthonormal, $\|B(x)\|_F^2 = \sum_{i=1}^l \|\widehat{\mathrm{grad}} h_i(x)\|_2^2 = \sum_{i=1}^l \|\mathrm{grad}\, h_i(x)\|_x^2$, which implies continuity. □

Proposition 6.4. *Consider (RCOP) and the linear operators $\mathcal{H}_x$ and $\mathcal{G}_x$ defined in (11). Then, $x \mapsto \|\mathcal{H}_x\|$ and $x \mapsto \|\mathcal{G}_x\|$ are continuous scalar fields on $\mathbb{M}$.*

Proof. We will only prove the result for $\|\mathcal{H}_x\|$. First, we claim that $\|\mathcal{H}_x\| = \|\hat{\mathcal{H}}_x\|_2$ for each $x \in \mathbb{M}$, where $\hat{\mathcal{H}}_x$ denotes the representing matrix of $\mathcal{H}_x$ with respect to an arbitrary orthonormal basis of $T_x\mathbb{M}$ and natural basis $\{e_i\}_{i=1}^l$ of $\mathbb{R}^l$. The proof of this claim is very similar to the context of Lemma 6.1, so we will omit it.

Next, by [22, Corollary 13.8], for each $\bar{x} \in \mathbb{M}$ there is a smooth, local orthonormal frame $\{E_i\}_{i=1}^d$ on a neighborhood $\mathcal{U}$ of $\bar{x}$. Choose such a local frame $\{E_i\}_{i=1}^d$ around $\bar{x}$. The representing matrix of $\mathcal{H}_x$ with respect to $\{E_i(x)\}_{i=1}^d$ and $\{e_i\}_{i=1}^l$ is $\hat{\mathcal{H}}_x \in \mathbb{R}^{d\times l}$ given by, for $1 \leqslant k \leqslant d, 1 \leqslant j \leqslant l$,

$$[\hat{\mathcal{H}}_x]_{kj} = \langle \mathcal{H}_x e_j, E_k(x) \rangle_x = \langle \mathrm{grad}\, h_j(x), E_k(x) \rangle_x . \tag{29}$$

Again, by the smoothness of the Riemannian metric, it follows that $x \mapsto \hat{\mathcal{H}}_x$ is a continuous function from $\mathcal{U} \subseteq \mathbb{M}$ to $\mathbb{R}^{d\times l}$. The remaining argument is essentially the same as that of Proposition 6.2. □

6.2 Convergence Theorem

Owing to the good theoretical properties of the exponential map, in this section we will assume that $R \equiv \exp$ in Algorithm 4 and denote $\bar{R}$ by $\bar{\exp}$.

Proposition 6.5 (Boundedness of the sequences)**.** *Let $\{w_k\}$ be a sequence generated by Algorithm 4 and suppose that Assumptions (C1)∼(C3) hold. If $\epsilon > 0$ and $w_k \in \Omega(\epsilon)$ for all k, then*

(a) $\{z_k^T s_k\}$ and $\{(z_k)_i (s_k)_i\}, i = 1, \ldots, m$, are all bounded above and below away from zero.

(b) $\{z_k\}$ and $\{s_k\}$ are bounded above and component-wise bounded away from zero;

(c) $\{w_k\}$ is bounded; (d) $\{\|\nabla F(w_k)^{-1}\|\}$ is bounded; (e) $\{\Delta w_k\}$ is bounded.

Proof. The proofs in [10, Lemma 6.1] and/or [3, Theorem 2 (a)] can be applied verbatim to (a) and (b).

(c) On the basis of the previous result (b), it suffices to prove that $\{y_k\}$ is bounded. The index k is omitted in what follows. By using the notation $\mathcal{H}_x$ and $\mathcal{G}_x$ as defined in (11), we have

$$\mathcal{H}_x y = \mathrm{grad}_x \mathcal{L}(w) - \mathrm{grad}\, f(x) - \mathcal{G}_x z =: b. \tag{30}$$

By (C1), $\mathrm{rank}\, \mathcal{H}_x = l$, i.e., $\mathcal{H}_x$ is an injection. There exists the unique solution to (30). We have

$$y = \left[(\mathcal{H}_x^* \mathcal{H}_x)^{-1} \mathcal{H}_x^* \right] (\mathrm{grad}_x \mathcal{L}(w) - \mathrm{grad}\, f(x) - \mathcal{G}_x z) . \tag{31}$$

Define $\mathcal{C}_x : T_x\mathbb{M} \to \mathbb{R}^l$ as $\mathcal{C}_x := \left(\mathcal{H}_x^* \mathcal{H}_x\right)^{-1} \mathcal{H}_x^*$. Under the basis of $T_x\mathbb{M}$ and natural basis $\{e_i\}_{i=1}^l$ of $\mathbb{R}^l$, if $\hat{\mathcal{H}}_x$ is the matrix corresponding to $\mathcal{H}_x$, then $\hat{\mathcal{C}}_x = (\hat{\mathcal{H}}_x^T \hat{\mathcal{H}}_x)^{-1} \hat{\mathcal{H}}_x^T$. It is easy to show that $\|\mathcal{C}_x\| = \|\hat{\mathcal{C}}_x\|_2$ for any x. Recall that in the proof of Proposition 6.4, (29) shows that for each $\bar{x} \in \mathbb{M}$ there is a neighborhood $\mathcal{U}$ of $\bar{x}$ such that $x \mapsto \hat{\mathcal{H}}_x$ is continuous over $\mathcal{U}$. Then, by function composition, $x \mapsto \hat{\mathcal{C}}_x$ is also continuous over $\mathcal{U}$. This shows that $\|\mathcal{C}_x\| = \|\hat{\mathcal{C}}_x\|_2$ is continuous at each $\bar{x}$, and hence, on $\mathbb{M}$. Finally, again by Proposition 6.4, $\|\mathcal{C}_x\|$, $\|\mathrm{grad}\, f(x)\|$, and $\|\mathcal{G}_x\|$ are all continuous on $\mathbb{M}$. Because $\{x_k\}$ is bounded, by (31) we have

$$\|y_k\| \leqslant \|\mathcal{C}_{x_k}\| \left(\|\mathrm{grad}_x\, \mathcal{L}(w_k)\| + \|\mathrm{grad}\, f(x_k)\| + \|\mathcal{G}_{x_k}\| \|z_k\|\right) \leqslant c_1 \left(\sqrt{\varphi_0} + c_2 + c_3 \|z_k\|\right),$$

for some positive constants c_1, c_2, c_3. Then, $\{y_k\}$ is bounded because $\{z_k\}$ is bounded.

(d) For each w_k, choose an arbitrary orthonormal basis of $T_{w_k}\mathscr{M}$. If the matrix $\widehat{\nabla} F(w_k)$ corresponds to $\nabla F(w_k)$, then $\widehat{\nabla} F(w_k)^{-1}$ corresponds to $\nabla F(w_k)^{-1}$. By Lemma 6.1, we have $\left\|\nabla F(w_k)^{-1}\right\| \leqslant \|\widehat{\nabla} F(w_k)^{-1}\|_F$; thus, it is sufficient to show that $\{\|\widehat{\nabla} F(w_k)^{-1}\|_F\}$ is bounded. For convenience, we will choose the basis of $T_{w_k}\mathscr{M}$ given in (28). Then, we have $\widehat{\nabla} F(w_k) = \begin{pmatrix} Q_k & B_k & C_k & 0 \\ B_k^T & 0 & 0 & 0 \\ C_k^T & 0 & 0 & I \\ 0 & 0 & S_k & Z_k \end{pmatrix}$. By Proposition 6.3, there is a continuous scalar field $T(w)$ on $\mathscr{M}$ such that $\|Q(w)\|_F \leqslant T(w)$; and $\|B(x)\|_F$, $\|C(x)\|_F$ are continuous on $\mathbb{M}$. It follows from the boundedness of $\{x_k\}$, $\{w_k\}$ that for all k, $\|Q_k\|_F = \|Q(w_k)\|_F \leqslant T(w_k) \leqslant c_4$, $\|B_k\|_F = \|B(x_k)\|_F \leqslant c_5$, and $\|C_k\|_F = \|C(x_k)\|_F \leqslant c_6$, for some positive constants c_4, c_5, and c_6.

On the other hand, whichever basis is used in the form of (28), the structure of $\widehat{\nabla} F(w_k)$ and the properties of its block submatrices remain unchanged, e.g., symmetry of Q_k; full rank of B_k; identity matrix I in the third row; all zero matrices; diagonal matrices S_k, Z_k; etc. This ensures that we can obtain the desired result by performing an appropriate decomposition of $\widehat{\nabla} F(w_k)$. Up to this point, we have created all the conditions needed in the proof of the Euclidean version. We can make the claim that $\{\|\widehat{\nabla} F(w_k)^{-1}\|_F\}$ is bounded by applying the proofs in [10, Lemma 6.2] and/or [3, Theorem 2 (c)] directly.

(e) By (23), we have $\|\Delta w_k\| \leqslant \|\nabla F(w_k)^{-1}\| \left(\|F(w_k)\| + \|\sigma_k \rho_k \hat{e}\|\right)$. □

Lemma 6.6. *Let $\{w_k\}$ be a sequence generated by Algorithm 4 and let (C1)∼(C3) hold. If $\epsilon > 0$ and $w_k \in \Omega(\epsilon)$ for all k, $\{\sigma_k\}$ is bounded away from zero, and ρ_k is as in (23). Then, $\{\bar{\alpha}_k\}$ is bounded away from zero.*

Proof. Since $\bar{\alpha}_k = \min(\alpha_k^I, \alpha_k^{II})$, it is sufficient to show that $\{\alpha_k^I\}$ and $\{\alpha_k^{II}\}$ are bounded away from zero. Let us suppress the subscript k. For α_k^I, see [10, Lemma 6.3] and/or [8, Theorem 3.1]. The proofs in those references apply verbatim to the Riemannian case. On the other hand, for α_k^{II}, we need to adapt the proofs in references [10, 8], since Lipschitz continuity on a Riemannian manifold is more complicated. Recall that $w(\alpha) = \bar{\mathrm{exp}}_w(\alpha \Delta w)$. Fix $\alpha \Delta w$ and let $\mathcal{P}_\gamma$ be the parallel transport along the geodesic $c(t) = \bar{\mathrm{exp}}_w(t\alpha\Delta w)$. By the fundamental theorem of calculus in the Riemannian case (Lemma 3.10), we obtain

$$\begin{aligned}
&\mathcal{P}_c^{0\leftarrow 1} F(w(\alpha)) = F(w) + \alpha \nabla F(w)\Delta w - \alpha \nabla F(w)\Delta w + \int_0^1 \mathcal{P}_c^{0\leftarrow t} \nabla F(c(t)) \mathcal{P}_c^{t\leftarrow 0} \alpha \Delta w dt \\
&= F(w) + \alpha\left(\sigma\rho\hat{e} - F(w)\right) + \alpha \int_0^1 \left[\mathcal{P}_c^{0\leftarrow t} \nabla F(c(t)) \mathcal{P}_c^{t\leftarrow 0} - \nabla F(w)\right] \Delta w dt \text{ (by (23))} \\
&= (1-\alpha) F(w) + \alpha\sigma\rho\hat{e} + \alpha \int_0^1 \left[\mathcal{P}_c^{0\leftarrow t} \nabla F(c(t)) \mathcal{P}_c^{t\leftarrow 0} - \nabla F(w)\right] \Delta w dt.
\end{aligned}$$

Taking the norm on both sides above gives

$$\begin{aligned}
&\|F(w(\alpha))\| = \left\|\mathcal{P}_c^{0\leftarrow 1} F(w(\alpha))\right\| \text{ (since parallel transport is isometric)} \\
&\leqslant (1-\alpha)\|F(w)\| + \alpha\sigma\rho\|\hat{e}\| + \alpha \int_0^1 \left\|\mathcal{P}_c^{0\leftarrow t} \nabla F(c(t)) \mathcal{P}_c^{t\leftarrow 0} - \nabla F(w)\right\| \|\Delta w\| dt \\
&\leqslant (1-\alpha)\|F(w)\| + \alpha\sigma\rho\sqrt{m} + \alpha \int_0^1 L\|t\alpha\Delta w\| \|\Delta w\| dt \text{ (by the Lipschitz continuity of } \nabla F) \\
&= (1-\alpha)\|F(w)\| + \alpha\sigma\rho\sqrt{m} + \frac{L}{2}\alpha_k^{II}\|\Delta w\|^2.
\end{aligned}$$

The rest of the proof is the same as [10, Lemma 6.3] and/or [8, Theorem 3.1], so we will omit it. □

Theorem 6.7 (Global Convergence of RIPM). *Let $\{w_k\}$ be generated by Algorithm 4 and $\{\sigma_k\} \subseteq (0,1)$ be bounded away from zero and one. Let $\operatorname{grad}\varphi$ be Lipschitz continuous with constant κ. If (C1)∼(C3) hold, then $\{\|F(w_k)\|\}$ converges to zero, and for any limit point $w^* = (x^*, y^*, z^*, s^*)$ of $\{w_k\}$, x^* is a Riemannian KKT point of problem (RCOP).*

Proof. Note that $\{\|F(w_k)\|\}$ is monotonically decreasing, hence convergent. Suppose that $\{\|F(w_k)\|\}$ does not converge to zero. Then, there exists $\epsilon > 0$ such that $\{w_k\} \subseteq \Omega(\epsilon)$ for infinitely many k.

Case 1. For infinitely many k, if Step (3b) is executed with $\alpha_k = \bar{\alpha}_k$, it follows from Proposition 5.2 that $\varphi(w_{k+1})/\varphi(w_k) \leqslant [1 - 2\bar{\alpha}_k\beta(1-\sigma_k)]$. Since $\{\bar{\alpha}_k\}$ is bounded away from zero (see Lemma 6.6) and $\{\sigma_k\}$ is bounded away from one, then $\varphi(w_k) \to 0$; this is a contradiction.

Case 2. On the other hand, for infinitely many k, if $\alpha_k < \bar{\alpha}_k$, we have that $\alpha_k \leqslant \theta\bar{\alpha}_k$. Then, condition (24) at Step (3b) fails for an $\tilde{\alpha}_k$ with $\alpha_k < \tilde{\alpha}_k \leqslant \alpha_k/\theta = \theta^{t-1}\bar{\alpha}_k$. Notice that α_k/θ is the value corresponding to the last failure. Recall that the derivative of the real function $\varphi(\alpha) = \varphi\left(\bar{R}_w(\alpha\Delta w)\right)$ at some value α is

$$\varphi'(\alpha) = \mathcal{D}\varphi\left(\bar{\exp}_{w_k}(\alpha\Delta w_k)\right)\left[\mathcal{D}\bar{\exp}_{w_k}(\alpha\Delta w_k)[\Delta w_k]\right].$$

Applying the mean value theorem to $\varphi(\alpha)$ on the interval $[0, \tilde{\alpha}_k]$ yields $\exists\xi \in (0,1) : \tilde{\alpha}_k\varphi'(\xi\tilde{\alpha}_k) = \varphi(\tilde{\alpha}_k) - \varphi(0)$. Hence,

$$\begin{aligned}
&\tilde{\alpha}_k\beta\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle < \varphi(\tilde{\alpha}_k) - \varphi(0) = \tilde{\alpha}_k\varphi'(\xi\tilde{\alpha}_k) \text{ (since condition (24) fails for } \tilde{\alpha}_k)\\
=&\tilde{\alpha}_k\mathcal{D}\varphi(\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k))\left[\mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\Delta w_k]\right]\\
=&\tilde{\alpha}_k\left\langle\operatorname{grad}\varphi(\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)), \mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\Delta w_k]\right\rangle. \text{ (by the definition of the gradient)} \quad (32)
\end{aligned}$$

Note that

$$\begin{aligned}
&\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle = \langle\operatorname{grad}\varphi_k, \xi\tilde{\alpha}_k\Delta w_k\rangle/\xi\tilde{\alpha}_k\\
&= \left\langle\mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\operatorname{grad}\varphi_k], \mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\xi\tilde{\alpha}_k\Delta w_k]\right\rangle/\xi\tilde{\alpha}_k \text{ (by Lemma 3.13)}\\
&= \left\langle\mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\operatorname{grad}\varphi_k], \mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\Delta w_k]\right\rangle. \quad (33)
\end{aligned}$$

Subtracting $\tilde{\alpha}_k\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle$ from both sides of (32) and using equalities (33) gives

$$\begin{aligned}
&\tilde{\alpha}_k(\beta-1)\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle\\
&< \tilde{\alpha}_k\left[\left\langle\operatorname{grad}\varphi(\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)), \mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\Delta w_k]\right\rangle - \langle\operatorname{grad}\varphi_k, \Delta w_k\rangle\right]\\
&= \tilde{\alpha}_k\left\langle\operatorname{grad}\varphi(\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)) - \mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\operatorname{grad}\varphi_k], \mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\Delta w_k]\right\rangle\\
&\leqslant \tilde{\alpha}_k\left\|\operatorname{grad}\varphi(\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)) - \mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\operatorname{grad}\varphi_k]\right\|\left\|\mathcal{D}\bar{\exp}_{w_k}(\xi\tilde{\alpha}_k\Delta w_k)[\Delta w_k]\right\|\\
&\leqslant \tilde{\alpha}_k\kappa\left\|\xi\tilde{\alpha}_k\Delta w_k\right\|\left\|\Delta w_k\right\| \text{ (by Lipschitz continuity of grad } \varphi \text{ and Lemma 3.13)}\\
&= \kappa\xi\tilde{\alpha}_k^2\left\|\Delta w_k\right\|^2.
\end{aligned}$$

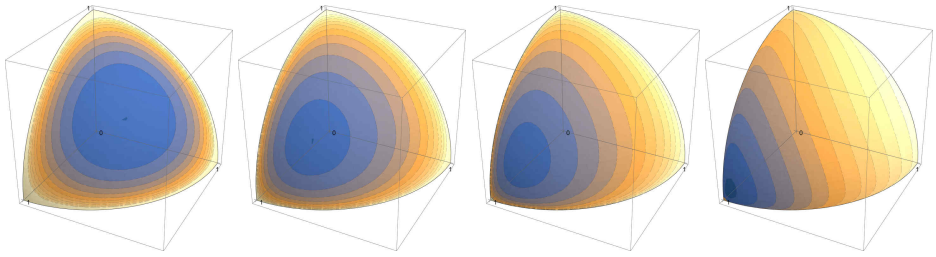
Regarding the above use of the Lipschitz continuity of a vector field $\operatorname{grad}\varphi$, we must note the fact that the differentiated retraction $\mathcal{T}^R$ defined by $\mathcal{T}^R_\eta(\xi) := \mathrm{D}R_x(\eta)[\zeta], x \in M, \eta, \zeta \in T_xM$, is a valid vector transport. In our case, $R = \bar{\exp}$. Finally, we obtain $\tilde{\alpha}_k(\beta-1)\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle < \kappa\xi\tilde{\alpha}_k^2\left\|\Delta w_k\right\|^2$. Consequently,

$$(\beta-1)\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle/(\kappa\xi\left\|\Delta w_k\right\|^2) < \tilde{\alpha}_k. \quad (34)$$

Because α_k satisfies (24) and $\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle < 0$, we have

$$\begin{aligned}
&\varphi_k(0) - \varphi_k(\alpha_k) \geqslant -\alpha_k\beta\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle \geqslant -\theta\beta\tilde{\alpha}_k\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle\\
&\geqslant -\theta\beta\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle(\beta-1)\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle/(\kappa\xi\left\|\Delta w_k\right\|^2) \text{ (by (34))}\\
&\geqslant [\theta\beta(1-\beta)/\kappa\xi]\left(\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle/\left\|\Delta w_k\right\|\right)^2 = \omega\left(\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle/\left\|\Delta w_k\right\|\right),
\end{aligned}$$

where $\omega(\cdot)$ is an F-function (see [28, Definition 14.2.1 & 14.2.2 in P479]). Since $\{\varphi_k\}$ is bounded below and $\varphi_k \geqslant \varphi_{k+1}$, it follows that $\lim_{k\to\infty}(\varphi_k - \varphi_{k+1}) = 0$. By the definition of F-functions, we obtain $\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle/\left\|\Delta w_k\right\| \to 0$. Since $\{\|\Delta w_k\|\}$ is bounded (Proposition 6.5), we have $\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle \to 0$. Choosing ρ_k with $z_k^T s_k/m \leqslant \rho_k \leqslant \|F(w_k)\|/\sqrt{m}$ implies that $\langle\operatorname{grad}\varphi_k, \Delta w_k\rangle/(-2) = \varphi_k - \sigma_k\rho_k z_k^T s_k \geqslant \varphi_k - \sigma_k\left\|F(w_k)\right\| z_k^T s_k/\sqrt{m} \geqslant \varphi_k - \sigma_k\left\|F(w_k)\right\|^2 \geqslant (1-\sigma_k)\varphi_k \to 0$. This shows that $\varphi(w_k) \to 0$, because $\{\sigma_k\}$ is bounded away from one; this is a contradiction. □



(a) (b) (c) (d)

Figure 1: Contour plots of logarithmic barrier function $B(x;\mu)$ of (SP) for (a) $\mu = 10$ (b) $\mu = 1$ (c) $\mu = 0.5$ (d) $\mu = 0.1$. The blue area indicates low values.

7 Numerical experiments

In the Euclidean setting, the interior point methods can be regarded as barrier methods. It would thus be beneficial to explore barrier methods in the Riemannian setting, although many nice results from the Euclidean setting may currently lack a corresponding proof in the Riemannian setting. Here, we will refer to the classic barrier method summarized in the review article [14].

7.1 An Intuitive Barrier Method on Manifolds

For simplicity, consider the all-inequality version of (RCOP) and let $c(x) \equiv -g(x)$,

$$\min_{x\in\mathbb{M}} \quad f(x) \quad \text{s.t.} \quad c(x) \geqslant 0. \tag{RCOP_Ineq}$$

Its logarithmic barrier function is given by

$$B(x;\mu) := f(x) - \mu \textstyle\sum_{i=1}^{m} \log c_i(x),$$

where $\mu > 0$ is the so-called barrier parameter. Note that the function $x \mapsto B(x;\mu)$ is differentiable on the open subset of $\mathbb{M}$, $\operatorname{strict}\mathcal{F} := \{x \in \mathbb{M} : c(x) > 0\}$. Its Riemannian gradient is

$$\operatorname{grad} B(x;\mu) = \operatorname{grad} f(x) - \textstyle\sum_{i=1}^{m} \frac{\mu}{c_i(x)} \operatorname{grad} c_i(x).$$

Intuition suggests that minimizing $B(x;\mu)$ for a sequence of $\mu > 0$ converging to zero will cause the unconstrained minimizers of $B(x;\mu)$ to converge to a minimizer of the original problem (RCOP_Ineq).

Algorithm 5 (Barrier Method on Manifolds).

(Step 0) Set $x_0 \in \mathbb{M}$ to a strictly feasible point, i.e., $c(x_0) > 0$, and set $\mu_0 > 0$ and $k \leftarrow 0$.
(Step 1) Check whether x_k satisfies a stopping test for (RCOP_Ineq).
(Step 2) Compute an unconstrained minimizer $x(\mu_k)$ of $B(x;\mu_k)$ with a warm starting point x_k.
(Step 3) $x_{k+1} \leftarrow x(\mu_k)$; choose $\mu_{k+1} < \mu_k$; $k \leftarrow k+1$. Return to Step 1.

Example 7.1. Consider the following simple problem on a sphere manifold, $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$,

$$\min_{x\in\mathbb{S}^2} \quad a^T x \quad \text{s.t.} \quad x \geqslant 0, \tag{SP}$$

where $a = [-1, 2, 1]^T$. Its solution is $x^* = [1, 0, 0]^T$. Now, check the KKT conditions at x (asterisks omitted below): $\operatorname{grad} f(x) = (I_n - xx^T)a = [0, 2, 1]^T$. The constraint $x \geqslant 0$ implies $c_i(x) = e_i^T x$ for $i = 1, 2, 3$; the active set $A(x) = \{2, 3\}$. $\operatorname{grad} c_1(x) = (I_n - xx^T)e_1 = [0, 0, 0]^T$; $\operatorname{grad} c_2(x) = (I_n - xx^T)e_2 = [0, 1, 0]^T$; $\operatorname{grad} c_3(x) = (I_n - xx^T)e_3 = [0, 0, 1]^T$. Clearly, the multipliers $z^* = [0, 2, 1]^T$, and LICQ and strict complementarity hold. Actually, (SP) satisfies the standard Riemannian assumptions 1.

Figure 1 shows the contour plots of $B(x;\mu)$ of (SP) as $\mu \to 0$. We can see that for sufficiently small μ, there is a unconstrained minimizer of the barrier function $B(x;\mu)$ in $\operatorname{strict}\mathcal{F} \subseteq \mathbb{M}$. Moreover, as shown in Figure 2, we successfully implemented Algorithm 5 for (SP). In particular, we set $x_0 = [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]^T$, $\mu_0 = 10$, $\mu_{k+1} = \mu_k/1.5$. In Step 2, for each subproblem, we use the trust regions

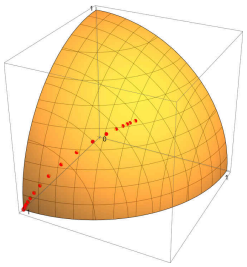


Figure 2: Iterates x_k of barrier method for (SP).

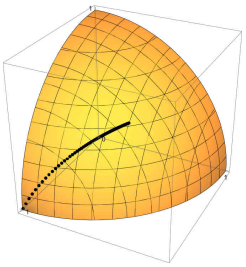


Figure 3: Existence of a central path for (SP).

solver in the Manopt toolbox [6] and set `options.tolgradnorm=1e-3; options.maxiter=10`. The outer iterations stop if $\mu_k < 10^{-10}$. Finally, we find that $\lim_{k\to\infty} x_k = x^*$ and that

$$\lim_{k\to\infty} \mu_k/c_1\,(x_k) = 0 = z^*_{(1)},\ \lim_{k\to\infty} \mu_k/c_2\,(x_k) = 2 = z^*_{(2)},\ \lim_{k\to\infty} \mu_k/c_3\,(x_k) = 1 = z^*_{(3)},$$

which are the notable features of the classical barrier method; see [14, Theorem 3.10 & 3.12]. Furthermore, if we denote the minimizer of $B(x;\mu)$ by either x_μ or $x(\mu)$, it must be that $\operatorname{grad} B(x_\mu;\mu) = 0$. A similar discussion to the one in Lemma 2.4 shows that there is a unique, smooth curve $x(\mu)$ on $\mathbb{M}$ for positive μ in a neighborhood of $\mu = 0$, and $\lim_{\mu\to 0_+} x(\mu) = x^*$. In particular, for (SP) we have that

$$\begin{cases} 0 = -\mu/x_{(1)} - x_{(1)}(-x_{(1)} + 2x_{(2)} + x_{(3)} - 3\mu) - 1, \\ 0 = -\mu/x_{(2)} - x_{(2)}(-x_{(1)} + 2x_{(2)} + x_{(3)} - 3\mu) + 2, \\ 0 = -\mu/x_{(3)} - x_{(3)}(-x_{(1)} + 2x_{(2)} + x_{(3)} - 3\mu) + 1. \end{cases}$$

Figure 3 plots the positive solutions of above $[x_{(1)}(\mu), x_{(2)}(\mu), x_{(3)}(\mu)]^T$ for different $\mu > 0$. Numerically, all solutions belong to the sphere and approach x^* as μ gets closer to zero. Not surprisingly, the iterative trajectory of x_k in Figure 2 is almost identical to the central path in Figure 3.

Here, we should emphasize that although we have not formally discussed barrier methods on manifold, these initial observations provide a motivation for expanding the scope of the interior point method to Riemannian manifolds.

7.2 Problems

Here, we model two problems within the framework of (RCOP) and use them to evaluate the performance of ours and various other algorithms. The problems involve three manifolds:

$$\begin{aligned} &\text{oblique manifold} \quad \mathrm{Ob}(n,k) \overset{\text{def}}{=} \left\{X \in \mathbb{R}^{n\times k} : \text{each column of } X \text{ has unit } l_2 \text{ norm}\right\}, \\ &\text{Stiefel manifold} \quad \mathrm{St}(n,k) \overset{\text{def}}{=} \left\{X \in \mathbb{R}^{n\times k} : X^TX = I_k\right\}, \\ &\text{fixed-rank manifold} \quad \mathcal{M}_r \overset{\text{def}}{=} \left\{X \in \mathbb{R}^{m\times n} : \operatorname{rank}(X) = r\right\}. \end{aligned}$$

We consider their embedded geometry of matrix spaces. Notice that although $\mathcal{M}_r$ is not complete, RIPM is still valid in practice.

Nonnegative low-rank matrix (NLRM) approximation. Recently, [32] proposed the NLRM approximation method, which is different from the classical nonnegative matrix factorization (NMF) method, i.e., $\min_{B,C\geqslant 0} \|A - BC\|_F^2$ for $B \in \mathbb{R}^{m\times r}$, $C \in \mathbb{R}^{r\times n}$. Mathematically, NLRM aims to solve

$$\min_{X\in\mathcal{M}_r} \|A - X\|_F^2 \quad \text{s.t. } X \geqslant 0. \tag{NLRM}$$

Clearly, NLRM can obtain a better nonnegative low-rank matrix approximation than that of NMF.

Input. We will consider three cases $m = 20, 30, 40$ and let $n = 0.8m, r = 0.1m$. For each (m, n, r), we randomly generated nonnegative matrices $L \in \mathbb{R}^{m\times r}$ and $R \in \mathbb{R}^{r\times n}$ whose entries follow a uniform distribution in [0,1]. The original nonnegative matrix $A := LR$ is then obtained, and $\operatorname{rank}(A) = r$ with a very high probability. In the same way, we can generate a random feasible initial point X_0. Moreover, just like in [32], we add Gaussian noise with zero mean and different standard deviation $(\sigma = 0, .001, .01)$ to the original A. When there is no noise (i.e., $\sigma = 0$), the input data matrix A itself is exactly a solution.

Projection onto nonnegative Stiefel manifold. Given $C \in \mathbb{R}^{n\times k}$, we compute its projection onto the nonnegative part of the Stiefel manifold. If the distance between C and X is measured in terms of the Frobenius norm, it can be equivalently formulated as

$$\min_{X\in \mathrm{St}(n,k)} -2\,\mathrm{trace}(X^T C) \quad \text{s.t. } X \geqslant 0. \tag{Model_St}$$

In [20], it is shown that (Model_St) can be reformulated into

$$\min_{X\in \mathrm{Ob}(n,k)} -2\,\mathrm{trace}(X^T C) \quad \text{s.t. } X \geqslant 0, \text{ and } \|XV\|_F = 1, \tag{Model_Ob}$$

where the positive integer p and matrix $V \in \mathbb{R}^{k\times p}$ can be arbitrary as long as $\|V\|_F = 1$ and VV^T is entrywise positive. In particular, the feasible region of (Model_St) and (Model_Ob) are *identical*, and this reformulation has no effect on the objective function. We will examine both models on different manifolds.

Input. We will consider the cases of $n = 40, 50, 60, 70$ and let $k = 0.2n$. For a general matrix C, it is always difficult to seek nonnegative projections globally. Fortunately, [20, Proposition 1] showed a way to construct matrix C such that (Model_St) has a unique and known solution X^*. First, we generate a random feasible point B of (Model_St); then, we obtained C by using the following Matlab code: `X1=(B>0).*(1+rand(n,k)); Xstar =X1./sqrt(sum(X1.*X1)); L=rand(k,k); L=L+k*eye(k); C=Xstar*L'`. The initial point is computed by projecting C onto the Stiefel manifold. The projection is computed with `[U,~,V]=svd(C,'econ'); X0=U*V'`. The same settings are applied to (Model_Ob), except for `p=1; V=ones(k,p); V=V/norm(V,"fro")`.

7.3 Implementation Details

The numerical experiments were performed in Matlab R2022a on a computer equipped with an Intel Core i7-10700 at 2.90GHz with 16GB of RAM. Our algorithms are built in the framework of Manopt 7.0 [6], a Riemannian optimization toolbox on Matlab. The code is freely available.[1]

Solve the Newton equation. The challenge of the interior point method is how to solve the condensed Newton equation (14) in an efficient manner. For simplicity, we consider the case of only inequality constraints, where Δy vanishes and only a linear *operator equation* $\mathcal{A}_w \Delta x = c$ needs to be solved. The Riemannian situation leaves us with no explicit matrix form available. A general approach is to first find the *representing matrix* of $\hat{\mathcal{A}}$ (subscript w omitted), which involves three steps: obtain $d := \dim \mathbb{M}$ random independent vectors on $T_x\mathbb{M}$ (it often needs d orthogonal projections operations onto $T_x\mathbb{M}$ when $\mathbb{M}$ is submanifold); obtain an orthonormal basis $\{u_1, \ldots, u_d\}$ of $T_x\mathbb{M}$ by the modified Gram-Schmidt algorithm; compute components $[\hat{\mathcal{A}}]_{ij} := \langle \mathcal{A}u_j, u_i\rangle_x$ for $1 \leqslant i \leqslant j \leqslant d$ due to symmetry. Then, $\hat{\mathcal{A}}\Delta\hat{x} = \hat{c}$ for $\Delta\hat{x} \in \mathbb{R}^d$ is solved using direct methods (e.g. LU decomposition). Clearly, this approach is expensive. For example, for (Model_St), at each iteration, computing the representing matrix $\hat{\mathcal{A}}$ costs $\sim 6n^3k^3$ flops, and we haven't even started solving the equations yet.

An ideal approach is to use an iterative method, such as a *Krylov subspace method* (e.g., the conjugate gradients method [5, Chapter 6.3]), on $T_x\mathbb{M}$ directly. Such a method does not explicitly require a coefficient matrix, and instead needs only a matrix-vector product. In general, it only needs to call an abstract linear operator $v \mapsto \mathcal{A}v$. A significant feature is that the iterates v_k, conjugate directions p_k, and residual vectors $r_k := \mathcal{A}v_k - c$ are all contained in $T_x\mathbb{M}$. Since the operator $\mathcal{A}$ in (14) is a self-adjoint but indefinite operator (so is $\mathcal{T}$), we use the conjugate residual (CR) method to solve it. Usually, the initial point v_0 is the zero element of $T_x\mathbb{M}$; the iteration terminates when the relative residual $\|r_k\|/\|c\| \leqslant \epsilon$ for some threshold $\epsilon > 0$, or some maximum number of iterates is reached. The discussion of the above two approaches can be naturally extended to the case containing equality constraints, where we consider $\mathcal{T}$ with the product space $T_x\mathbb{M} \times \mathbb{R}^l$ instead of $\mathcal{A}$ with $T_x\mathbb{M}$.

Parameters. Our RIPM implementation (Algorithm 4) chooses the initial z_0 and s_0 from a uniform distribution in [0,1] and sets y_0 to zero if y exists. In Step 2, $\rho_k = z_k^T s_k/m$ and $\sigma_k = \min\{0.5, \|F(w_k)\|^{1/2}\}$. If the method is not specified in advance, we use the CR method to solve equation (14) and terminate it if the relative residual is smaller than 10^{-9} or the maximum number (1000) of iterates is reached. In Step 3, instead of finding the exact values of $\alpha_k^i, i = I, II$, we use a backtracking line search simultaneously for the central conditions and the sufficient decreasing condition. As a slight simplification (see [10]), we do not enforce the second central condition. Here, we set $\gamma_0 = 0.9, \gamma_{k+1} = (\gamma_k + 0.5)/2$; and $\beta = 10^{-4}, \theta = 0.5$.

We compared our method with the following Riemannian methods [23, 27]:

- RALM [23]: Riemannian augmented Lagrangian method.

[1] https://github.com/GALVINLAI/RIPM

- REPM(LQH) [23]: Riemannian exact penalty method with smoothing functions (linear- quadratic and pseudo-Huber).
- REPM(LSE) [23]: Riemannian exact penalty method with smoothing functions (log-sum exp).
- RSQP [27]: Riemannian sequential quadratic programming.
- RIPM (Our method): Riemannian interior point method (Algorithm 4).

Experimental setting. Our experimental settings followed those of Obara et al. [27], where they used residuals based on the KKT conditions (1) to measure the deviation of an iterate from the set of KKT points. The *KKT residual* is defined by

$$\sqrt{\|\mathrm{grad}_x\, \mathcal{L}(w)\|^2 + \sum_{i=1}^{m}\{\min(0, z_i)^2 + \max(0, g_i(x))^2 + |z_i g_i(x)|^2\} + \sum_{i=1}^{l} |h_i(x)|^2} + \mathrm{Manvio}(x),$$

where Manvio measures the violation of the manifold constraints. If $\mathbb{M} \stackrel{\text{def}}{=} \{x : f_j^{Man}(x) = 0, j = 1, \ldots, p\}$, then $\mathrm{Manvio}(x) := \sum_{j=1}^{p} |f_j^{Man}(x)|$. Only for a fixed-rank manifold $\mathcal{M}_r$, we define $\mathrm{Manvio}(X) := 0$ if $\mathrm{rank}(X) = r$ and $+\infty$, otherwise. For the parameters of RALM, REPMs and RSQP, we utilize the experimental setting and Matlab codes provided by [27].

To measure the stability and speed of the algorithms, we conducted 20 random trials of each problem and model. In each trial, all the algorithms ran with the same initial point. The stopping criteria were based on the KKT residual, maximum iteration, maximum time, and changes in parameters. Concretely, each experiment terminated successfully if a solution with a KKT residual ϵ_{kkt} was found. For RALM and REPMs, if the spent time exceeded $t_{\max}$ seconds, or the outer iteration number was over 1,000, or the algorithm did not update any parameters, we considered that the algorithm had terminated unsuccessfully. For RSQP and RIPM, if the spent time exceeded $t_{\max}$ seconds, or the iteration number was over 10,000, we considered that the algorithm had terminated unsuccessfully. The values of $t_{\max}$ and ϵ_{kkt} depended on the problem to be solved.

Table 1: Performance of various Riemannian methods on problem (NLRM).

(m,n,r)	(20,16,2)			(30,24,3)			(40,32,4)		
no noise	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.
RALM	0.4	1.115	31	0.65	1.813	31	0.75	2.800	31
REPM(LQH)	**1**	**5.165**$\times 10^{-1}$	31	**1**	**1.009**	31	**1**	**1.747**	31
REPM(LSE)	**1**	2.242	31	**1**	4.041	31	**0.95**	6.952	31
RSQP	0.9	6.429	7	0.9	3.944$\times 10$	8	0.9	1.254$\times 10^2$	8
RIPM	**1**	**4.920**$\times 10^{-1}$	19	**1**	**2.247**	27	**1**	**5.277**	32
(m,n,r)	(20,16,2)			(30,24,3)			(40,32,4)		
$\sigma = 0.001$	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.
RALM	0.2	1.001	31	0.15	2.050	31	0.05	2.758	31
REPM(LQH)	0.1	4.983$\times 10^{-1}$	32	0.25	1.035	31	0.15	1.787	31
REPM(LSE)	0.15	2.444	31	0.1	4.867	31	0.05	8.371	31
RSQP	**0.95**	**6.619**	7	**0.95**	**3.848**$\times 10$	8	0.9	1.299$\times 10^2$	8
RIPM	**1**	**5.376**$\times 10^{-1}$	20	**1**	**2.342**	27	**1**	**4.631**	29
(m,n,r)	(20,16,2)			(30,24,3)			(40,32,4)		
$\sigma = 0.01$	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.	Rate	Time [s]	Iter.
RALM	0	-	-	0	-	-	0	-	-
REPM(LQH)	0	-	-	0	-	-	0	-	-
REPM(LSE)	0	-	-	0	-	-	0	-	-
RSQP	**1**	**7.295**	8	**0.95**	**4.114**$\times 10$	8	**0.95**	**1.430**$\times 10^2$	9
RIPM	**1**	**5.980**$\times 10^{-1}$	21	**0.95**	**1.883**	25	**0.95**	**4.602**	29

7.4 Results and Analysis

The tables of this subsection report the success rate (Rate) divided by the total number of trials, the average time in seconds (Time [s]), and the average iteration number (Iter.) among the successful trials. Boldface highlights the best results under the combined considerations of stability and speed; that is, for each setting,

the Rate column shows in bold the success rates that are $\geqslant 0.95$ and the Time column shows in bold the two (if any) fastest results among the algorithms with success rates $\geqslant 0.95$.

7.4.1 Nonnegative low-rank matrix approximation

Here, we set $t_{\max} = 180$ and $\epsilon_{kkt} = 10^{-8}$. The numerical results are shown in Table 1. RIPM performed the best, while the first-order algorithms (including RALM and the REPMs) were as fast but less stable. The time spent by RALM and the REPMs grew slowly with the problem size, but their success rate of convergence decreased rapidly as the noise level became severe, until they did not converge at all.

In contrast, the convergence of the second-order algorithms (including RSQP, RIPM) was more stable, with RIPM being much faster than RSQP. The cost of a single iteration of RSQP drastically increased with the problem size. This is because RSQP requires solving a quadratic programming problem on the tangent space of x_k in each iteration. As with RIPM, there is no explicit matrix form available. RSQP transforms it into a representing matrix form before using a quadratic programming solver. Instead, our RIPM avoids the expensive computation of the representing matrix by using the Krylov subspace methods. As can be seen from the table, RIPM takes the same amount (order of magnitude) of time as RALM and the REPMs.

7.4.2 Projection onto nonnegative Stiefel manifold

We set $t_{\max} = 600$ and $\epsilon_{kkt} = 10^{-6}$ for both models (Model_St) and (Model_Ob). Since the true solution is known, we added an Error column showing the average error $\|X_{final} - X^*\|_F$. Here, X_{final} denotes the final iterate point of a successful trial. As a full demonstration, in the second model (Model_Ob), we also tested RIPM by using the representing matrix method to solve the Newton equation; this is denoted as RIPM_RepMat.

The numerical results are listed in Table 2 and 3. The Error columns show that if the KKT residual is sufficiently small, then X_{final} does approximate the true solution. In particular, the second-order algorithms (including RSQP, RIPM) yield a more accurate solution (the error is less than 10^{-7}). From Table 2, we can see that RALM is stable and fast for (Model_St). However, from Table 3, RALM's success rate of convergence for (Model_Ob) decreases as the problem size becomes larger. The REPMs do not work at all on either model. RSQP also does not perform well on both models. RIPM_RepMat and RIPM successfully solved all instances of both models, though the time taken by RIPM_RepMat grew explosively. Overall, RIPM using the iterative method was fast and the most stable.

Table 2: Performance of various Riemannian methods on (Model_St).

(n,k)	(40,8)				(50,10)			
	Rate	Time [s]	Iter.	Error	Rate	Time [s]	Iter.	Error
RALM	**1**	**2.347**	45	5.41×10^{-7}	**1**	**4.344**	54	5.21×10^{-7}
REPM(LQH)	0	-	-	-	0	-	-	-
REPM(LSE)	0	-	-	-	0	-	-	-
RSQP	0.9	1.352×10	7	2.05×10^{-9}	0.7	3.097×10	6	2.47×10^{-9}
RIPM	**1**	**2.225**	31	3.72×10^{-8}	**1**	**3.785**	32	3.38×10^{-8}
(n,k)	(60,12)				(70,14)			
	Rate	Time [s]	Iter.	Error	Rate	Time [s]	Iter.	Error
RALM	**1**	**4.097**	34	4.93×10^{-7}	**1**	**6.234**	37	5.34×10^{-7}
REPM(LQH)	0	-	-	-	0	-	-	-
REPM(LSE)	0	-	-	-	0	-	-	-
RSQP	0.65	7.802×10	7	6.48×10^{-9}	0.85	1.661×10^{2}	7	2.64×10^{-9}
RIPM	**1**	**5.555**	32	2.81×10^{-8}	**1**	**7.574**	33	2.45×10^{-8}

8 Concluding remarks

We proposed a Riemannian version of the classical interior point method (IPM) and established its local and global convergence. To our knowledge, this is the first study to apply the primal-dual interior point method to the constrained optimization problem on a Riemannian manifold. Numerical experiments show the stability and efficiency of our method. In closing, let us make a comparison with Euclidean IPM to

Table 3: Performance of various Riemannian methods on (Model_Ob).

(n,k)	(40,8)				(50,10)			
	Rate	Time [s]	Iter.	Error	Rate	Time [s]	Iter.	Error
RALM	**1**	**2.510**	51	5.04×10^{-7}	**0.95**	**4.727**	64	4.94×10^{-7}
REPM(LQH)	0	-	-	-	0	-	-	-
REPM(LSE)	0	-	-	-	0	-	-	-
RSQP	0.65	8.618	5	2.30×10^{-10}	0.7	2.782×10	6	1.12×10^{-10}
RIPM	**1**	**3.791**	22	5.62×10^{-9}	**1**	**5.880**	23	7.93×10^{-9}
RIPM_RepMat	**1**	1.954×10	31	4.34×10^{-8}	**1**	4.718×10	32	3.56×10^{-8}
(n,k)	(60,12)				(70,14)			
	Rate	Time [s]	Iter.	Error	Rate	Time [s]	Iter.	Error
RALM	0.6	5.725	49	3.82×10^{-7}	0.6	8.223	52	3.85×10^{-7}
REPM(LQH)	0	-	-	-	0	-	-	-
REPM(LSE)	0	-	-	-	0	-	-	-
RSQP	0.7	4.446×10	5	1.17×10^{-9}	0.5	9.138×10	5	1.82×10^{-9}
RIPM	**1**	**7.134**	23	9.69×10^{-9}	**1**	**9.268**	24	1.06×10^{-8}
RIPM_RepMat	**1**	$\mathbf{1.018}\times10^{2}$	32	3.20×10^{-8}	**1**	$\mathbf{1.861}\times10^{2}$	33	2.75×10^{-8}

illustrate the theoretical advantages of our RIPM and discuss two future directions of research on more advanced RIPM methods.

Riemannian IPM (RIPM) vs. Euclidean IPM (EIPM) (1) EIPM is a special case of RIPM when $\mathbb{M} \equiv \mathbb{R}^n$ or $\mathbb{R}^{n\times k}$. (2) If the equality constraints are considered to be a manifold $\mathbb{M}$, RIPM can solve a condensed equation (14) of smaller order. For example, in (Model_St), the Stiefel manifold can be used as the equality constraints; i.e., we set $h : \mathbb{M} \equiv \mathbb{R}^{n\times k} \to \mathrm{Sym}(k)$, where $h(X) = X^\top X - I_k$. Here, EIPM requires us to solve (14) of order $nk + k(k+1)/2$, but RIPM only requires us to solve a problem of order $nk - k(k+1)/2$, i.e., the dimension of $\mathrm{St}(n,k)$. (3) Not all manifolds are equivalent to smooth constraints. For example, suppose a fixed rank manifold to be the equality constraints. Since $\mathrm{rank}(\cdot)$ is not continuous, we can not apply EIPM.

Preconditioner for linear operator equation. Due to the complementary condition, as $k \to \infty$, the values of $S_k^{-1}Z_k$ display a huge difference in magnitude: some of them tend to zero while others go to infinity. Hence, the presence of the operator $\Theta := G_x S^{-1} Z G_x^*$ in the system (14) makes it very ill-conditioned, so the iterative method will likely fail unless it is carefully preconditioned. Unfortunately, *operator equation* has no explicit matrix form available, which makes the most common preconditioner methods based on matrix decomposition techniques useless. A possible way around this is to find another nonsingular operator $\mathcal{P}$ such that the condition number of the new operator $\mathcal{P}^{-1}\mathcal{T}$ becomes smaller.

Sophisticated global strategies. In the Euclidean setting, it has been shown that interior point methods employing a simple line search along Newton steps may manifest a weakness of convergence. The more sophisticated and robust global strategies are often based on the trust region or filter line-search method.

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A Proof of Lemma 2.2

For simplicity, we will consider the case in which (RCOP) contains only inequality constraints. Then, $\mathcal{L}(x,z) = f(x) + z^T g(x)$ and the KKT conditions without defining slack variables are

$$\mathrm{grad}_x\, \mathcal{L}(x,z) = \mathrm{grad}\, f(x) + \textstyle\sum_{i=1}^m z_i\, \mathrm{grad}\, g_i(x) = 0_x;\;\; g_i(x) z_i = 0, i = 1, \cdots, m,$$

and $-g(x) \geqslant 0, z \geqslant 0$. Let $G(x)$ be a diagonal matrix of vectors $g(x)$; the KKT vector field $F : \mathbb{M} \times \mathbb{R}^m \to T\mathbb{M} \times T\mathbb{R}^m$ is defined by $F(x,z) := (\mathrm{grad}_x\, \mathcal{L}(x,z), G(x)z)$. The next lemma establishes a way of defining a connection on product manifolds. It can easily be extended to a product of more than two manifolds.

Lemma A.1 (product connection [5, Exercise 5.4 & 5.13]). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Riemannian manifolds, respectively equipped with Riemannian connections ∇^1 and ∇^2. Consider the product manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. Let (u_1, u_2) be tangent to $\mathcal{M}$ at (x_1, x_2). Then, the map $\nabla : T\mathcal{M} \times \mathfrak{X}(\mathcal{M}) \to T\mathcal{M}$ defined by*

$$\nabla_{(u_1,u_2)}(F_1, F_2) = \left(\nabla^1_{u_1} F_1(\cdot, x_2) + \mathcal{D}F_1(x_1, \cdot)(x_2)[u_2], \nabla^2_{u_2} F_2(x_1, \cdot) + \mathcal{D}F_2(\cdot, x_2)(x_1)[u_1]\right)$$

is a Riemannian connection on $\mathcal{M}$; we call it the product connection. *The notation $F_1(\cdot, x_2)$ represents the map obtained from $F_1 : \mathcal{M}_1 \times \mathcal{M}_2 \to T\mathcal{M}_1$ by fixing the second input to x_2. In particular, $F_1(\cdot, x_2)$ is a vector field on $\mathcal{M}_1$, while $F_1(x_1, \cdot)$ is a map from $\mathcal{M}_2$ to $T_{x_1}\mathcal{M}_1$.*

Proof of Lemma 2.2. We apply Lemma A.1 directly. $\mathbb{R}^m$ is equipped with the canonical Euclidean connection, and we will not distinguish those connections with superscripts, as they should be clear from context. Let $F(x, z) = (F_1(x, z), F_2(x, z))$, where $F_1 : \mathbb{M} \times \mathbb{R}^m \to T\mathbb{M}$, $F_1(x, z) = \operatorname{grad}_x \mathcal{L}(x, z)$, and $F_2 : \mathbb{M} \times \mathbb{R}^m \to T\mathbb{R}^m$, $F_2(x, z) = G(x)z$. We will compute the covariant derivative of F at $(x, z) \in \mathbb{M} \times \mathbb{R}^m$ step by step in accordance with Lemma A.1. Let $(u_x, u_z) \in T_x\mathbb{M} \times \mathbb{R}^m$ be the tangent vector at (x, z).

1. $F_1(\cdot, z) : \mathbb{M} \to T\mathbb{M}$ (a vector field on M). From the linearity of the connection ∇ on $\mathbb{M}$, we have $\nabla_{u_x} F_1(\cdot, z) = \nabla_{u_x}(\operatorname{grad} f(x) + \sum_{i=1}^m z_i \operatorname{grad} g_i(x)) = \nabla_{u_x} \operatorname{grad} f(x) + \sum_{i=1}^m z_i \nabla_{u_x} \operatorname{grad} g_i(x) = \operatorname{Hess} f(x)[u_x] + \sum_{i=1}^m z_i \operatorname{Hess} g_i(x)[u_x] = \operatorname{Hess}_x \mathcal{L}(x, z)[u_x]$.
2. $F_1(x, \cdot) : \mathbb{R}^m \to T_x\mathbb{M}$ (a map between two vector spaces).
   $\mathcal{D}F_1(x, \cdot)(z)[u_z] = \lim_{t \downarrow 0} \{F_1(x, z + tu_z) - F_1(x, z)\}/t = \sum_{i=1}^m (u_z)_i \operatorname{grad} g_i(x)$.
3. $F_2(x, \cdot) : \mathbb{R}^m \to T\mathbb{R}^m \cong \mathbb{R}^m$ (a trivial function).
   $\nabla_{u_z} F_2(x, \cdot) = \mathcal{D}F_2(x, \cdot)(z)[u_z] = G(x)u_z = [g_1(x)(u_z)_1, \cdots, g_m(x)(u_z)_m]^T$.
4. $F_2(\cdot, z) : \mathbb{M} \to T_z\mathbb{R}^m \cong \mathbb{R}^m$ (a map from $\mathbb{M}$ to $\mathbb{R}^m$). Let $F_2^i(\cdot, z) = g_i(x)z_i$ be the component function for $i = 1, \ldots, m$. Since $\mathcal{D}F_2^i(\cdot, z)(x)[u_x] = \langle \operatorname{grad}_x F_2^i(\cdot, z), u_x \rangle_x = \langle z_i \operatorname{grad} g_i(x), u_x \rangle_x$, we have

$$\begin{aligned}\mathcal{D}F_2(\cdot, z)(x)[u_x] &= \left[\mathcal{D}F_2^1(\cdot, z)(x)[u_x], \cdots, \mathcal{D}F_2^m(\cdot, z)(x)[u_x]\right]^T \\ &= \left[z_1 \langle \operatorname{grad} g_1(x), u_x \rangle_x, \cdots, z_m \langle \operatorname{grad} g_m(x), u_x \rangle_x\right]^T.\end{aligned}$$

Finally, by combining 1-4, we obtain

$$\nabla F(x, z)[(u_x, u_z)] = \left(\operatorname{Hess}_x \mathcal{L}(x, z)[u_x] + \sum_{i=1}^m (u_z)_i \operatorname{grad} g_i(x), \begin{bmatrix} z_1 \langle \operatorname{grad} g_1(x), u_x \rangle_x + g_1(x)(u_z)_1 \\ \vdots \\ z_m \langle \operatorname{grad} g_m(x), u_x \rangle_x + g_m(x)(u_z)_m \end{bmatrix}\right).$$

□

B Proof of Proposition 2.3

Proof of Proposition 2.3. This proof omits all the asterisks of the variables. Let $\mathscr{E} = \{1, \ldots, l\}$, $\mathscr{I} = \{1, \ldots, m\}$ and $A = A(x) \subseteq \mathscr{I}$. Take a triple (x, y, z) satisfying (A1)-(A5). Set $s_i := -g_i(x)$ for $i \in \mathscr{I}$, and define $w = (x, y, z, s)$. Suppose that $\nabla F(w)\Delta w = 0$ for some $\Delta w \in T_x\mathscr{M}$. To prove its nonsingularity, we will show that $\Delta w = (\Delta x, \Delta y, \Delta s, \Delta z) = 0$. Expanding $\nabla F(w)\Delta w = 0$ gives

$$\begin{cases} 0_x = \operatorname{Hess}_x \mathcal{L}(w)\Delta x + \sum_{i \in \mathscr{E}} \Delta y_i \operatorname{grad} h_i(x) + \sum_{i \in \mathscr{I}} \Delta z_i \operatorname{grad} g_i(x), \\ 0 = \langle \operatorname{grad} h_i(x), \Delta x \rangle, \text{ for all } i \in \mathscr{E}, \\ 0 = \langle \operatorname{grad} g_i(x), \Delta x \rangle + \Delta s_i, \text{ for all } i \in \mathscr{I}, \\ 0 = z_i \Delta s_i + s_i \Delta z_i, \text{ for all } i \in \mathscr{I}. \end{cases} \tag{35}$$

Strict complementarity (A4) and the last equalities above imply that $\Delta s_i = 0$ for $i \in A$ and $\Delta z_i = 0$ for $i \in \mathscr{I} \backslash A$. Substituting those values into the system (35) reduces it to

$$\begin{cases} 0_x = \operatorname{Hess}_x \mathcal{L}(w)\Delta x + \sum_{i \in \mathscr{E}} \Delta y_i \operatorname{grad} h_i(x) + \sum_{i \in A} \Delta z_i \operatorname{grad} g_i(x), \\ 0 = \langle \operatorname{grad} h_i(x), \Delta x \rangle, \text{ for all } i \in \mathscr{E}, \\ 0 = \langle \operatorname{grad} g_i(x), \Delta x \rangle, \text{ for all } i \in A, \end{cases} \tag{36}$$

and $\Delta s_i = -\langle \operatorname{grad} g_i(x), \Delta x \rangle$ for all $i \in \mathscr{I} \backslash A$. It follows from system (36) that

$$0 = \langle \operatorname{Hess}_x \mathcal{L}(w)\Delta x + \sum_{i \in \mathscr{E}} \Delta y_i \operatorname{grad} h_i(x) + \sum_{i \in A} \Delta z_i \operatorname{grad} g_i(x), \Delta x \rangle$$

$$= \langle \operatorname{Hess}_x \mathcal{L}(w)\Delta x, \Delta x\rangle + \textstyle\sum_{i\in\mathscr{E}} \Delta y_i \langle \operatorname{grad} h_i(x), \Delta x\rangle + \sum_{i\in A} \Delta z_i \langle \operatorname{grad} g_i(x), \Delta x\rangle$$
$$= \langle \operatorname{Hess}_x \mathcal{L}(w)\Delta x, \Delta x\rangle,$$

which is a contradiction by second-order sufficiency (A5). Thus, Δx must be zero, and then $\Delta s_i = 0$ for all $i \in \mathscr{I}\backslash A$. Next, substituting $\Delta x = 0$ into the first equation in (36) yields $0_x = \sum_{i\in\mathscr{E}} \Delta y_i \operatorname{grad} h_i(x) + \sum_{i\in A} \Delta z_i \operatorname{grad} g_i(x)$. The linear independence of the gradients in $T_x\mathbb{M}$ of (A3) implies that Δy and Δz_i for $i \in A$ must be zero. This completes the proof. □

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