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A decomposition of strategy-proofness in discrete resource allocation problems

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A decomposition of strategy-proofness in discrete resource allocation problems *

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Abstract

This note provides a characterization of strategy-proofness in discrete resource allocation problems. Based on it, both the student-proposing deferred-acceptance (DA) rule for college admission problems and the top-trading cycles (TTC) rule for housing markets are shown to be strategy-proof by a single proof. The identical argument also works to prove strategy-proofness of the cumulative-offer process rule for matching with contracts under weakened substitutes conditions.

Journal of Economic Literature Classification Numbers : C78, D47, D71.

Keywords: Market design; Two-sided matching market; Indivisible goods allocation problem; Core; Strategy-proofness.

1 Introduction

Matching market design has mainly worked on two types of problems: two-sided matching markets (Gale and Shapley, 1962) and indivisible goods allocation problems (Shapley and Scarf, 1974). For the former, the deferred-acceptance (DA) algorithm that generates a stable matching plays the central role to operate the markets. On the other hand, for the latter, the top-trading cycles (TTC) algorithm is utilized to generate an efficient allocation. It is well-known that these two rules have a good feature for the incentives. Dubins and Freedman (1981) and Roth (1982a) first show that the men-proposing (resp. women-proposing) DA rule is strategy-proof for men (resp. women) in the sense that no proposer can profitably manipulate the rule by misreporting preferences.[1] The corresponding result for the TTC rule in housing markets is first proved by Roth (1982b).

In this paper, we provide a necessary and sufficient condition for a rule in discrete resource allocation problems to be strategy-proof for a fixed group of the market participants. The condition is a combination of three invariance properties under preference transformation. The characterization enable us to provide an elementary and transparent proof for strategy-proofness of one-sided optimal core rules for generalized indivisible goods allocation problems (Sönmez, 1999). Since our model includes both college admission problems and housing markets, strategy-proofness of the student-proposing DA rule and the TTC rule is obtained by a single proof. Moreover, the proof technique also works to prove strategy-proofness of the cumulative-offer process rule for matching with contracts under weakened substitutes conditions (Hatfield and Kojima, 2010).

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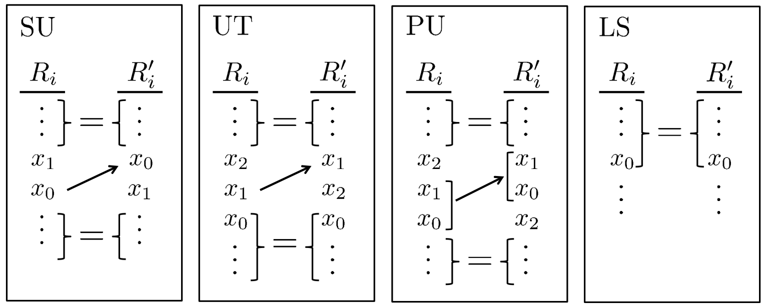
[1] Gale and Sotomayor (1985) and Hatfield and Milgrom (2005) provide an alternative proof for this result.

Figure 1: Four types of preference transformation



Note: In each box, R'_i denotes the (a) transformed preference of R_i at x_0. The type of transformation is indicated on the upper left corner of the box.

2 Model and a characterization of strategy-proofness

We describe a version of the model in Sönmez (1999). A generalized indivisible goods allocation problem (GIGAP) is a 4-tuple $(\mathcal{N}, \omega, \mathcal{A}^f, R)$: $\mathcal{N} = \{1, \ldots, n\}$ denotes the set of agents. For each $i \in \mathcal{N}$, let $\omega(i)$ be the object initially owned by i. For each $S \subseteq \mathcal{N}$, let $\omega(S) := \{\omega(i) | i \in S\}$. An allocation is a function from $\mathcal{N}$ to $2^{\omega(\mathcal{N})}$. Throughout the paper, fix a non-empty subset of agents $N \subseteq \mathcal{N}$ such that each member of N is necessarily unit-demand.[2] Let $\mathcal{A} := \{a : \mathcal{N} \to 2^{\omega(\mathcal{N})} \mid \forall i \in N, |a(i)| = 1\}$. For each $a \in \mathcal{A}$, we call $a(i)$ the assignment of $i \in \mathcal{N}$ at a. We abuse the notation $a(i)$ for $i \in N$ to denote both a singleton and the object contained in it, interchangably. The set of feasible allocations, $\mathcal{A}^f$, is a subset of $\mathcal{A}$. We assume that $\mathcal{A}^f$ contains an allocation such that each $i \in N$ receives her endowment $\omega(i)$. For each $i \in \mathcal{N}$, letting $X_i := \{a(i) \mid a \in \mathcal{A}^f\}$, let $\mathcal{R}_i$ be the set of complete, transitive and anti-symmetric binary relations on X_i.[3] Let $\mathcal{D}_i \subseteq \mathcal{R}_i$ be the set of feasible preference relations of i. We assume that $\mathcal{D}_i = \mathcal{R}_i$ for each $i \in N$. Letting $\mathcal{D} := \prod_{i \in \mathcal{N}} \mathcal{D}_i$, the symbol R denotes a preference profile belonging to $\mathcal{D}$. For notational simplicity, given $R = (R_1, \ldots, R_n) \in \mathcal{D}$ and $i \in N$, it is convenient to use the notation R_{-i} to represent the $(n-1)$-fold preference profile obtained by deleting R_i from R. Without any confusion, we use the notation (R_i, R_{-i}) to denote R even when $i \neq 1$. For each $i \in N$, each $x \in X_i$, and each $R_i \in \mathcal{D}_i$, let $UC(R_i, x) := \{y \in X_i | y \; R_i \; x\}$, $SUC(R_i, x) := UC(R_i, x) \backslash \{x\}$ and $r_{R_i}(x) := |UC(R_i, x)|$.

Hereafter, we fix $\mathcal{N}, \omega$ and $\mathcal{A}^f$. Thus a GIGAP, or a problem for short, is identified with a preference profile $R \in \mathcal{D}$.

Remark 1. **A college admission problem** is a special case of GIGAPs. Let S and C be disjoint sets of students and colleges. Namely, the set of agents in this problem is $S \cup C$. Let $\omega(i) := i$ for each $i \in S \cup C$. Assume that a capacity vector $(q_c)_{c \in C} \in \mathbb{Z}^C_{++}$ is given. Assume also that for each $c \in C$, the set of feasible preferences of college c consists of responsive preferences on 2^S (Roth and Sotomayor, 1990). Let $\mathcal{A}^f := \{a \in \mathcal{A} | \forall (s, c) \in S \times C, a(s) \in C \cup \{s\}$ and $a(c) \in 2^S$ with $|a(c)| \leq q_c$ and $a(s) = c \Leftrightarrow a(c) \ni s\}$. Obviously, any college admission problem can be represented by the above specification along with a preference profile. **A housing market** is also a special case of GIGAPs. Letting $\omega(i) := i$ for each $i \in \mathcal{N}$, assume that $N = \mathcal{N}$. Let $\mathcal{A}^f = \mathcal{A}$. Obviously, any housing market can be represented by the above specification along with a preference profile. ♢

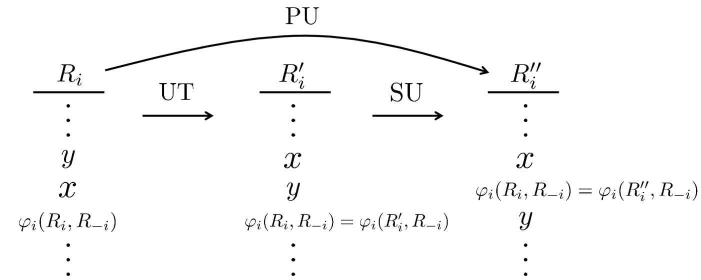
A rule is a function from $\mathcal{D}$ to $\mathcal{A}^f$. Our generic notation for a rule is φ. *Strategy-proofness* for N requires that no agent in N can profitably manipulate a rule by misreporting her preferences. Formally, a rule φ is **strategy-proof for N** *if* for each $i \in N$, each $(R_i, R_{-i}) \in \mathcal{D}$, and each $R'_i \in \mathcal{D}_i, \varphi_i(R_i, R_{-i}) \; R_i \; \varphi_i(R'_i, R_{-i})$. To provide a characterization

[2]In concrete problems, an interpretation is given to N. For example, N denotes the set of students in college admission problems. In housing markets, $N(= \mathcal{N})$ denotes the set of all agents in the economy.

[3]A binary relation R_i on X_i is complete *if* for each $\{x, y\} \subseteq X_i, x \; R_i \; y$ or $y \; R_i \; x$. A binary relation R_i on X_i is transitive *if* for each $\{x, y, z\} \subseteq X_i, [x \; R_i \; y$ and $y \; R_i \; z] \Rightarrow x \; R_i \; z$. A binary relation R_i on X_i is anti-symmetric *if* for each $\{x, y\} \subseteq X_i, [x \; R_i \; y$ and $y \; R_i \; x] \Rightarrow x = y$.

Given $i \in \mathcal{N}$ and a preference relation R_i, the anti-symmetric part of R_i is denoted as P_i.

Figure 2: Proof of Remark 2.



of this property, we define four types of preference transformation. Let $i \in N, R_i \in \mathcal{D}_i$ and $x_0 \in X_i$.

- Assume that $r_{R_i}(x_0) \geq 2$. Let $x_1 \in X_i$ be such that $r_{R_i}(x_1) = r_{R_i}(x_0) - 1$. The preference $R'_i \in \mathcal{D}_i$ satisfying the following condition is called **the single-upgrade (SU) of $\mathbf{R_i}$ at $\mathbf{x_0}$**: For each $x \in X_i$, $r_{R'_i}(x) = r_{R_i}(x) - 1$ *if* $x = x_0, r_{R'_i}(x) = r_{R_i}(x) + 1$ *if* $x = x_1$, and $r_{R'_i}(x) = r_{R_i}(x)$ *if* $x \notin \{x_0, x_1\}$.

- Assume that $r_{R_i}(x_0) \geq 3$. Let $x_1 \in X_i$ and $x_2 \in X_i$ be such that $r_{R_i}(x_1) = r_{R_i}(x_0) - 1$ and $r_{R_i}(x_2) = r_{R_i}(x_0) - 2$. The preference $R'_i \in \mathcal{D}_i$ satisfying the following condition is called **the upper-transposition (UT) of $\mathbf{R_i}$ at $\mathbf{x_0}$**: For each $x \in X_i$, $r_{R'_i}(x) = r_{R_i}(x) - 1$ *if* $x = x_1, r_{R'_i}(x) = r_{R_i}(x) + 1$ *if* $x = x_2$, and $r_{R'_i}(x) = r_{R_i}(x)$ *if* $x \notin \{x_1, x_2\}$.

- Assume that $r_{R_i}(x_0) \geq 3$. Let $x_1 \in X_i$ and $x_2 \in X_i$ be such that $r_{R_i}(x_1) = r_{R_i}(x_0) - 1$ and $r_{R_i}(x_2) = r_{R_i}(x_0) - 2$. The preference $R'_i \in \mathcal{D}_i$ satisfying the following condition is called **the pairwise-upgrade (PU) of $\mathbf{R_i}$ at $\mathbf{x_0}$**: For each $x \in X_i$, $r_{R'_i}(x) = r_{R_i}(x) - 1$ *if* $x \in \{x_0, x_1\}, r_{R'_i}(x) = r_{R_i}(x) + 2$ *if* $x = x_2$, and $r_{R'_i}(x) = r_{R_i}(x)$ *if* $x \notin \{x_0, x_1, x_2\}$.

- A preference $R'_i \in \mathcal{D}_i$ is called **a lower-shuffle (LS) of $\mathbf{R_i}$ at $\mathbf{x_0}$** *if* for each $x \in UC(R_i, x_0), r_{R'_i}(x) = r_{R_i}(x)$.

We say that a rule φ satisfies the **single-upgrade invariance (SUI)** (resp. **upper-transposition invariance (UTI)**, **pairwise-upgrade invariance (PUI)**, **lower-shuffle invariance (LSI)**) **on N** *if* for each $i \in N$ and each $(R_i, R_{-i}) \in \mathcal{D}$, the assignment of i does not change when she changes her reporting to the SU (resp. the UT, the PU, a LS) of R_i at $\varphi_i(R_i, R_{-i})$. Note that SU, UT, PU and LS are special cases of Maskin monotonic transformation (Maskin, 1999). Thus, so-called **Maskin monotonicity** implies SUI, UTI, PUI and LSI. However, the converse is not true in general because our properties require invariance of the individual assignment while Maskin monotonicity does the allocation.

Note that the PU at the selected object is obtained by the combination of the UT and SU at the same object (Figure 2).

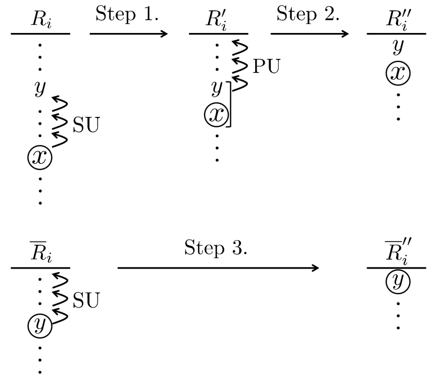
Remark 2. Suppose that a rule φ satisfies SUI and UTI on N. Then, φ satisfies PUI on N.

Theorem 1. *A rule φ is strategy-proof for N if and only if φ satisfies SUI, UTI and LSI on N.*

Proof. We only show the sufficiency part. Suppose to the contrary that there are $i \in N$, $(R_i, R_{-i}) \in \mathcal{D}$ and $\overline{R}_i \in \mathcal{D}_i$ such that $\varphi_i(\overline{R}_i, R_{-i})\ P_i\ \varphi_i(R_i, R_{-i})$. Let $x := \varphi_i(R_i, R_{-i})$ and $y := \varphi_i(\overline{R}_i, R_{-i})$. Notice that $x \neq y$.

Step 1. If $r_{R_i}(y) = r_{R_i}(x) - 1$, let $R'_i := R_i$. Otherwise, apply SU to R_i at x succesively until x becomes the immediate successor of y. Then, define R'_i as the preference found at the very last step. Since φ satisfies SUI on N, $\varphi_i(R'_i, R_{-i}) = x$.

Figure 3: Proof of Theorem 1.



Note: Fixing other agents' reporting R_{-i}, the assignment for i is indicated by the circled object. In step 1, successive applications of SU to R_i leads to R_i'. Then, in step 2, successive applications of PU to R_i' leads to R_i''. In step 3, successive applications of SU to $\overline{R}_i$ leads to $\overline{R}_i''$. Since R_i'' is a LS of $\overline{R}_i''$ at y, $\varphi_i(\overline{R}_i'', R_{-i}) = \varphi_i(R_i'', R_{-i})$, a contradiction.

Step 2. If $r_{R_i'}(y) = 1$, let $R_i'' := R_i'$. Otherwise, apply PU to R_i' at x succesively until y becomes the most preferred object. Then, define R_i'' as the preference found at the very last step. Since φ satisfies PUI on N ($\because$ Remark 2), $\varphi_i(R_i'', R_{-i}) = x$.

Step 3. If $r_{\overline{R}_i}(y) = 1$, let $\overline{R}_i'' := \overline{R}_i$. Otherwise, apply SU to $\overline{R}_i$ at y succesively until y becomes the most preferred object. Then, define $\overline{R}_i''$ as the preference found at the very last step. Since φ satisfies SUI on N, $\varphi_i(\overline{R}_i'', R_{-i}) = y$.

Note that R_i'' is a LS of $\overline{R}_i''$ at y. Since φ satisfies LSI on N, $x = \varphi_i(R_i'', R_{-i}) = \varphi_i(\overline{R}_i'', R_{-i}) = y$, a contradiction. □

3 Application: A unified approach to strategy-proofness of the DA and the TTC

As an application of Theorem 1, we prove that N-optimal core rules for GIGAPs are strategy-proof. Let $R \in \mathcal{D}$. An allocation $a \in \mathcal{A}^f$ is **blocked** at R by a coalition $S \in 2^{\mathcal{N}} \setminus \{\emptyset\}$ via an allocation $b \in \mathcal{A}^f$ *if* (i) $b(S) \subseteq \omega(S)$, (ii) $b(i)\ R_i\ a(i)$ for each $i \in S$, and (iii) $b(i)\ P_i\ a(i)$ for some $i \in S$. An allocation $a \in \mathcal{A}^f$ belongs to the **core** at R, written as $a \in \mathcal{C}(R)$, *if* it is not blocked at R by any coalition via any allocation. Letting $R_N \in \prod_{i \in N} D_i$, the common preference of N-agents, written as $\geq_{R_N}$, is defined as follows: For each $\{a, a'\} \subseteq \mathcal{A}^f$, $a \geq_{R_N} a'$ *if and only if* $a(i)\ R_i\ a'(i)$ for all $i \in N$. Note that $\geq_{R_N}$ is a reflexive, anti-symmetric and transitive,[4] but not necessarily complete binary relation on $\mathcal{A}^f$.

Assumption 1. *[Existence of N-optimal core allocations]* $\forall R = (R_N, R_{-N}) \in \mathcal{D}, \exists a \in \mathcal{C}(R)$ *s.t.* $\forall a' \in \mathcal{C}(R), a \geq_{R_N} a'$.

Note that every N-optimal core allocation gives the same assignment for N-agents.[5] We call a rule that assigns an N-optimal core allocation for each $R \in \mathcal{D}$ an **N-optimal core (NOC) rule**.

Assumption 2. *[A version of rural hospital theorem]* $\forall R \in \mathcal{D}, \forall \{a, a'\} \subseteq \mathcal{C}(R), \forall i \in N, [a(i) = \omega(i) \Leftrightarrow a'(i) = \omega(i)]$.

Under Assumption 1 and 2, we show the following theorem.[6] A proof is given after we prove two lemmas.

[4] This is a direct consequence of our setting where each agent in N has a complete, transitive and anti-symmetric preference.

[5] This is because $\geq_{R_N}$ is anti-symmetric. Note that N-optimal core allocation is unique if $\mathcal{A}^f$ consists of allocations that exclude a match between $(\mathcal{N} \setminus N)$-agents. College admission problems and housing markets are included in this class.

[6] A closely related paper by Takamiya (2003) establishes that any selection from the core is coalition strategy-proof under an assumption, called **essentially single-valuedness (ESV)** of the core, that requires that all core allocations be indifferent for all agents. Note that, in our setting with strict preferences, ESV of the core combined with the existence of a selection from the core implies Assumption 1 and 2. Since Takamiya's setting includes indifference and externalities in agents' preferences, his theorem and ours do not imply each other.

Figure 4: For the last part of the proof of Claim 2 in Lemma 1.

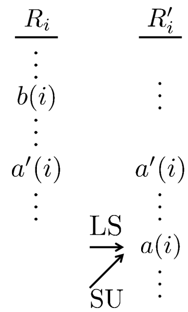
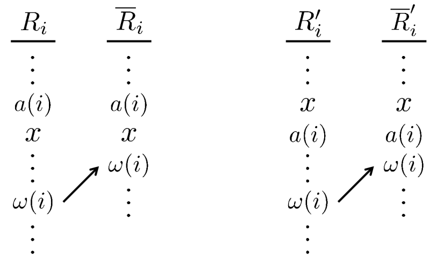


Figure 5: The construction of $\overline{R}_w$ and $\overline{R}'_w$ in the proof of Lemma 2.



Theorem 2. *Every NOC rule is strategy-proof for N.*

Lemma 1. *Every NOC rule satisfies SUI and LSI on N.*

Proof. Let φ be an NOC rule. Let $i \in N$, $R = (R_i, R_{-i}) \in \mathcal{D}$ and $a := \varphi(R)$. Let $R'_i \in \mathcal{D}_i$ be the SU or a LS of R_i at $a(i)$. Let $R' := (R'_i, R_{-i})$ and $a' := \varphi(R')$. Suppose to the contrary that $a'(i) \neq a(i)$.

Claim 1. $a \in \mathcal{C}(R')$. Suppose not. Then, there is $S \subseteq \mathcal{N}$ that blocks a at R' via $b \in \mathcal{A}^f$. Since the only difference between R and R' is i's preference, i must be a member of S who exibits $b(i)\ P'_i\ a(i)$. Since $SUC(R'_i, a(i)) \subseteq SUC(R_i, a(i))$, $b(i)\ P_i\ a(i)$. This implies that S blocks a at R via b, i.e., $a \notin \mathcal{C}(R)$, a contradiction. This completes the proof of Claim 1.

By Claim 1, $\{a, a'\} \subseteq \mathcal{C}(R')$. Since φ is an NOC rule, $a'(i)\ R'_i\ a(i)$. Thus $a'(i)\ P'_i\ a(i)$.

Claim 2. $a' \in \mathcal{C}(R)$. Suppose not. Then, there is $S \subseteq \mathcal{N}$ that blocks a' at R via $b \in \mathcal{A}^f$. Similar to Claim 1, i exibits $b(i)\ P_i\ a'(i)$. Since R'_i is either the SU or a LS of R_i at $a(i)$, the preference between $b(i)$ and $a'(i)$ is preserved, i.e., $b(i)\ P'_i\ a'(i)$ (See Figure 4).[7] Thus, S blocks a' at R' via b, i.e., $a' \notin \mathcal{C}(R')$, a contradiction. This completes the proof of Claim 2.

By Claim 2, $\{a, a'\} \subseteq \mathcal{C}(R)$. Since φ is an NOC rule, $a(i)\ R_i\ a'(i)$. On the other hand, $a'(i)\ P'_i\ a(i)$ and $SUC(R'_i, a(i)) \subseteq SUC(R_i, a(i))$ together imply that $a'(i)\ P_i\ a(i)$, a contradiction. □

Lemma 2. *Let φ be an NOC rule. Let $i \in N, R = (R_i, R_{-i}) \in \mathcal{D}$ and $a := \varphi(R)$. Let $x \in X_i$ be such that $r_{R_i}(x) = r_{R_i}(a(i)) + 1$. Then, for the SU of R_i at x, denoted as R'_i, $\varphi_i(R'_i, R_{-i}) \in \{a(i), x\}$.*

Proof. Let $R' := (R'_i, R_{-i})$ and $a' := \varphi(R')$. Suppose to the contrary that $a'(i) \notin \{a(i), x\}$. First note that $a(i)\ P'_i\ a'(i)$.[8] Thus we have $x\ P_i\ a'(i)\ R_i\ \omega(i)$ and $a(i)\ P'_i\ a'(i)\ R'_i\ \omega(i)$. Define $\overline{R}_i$ (resp. $\overline{R}'_i$) by upgrading $\omega(i)$ to be the immediate successor of x (resp. $a(i)$) at R_i (resp. R'_i) without affecting the preferences between other objects (Figure 5). Let $\overline{R} := (\overline{R}_i, R_{-i}), \overline{R}' := (\overline{R}'_i, R_{-i})$ and $\overline{a}' := \varphi(\overline{R}')$.

[7] Since $a'(i)\ P'_i\ a(i)$, $a'(i)\ P_i\ a(i)$. Thus, $b(i)$ and $a'(i)$ do not move at the transformation from R_i to R'_i.
[8] If this is not true, $a'(i)\ P'_i\ x$. However, in this case, R_i is a LS of R'_i at $a'(i)$. Thus, $a(i) = a'(i)$ by Lemma 1, a contradiction.

Figure 6: The preference relations R_i and R'_i in the proof of Theorem 2.

R_i	R'_i
$\vdots$	$\vdots$
y	x
x	y
$a(i)$	$a(i)$
$\vdots$	$\vdots$

Claim 1.$\varphi_i(\overline{R}) = a(i)$.Since $\overline{R}_i$ is a LS of R_i at $a(i)$, $\varphi_i(\overline{R}) = \varphi_i(R)$ by Lemma 1.This completes the proof of Claim 1.

Claim 2. $\overline{a}'(i) = \omega(i)$. Suppose to the contrary that $\overline{a}'(i)\ \overline{P}'_i\ \omega(i)$.[9] Note that $\overline{a}'(i)\ \overline{P}'_i\ \omega(i)\ \overline{R}'_i\ a'(i)$.[10] This implies that $\overline{a}'(i)\ P'_i\ a'(i)$ since the only difference between R'_i and $\overline{R}'_i$ is the position of $\omega(i)$. On the other hand, $\overline{a}' \in \mathcal{C}(R')$. This contradicts that a' is an N-optimal core allocation at R'. This completes the proof of Claim 2.

Claim 3. $\overline{a}' \in \mathcal{C}(\overline{R})$. Suppose to the contrary that $\overline{a}' \notin \mathcal{C}(\overline{R}_i)$. Then, there is $S \subseteq \mathcal{N}$ that blocks $\overline{a}'$ at $\overline{R}$ via $b \in \mathcal{A}^f$. Since the only difference between $\overline{R}$ and $\overline{R}'$ is i's preference, $i \in S$ exibits $b(i)\ \overline{P}_i\ \overline{a}'(i)$. By Claim 2, $\overline{a}'(i) = \omega(i)$. As $SUC(\overline{R}_i, \omega(i)) \subseteq SUC(\overline{R}'_i, \omega(i))$, $b(i)\ \overline{P}'_i\ \omega(i) = \overline{a}'(i)$. This implies $\overline{a}' \notin \mathcal{C}(\overline{R}')$, a contradiction. This completes the proof of Claim 3.

Now we complete the proof of Lemma 2. By Claim 1, the assignment of i at $\varphi(\overline{R}) \in \mathcal{C}(\overline{R})$ is $a(i) \neq \omega(i)$. On the other hand, by Claim 2 and 3, the assignment of i at $\overline{a}' \in \mathcal{C}(\overline{R})$ is $\overline{a}'(i) = \omega(i)$. This contradicts Assumption 2. □

Proof of Theorem 2. Let φ be an NOC rule. We show that φ satisfies UTI on N. Let $i \in N$ and $R = (R_i, R_{-i}) \in \mathcal{D}$. Letting $a := \varphi(R)$, let $R'_i \in \mathcal{D}_i$ be the UT of R_i at $a(i)$. Let $R' := (R'_i, R_{-i})$ and $a' := \varphi(R')$. Let $x, y \in X_i$ be such that $r_{R_i}(x) = r_{R_i}(a(i)) - 1$ and $r_{R_i}(y) = r_{R_i}(a(i)) - 2$. Suppose to the contrary that $a'(i) \neq a(i)$. We first show that $a'(i) = x$ through the following three claims.

Claim 1. $x\ R'_i\ a'(i)$. If $a'(i)\ P'_i\ x$, $a'(i) = a(i)$ ($\because$ R_i is a LS of R'_i at $a'(i)$), a contradiction. This completes the proof of Claim 1.

Claim 2. $a'(i) \neq y$.If $a'(i) = y$, $a'(i) = a(i)$ ($\because$ R_i is the SU of R'_i at $a'(i)$), a contradiction. This completes the proof of Claim 2.

Claim 3. $a'(i)\ R'_i\ a(i)$. Suppose to the contrary that $a(i)\ P'_i\ a'(i)$. Since a' is an NOC allocation at R', $a \notin \mathcal{C}(R')$. Thus, there exists $S \subseteq \mathcal{N}$ that blocks a at R' via $b \in \mathcal{A}^f$. Since the only difference between R and R' is i's preference, $i \in S$ and $b(i)\ P'_i\ a(i)$. Since $SUC(R_i, a(i)) = SUC(R'_i, a(i))$, $b(i)\ P_i\ a(i)$. Thus $a \notin \mathcal{C}(R)$, a contradiction. This completes the proof of Claim 3.

Since $a'(i) \neq a(i)$, Claim 1,2 and 3 imply that the remaining possibility is $a'(i) = x$, i.e., $\varphi_i(R'_i, R_{-i}) = x$. Noting that R_i is the SU of R'_i at y, $\varphi_i(R_i, R_{-i}) \in \{x, y\}$ ($\because$ Lemma 2). This contradicts that $a(i) \notin \{x, y\}$. □

Corollary 1. *The student-proposing deferred-acceptance rule for college admission problems is strategy-proof for students.*

Proof. Theorem 2 in Gale and Shapley (1962) shows that the student-proposing DA algorithm hits the student-optimal "stable" matching. Since the set of stable matchings in college admission problems coincides with the core by

[9] Since φ is an NOC rule, $\overline{a}' \in \mathcal{C}(\overline{R}'_i, R_{-i})$. Thus, $\overline{a}'(i)\ \overline{R}'_i\ \omega(i)$.

[10] $a(i)\ P'_i\ a'(i)$ and the construction of $\overline{R}'_i$ together imply $a(i)\ \overline{P}'_i\ a'(i)$. Note that $\omega(i)$ is the best object at $\overline{R}'_i$ among the objects worse than $a(i)$. Thus, $\omega(i)\ \overline{R}'_i\ a'(i)$.

Proposition 5.36 in Roth and Sotomayor (1990), Assumption 1 is satisfied. Gale and Sotomayor (1985) provide a brief proof of Assumption 2 in college admission problems. □

Corollary 2. *The top-trading cycles rule for housing markets is strategy-proof for $\mathcal{N}$.*

Proof. Roth and Postlewaite (1977) provide a brief proof for the uniqueness of the core allocation in housing markets. This fact implies that Assumption 1 and 2 are satisfied in housing markets. Note that Shapley and Scarf (1974) point out that Gale's TTC rule hits the unique core allocation for each housing market. □

4 Conclusion

In this paper, we characterize strategy-proofness with three invariance properties under preference transformation (Theorem 1). Utilizing it, we gave an elementary and transparent proof for strategy-proofness of N-optimal core rules for GIGAPs (Theorem 2). Consequently, strategy-proofness of both the student-proposing DA rule for college admission problems and the TTC rule for housing markets is obtained. Before concluding the paper, we point out that the identical argument works to establish strategy-proofness of the worker-optimal stable rule, also known as the cumulative-offer process rule, for matching with contracts under weakened substitutes conditions.

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Appendix: Not intended for publication

A new proof for strategy-proofness of the worker-optimal stable rule for matching with contracts under weakened substitutes conditions

In the main text, we claim that the proof technique developed in this paper works to establish strategy-proofness of the worker-optimal stable rule, also known as the cumulative-offer process rule, for matching with contracts under weakened substitutes conditions (Hatfield and Kojima, 2010). This appendix shows that the proof technique for Theorem 2 indeed works to establish the result. Thus it offers an alternative proof for the strategy-profness result in Theorem 7 of Hatfield and Kojima (2010).

First, we introduce the matching with contracts model under weakened substitutes conditions (Hatfield and Kojima, 2010). Let $W = \{w_1, \dots, w_p\}$ be the set of workers, and $F = \{f_0, f_1, \dots, f_q\}$ be the set of firms ($1 \leq p, q < +\infty$). Let X be a finite set of potentially possible contracts. Let a function $g : X \to W \times F$ be such that for each contract $x \in X$, g assigns the worker-firm pair who are concerned in the contract x. For each $x \in X$, let $x_W := g_W(x)$ and $x_F := g_F(x)$. For each $Y \subseteq X$, let $Y_W := \{w \in W \mid \exists x \in Y \text{ s.t. } w = x_W\}$ and $Y_F := \{f \in F \mid \exists x \in Y \text{ s.t. } f = x_F\}$. For each $w \in W$ and each $f \in F$, let $X_w := \{x \in X \mid x_W = w\}$ and $X_f := \{x \in X \mid x_F = f\}$.

We call f_0 the null firm. For each worker $w \in W$, we assume that there exists exactly one contract between w and f_0, i.e., for each $w \in W$, there exists unique contract $x_{wf_0} \in X$ such that $g(x_{wf_0}) = (w, f_0)$. Thus, $X_{f_0} = \{x_{w_1 f_0}, \dots, x_{w_p f_0}\}$.

Next, we define the choice criterion of agents. For each $w \in W$, let $\mathcal{D}_w$ be the set of preferences of w on X_w.[11] For each $f \in F$, the set of feasible choice functions of firm f is denoted as $\mathcal{D}_f$. We assume that a firm cannot have multiple contracts with a single worker. Namely, we assume that for each $f \in F, \mathcal{D}_w \subseteq \{C_f : 2^{X_f} \to 2^{X_f} \mid (i) \forall Y \in 2^{X_f}, C_f(Y) \subseteq Y, \text{ and } (ii) \forall Y \in 2^{X_f}, \forall x, y \in C_f(Y) \text{ with } x_W = y_W, x = y\}$. Specifically, we assume that for the null firm $f_0 \in F$, the only feasible choice function is identity mapping $\mathcal{D}_{f_0} = \{id\}$, i.e., it always accepts all contracts offered to it. Let $\mathcal{D} := \left(\prod_{w \in W} \mathcal{D}_w\right) \times \left(\prod_{f \in F} \mathcal{D}_f\right)$.

Next, we introduce the concept of feasible allocations in this appendix. A matching is a function $\mu : W \cup F \to 2^X$ such that (i) for each $w \in W, \mu(w) \subseteq X_w$ and $|\mu(w)| = 1$,[12] (ii) for each $f \in F, \mu(f) \subseteq X_f$, (iii) for each $f \in F$ and each $\{x, y\} \subseteq \mu(f)$ with $x_W = y_W, x = y$, and (iv) for each $(w, f) \in W \times F$ with $\mu(w)_F = f, \mu(w) \subseteq \mu(f)$. Letting $\mathcal{M}$ be the set of matchings, the set of feasible allocations in this appendix is $\mathcal{M}$.

Finally, we define the central concept in two-sided matching problems, *stability*. To this end, we need to introduce several types of concepts of blocking. Let $(R, C) \in \mathcal{D}$ and $\mu \in \mathcal{M}$ be given. A worker $w \in W$ blocks μ at (R, C) *if* $x_{wf_0} \; P_w \; \mu(w)$. A firm $f \in F$ blocks μ at (R, C) *if* $C_f(\mu(f)) \neq \mu(f)$. A worker-firm pair $(w, f) \in W \times F$ blocks μ at (R, C) *if* there exists $x \in X_w \cap X_f$ such that $x \; P_w \; \mu(w)$ and $C_f(\mu(f) \cup \{x\}) \neq \mu(f)$. The matching μ is **individually rational (IR) at $(\mathbf{R}, \mathbf{C})$** *if* no agent $i \in W \cup F$ blocks μ at (R, C). The matching μ is **stable at $(\mathbf{R}, \mathbf{C})$** *if* i) μ is IR at (R, C), and ii) no worker-firm pair blocks μ at (R, C). Let $\mathcal{I}(R, C) := \{\mu \in \mathcal{M} \mid \mu \text{ is IR at } (R, C)\}$ and $\mathcal{S}(R, C) := \{\mu \in \mathcal{M} \mid \mu \text{ is stable at } (R, C)\}$.

[11] A preference relation of a worker w is a complete, transitive and anti-symmetric binary relation on X_w.

[12] Hereafter, abusing the notation, we also use $\mu(w)$ to denote the contract contained in $\mu(w)$.

A rule is a function that assigns a feasible allocation for each problem, i.e., a function from $\mathcal{D}$ to $\mathcal{M}$. Our generic notation for a rule is φ. *Strategy-proofness* for W requires that no agent in W can profitably manipulate a rule by misreporting her preferences. Formally, a rule φ is **strategy-proof for W** *if* for each $w \in W$, each $(R_w, R_{-w}, C) \in \mathcal{D}$, and each $R'_w \in \mathcal{D}_w$, $\varphi_w(R_w, R_{-w}, C)\ R_w\ \varphi_w(R'_w, R_{-w}, C)$.

We define four types of preference transformation SU, UT, PU and LS in the similar way to those in the main text. We omit the details of them to avoid the repetition. The invariance properties of a rule is defined by using them. We say that a rule φ satisfies the **single-upgrade invariance (SUI)** (Resp. **upper-transposition invariance (UTI)**, **pairwise-upgrade invariance (PUI)**, **lower-shuffle invariance (LSI)**) **on W** *if* for each $w \in W$ and each problem $(R_w, R_{-w}, C) \in \mathcal{D}$, the assignment of w does not change when she changes her reporting to the SU (Resp. the UT, the PU, a LS) of R_w at $\varphi_w(R_w, R_{-w}, C)$. The following theorem is an adaptation of Theorem 1 to the current model. The proof is identical to that of Theorem 1, thus omitted.

Theorem 3. *A rule φ is strategy-proof for W if and only if φ satisfies SUI, UTI and LSI on W.*

We need two assumptions for our strategy-proofness result as was in the main text. Let $R \in \prod_{w\in W} D_w$ be given. The common preference of workers, $\geq_R$, is defined as follows: For each $\{\mu, \mu'\} \subseteq \mathcal{M}$, $\mu \geq_R \mu'$ *if and only if* $\mu(w)\ R_w\ \mu'(w)$ for all $w \in W$. Note that $\geq_R$ is a reflexive, anti-symmetric and transitive,[13] but not necessarily complete binary relation on $\mathcal{M}$.

Assumption 3. *[Existence of the worker-optimal stable matching]*

$$\forall (R, C) \in \mathcal{D}, \exists \mu \in \mathcal{S}(R, C)\ \ s.t.\ \ \mu' \in \mathcal{S}(R, C), \mu \geq_R \mu'.$$

Note that the worker-optimal stable matching defined in Assumption 3 is unique for each problem.[14] Thus, we could define a rule that assigns the worker-optimal stable matching for each $(R, C) \in \mathcal{D}$. We call it the **worker-optimal stable (WOS) rule**.

Assumption 4. *[A version of rural hospital theorem]*

$$\forall (R, C) \in \mathcal{D}, \forall \{\mu, \mu'\} \subseteq \mathcal{S}(R, C), \forall w \in W, [\mu(w) = x_{wf_0} \Leftrightarrow \mu'(w) = x_{wf_0}].$$

Under Assumption 3 and 4, we provide a new proof for the following theorem. Note that we made no direct assumption on the firms' choice functions. A well-known sufficient condition for Assumption 3 and 4 is given in Hatfield and Kojima (2010). According to their Theorem 5 and 6, the combination of the unilateral substitute condition and the law of aggregate demand is sufficient for Assumption 3 and 4.[15] Under Assumption 3 and 4, we show the following theorem. A proof of the theorem is given after we prove two lemmas.

Theorem 4. *The WOS rule is strategy-proof for W.*

Lemma 3. *The WOS rule satisfies SUI and LSI on W.*

Proof. Let φ be the WOS rule. Let $w \in W, (R, C) = (R_w, R_{-w}, C) \in \mathcal{D}$ and $\mu := \varphi(R, C)$. Let $R'_w \in \mathcal{D}_w$ be the SU or a LS of R_w at $\mu(w)$. Let $R' := (R'_w, R_{-w})$ and $\mu' := \varphi(R', C)$. Suppose to the contrary that $\mu'(w) \neq \mu(w)$.

Claim 1. μ is stable at (R', C). Suppose not. Then, there is $(\overline{w}, \overline{f}) \in W \times F$ that blocks μ at (R', C).[16] Namely, there is $\overline{x} \in X_{\overline{w}} \cap X_{\overline{f}}$ such that $\overline{x}\ P'_{\overline{w}}\ \mu(\overline{w})$ and $C_{\overline{f}}(\mu(\overline{f}) \cup \{\overline{x}\}) \neq \mu(\overline{f})$. Obviously, $\overline{w} = w$. Thus $\overline{x}\ P'_w\ \mu(w)$. Since

[13] Note that workers' preferences are reflexive, anti-symmetric and transitive.
[14] This is because $\geq_R$ is anti-symmetric.
[15] See also Aygün and Sönmez (2012) for an additionally important property of *the irrelevance of rejected contracts*.
[16] Note that $\mu \in \mathcal{I}(R', C)$ by the construction of R'_w.

Figure 7: For the last part of the proof of Claim 2 in Lemma 3.

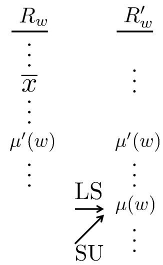
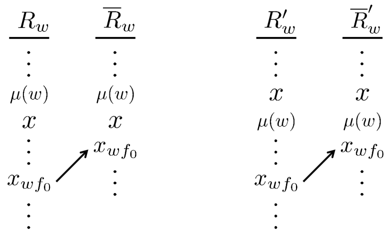


Figure 8: The construction of $\overline{R}_w$ and $\overline{R}'_w$ in the proof of Lemma 4.



$SUC(R'_w, \mu(w)) \subseteq SUC(R_w, \mu(w))$, $\overline{x}\ P_w\ \mu(w)$. Thus, the pair $(w, \overline{f})$ blocks μ at (R, C). This contradicts that $\mu \in \mathcal{S}(R, C)$. This completes the proof of Claim 1.

By Claim 1, $\{\mu, \mu'\} \subseteq \mathcal{S}(R', C)$. Since φ is the WOS rule, $\mu'(w)\ R'_w\ \mu(w)$. Thus $\mu'(w)\ P'_w\ \mu(w)$.

Claim 2. μ' is stable at (R, C). Suppose not. Thus, there is $(\overline{w}, \overline{f}) \in W \times F$ that blocks μ' at (R, C).[17] Namely, there exists $\overline{x} \in X_{\overline{w}} \cap X_{\overline{f}}$ such that $\overline{x}\ P_{\overline{w}}\ \mu'(\overline{w})$ and $C_{\overline{f}}(\mu'(\overline{f}) \cup \{\overline{x}\}) \neq \mu'(\overline{f})$. Obviously, $\overline{w} = w$. Thus $\overline{x}\ P_w\ \mu'(w)$. Since R'_w is either the SU or a LS of R_w at $\mu(w)$, the preference between $\overline{x}$ and $\mu'(w)$ is preserved, i.e., $\overline{x}\ P'_w\ \mu'(w)$ (See Figure 7).[18] Thus, the pair $(w, \overline{f})$ blocks μ' at (R', C). This contradicts that $\mu' \in \mathcal{S}(R', C)$. This completes the proof of Claim 2.

By Claim 2, $\{\mu, \mu'\} \subseteq \mathcal{S}(R, C)$. Since φ is the WOS rule, $\mu(w)\ R_w\ \mu'(w)$. On the other hand, $\mu'(w)\ P'_w\ \mu(w)$ and $SUC(R'_w, \mu(w)) \subseteq SUC(R_w, \mu(w))$ together imply that $\mu'(w)\ P_w\ \mu(w)$, a contradiction. □

Lemma 4. *Let φ be the WOS rule. Let $w \in W, (R, C) = (R_w, R_{-w}, C) \in \mathcal{D}$ and $\mu := \varphi(R, C)$. Let $x \in X_w$ be such that $r_{R_w}(x) = r_{R_w}(\mu(w)) + 1$. Then, for the SU of R_w at x, denoted as R'_w, $\varphi_w(R'_w, R_{-w}, C) \in \{\mu(w), x\}$.*

Proof. Let $R' := (R'_w, R_{-w})$ and $\mu' := \varphi(R', C)$. Suppose to the contrary that $\mu'(w) \notin \{\mu(w), x\}$. First note that $\mu(w)\ P'_w\ \mu'(w)$.[19] Thus $x\ P_w\ \mu'(w)\ R_w\ x_{wf_0}$ and $\mu(w)\ P'_w\ \mu'(w)\ R'_w\ x_{wf_0}$. Define $\overline{R}_w$ (resp. $\overline{R}'_w$) by upgrading x_{wf_0} to be the immediate successor of x (resp. $\mu(w)$) at R_w (resp. R'_w) without affecting the preferences between other contracts (Figure 8). Let $\overline{R} := (\overline{R}_w, R_{-w}), \overline{R}' := (\overline{R}'_w, R_{-w})$ and $\overline{\mu}' := \varphi(\overline{R}', C)$.

Claim 1. $\varphi_w(\overline{R}, C) = \mu(w)$. Since $\overline{R}_w$ is a LS of R_w at $\mu(w)$, $\varphi_w(\overline{R}, C) = \varphi_w(R, C)$ by Lemma 3. This completes the proof of Claim 1.

Claim 2. $\overline{\mu}'(w) = x_{wf_0}$. Suppose to the contrary that $\overline{\mu}'(w)\ \overline{P}'_w\ x_{wf_0}$.[20] Note that $\overline{\mu}'(w)\ \overline{P}'_w\ x_{wf_0}\ \overline{R}'_w\ \mu'(w)$.[21] This

[17] Note that $\mu' \in \mathcal{I}(R, C)$ by the construction of R'_w.

[18] Since $\mu'(w)\ P'_w\ \mu(w)$, $\mu'(w)\ P_w\ \mu(w)$ ($\because$ the transformation is either SU or LS). Thus, $\overline{x}$ and $\mu'(w)$ do not move at the transformation from R_w to R'_w.

[19] If this is not true, $\mu'(w)\ P'_w\ x$. However, in this case, R_w is a LS of R'_w at $\mu'(w)$. Thus, $\mu(w) = \mu'(w)$ by Lemma 3, a contradiction.

[20] Since φ is the WOS rule, $\overline{\mu}' \in \mathcal{I}(\overline{R}'_w, R_{-w}, C)$. Thus, $\overline{\mu}'(w)\ \overline{R}'_w\ x_{wf_0}$.

[21] $\mu(w)\ P'_w\ \mu'(w)$ and the construction of $\overline{R}'_w$ together imply $\mu(w)\ \overline{P}'_w\ \mu'(w)$. Note that x_{wf_0} is the best contract at $\overline{R}'_w$ among the contracts worse than $\mu(w)$. Thus, $x_{wf_0}\ \overline{R}'_w\ \mu'(w)$.

Figure 9: The preference relations R_w and R'_w in the proof of Theorem 4.

R_w	R'_w
$\vdots$	$\vdots$
y	x
x	y
$\mu(w)$	$\mu(w)$
$\vdots$	$\vdots$

implies that $\overline{\mu}'(w)\ P'_w\ \mu'(w)$ since the only difference between R'_w and $\overline{R}'_w$ is the position of x_{wf_0}. On the other hand, $\overline{\mu}' \in \mathcal{S}(R', C)$. This contradicts that μ' is worker-optimal stable at (R', C). This completes the proof of Claim 2.

Claim 3. $\overline{\mu}' \in \mathcal{S}(\overline{R}, C)$. Suppose to the contrary that $\overline{\mu}' \notin \mathcal{S}(\overline{R}, C)$. Then, there is $(\overline{w}, \overline{f}) \in W \times F$ that blocks $\overline{\mu}'$ at $(\overline{R}, C)$.[22] Namely, there exists $y \in X_{\overline{w}} \cap X_{\overline{f}}$ such that $y\ \overline{P}_{\overline{w}}\ \overline{\mu}'(\overline{w})$ and $C_{\overline{f}}(\overline{\mu}'(\overline{f}) \cup \{y\}) \neq \overline{\mu}'(\overline{f})$. Obviously, $\overline{w} = w$. By Claim 2, $\overline{\mu}'(w) = x_{wf_0}$. As $SUC(\overline{R}_w, x_{wf_0}) \subseteq SUC(\overline{R}'_w, x_{wf_0})$, $y\ \overline{P}'_w\ x_{wf_0} = \overline{\mu}'(w)$. This implies that $\overline{\mu}' \notin \mathcal{S}(\overline{R}', C)$, a contradiction. This completes the proof of Claim 3.

Now we completes the proof of Lemma 4. By Claim 1, the assignment of w at $\varphi(\overline{R}, C) \in \mathcal{S}(\overline{R}, C)$ is $\mu(w) \neq x_{wf_0}$. On the other hand, by Claim 2 and 3, the assignment of w at $\overline{\mu}' \in \mathcal{S}(\overline{R}, C)$ is $\overline{\mu}'(w) = x_{wf_0}$. This contradicts Assumption 4. □

Proof of Theorem 4. Let φ be the WOS rule. We only show that φ satisfies UTI on W. Let $w \in W$ and $(R, C) = (R_w, R_{-w}, C) \in \mathcal{D}$. Letting $\mu := \varphi(R, C)$, let $R'_w \in \mathcal{D}_w$ be the UT of R_w at $\mu(w)$. Let $R' := (R'_w, R_{-w})$ and $\mu' := \varphi(R', C)$. Let $x, y \in X_w$ be such that $r_{R_w}(x) = r_{R_w}(\mu(w)) - 1$ and $r_{R_w}(y) = r_{R_w}(\mu(w)) - 2$. Suppose to the contrary that $\mu'(w) \neq \mu(w)$. We first show that $\mu'(w) = x$ through the following three claims.

Claim 1. $x\ R'_w\ \mu'(w)$. If $\mu'(w)\ P'_w\ x$, $\mu'(w) = \mu(w)$ ($\because$ R_w is a LS of R'_w at $\mu'(w)$), a contradiction. This completes the proof of Claim 1.

Claim 2. $\mu'(w) \neq y$. If $\mu'(w) = y$, $\mu'(w) = \mu(w)$ ($\because$ R_w is the SU of R'_w at $\mu'(w)$), a contradiction. This completes the proof of Claim 2.

Claim 3. $\mu'(w)\ R'_w\ \mu(w)$. Suppose to the contrary that $\mu(w)\ P'_w\ \mu'(w)$. Then, there is $(\overline{w}, \overline{f}) \in W \times F$ that blocks μ at (R', C).[23] Namely, there exists $z \in X_{\overline{w}} \cap X_{\overline{f}}$ such that $z\ P'_{\overline{w}}\ \mu(\overline{w})$ and $C_{\overline{f}}(\mu(\overline{f}) \cup \{z\}) \neq \mu(\overline{f})$. Obviously, $\overline{w} = w$. Since $SUC(R_w, \mu(w)) = SUC(R'_w, \mu(w))$, $z\ P_w\ \mu(w)$. Thus, $\mu \notin \mathcal{S}(R, C)$, a contradiction. This completes the proof of Claim 3.

Since $\mu'(w) \neq \mu(w)$, Claim 1,2 and 3 imply that the remaining possibility is $\mu'(w) = x$, i.e., $\varphi_w(R'_w, R_{-w}, C) = x$. Noting that R_w is the SU of R'_w at y, $\varphi_w(R_w, R_{-w}, C) \in \{x, y\}$ ($\because$ Lemma 4). This contradicts that $\mu(w) \notin \{x, y\}$. □

Corollary 3. *Suppose that for each $f \in F$, each $C_f \in \mathcal{D}_f$ satisfies the irrelevance of rejected contracts, unilateral substitute and the law of aggregate demand. Then, the cumulative-offer process rule is strategy-proof.*

Proof. Theorem 5 in Hatfield and Kojima (2010) states that the cumulative-offer process rule hits the worker-optimal stable matching. Thus Assumption 3 is satisfied. Theorem 6 in Hatfield and Kojima (2010) states that Assumption 4 is satisfied. □

[22] Note that $\overline{\mu}' \in \mathcal{I}(\overline{R}_w, R_{-w}, C)$.

[23] Since μ' is the worker-optimal stable matching at (R', C), μ is not stable at (R', C). Moreover, μ is individually rational at (R', C). Thus, there must be a blocking pair.