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Dual Organ Markets: Coexistence of Living and Deceased Donors

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Dual-Organ Markets: Coexistence of Living and Deceased Donors *

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Abstract

To overcome the worldwide shortage of deceased-donor organs, the medical community has developed various modalities of transplantation. For dual-organ transplantation, the shortage is serious as the modality requires two donors for a single patient. Lung transplantation, a representative example of dual-organ transplantation, is the only treatment for patients in the final stage of a chronic lung disease. Prior to April 2015, there were only two types of transplantation available: deceased-donor transplants and living-donor transplants. Ergin, Sönmez, and Ünver (2017) have proposed the idea of exchanging donors exclusively for living-donor lung transplantation. The new technology, called hybrid transplantation, is now available as evidenced as Dr. Oto and his team at Okayama University Hospital successfully transplanted a deceased lung and a lobe of live lung to one patient at the same time. As the modality itself reveals the importance of simultaneously operating the deceased- and living-donor markets, we study the market with both deceased- and living-donors. In particular, we point

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out that the hybrid transplantation opens up a new type of donor exchange. We investigate a mechanism of organizing transplants in terms of efficiency, fairness, and incentive-compatibility.
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1 Introduction

To overcome the worldwide shortage of deceased-donor organs, the medical community has developed various modalities of transplantation. For dual-organ transplantation including lung transplantation, dual-graft liver transplantation, and simultaneous liver-kidney transplantation, the shortage is serious as the modality requires two donors for a single patient. This paper studies how to operate these markets when both deceased- and living-donors are available.

Lung transplantation, a representative example of dual-organ transplantation, is the only treatment for patients with end-stage lung diseases. As of October 31, 2020, 430 patients were registered on the waiting list for deceased-donor lung transplants in Japan, while 79 patients received transplants in 2019.[1] Deceased-donor transplantation uses one or two lungs of a deceased donor to replace the diseased lungs of a patient.[2] Because the number of deceased donors is historically low for various reasons, the Japanese medical community has developed live donor lung transplantation which needs two living donors each of whom donates one lobe out of five to their intended recipient. These transplants have been conducted on approximately 10 to 20 patients a year in recent years.[3] For the liver case, it is reported that about 400 dual-graft transplants had been conducted till 2017 in South Korea (Song et al., 2017). Although they are a substantial source for transplantation, living donors are conventionally constrained to being relatives of patients.[4] Moreover, they are medically

[1] The cited data are available on the Japan Organ Transplant Network homepage. In particular, the former is on `https://www.jotnw.or.jp/data/`, while the latter is on `https://www.jotnw.or.jp/data/offer.php?year=2019`. These were retrieved on November 18, 2020.

[2] We refer to the donor by the male personal pronoun and to the patient by the female personal pronoun.

[3] See the Factbook by the Japan Society for Transplantation, `http://www.asas.or.jp/jst/pdf/factbook/factbook2019.pdf`, accessed on September 15, 2020.

[4] "Conventionally" means that this practice is not illegal, but it is followed by the medical community as the ethical guideline published in the Japan Society for Transplantation. A transplant from non-relatives of a patient needs special permission in hospitals conducting the transplant. The guideline is available on `http://www.asas.or.jp/jst/about/about12.html`, accessed on March 5, 2018.

constrained to being compatible with patients of blood type, tissue type, sizes, etc. These constraints are difficult to meet as a patient needs two compatible relatives. To overcome such difficulties, Ergin, Sönmez, and Ünver (2017) recently proposed a novel transplantation modality of donor exchange — a between-patients exchange of the incompatible donors — by applying the same idea for kidney exchange (Rapaport, 1986; Roth, Sönmez, and Ünver, 2004, 2005). Due to the multi-unit demand, the application is theoretically non-trivial and is practically important as Ergin, Sönmez, and Ünver (2017) show the potential increase of lung transplants from roughly 80 to 260%, depending on the size of the exchange, using the Japanese data.[5]

Another innovative transplant, called hybrid transplantation, has been successfully conducted for liver by Dr. Lee and his colleagues in 2000 (Lee et al., 2001) and for the lung by Dr. Oto and his colleagues in 2015.[6] The hybrid transplantation uses one graft of a living donor *and* one graft of a deceased donor,[7] which is one of a key modality for patients who could not receive transplants through conventinal modalities.

We incorporate the possibility of hybrid transplantation for donor exchange in lung transplants, extending the Ergin, Sönmez, and Ünver (2017) model which exclusively takes up living-donor transplantation for donor exchange. This extension not only practically increases the number of saved patients but it also opens up a new type of theoretical challenge. Let us discuss its practical importance and then the theoretical challenge.

A patient with only one compatible donor has an obvious benefit from hybrid transplantation. There might also be a situation in which two patients, each with two incompatible donors, cannot exchange donors for living-donor transplants but can do so under hybrid transplantation. Hybrid transplantation can enhance a living-donor exchange in the sense that patients who could not participate in the swapping of donors can have a living-donor transplant (see Figure 1). Moreover, if we consider only two-way exchange under a living-donor transplantation, O-blood type patients do not benefit from donor exchange (Lemma 1 in Ergin, Sönmez, and Ünver, 2017). However, with the introduction of hybrid transplantation, such patients can receive transplants (see Figure 1).

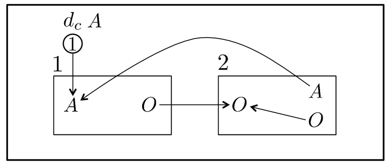
Taking into account hybrid transplantation poses several new theoretical challenges. The model we introduce has one deceased donor and finitely many patients who bring a number of

[5] The numbers are for 50 patients. See Table II in Ergin, Sönmez, and Ünver (2017).

[6] See the official news at Okayama University, `https://www.okayama-u.ac.jp/eng/news/index_id4469.html`. The web was accessed on January 6, 2019.

[7] One of the biggest difference between lung and liver transplantation is that the main modality for lung is the dual-graft transplant while that of liver is the single-graft one.

Figure 1: Benefits from hybrid transplantation.



Note: In the figure, one lung from the deceased donor, d_c, is available, patient 1 of blood type A has a donor of blood type O, and patient 2 of blood type O has two donors with blood types A and O. The allocation indicated by the arrows shows that with donor exchange, patient 1 receives the hybrid transplant, while patient 2 the living donor transplant. Note that without the hybrid transplantation, patient 2 has no transplant opportunity.

their relatives as (compatible or incompatible) living donors.[8] A patient may have multi-unit demand as she is assigned two living donors for living-donor transplantation, or is assigned one lung from a deceased donor and one graft from a living donor for hybrid transplantation, unlike the standard matching model with unit demand. Organs are indivisible goods, a deceased organ is taken to be a social endowment (common ownership), and living donors are those owned by patients (private ownership). Thus, a patient, who forms preferences over the transplantation types, can simultaneously participate in the two "dual" markets for deceased donors and living donors.[9] Thus, a hybrid transplantation treatment naturally leads to the model with mixed ownership. Our model is the first real-life application of a matching problem with multi-unit demand and mixed ownership.[10] A desirable mechanism in such a model has not been discussed.

We search for a matching mechanism — a procedure of assigning donors to patients for transplants based on their medical types and preferences — which should have the desirable properties of individual rationality, Pareto efficiency, fairness, and the incentive compatibility of strategy-proofness. We focus on individually rational allocations in which each patient receives an assignment at least as good as the no-transplant. The Pareto efficiency is the standard one, requiring that no one could be improved without hurting others. We propose

[8]The assumption of one deceased donor reflects the situation of countries with relatively few deceased donors. Such an example is Japan which has at most 97 deceased donors a year from 1997 to 2019. Since the arrival date of a deceased donor is not controllable, the assumption is appropriate if we operate a mechanism every day or every time when the pool of patients and donors is updated.

[9]We use the term "dual-organ markets" for the following two meanings. First, it expresses the multi-unit demand used by Ergin, Sönmez, and Ünver (2017). Second, patients simultaneously participate in the two markets for deceased donors and living donors.

[10]An exception is Roth, Sönmez, and Ünver (2004) for mixed ownership. In their kidney exchange model, they have living donors and the waiting option of putting patients on higher priority in the waiting list of patients. The waiting option implicitly expresses the treatment of deceased donors. On the other hand, we explicitly model the situation in which a patient can have access to and use resources in both the markets for living donors and those for deceased donors.

a notion of $\succeq$-fairness, based on a priority ordering $\succeq$ over patients, which represents the priority to receive the organ from the deceased donor. A $\succeq$-fair allocation is the one without justified envy, adjusted to our environment from the school choice problem (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Roughly speaking, justified envy means that some patient can improve with the deceased donor's organ assigned to a lower-priority patient. Finally, the strategy-proofness we use is also the standard one where truth-telling is a weakly dominant strategy of each patient in the induced preference revelation game.

We examine the current Japanese mechanism that is only for assigning deceased lungs. We show in Proposition 3 that it is neither strategy-proof nor $\succeq$-fair, though it is individually rational and Pareto efficient. Instead, we adapt a priority mechanism considered in the literature, in particular Roth, Sönmez, and Ünver (2005)'s mechanism for a kidney exchange. In the priority mechanism, the highest-priority patient selects all of her favorite individually rational allocations, the second-highest patient selects all of her favorites among those selected by the highest-priority patient, and so on. Then, we show in Theorem 1 that in regimes without donor exchange, including the current Japanese one, the priority mechanism recovers strategy-proofness and $\succeq$-fairness, still keeping individual rationality and Pareto efficiency. However, once we allow for donor exchange, the strategy-proofness will be violated (Proposition 4) because a lower-priority patient can, by misreporting, narrow down the higher-priority patient's allocations by hiding her own donors and then obtain more preferable deceased transplants. It turns out that such a negative result is from a general impossibility result (Theorem 2) under any regime with donor exchange: no mechanism is individually rational, Pareto efficient, $\succeq$-fair, and strategy-proof. Note that the priority mechanism satisfies all of the properties except for strategy-proofness. We observe that misreporting is risky in the priority mechanism in that if the medical types were different, a misreporting patient would get no-transplant instead of some transplant. With this observation, we introduce uncertainty over the medical type of the deceased donor as well as other patients' medical types and preferences. Then, the priority mechanism is shown to be robust against manipulation, that is, the truth-telling profile is a Bayesian Nash equilibrium (Theorem 3).

We introduce the dual-organ markets in Section 2. The model covers dual-donor organ exchange (Ergin, Sönmez, and Ünver, 2017) for dual-graft liver transplantation, bilateral living-donor lung transplantation, and simultaneous liver-kidney transplantation, with a new structure of hybrid transplantation when both deceased and living donors coexist. Our description in the paper is for lung for simplicity, but the model is applicable to any other organ by selectively ignoring some parts of the model. In Section 3, we provide and investigate several properties of mechanisms. Section 4 discusses our main results regarding the current Japanese mechanism and the priority mechanism. Finally, Section 5 concludes. Omitted proofs are given in Appendix.

1.1 Related literature

In the matching problem with unit demand, a model with social endowments is called a house allocation problem (Hylland and Zeckhauser, 1979); the one with private ownership is called a housing market (Shapley and Scarf, 1974); the one with mixed ownership is a house allocation problem with existing tenants (Abdulkadiroğlu and Sönmez, 1999) which Roth, Sönmez, and Ünver (2004) apply to the donor exchange for kidney transplantation. In dichotomous preferences, Roth, Sönmez, and Ünver (2005) further investigate a priority mechanism under private ownership in which higher-priority patients narrow down their favorite allocations. For the liver market, Ergin, Sönmez, and Ünver (2018) consider the unit-demand of a patient with the possibility that a donor can choose one of her left and right lobes. Thus, in this regard, their model is different from the kidney exchange model as well as ours.

For multi-unit demand, a matching problem with social endowment is studied by, for example, Klaus and Miyagawa (2001), Pápai (2001) and Budish and Cantillon (2012), while the one with private endowment is studied by Pápai (2007). Its special case where objects are exogenously separated by types is a multiple-type market (e.g., Moulin 1995; Konishi, Quint, and Wako 2001; Anno and Kurino 2016).

The most related paper is Ergin, Sönmez, and Ünver (2017) which is modeled as a matching problem with multi-unit demand and private endowment. Their model is a special case of our model when a deceased donor is not compatible with any patients. In this case, they focus on maximal matching instead of a mechanism. Our priority mechanism can achieve their maximal matching for a two-way donor exchange, too.

2 Model: Dual-Organ Markets

2.1 Basics

We describe the model for dual-organ markets. Although our description of the model is best fit for a more complex lung market with multi-unit demand, markets for the kidney and liver with single-unit demand can be easily adjusted with a trivial modification.[11]

There is one deceased donor who donates (one or two grafts of) her organs and finitely many patients each of whom accompanies living donors. All of the patients and donors have (medical) types for transplantation which determines which donor can donate to which patient. All patients have a priority for the organ(s) from a deceased donor which is typically determined by their waiting time. Formally, a model is a list $(N, \{D_i^L\}_{i\in N}, D^C, (T, \trianglerighteq), \theta, \succeq)$ which satisfies the following conditions:

1. $N := \{1, \ldots, n\}$ is a finite set of patients. We suppose that with enough medical knowledge each patient has the authority to decide for transplantation.[12] We assume that N contains at least two patients.

2. For each $i \in N$, D_i^L is the finite set of patient i's living donors. With this condition, patients may have various number of living donors.[13] We assume that $D_i^L \cap D_j^L = \emptyset$ for all $i, j \in N$ with $i \neq j$. That is, no living donor is shared by two patients. Since the main focus of this paper involves living-donor exchange, we assume that at least two patients have multiple living donors. Let $D^L := \cup_{i\in N} D_i^L$ be the set of all living donors in the market.

3. $D^C := \{d_c\}$ where d_c is the deceased (cadaveric) donor.[14] We assume that the deceased donor is not one of the donors registered as a living donor of a patient, i.e., $D^C \cap D^L = \emptyset$. We denote $D := D^L \cup D^C$.

4. $(T, \trianglerighteq)$ is the medical type space defined as the product space of given K kinds of component medical type spaces $\{(T_k, \trianglerighteq_k)\}_{k=1}^K$. For each $k \in \{1, \ldots, K\}$, T_k is a finite set of k-th component types equipped with a reflexive binary relation $\trianglerighteq_k$ where $t_k \trianglerighteq_k t'_k$ means that t_k is medically compatible with t'_k. For example, a donor with blood type O can donate to a patient with blood type A, not vice versa. This medical

[11]Several exceptional cases with dual-graft liver transplants are reported. For example, see Lee et al. (2001) for hybrid transplant and heterodoxical dual-graft living donor transplant, and Soejima et al. (2008) for orthodoxical dual-graft living donor transplant. Here, however, we understand the liver market as the market with single-unit demand, since it is the main modality in that market.

[12]This is because a patient can be considered to represent a team with her doctor. She does not necessarily know nor understand her own health status for transplantation, but she makes decision about whether to take a transplant. On the other hand, a medical doctor knows her health status, but cannot force her to agree to a transplant.

[13]Note that this setting generalizes the assumption of Ergin, Sönmez, and Ünver (2017) in which each patient has exactly two living donors.

[14]See Footnote 8 for the justification of this assumption of one deceased donor.

We mainly use the adjective "deceased" instead of "cadaveric" for the status of the donor. However, we use the notation d_c to denote the deceased donor to avoid the puzzling notation d_d.

compatibility is written as $O \trianglerighteq_k A$ and $A \not\trianglerighteq_k O$. For each $\{t, t'\} \subseteq T = \prod_{k=1}^{K} T_k$, a donor of type t is medically compatible with a patient of type t' if and only if for each $k \in \{1, \ldots, K\}$, $t_k \trianglerighteq_k t'_k$. This is denoted as $t \trianglerighteq t'$.

In case a component medical type space is defined by blood types, written as $(T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$,

$T_{\mathcal{B}} := \{O, A, B, AB\}$, and

$\trianglerighteq_{\mathcal{B}} := \{(O, A), (O, B), (O, AB), (A, AB), (B, AB)\} \cup \{(X, X) \mid X \in \mathcal{B}\}$.

We call $(T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$ **the ABO type space**. We assume for exposition that the collection of component type spaces $\{(T_k, \trianglerighteq_k)\}_{k=1}^{K}$ contains the ABO type space.

5. $\theta = (\theta_{d_c}, \theta_1, \ldots, \theta_n)$ represents the medical status of all agents. For brevity but slight confusion, we call it a type profile.

   - $\theta_{d_c} := (\theta_{d_c q}, \theta_{d_c T}) \in \{1, 2\} \times T$ is the medical status, or type, of the deceased donor where the deceased donor can supply $\theta_{d_c q}$ units of grafts and is of medical type $\theta_{d_c T}$.[15] Let $\Theta_{d_c} := \{1, 2\} \times T$ denote the set of types of the deceased donor.
   - For each $i \in N$, $\theta_i = (\theta_i(i), (\theta_i(d))_{d \in D_i^L}) \in T^{\{i\} \cup D_i^L}$ is the medical status, or type, of patient i which indicates her own medical type $\theta_i(i) \in T$ and her living donors' medical types $(\theta_i(d))_{d \in D_i^L} \in T^{D_i^L}$. We assume that the set of living donors of patient i contains at most one donor whose type is compatible with patient i. That is, for all $d, d' \in D_i^L, \theta_i(d) \trianglerighteq \theta_i(i)$ and $\theta_i(d') \trianglerighteq \theta_i(i)$ imply $d = d'$. This is because a patient would conduct a living-donor transplant with her own compatible donors if she had at least two compatible donors.[16] In other words, our model captures the market with patients who cannot have a transplant with their own donors. Note that this simplification is also employed in Ergin, Sönmez, and Ünver (2017).

[15] A unit of donated organ, called a "graft," is different according to the kind or source of the organ under consideration. As for the lung, the human body is equipped with a left and right lung, where each of them is composed of smaller parts called "lobe." In practice, a deceased donor donates both the left and right lung, except for the infected side if one exists, where each of them can be transplanted to different patients, while a living donor cannot donate more than a lobe of her lungs for health reasons. Following the convention, by a *unit of graft donated from the deceased donor*, we mean a lung (left lung or right lung), while by a *unit of graft donated from a living donor*, we mean a lobe of lung. As for the liver, the human body is equipped with a liver consisting of a left and right lobe. In practice, a deceased donor donates her whole liver to one recipient in Japan, while a living donor cannot donate more than a lobe of her liver for health reasons. Following the convention, by a *unit of graft donated from the deceased donor*, we mean a whole liver, while a *unit of graft donated from a living donor*, we mean a lobe of liver. As for the kidney, the human body is equipped with two kidneys, and its unit is understood in an obvious way.

[16] The modality of transplantation will be introduced in the next subsection.

- Let Θ_i be the set of types of patient i. Let $\Theta := \Theta_{d_c} \times \Theta_1 \times \ldots \times \Theta_n$ be the set of type profiles. For each $i \in N$, let $\Theta_{-i} := \Theta_{d_c} \times \prod_{j \neq i} \Theta_j$. For notational simplicity, given $\theta \in \Theta$, let $\theta(i)$ denote $\theta_i(i)$ for each $i \in N$, and $\theta(d)$ denote $\theta_i(d)$ for each $i \in N$ and each $d \in D_i^L$.

6. The symbol $\succeq$ represents a priority order for patients. Mathematically, it is a complete, transitive and anti-symmetric binary relation over N.

A dual-organ market is the model above together with a preference profile of patients. We will introduce the preferences after defining allocations in the next subsection.

2.2 Allocation

To describe the notion of assignments, we first clarify what kinds of transplants are potentially available for each patient $i \in N$ under a given type profile θ. A **transplant** for patient i is expressed by a pair $x_i = (x_i^C, x_i^L)$ where x_i^C is the number of grafts from the deceased donor and x_i^L is the set of living donors who contribute the transplant. Given $\theta \in \Theta$, we classify available transplants as follows.

1. **Deceased-donor dual-graft transplantation**: Transplanting two grafts from the deceased donor to a patient. The set of such transplants for patient i is denoted by

$$X_i^{20}(\theta) := \begin{cases} \{(2, \emptyset)\} & \text{if } \theta_{d_c q} = 2 \text{ and } \theta_{d_c T} \trianglerighteq \theta(i), \\ \emptyset & \text{otherwise.} \end{cases}.$$

2. **Deceased-donor single-graft transplantation**: Transplanting a single graft from the deceased donor to a patient. The set of such transplants for patient i is denoted by

$$X_i^{10}(\theta) := \begin{cases} \{(1, \emptyset)\} & \text{if } \theta_{d_c T} \trianglerighteq \theta(i), \\ \emptyset & \text{otherwise.} \end{cases}$$

3. **Living-donor dual-graft transplantation**: Transplanting two grafts from two living donors, one for each, to a patient. The set of such transplants for patient i is denoted by

$$X_i^{02}(\theta) := \left\{ (0, x^L) \,\middle|\, \begin{array}{l} i)\ x^L \in 2^{D^L} \text{ and } |x^L| = 2, \text{ and} \\ ii)\ \forall d \in x^L, \theta(d) \trianglerighteq \theta(i) \end{array} \right\}$$

4. **Hybrid transplantation**: Transplanting two grafts, one from the deceased donor and another from a living donor, to a patient. The set of such transplants for patient i is denoted by

$$X_i^{11}(\theta) := \begin{cases} \left\{ (1, x^L) \,\middle|\, \begin{array}{l} i)\ x^L \in 2^{D^L} \text{ and } |x^L| = 1, \text{ and} \\ ii)\ \forall d \in x^L, \theta(d) \trianglerighteq \theta(i) \end{array} \right\} & \text{if } \theta_{d_cT} \trianglerighteq \theta(i), \\ \emptyset & \text{otherwise.} \end{cases}$$

This was newly conducted at Okayama University Hospital for lung transplantation. In particular, let $\tilde{X}_i^{11}(\theta)$ be the set of hybrid transplants with i's own donor, i.e., $\tilde{X}_i^{11}(\theta) := \{(1, x^L) \in X_i^{11}(\theta) \mid x^L \subseteq D_i^L\}$.

5. **Null transplantation**: The transplant $(0, \emptyset)$, called the **null transplant**, means that patient i will not receive any transplant. The set $X_i^{00}(\theta) = \{(0, \emptyset)\}$ denotes the one that contains only the null transplant.[17]

Then, the set of potentially possible transplants for patient i under θ, $X_i(\theta)$, is defined as follows.[18]

$$X_i(\theta) := X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup X_i^{11}(\theta) \cup X_i^{02}(\theta) \cup X_i^{00}(\theta).$$

We will later describe by preferences which transplant is sufficient or not for saving a patient.

Given a type profile $\theta \in \Theta$, an **allocation**, $a^\theta = (a_i^\theta)_{i \in N} = \left((a_i^{\theta C}, a_i^{\theta L})\right)_{i \in N} \in \prod_{i \in N} X_i(\theta)$, describes a distribution of transplants among patients. In particular, we say that $a_i^\theta \in X_i(\theta)$ is an **assignment** of patient i at a^θ, or a **transplant** of patient i at a^θ. We impose three conditions on allocations. The first is a physical constraint: the number of grafts of d_c used at the allocation cannot exceed the number of grafts supplied by d_c. That is,

$$\sum_{i \in N} a_i^{\theta C} \leq \theta_{d_c q}. \tag{1}$$

The second is also a physical constraint: the number of grafts that a living donor can donate is at most one. Note that for each patient i, $a_i^{\theta L} \in 2^{D^L}$ describes who provides a

[17] Actually, the set $X_i^{00}(\theta)$ does not depend on the type profile θ. However, for notational consistency, we do not use the notation without the reference for the type profile such as X_i^{00}.

[18] When we interpret our model as a kidney transplant market, the set of potentially possible transplants is defined as $X_i(\theta) := X_i^{10}(\theta) \cup X_i^{01}(\theta) \cup X_i^{00}(\theta)$ while for the liver market $X_i(\theta) := X_i^{20}(\theta) \cup X_i^{01}(\theta) \cup X_i^{00}(\theta)$, where $X_i^{01}(\theta)$ denotes the set of living-donor transplants by using a single living donor. Note that the deceased donor in the kidney market can generate two transplants for the market since two kidneys can be delivered to different recipients, while the deceased donor in the liver market can generate only one transplant since the left and right lobes from the donor are conventionally transplanted to a single recipient.

Figure 2: Inflow and outflow are not balanced.

graft to i. Thus, to formalize the second condition, we just need the requirement that a living donor should not be included in two living-donor assignments at an allocation. That is,

$$\forall i, j \in N \text{ with } i \neq j, a_i^{\theta L} \cap a_j^{\theta L} = \emptyset. \tag{2}$$

The last condition is motivated by the allocations described in Figure 2. At the allocation in the left figure, patient i's donor d_i provides a graft to patient j, even though his own patient i does not receive a transplant. This allocation would fail to achieve the goal of donor d_i who participates in the market to relieve patient i. At the allocation in the right figure, both patients i and j receive transplants. However, their treatments are very different. Patient i receives a donation from d_j in exchange for the donation by donors' d_{i2} and d_{i3}. This situation might arise when the type of patient i is so rare that she cannot find any compatible donor except for d_{i1}. In this case, if i finally finds d_j as her compatible donor, she and her own donors might be willing to accept d_j in exchange for donors d_{i2} and d_{i3}. However, this type of exchange has a flavor of price mechanism which in most countries is strictly prohibited for the distribution of organs.

In this paper, we will not be involved in a radical interpretation of an allocation that calls for a drastic change in organ transplant legislation. To construct an allocation system without controversial concepts, we employ the cautious condition, requiring that each patient's benefit from the market in terms of the number of grafts from others' living donors should not exceed the contribution of her own living donors to other patients in terms of the number of grafts. That is,

$$\forall i \in N, \left| a_i^{\theta L} \setminus D_i^L \right| \leq \left| D_i^L \cap \left(\cup_{j \neq i} a_j^{\theta L} \right) \right|. \tag{3}$$

The left-hand side of the inequality is the number of grafts from other patients' living donors, while the right-hand side is the number of i's own donors who donate to other patients. We call this condition the **balanced condition**. Note that the allocations in Figure 2 are excluded because patient j violates the condition. Let us emphasize that this condition is implicitly employed in the Ergin, Sönmez, and Ünver's (2017) living-donor exchange model. Thus, our model with deceased and living donors under the balanced condition is a natural extension of their model. We denote the set of all allocations under θ by $\mathcal{A}(\theta)$. Moreover, let $\mathcal{A}$ be the set of all potentially possible allocations, i.e., $\mathcal{A} := \cup_{\theta\in\Theta}\mathcal{A}(\theta)$.

Our notion of allocations describes not only which patients are assigned organs and lobes from donors, but also which transplants are conducted. This point is technically important. For example, if the assignment $(1, \{d_i\})$ for patient i just described the former, it would not be clear whether the actual transplant is single-graft or hybrid. For this reason, we interpret the assignment a_i^θ of agent i as the conducted transplant, and assume that the transplant using all of the grafts described in a_i^θ will be conducted.

Remark 1 (Organs not described in an allocation). Based on the above interpretation, we explain how organs that do not appear in an allocation are treated.

1. At an allocation $a^\theta \in \mathcal{A}(\theta)$, if $\sum_{i\in N} a_i^{\theta C} < \theta_{d_c q}$, then $\theta_{d_c q} - \sum_{i\in N} a_i^{\theta C}$ units of deceased grafts are disposed of at allocation a^θ.

2. At an allocation $a^\theta \in \mathcal{A}(\theta)$, if a living donor $d_i \in D_i^L$ does not appear in any patient's assignment i.e., $d_i \notin a_1^{\theta L} \cup \ldots \cup a_n^{\theta L}$, then d_i does not undergo surgery at the allocation.

As a consequence of conditions imposed on allocations, we have the following simple pattern of allocations, i.e., the balanced condition holds with equality.

Proposition 1. *Under any type profile $\theta \in \Theta$, every allocation $a^\theta \in \mathcal{A}(\theta)$ is balanced in the following sense.*

$$\forall i \in N, \left|a_i^{\theta L} \setminus D_i^L\right| = \left|D_i^L \cap \left(\cup_{j\neq i}\, a_j^{\theta L}\right)\right|.$$

Namely, for each $i \in N$, the number of grafts donated to i from other patients' living donors is equal to the number of i's living donors who donate a graft to other patients.

2.3 Preference

We formulate the preferences of patients. To this end, it is useful to have the notations $\mathcal{R}(Z)$ and $\mathcal{P}(Z)$ for any finite set Z: $\mathcal{R}(Z)$ is the set of complete and transitive binary relations on Z, while $\mathcal{P}(Z)$ is the set of complete, transitive, and anti-symmetric binary relations on Z.

We assume that each patient has a preference in the set of transplantation types $\{20, 10, 11, 02, 00\}$, where $20, 10, 11, 02$, and 00 stand for deceased-donor dual-graft, deceased-donor single-graft,

		Donation of living donors to non-relatives	
		unacceptable	acceptable
Hybrid transpl.	unacceptable	Regime O	Regime E
	acceptable	Regime H	Regime EH

Table 1: Four regimes

hybrid, living-donor dual-graft, and null transplantation, respectively. That is, each patient $i \in N$ has a strict preference $R_i \in \mathcal{P}(\{20, 10, 11, 02, 00\})$ over the set of transplantation types. Note that the formulation of a preference is free from a given type profile θ. Let $\mathcal{R}$ be the set of preferences, i.e., $\mathcal{R} := \mathcal{P}(\{20, 10, 11, 02, 00\})$. For each $R_i \in \mathcal{R}$, the anti-symmetric part and symmetric part of R_i are denoted by P_i and I_i, respectively. For each $R_i \in \mathcal{R}$, a transplantation type α is **acceptable** at R_i if she prefers α to the null transplantation 00. Let $Ac_i(R_i)$ be the set of acceptable transplantation types at R_i, i.e., $Ac_i(R_i) := \{\alpha \mid \alpha\ P_i\ 00\}$.

Based on a preference $R_i \in \mathcal{R}$ over transplantation types, we induce a preference over available transplants under a type profile θ. Namely, given $\theta \in \Theta$, we assume that a patient $i \in N$ with her preference $R_i \in \mathcal{R}$ has a preference $R_i(\theta) \in \mathcal{R}\left(X_i(\theta)\right)$ defined as follows;

$$\forall \alpha, \beta \in \{20, 10, 11, 02, 00\}, \forall x_i \in X_i^{\alpha}(\theta), \forall y_i \in X_i^{\beta}(\theta), x_i\ R_i(\theta)\ y_i \Leftrightarrow \alpha\ R_i\ \beta.$$

Note that in the induced preference $R_i(\theta)$ patients are indifferent between two transplants in a same-type transplantation. Without any confusion, we abuse the notation R_i to represent $R_i(\theta)$. That is, we write $x_i\ R_i\ y_i$ instead of $x_i\ R_i(\theta)\ y_i$ for two transplants $x_i, y_i \in X_i(\theta)$ under a type profile θ.

Given a type profile θ, a transplant $x_i \in X_i(\theta)$ is called **acceptable** for agent i with her preference R_i if she prefers s transplant x_i to nothing, i.e., $x_i\ P_i\ (0, \emptyset)$. Let $Ac_i(R_i; \theta)$ be the set of acceptable transplants for agent i with R_i under θ.

A **preference profile** is a list $R = (R_i)_{i \in N} \in \mathcal{R}^N$ consisting of the preferences of all patients. The set of all preference profiles $\mathcal{R}^N$ is called the **preference domain**.

2.4 Regimes: Legal constraints

The feasibility of an allocation is determined not only by medical technologies but also by social environments. By social environment we mean legal and ethical ones that stipulate whether a medically possible transplant is socially acceptable and implementable without much of an administrative and monetary burden. In our paper we examine two types of technologies that are constrained by social environments: (i) hybrid transplantation, and (ii) donation from living donors to non-relative patients. According to how these constraints are treated, we consider the following 2×2 kinds of regimes (see Table 1).

2.4.1 Regime *O*

Regime *O* is the environment before the introduction of the living-donor exchange and hybrid transplantation technology.[19] Since each patient cannot have a living-donor transplant with her own donors, only deceased-donor (dual-graft or single-graft) transplantation is possible. Thus, given $\theta \in \Theta$, an allocation $a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime O if for each $i \in N$, $a_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup X_i^{00}(\theta)$. Let $\mathcal{A}^O(\theta)$ be the set of feasible allocations at θ under regime O.

2.4.2 Regime *E*

Regime *E* is the environment in which a living-donor exchange is introduced to the original market. Donor exchanges (Ergin, Sönmez, and Ünver, 2017) are allowed, in addition to deceased-donor (dual-graft or single-graft) and living-donor transplantation. Thus, given $\theta \in \Theta$, an allocation $a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime E if for each $i \in N$, $a_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup X_i^{02}(\theta) \cup X_i^{00}(\theta)$. Let $\mathcal{A}^E(\theta)$ be the set of feasible allocations at θ under regime E.

2.4.3 Regime *H*

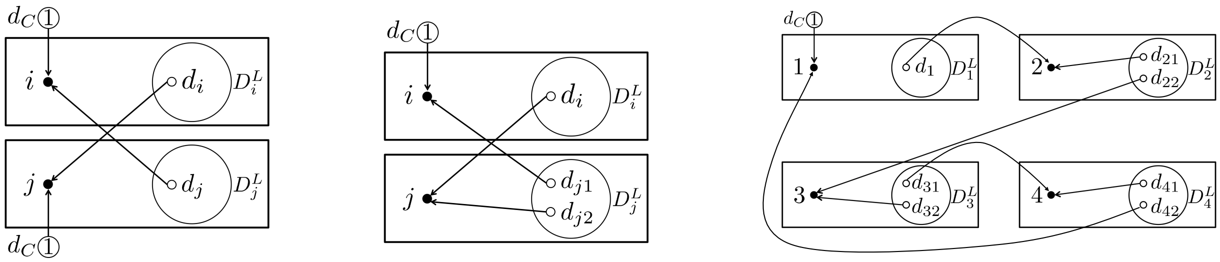
Regime *H* is the environment where hybrid transplantation technology is introduced to the original market. The hybrid transplant between a patient and one of her own donors is allowed, in addition to deceased-donor (dual-graft or single-graft) transplantation. Thus, given $\theta \in \Theta$, an allocation $a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime H if for each $i \in N$, $a_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$. Let $\mathcal{A}^H(\theta)$ be the set of feasible allocations at θ under regime H.

2.4.4 Regime *HE*

Regime *HE* is the environment in which both the living-donor exchange and hybrid transplantation technology is introduced to the original market. All types of (deceased-donor, living-donor, hybrid) transplantation are possible. Thus, given $\theta \in \Theta$, every allocation

[19] Because this is the "original" state of the market, we call the regime "O".

Figure 3: Hybrid exchange opens up new patterns of allocation.



$a^\theta \in \mathcal{A}(\theta)$ is **feasible** under regime *HE*. Let $\mathcal{A}^{HE}(\theta)$ be the set of feasible allocations at θ under regime *HE*. The new patterns of exchange that our paper advocates are exchanges of donors for hybrid transplantation. In the left allocation in Figure 3, both patients i and j receive a hybrid transplant with the other's donor. In the middle in Figure 3, patient i receives a hybrid transplant while j receives a living-donor transplant. This allocation suggests that the hybrid transplant makes the living-donor transplant possible.[20] Theoretically, the number of living-donor transplants caused by a hybrid transplant can be any large number. On the right in Figure 3, three living-donor transplants are implemented.

Remark 2. By definition, we have the following relations among the sets of feasible allocations under various regimes.

$$\begin{array}{ccc} \mathcal{A}^O(\theta) & \subseteq & \mathcal{A}^E(\theta) \\ \cap\!\mid & & \cap\!\mid \\ \mathcal{A}^H(\theta) & \subseteq & \mathcal{A}^{HE}(\theta) \end{array}$$

2.5 Dual-organ markets under various regimes

Let us summarize our model. A **dual-organ market under regime Y** $\in \{O, E, H, HE\}$ consists of the following components:

1. $(N, \{D_i^L\}_{i\in N}, D^C, (T, \trianglerighteq), \theta, \succeq)$ as described in Section 2.1.

2. $R = (R_i)_{i\in N} \in \mathcal{R}^N$, a preference profile as described in Section 2.3;

3. $\mathcal{A}^Y(\theta)$, the set of feasible allocations under regime Y as described in Section 2.4.

[20] The literature on kidney exchange considers the system that allows patients to exchange their donors with the right to receive a deceased kidney (Roth, Sönmez, and Ünver, 2004). Note that it contains a living-donor kidney and deceased-donor kidney exchange. What is new in the middle allocation in Figure 3 is that patient i who receives a hybrid transplant exchanges her living donor with patient j's living donor.

We assume that there is a clearinghouse whose goal is to distribute transplants among agents in a "desirable" manner.[21] To do so, it needs to collect the decentralized information on the

- market participants' type $\theta = (\theta_i)_{i \in N}$ and
- preferences of patients R.

Together with the type of the deceased donor, the clearinghouse processes the information in determining a feasible allocation.[22] The procedure is called a mechanism. Formally, a **mechanism under regime** Y is a function φ from $\mathcal{R}^N \times \Theta$ to $\cup_{\theta \in \Theta} \mathcal{A}^Y(\theta)$ such that for each $(R; \theta) \in \mathcal{R}^N \times \Theta$, $\varphi(R; \theta) \in \mathcal{A}^Y(\theta)$. That is, the clearinghouse has patients report their types and preferences. Expressing the information by type and preference profile $(R; \theta) \in \mathcal{R}^N \times \Theta$, the clearinghouse uses a mechanism φ to determine an allocation $\varphi(R; \theta) \in \mathcal{A}^Y(\theta)$ that is feasible under regime Y.

3 Properties of Mechanisms

In a general indivisible goods allocation problem, the desirable properties for a mechanism to satisfy are individual rationality, efficiency, fairness, and incentive compatibility. We introduce these properties for our dual-organ market. We fix a regime $Y \in \{O, E, H, HE\}$ throughout this section.

3.1 Individual rationality

We introduce individual rationality for our model. It is a condition on allocations in which any patient does not hurt from participating in a market. Formally, we define:

Definition 1. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **individually rational** at $(R; \theta) \in \mathcal{R}^N \times \Theta$ if for each patient $i \in N$, a_i^θ is at least as good as the null transplant at R_i. We denote by $\mathcal{I}^Y(R; \theta)$ the set of all individually rational allocations at $(R; \theta)$. Moreover, a mechanism under regime Y, φ, is **individually rational** if for each $(R; \theta) \in \Theta \times \mathcal{R}^N$, the selected allocation $\varphi(R; \theta) \in \mathcal{A}^Y(\theta)$ is individually rational at $(R; \theta)$.

[21] In the next section, we will discuss what allocations are desirable.

[22] We assume that the type information of the deceased donor is known by the clearinghouse whenever a deceased donor appears. This is natural because it is reported by a medical doctor who is in charge of the deceased donor. The clearinghouse shares the information with the medical doctors in charge who are out of our model.

3.2 Efficiency

We define three notions of efficiency. The first is the standard notion of Pareto efficiency. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **Pareto efficient** at $(R;\theta) \in \mathcal{R}^N \times \Theta$ if there is no allocation $b^\theta \in \mathcal{A}^Y(\theta)$ such that for each $i \in N$, $b_i^\theta \; R_i \; a_i^\theta$, and for some $i \in N$, $b_i^\theta \; P_i \; a_i^\theta$. A mechanism under regime Y, φ, is **Pareto efficient** if for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, the selected allocation $\varphi(R;\theta)$ is Pareto efficient at $(R;\theta)$.

We next introduce the notions of non-wastefulness for our dual-organ market. The standard notion of non-wastefulness is defined for indivisible goods allocation problems under common ownership and unit demand where no agent initially owns an object and an agent consumes one object. The non-wastefulness means that "unused" objects cannot be assigned to benefit some agent without affecting anybody else's assignment (Balinski and Sönmez, 1999). On the other hand, our dual-organ market has mixed ownership and multi-unit demand. Thus, the unused objects available for a patient are disposed deceased lungs and a lobe from one of her own donors who did not donate at the original allocation. Thus, each patient could potentially access to the unused objects in addition to the original assignment without affecting anybody else's assignment. Although the set of better opportunities can be well captured by the set of unused objects under unit demand, the appropriate extension to our model with multi-unit demand and mixed ownership should include both unused objects and the original individual assignment. If a patient finds a better transplant within her potentially "accessible" objects, she might be better off by herself. The accessibility can be captured by the following notion of induced allocations.

Definition 2 (Induced allocation)**.** Given a type profile $\theta \in \Theta$ and an allocation $a^\theta = \big((a_i^{\theta C}, a_i^{\theta L})\big)_{i \in N} \in \mathcal{A}(\theta)$, we define the **induced allocation** $\overline{a^\theta} = \big((\overline{a}_i^{\theta C}, \overline{a}_i^{\theta L})\big)_{i \in N}$ as follows: For each $i \in N$,

$$(i) \quad \overline{a}_i^{\theta C} := a_i^{\theta C} + \Big(\theta_{d_c q} - \textstyle\sum_{j \in N} a_j^{\theta C}\Big), \text{ and}$$
$$(ii) \quad \overline{a}_i^{\theta L} := a_i^{\theta L} \cup \{d \in D_i^L \big| d \notin \cup_{j \neq i} a_j^{\theta L}\}.$$

In words, patient i's induced grafts assignment $\overline{a}_i^{\theta C}$ from the deceased donor is the sum of the number of grafts from d_c she receives at a^θ and the number of grafts from d_c disposed at a^θ. Patient i's induced living-donor assignment $\overline{a}_i^{\theta L}$ denotes the union of her assignment at a^θ and her own donors who do not donate to other patient at a^θ. Thus, $\overline{a}_i^\theta$ formalizes the potentially accessible resources of patient i at a^θ without changing the other patients' assignment. Note that $\overline{a}_i^\theta$ may not be an assignment. Now, we are ready to introduce non-wastefulness.

Definition 3. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **wasteful** at $(R;\theta) \in \Theta \times \mathcal{R}^N$ if there exist $i \in N$ and $b_i \in X_i(\theta)$ such that (i) $b_i \; P_i \; a_i^\theta$, (ii) $(b_i; a_{-i}^\theta) \in \mathcal{A}^Y(\theta)$, and (iii) $b_i^C \leq \overline{a}_i^{\theta C}$ and $b_i^L \subseteq \overline{a}_i^{\theta L}$. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **non-wasteful** at $(R;\theta) \in \Theta \times \mathcal{R}^N$ if it is not wasteful at $(R;\theta) \in \Theta \times \mathcal{R}^N$.

The condition says that patient i cannot find (i) a better transplant (ii) which is allowed under regime Y without affecting others' assignments, and (iii) which can be constructed within patient i's accessible resources at a^θ. For example, if $a_i^\theta = (1, \emptyset)$ is a deceased-donor single-graft transplant, then $b_i = (1, \{d_i\})$ can be a hybrid transplant if patient i's compatible own donor d_i does not donate at a^θ, i.e., $d_i \in \overline{a}_i^L$.

In addition to being non-wasteful, an allocation is strongly non-wasteful if it maximizes the number of grafts from the deceased donor used at the allocation within individually rational allocations. Such an allocation is desirable because it expresses the view that donated organs should be fully utilized, leading to respect for deceased donors. That is,

Definition 4. Under regime Y, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **strongly non-wasteful** at $(R;\theta) \in \mathcal{R}^N \times \Theta$ if a^θ is non-wasteful, and

$$\sum_{i \in N} a_i^{\theta C} = \max_{b^\theta \in \mathcal{I}^Y(R;\theta)} \sum_{j \in N} b_j^{\theta C}.$$

We say that a mechanism under regime Y, φ, is **(strongly) non-wasteful** if for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, the selected allocation $\varphi(R;\theta) \in \mathcal{A}^Y(\theta)$ is (strongly) non-wasteful at $(R;\theta)$.

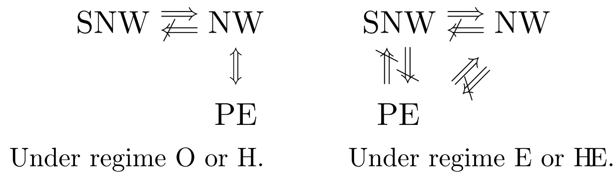
Example 1 (Non-wasteful but not strongly non-wasteful allocation)**.** Let $(T, \trianglerighteq) = (T_\mathcal{B}, \trianglerighteq_\mathcal{B})$.[23] Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, A)$ and $\theta(1) = \theta(2) = A$. Suppose also that for each $i \in N \backslash \{1, 2\}$ and each $d \in D^L$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be a preference profile described by the following table.

R_1	10	00	$\cdots$
R_2	20	00	$\cdots$

Thus, patients 1 and 2 can only receive a donation from the deceased donor. Moreover, patient 1 only finds the single-graft transplant acceptable, and patient 2 only the dual-graft transplant acceptable. Consider an allocation $a^\theta := ((1, \emptyset), (0, \emptyset))$ where one graft is transplanted to patient 1, but the other is disposed so that patient 2 receives the null

[23] This assumption simplifies the description without any loss of generality as it corresponds to the assumption that all patients and donors have the identical medical type except for the ABO blood type.

Figure 4: Logical relationship among efficiency concepts



Note: SNW stands for strong non-wastefulness, NW for non-wastefulness, and PE for Pareto efficiency.

transplant.[24] Obviously, a^θ is non-wasteful. However, one graft is disposed and the two grafts can be transplanted to patient 2 in another individually rational allocation $b^\theta := ((0, \emptyset), (2, \emptyset)) \in \mathcal{I}^Y(R; \theta)$, and thus allocation a^θ is not strongly non-wasteful. Note that $\sum_{i \in N} a_i^{\theta C} = 1 < 2 = \sum_{i \in N} b_i^{\theta C}$. $\Diamond$

The following remark clarifies the logical relationship among our efficiency notions (see Figure 4). For a proof, see the online appendix.

Remark 3. We have the following three statements. Sentences without a reference for regime hold under any regime. Let $(R; \theta) \in \mathcal{R}^N \times \Theta$.

1. If an allocation is strongly non-wasteful at $(R; \theta)$, then it is non-wasteful at $(R; \theta)$. The converse is not true.

2. If an allocation is Pareto efficient at $(R; \theta)$, then it is non-wasteful at $(R; \theta)$. The converse is also true only if $Y \in \{O, H\}$.

3. There is no logical relationship between Pareto efficiency and strong non-wastefulness under any regime $Y \in \{E, HE\}$.

3.3 Fairness

In our dual-organ market, we have a priority $\succeq$ given as one component of the market. The priority expresses the right of patients receiving grafts from the deceased donor. In this subsection we introduce the notion of fairness with respect to the priority.

We say that a patient i has a justified envy at an allocation *if* some lower-priority patient j is assigned an organ from the deceased donor, and patient i can be made better off with the deceased donor's graft while keeping the same welfare except j. A fair allocation has no patient having such justified envies. More formally,

[24]The assignments for $i \in N \backslash \{1, 2\}$ are necessarily the null transplant $(0, \emptyset)$, and they are thus omitted in the description of the allocations.

Definition 5. Under regime Y, patient $i \in N$ has a **justified envy** under an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ at $(R;\theta) \in \mathcal{R}^N \times \Theta$ *if* there is $b^\theta \in \mathcal{I}^Y(R;\theta)$ such that (i) $b_i^\theta \; P_i \; a_i^\theta$, (ii) $b_i^{\theta C} > a_i^{\theta C}$ and for each $j \in N$ with $j \succ i$, $b_j^{\theta C} = a_j^{\theta C}$ and (iii) for each $j \in N$, $a_j^\theta \; P_j \; b_j^\theta$ implies $a_j^{\theta C} > b_j^{\theta C}$. Moreover, an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ is **$\succeq$-fair** at $(R;\theta) \in \mathcal{R}^N \times \Theta$ if no patient has a justified envy under a^θ at $(R;\theta)$.

We say that a mechanism under regime Y, φ, is **$\succeq$-fair** if for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, the selected allocation $\varphi(R;\theta) \in \mathcal{A}^Y(\theta)$ is $\succeq$-fair at $(R;\theta)$.

The following remark clarifies the logical relationship between $\succeq$-fairness and axioms previously introduced. For a proof, see the online appendix.

Remark 4. We have the following two statements. Sentences without a reference for regime hold under any regime.

1. $\succeq$-fairness does not imply individual rationality. Under any regime $Y \in \{E, HE\}$, even the combination of Pareto efficiency and $\succeq$-fairness does not imply individual rationality.

2. There is no logical relationship between $\succeq$-fairness and any one of the three efficiency notions of non-wastefulness, strong non-wastefulness, and Pareto efficiency.

The following impossibility asserts that strong non-wastefulness is too demanding as an efficiency notion as long as we employ individual rationality and $\succeq$-fairness as basic axioms for a mechanism.

Proposition 2. *Under any regime $Y \in \{O, E, H, HE\}$, there exists $(R;\theta) \in \mathcal{R}^N \times \Theta$ such that no allocation is individually rational, strongly non-wasteful, and $\succeq$-fair at $(R;\theta)$.*

3.4 Incentive compatibility

We employ strategy-proofness as our incentive compatibility condition. In our model, each patient has two pieces of private information about herself: type θ_i and preference R_i. These information is not treated symmetrically, since the former is verifiable but the latter is not. Thus, we assume that her reporting type θ_i is sincerely transmitted without manipulation, while her reporting preference R_i may not be. In other words, preference reporting is the only source of strategic manipulation.

Assumption 1 (Sincere reporting about types)**.** *Each patient reports her type sincerely.*

Under the above assumption, the following is a standard definition of *strategy-proofness*. We discuss the importance of strategy-proofness in section 4.2.2.

Definition 6. A mechanism under regime Y, φ, is **strategy-proof** if for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R;\theta)\ R_i\ \varphi_i(R'_i, R_{-i};\theta)$.

4 Main Results

In this section, we first investigate the mechanism currently used in the Japanese lung market under regime O.[25] We show that the Japanese mechanism is individually rational and Pareto efficient, but it is neither $\succeq$-fair nor strategy-proof. Then, we propose priority mechanisms that can be used for any regime, and show its prominence in our dual-organ market. To define these mechanisms, we use the notation $Top(\succeq, M)$ which is the highest-priority patient among those in the non-empty set M of patients. That is, for each $i \in M$, $Top(\succeq, M) \succeq i$.

4.1 Case study: The Japanese mechanism in the lung market

We describe the Japanese mechanism, denoted as φ^J, that works only under regime O.[26] To this end, we introduce the following notation. For each $(R;\theta) \in \mathcal{R}^N \times \Theta$, let

$$N(R;\theta) := \{i \in N \mid (2,\emptyset) \in Ac_i(R_i;\theta) \text{ or } (1,\emptyset) \in Ac_i(R_i;\theta)\}$$

be the set of all candidates of the deceased-donor transplantation.

The Japanese mechanism first looks at the preference of the highest-priority patient in $N(R;\theta)$ when the deceased donor provides two units of lungs. Let us call the highest-priority patient i_1. If i_1 has multi-unit demand, the mechanism gives two units to the patient (Case 1). On the other hand, if i_1 has single-unit demand, the mechanism tries to pick another patient with single-unit demand to allocate one unit for each. If the market has another patient with single-unit demand, the mechanism selects the highest-priority patient among them (Case 2). If i_1 is the only patient with single-unit demand, the mechanism looks at the second-most preferred transplant of patients. If there is at least one patient who can accept a single-graft transplant (as the second-most preferred transplant), the mechanism

[25] We formalize the mechanism based on the recipient selection rule described in `http://www.jotnw.or.jp/jotnw/law_manual/pdf/rec-lungs.pdf`. A formal description of the mechanism is given in the online appendix.

[26] For a rigorous description of the Japanese mechanism, see Appendix A in online appendix.

picks the highest-priority patient among them, giving one unit to her and i_1 for each (Case 3). Finally, if i_1 is the only patient with single-unit demand even when we consider the first- and second-most preferred transplant of all patients, then there are two subcases. If i_1 can only accept a single-graft transplant and there is at least one patient who needs a dual-graft transplant, then the mechanism skips i_1, giving two units to the highest-priority patient among the rest of the patients (Case 3.2.1). Otherwise, i.e., when i_1 is the only patient who is willing to accept a graft from the deceased donor, the mechanism gives one unit to i_1, disposing of one unit (Case 3.2.2). Next we consider the case when only one unit of lung is donated by the deceased donor. The Japanese mechanism selects the highest-priority patient among the unit-demand patients (Case 4).[27] If there is no unit-demand patient then the mechanism picks the highest-priority patient who can accept a single-graft transplant as the second-most preferred transplant (Case 5).

The Japanese mechanism aspires to be a strongly non-wasteful allocation in a situation where the compatible highest-priority patient has the unit demand after rejecting two units. In that situation, if (1) there is no lower-priority patient with unit demand, and (2) at least one lower-priority patient is demanding two units, then the mechanism skips the compatible highest-priority patient and assigns two grafts to the compatible highest-priority patient who needs two units to reduce the number of disposed grafts (see Case 3.2.1 in the definition of the Japanese mechanism).[28]

However, the attempt to reduce the number of disposed grafts is not fully achieved by the current mechanism, i.e., there is another situation where it fails to assign grafts in a strongly non-wasteful manner. Moreover, the anomalistic manner of the mechanism can be a source of unfairness and strategic manipulation. These points are captured by the following example.

Example 2 (Flaws of the Japanese mechanism)**.** Suppose that there are two units of deceased grafts available, and also that only patients 1 and 2 have the same type as the deceased grafts. That is, let $\theta \in \Theta$ be such that $\theta_{d_c q} = 2$, $\theta(1) = \theta(2) = \theta_{d_c T}$, and for each $i \in N \backslash \{1, 2\}$, $\theta_{d_c T} \not\trianglerighteq \theta(i)$. Suppose also that patient 1 has a higher priority than patient 2, i.e., $1 \succ 2$.

Flaw 1 (φ^J is not strongly non-wasteful). Let $R \in \mathcal{R}^N$ be a preference profile in Table 2. Then, $N(R; \theta) = \{1, 2\}$ and $i_1 = 1$. Note that the highest-priority patient demands one

[27] Not necessarily, the highest-priority patient in $N(R; \theta)$.

[28] More precisely, this type of skip is not applied if the deceased donor is not one of the relatives of the compatible highest-priority patient at Case 3.2.1. Since we assume that $D^C \cap D^L = \emptyset$, this case is omitted from the description of the Japanese mechanism.

Table 2: Preferences for Example 2

R_1	10	20	00	$\cdots$
R_2	20	00	$\cdots$	
R'_1	10	00	$\cdots$	

unit. Thus, the mechanism tries to pick another patient who can accept a deceased-donor single-graft transplant. Since 1) patient 1 is the only patient who can accept single-unit transplantation even when we look at the first- and second-most preferred transplantation of all patients, and 2) the second-most preferred transplant is a deceased-donor dual-graft transplant, Case 3.2.2 in the definition is applied. Thus, $\varphi^J(R;\theta) = \big((1,\emptyset),(0,\emptyset)\big)$, disposing of one unit of lung. However, since $\big((0,\emptyset),(2,\emptyset)\big)$ is individually rational at $(R;\theta)$, $\varphi^J(R;\theta)$ is not strongly non-wasteful at $(R;\theta)$.

Flaw 2 (φ^J is not $\succeq$-fair nor strategy-proof). Let $R'_1 \in \mathcal{R}$ be a preference in Table 2. Let $R' := (R'_1, R_2)$. Then, $N(R';\theta) = \{1,2\}$ and $i_1 = 1$. Again, the market does not have a patient, except for i_1, who can accept a deceased-donor single-graft transplant. Thus, the Japanese mechanism considers the second-most preferred transplantation of patient 1. Since on this occasion it is a null transplantation, Case 3.2.1 is applied. Thus, $\varphi^J(R';\theta) = ((0,\emptyset),(2,\emptyset))$. Higher-priority patient 1 would be better off from a graft assigned to lower-priority patient 2 at $\varphi^J(R';\theta)$. Thus, $\varphi^J(R';\theta)$ is not $\succeq$-fair at $(R';\theta)$.

Now consider the strategic deviation of patient 1 from R'_1 to R_1. Then, the allocation selected at $(R_1,R_2;\theta) = (R;\theta)$ is $\big((1,\emptyset),(0,\emptyset)\big)$ (see the calculation at Flaw 1). Thus, $\varphi^J_1(R_1,R_2;\theta) = (1,\emptyset)\ P'_1\ (0,\emptyset) = \varphi^J_1(R'_1,R_2;\theta)$, which violates strategy-proofness. $\Diamond$

The following proposition summarizes the properties of the Japanese mechanism.

Proposition 3. *Under regime O, the Japanese mechanism φ^J is (i) individually rational, (ii) Pareto efficient, (iii) not strongly non-wasteful, (iv) not $\succeq$-fair, and (v) not strategy-proof.*

4.2 Priority mechanism

We introduce a priority mechanism which extends Roth, Sönmez, and Ünver (2005)'s mechanism to our setting. Roughly speaking, in the mechanism agents select their favorite and individually rational allocations among those selected by higher-priority agents. For its definition, we need the following notation. Let $\sigma : \{1,\ldots,n\} \to N$ be the bijection that represents the priority order. That is, for each $i \in \{1, \ldots, n\}$, σ selects the i-th highest-priority patient, i.e., $\sigma(1) \succ \sigma(2) \succ \cdots \succ \sigma(n)$.

Definition 7 (Priority mechanism)**.** The **priority correspondence under regime** Y, Φ^Y, is the nonempty-valued correspondence from $\mathcal{R}^N \times \Theta$ to $\cup_{\theta\in\Theta}\mathcal{A}^Y(\theta)$ that selects feasible allocations in $\mathcal{A}^Y(\theta)$ for each $(R;\theta) \in \mathcal{R}^N \times \Theta$ as follows.
Round 0: $\Phi_0^Y(R;\theta)$ selects all individually rational allocations at $(R;\theta)$. That is, $\Phi_0^Y(R;\theta) := \mathcal{I}^Y(R;\theta)$.
Round $k \in \{1, \ldots, n\}$: $\Phi_k^Y(R;\theta)$ selects all allocations that k^{th}-priority patient, $\sigma(k)$, prefers the most among those in $\Phi_{k-1}^Y(R;\theta)$. That is, $\Phi_k^Y(R;\theta) := \Big\{a^\theta \in \Phi_{k-1}^Y(R;\theta) \mid \forall b^\theta \in \Phi_{k-1}^Y(R;\theta), a^\theta_{\sigma(k)}\ R_{\sigma(k)}\ b^\theta_{\sigma(k)}\Big\}$. Let $\Phi^Y(R;\theta) := \Phi_n^Y(R;\theta)$.

A **priority mechanism** under regime Y, φ^P, is a selection from the priority correspondence, i.e., for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, $\varphi^P(R;\theta) \in \Phi^Y(R;\theta)$.

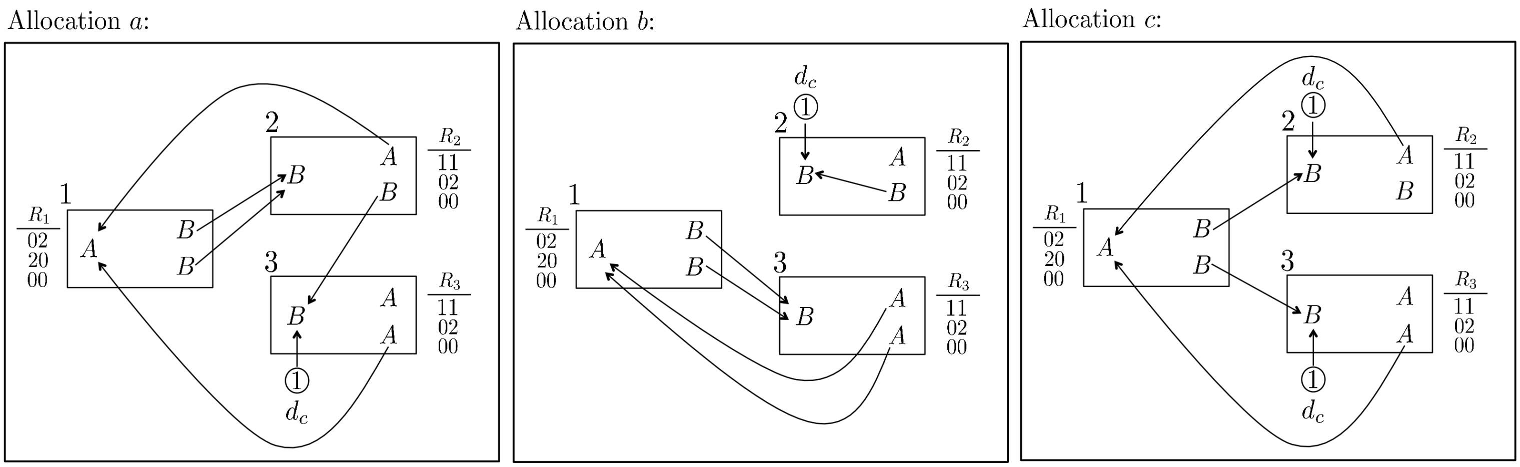
Example 3 (Priority mechanism)**.** This example describes the procedure of the priority mechanism (Figure 5). There are three patients, each of whom has exactly two living donors. For each $i \in \{1, 2, 3\}$, the types of patient i and her donors are shown in box i in the figures. We assume that only the blood type accounts for the medical type of agents. For example, at allocation a, patient 1's type is shown in the left-hand side of the box, i.e., $\theta_1(1) = A$, while her two donors have the identical type B as shown in the right-hand side of the same box. Suppose that patient 1 has the highest priority, patient 2 the second, and patient 3 the third. The preferences of patients are given as follows. $R_1 : 02, 20, 00$, $R_2 : 11, 02, 00$, $R_3 : 11, 02, 00$.

In the first round, patient 1 chooses her assignment. Since her best is a living-donor dual-graft transplantation, all feasible allocations in which patient 1 receives a living-donor dual-graft transplant are selected by the priority correspondence. Such allocations are abundant, and some of them are given in Figure 5: allocations a, b, and c. Note that all three allocations are indifferent for patient 1 but not for patients 2 and 3.

In the second round, patient 2 chooses her assignment. Since her best assignment is a hybrid transplantation, some allocations selected in the first round are rejected. Among allocations a, b, and c, allocation a is rejected, since it assigns a living-donor dual-graft transplant to patient 2. On the other hand, allocations b and c remain selected at the second round, since they assign a living-donor dual-graft transplant to patient 1 and a hybrid transplant to patient 2. Note that patient 3 is not indifferent between b and c.

In the third round, patient 3 chooses her assignment. Since her best choice is a hybrid transplantation, some allocations selected in the second round are rejected. Among alloca-

Figure 5: Allocations a, b and c in Example 3



tions b and c, allocation b is rejected, since it assigns a living-donor dual-graft transplant to patient 3.

Consequently, the priority correspondence contains allocation c. As shown in this example, in each round, the priority mechanism selects the best feasible allocations for the selecting patient in this round so as not to harm the higher-priority patients.[29] ◊

Remark 5. The difference in regimes makes some differences in the nature of priority mechanisms. The following items 2 and 3 are basic. For a proof, see the online appendix.

1. Under any regime $Y \in \{O, E, H, HE\}$, if both φ and $\tilde{\varphi}$ are priority mechanisms then they are welfare-equivalent. That is, for each $(R; \theta) \in \mathcal{R}^N \times \Theta$ and each $i \in N$, $\varphi_i(R; \theta)\ I_i\ \tilde{\varphi}_i(R; \theta)$.

2. Under any regime $Y \in \{O, H\}$, Φ^Y is single-valued. Thus, a priority mechanism is unique.

3. Under any regime $Y \in \{E, HE\}$, Φ^Y may not be single-valued. Thus, a priority mechanism may not be unique.

[29]Rigorously speaking, a variant of c is also selected by the priority correspondence. It is the allocation in which the donation from patient 3's second donor is replaced by the one from patient 3's first donor. Note that c and its variant are indifferent for all patients.

4.2.1 Performance of priority mechanism under regimes O and H

Under regimes without donor exchange (regime O or H), the performance of the priority mechanism is quite good:

Theorem 1. *Under any regime $Y \in \{O, H\}$, the priority mechanism is individually rational, Pareto efficient, $\succeq$-fair, and strategy-proof.*

The priority mechanism overcomes the two major flaws of the Japanese mechanism — the violation of $\succeq$-fairness and strategy-proofness. Moreover, the prominence is kept even under the introduction of hybrid transplantation. However, by Proposition 2, strong non-wastefulness is not overcome. This is viewed as an inevitable cost for a mechanism to be individually rational and $\succeq$-fair.

4.2.2 Performance of priority mechanism under regimes E and HE

Under regimes with donor exchange (regime E or HE), each priority mechanism keeps its good performance for the normative properties.

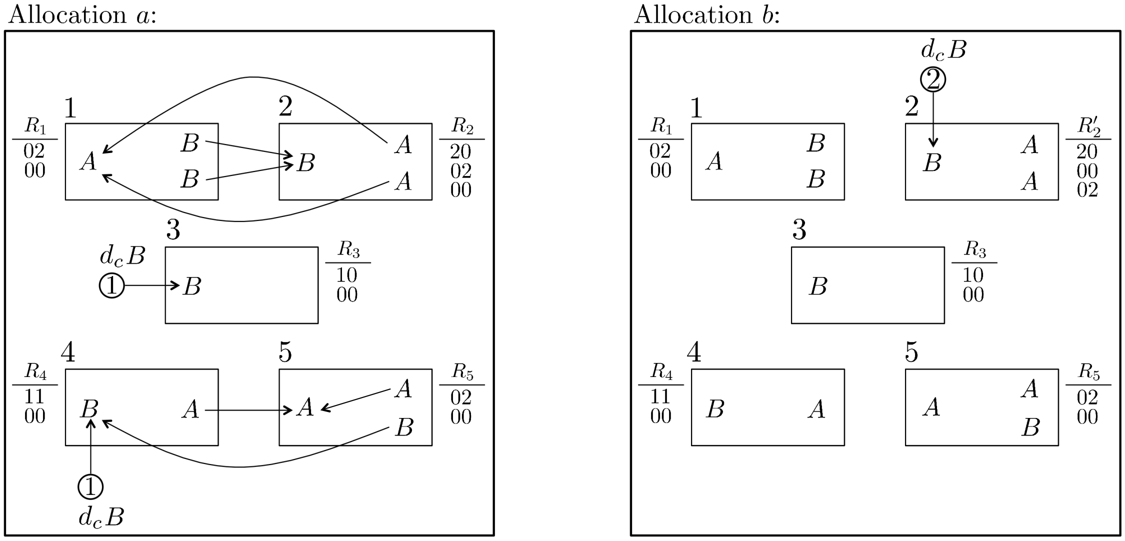
Proposition 4. *Under any regime $Y \in \{E, HE\}$, each priority mechanism is individually rational, Pareto efficient, and $\succeq$-fair. However, it is not strategy-proof.*

One of the critical differences between regimes with and without donor exchange lies in the degree of manipulability of the mechanisms. In particular, under regimes with donor exchange, a patient can hide her own donors by rejecting hybrid and living-donor transplants in her preference.[30] Recall that we focus on balanced allocations (Proposition 1). Thus, if a patient pretends that she cannot accept any hybrid and living-donor transplants then no other patients can use her living donors. This type of strategic behavior is a source for the manipulation of a mechanism. Consequently, every priority mechanism is manipulable. To scrutinize the problem evoked by strategic behavior in dual-organ markets, we consider the following example.

Example 4 (Manipulation of the priority mechanism)**.** There are five patients, patients 1 to 5, who are prioritized as $1 \succ 2 \succ 3 \succ 4 \succ 5$. Patients 1, 2, and 5 have two living donors, patient 4 has one while patient 3 has nothing. Their medical types and preferences are illustrated in Figure 6.

[30]Note that hiding donors has no positive effect for that patient under the regimes without donor exchange.

Figure 6: Allocations a and b in Example 4



Under true preferences $(R_2; R_{-2})$, patients 1,3,4, and 5 have just one acceptable surgery type, while patient 2 prefers a deceased-donor dual-graft transplant to a living-donor dual-graft transplant. This market has individually rational allocations in which patient 1 receives a living-donor dual-graft transplant, for example, allocation a, the priority mechanism selects one of them. Now we claim that patient 2 must receive a living-donor dual-graft transplant in the selected allocation. To show this, suppose to the contrary that patient 2 does not. Since we assume the balanced condition, the living donors of patient 2 are not used for the selected allocation. Thus, patient 1 receives A-type living donors from patients 4 and 5. Thus, by the balanced condition, patient 5 must receive a living-donor dual-graft transplant under the selected allocation. However, this is impossible because no available A-type living donors remain in the market. Therefore, we have proved that patients 1 and 2 receive living-donor dual-graft transplants at the selected allocation. At this point, it is obvious that patients 3,4, and 5 are assigned their most preferred transplants at the selected allocation. Thus, the priority mechanism selects allocation a (or, its welfare equivalent variant) in the left figure.[31]

Now suppose that patient 2 deviates from R_2 to R'_2. At R'_2, patient 2 hides the fact that she can accept a living-donor dual-graft transplant. This is a kind of "truncated strategy" well known in two-sided matching models (Roth and Sotomayor, 1990). Note that for patient 1 to get a living-donor dual-graft transplant, the living donors of patient 2 are

[31] Since we assume that patients are indifferent to identical surgery-type transplants, the welfare-equivalent variant of a can be easily obtained by changing some of the patient-donor combinations at a.

critical. Consequently, since the hiding strategy of patient 2 narrows down the opportunity for patient 1, she cannot help but choose the null transplant in the first round of the priority mechanism. This enables patient 2 to get a more preferred transplant, i.e., the deceased-donor dual-graft transplant. ◊

In practice, it might be realistic to operate the mechanism by letting patients register their preferences once within a certain fixed period, such as three or six months. Since the priority order is mainly based on waiting time, it should be noted that even the patient with the highest-priority in our model must have experienced low-priority positions before. Although the dual-organ market in this paper is defined as a static model, the switch from R_2 to R'_2 in Example 4 can be interpreted as the free-riding behavior of patient 2. By hiding her living-donors, not only does she prevent the transplants of other four patients, but she also eliminates the opportunity of having her living donors contribute to the market, although she positions herself with the potential possibility for a living-donor transplantation through the previous interaction with higher-priority patients when she was a lower-priority patient. This point enhances the importance of incentive property for a mechanism in the dual-organ market.

A natural question arising from Proposition 4 is: Is there a mechanism satisfying all axioms listed in the proposition? The answer is negative.

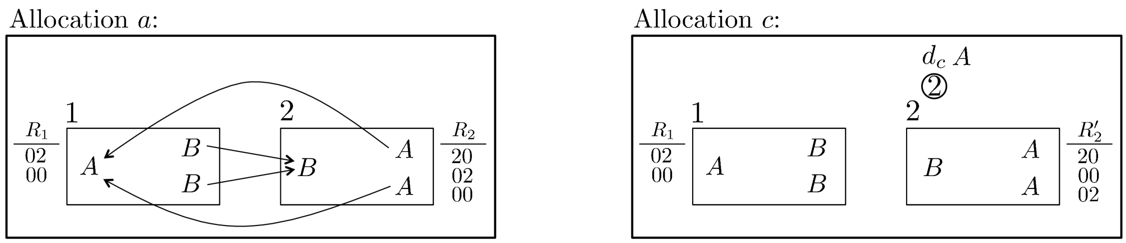
Theorem 2. *Under any regime $Y \in \{E, HE\}$, no mechanism is individually rational, Pareto efficient, $\succeq$-fair, and strategy-proof.*

Since no mechanism satisfies all axioms listed in Theorem 2, we have to give up at least one of the axioms to design a plausible mechanism under a regime with donor exchange. Since Proposition 4 says that under regimes with donor exchange, each priority mechanism satisfies three of our four basic axioms, priority mechanisms attain one of the best we can choose. Moreover, we will see a positive aspect of priority mechanisms which are robust against strategic manipulation under incomplete information, although they are not strategy-proof under complete information.

4.3 Preference revelation game with incomplete information under a priority mechanism

The successful manipulation of the priority mechanism in Example 4 heavily depends on the complete information setup. In practice, the type of deceased donors is uncertain, and the types of other patients and living donors are usually not open to the public. Thus, the

Figure 7: Allocations a and c in Example 5



manipulation of the mechanism pertains to the risk of missing a transplant opportunity. To see this point, let us consider the previous example under modification.

Example 5 (Manipulation of the priority mechanism is risky)**.** Consider Example 4 where patient 2 of blood type B gets the deceased donor of the same blood type by manipulating the priority mechanism with the truncation strategy. Here, for simplicity, we focus on the submarket with patients 1 and 2. Let us consider a slightly modified case in which the blood type of the deceased donor is type A instead of type B. Note that patient 2's truncation strategy, in this case, ends up with no transplants for both patients (Figure 7). In other words, she cannot receive a living-donor dual-graft transplant plus a deceased-donor dual-graft transplant because the truncation strategy narrows down not only the opportunity for the higher-priority patient but also the one for herself. In this sense, the manipulation strategy is risky.

Summing up, without accurate information about resources, i.e., types of deceased-donor and other patients' living donors, the manipulation behavior may be harmful to the manipulator, too. In that sense, the truncation strategy is a "double-edge sword" for the manipulator.[32] $\Diamond$

Motivated by the above example, we introduce incomplete information into our model. We assume that each patient can observe only her own medical type and preference, not the others'. That is, she knows her own preference and type (R_i, θ_i), but does not know other patients' $(R_{-i}; \theta_{-i})$, including the deceased donor's.[33] Formally, we consider a preference revelation game $G = (N, D^C, \{D_i^L\}_{i \in N}, (T, \trianglerighteq), \Theta_{d_c}, \{\mathcal{R} \times \Theta_i\}_{i \in N}, \{u_i^*\}_{i \in N}, Y, \varphi^P, \{p_i\}_{i \in N})$, where

[32]In Lemma 1 of the Appendix, we show that all successful manipulation strategies of the priority mechanism are necessarily double-edge in the sense that they always narrow down the possible assignment for the manipulator.

[33]This setup is suitable for Japan, because patients simply register for the Japan Organ Transplant Network without any communication with other patients.

1. The symbols $N, D^C, \{D_i^L\}_{i\in N}, (T, \trianglerighteq), \Theta_{d_c}$ are the same as in the complete information model: Each of them represents the set of patients, the set of deceased donors, the collection of the set of living donors, the type space for the medical status of patients and donors, and the type space of the deceased donor, respectively.

2. For each $i \in N$, $\mathcal{R} \times \Theta_i$ denotes patient i's action set. It also represents patient i's "type space" in the standard Bayesian game terminology.

3. For each $i \in N$, $u_i^* : \{20, 10, 11, 02, 00\} \times \big(\mathcal{R} \times \Theta_i\big) \to \mathbb{R}$ is a state-dependent utility function. For each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, $u_i^*(\cdot \mid R_i, \theta_i)$ represents R_i. Without any confusion, given $\theta \in \Theta$, for each $x_i^\theta \in X_i(\theta)$, $u_i^*(x_i^\theta \mid R_i, \theta_i)$ denotes the value of $u_i^*(\cdot \mid R_i, \theta_i)$ for the transplantation type to which x_i^θ belongs. Note that the above setup does not exclude the state-independent utility case.

4. A priority mechanism φ^P under regime Y is fixed.

5. For each $i \in N$, $p_i : \mathcal{R}^N \times \Theta \to [0, 1]$ is a probability distribution that represents patient i's prior belief. We assume that p_i has full support, i.e., for each $(R; \theta) \in \mathcal{R}^N \times \Theta$, $p_i(R; \theta) > 0$. Note that we do not place the common prior assumption. For each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, the posterior belief is denoted by $p_i(\cdot \mid R_i, \theta_i)$, i.e., it is the function from $\mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ to $[0, 1]$ defined as $p_i(R_{-i}; \theta_{-i} \mid R_i, \theta_i) := \frac{p_i(R_i, R_{-i}; \theta_i, \theta_{-i})}{\sum_{(R'_{-i}; \theta'_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}} p_i(R_i, R'_{-i}; \theta_i, \theta'_{-i})}$.

Now we make an assumption about the players' utility functions that reflects the huge gap in utilities between acceptable transplants and unacceptable ones. To describe it, for each $i \in N$ and each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, we introduce the following two notations:

- $\overline{UD}(u_i^*; R_i, \theta_i) := \begin{cases} \max\limits_{\alpha \in Ac_i(R_i)} u_i^*(\alpha \mid R_i, \theta_i) - \min\limits_{\alpha \in Ac_i(R_i)} u_i^*(\alpha \mid R_i, \theta_i) & \text{if } Ac_i(R_i) \neq \emptyset, \\ 0 \quad \text{if } Ac_i(R_i) = \emptyset. \end{cases}$

- $\underline{p_i}(R_i, \theta_i) := \min\limits_{(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}} p_i(R_{-i}; \theta_{-i} \mid R_i, \theta_i)$.

In words, $\overline{UD}(u_i^*; R_i, \theta_i)$ denotes the utility difference between the best acceptable transplantation and the worst acceptable one which in turn shows the maximal gain from the quality improvement when a patient gets an acceptable transplant instead of another. Given patient i's own type $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, the most unlikely event occurs with probability $\underline{p_i}(R_i, \theta_i)$ in patient i's perspective. Note that $\underline{p_i}(R_i, \theta_i) > 0$, since we assume that p_i has full support.

Assumption 2 (Huge utility gap between acceptable and unacceptable transplants)**.** *Even at the most unlikely event, the expected utility loss from the worst acceptable transplant to the best unacceptable one is so huge that a patient cannot recover it even if she gets an additional utility $\overline{UD}(u_i^*; R_i, \theta_i)$ in every other event. Formally, for each $i \in N$ and each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$ with at least one acceptable transplantation type at R_i, we have the following inequality:*

$$\underline{p}_i(R_i, \theta_i)\Big(\min_{\alpha \in Ac_i(R_i)} u_i^*(\alpha \mid R_i, \theta_i) - u_i^*(00 \mid R_i, \theta_i)\Big) > (1 - \underline{p}_i(R_i, \theta_i))\Big(\Big|\mathcal{R}^{N \setminus \{i\}} \times \Theta_{-i}\Big| - 1\Big)\overline{UD}(u_i^*; R_i, \theta_i).$$

Since an assignment in our model represents a transplant, an unacceptable transplant can be interpreted as the death of the patient. Thus, it is natural that there is a huge utility gap between acceptable and unacceptable transplants.

For each player $i \in N$, a strategy is a function $s_i : \mathcal{R} \times \Theta_i \to \mathcal{R} \times \Theta_i$ such that for each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, the submitted type is a sincere one, i.e., the second coordinate of $s_i(R_i, \theta_i)$ is θ_i. Recall that the medical condition of a patient and her donors θ_i is verifiable by medical doctors (see Assumption 1).[34] Let $\mathcal{S}_i$ be the set of patient i's strategies. The identity mapping $s_i^* \in \mathcal{S}_i$ is called **the truth-telling strategy**.

Before introducing the equilibrium concept of the game G, we use the following simplifying notation. For each $(R; \theta) = (R_1, \ldots, R_n; \theta_{d_c}, \theta_1, \ldots, \theta_n) \in \mathcal{R}^N \times \Theta$, we sometimes denote it as $(\theta_{d_c}; (R_1, \theta_1), \ldots, (R_n, \theta_n))$. Moreover, when we focus on a patient i, we denote it as $(\theta_{d_c}; (R_i, \theta_i); (R_j, \theta_j)_{j \neq i})$. A strategy profile $s = (s_1, \ldots, s_n) \in \prod_{i \in N} \mathcal{S}_i$ is a **Bayesian Nash equilibrium** in G if for each $i \in N$, each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, and each $R_i' \in \mathcal{R}$,

$$\sum_{(R_{-i};\theta_{-i}) \in \mathcal{R}^{N \setminus \{i\}} \times \Theta_{-i}} p_i(R_{-i}; \theta_{-i} \mid R_i, \theta_i) u_i^* \left(\varphi_i^P(\theta_{d_c}; s_i(R_i, \theta_i); \big(s_j(R_j, \theta_j)\big)_{j \neq i}) \,\Big|\, R_i, \theta_i\right)$$

$$\geq \sum_{(R_{-i};\theta_{-i}) \in \mathcal{R}^{N \setminus \{i\}} \times \Theta_{-i}} p_i(R_{-i}; \theta_{-i} \mid R_i, \theta_i) u_i^* \left(\varphi_i^P(\theta_{d_c}; (R_i', \theta_i); \big(s_j(R_j, \theta_j)\big)_{j \neq i}) \,\Big|\, R_i, \theta_i\right).$$

The main result of this subsection shows that each priority mechanism is robust against strategic manipulation even under regimes with living-donor exchange under incomplete information.[35] To show that positive result, we need some specifications and simplifications that decently approximate the real world. We make the following three assumptions.

[34] Each patient knows that other patients submit own medical type honestly. However, she does not know which types are realized.

[35] Ehlers and Massó (2007, 2015) introduce incomplete information to the many-to-one matching problems to characterize the Bayesian Nash equilibrium profiles under the stable mechanism. Our problem is different from theirs in 1) the uncertainty in our model is wider in the sense that the other players' and deceased donor's medical types are included, and 2) patients' do not necessarily have unit demand.

Assumption 3 (The number of component type spaces are two)**.** *The collection of component type spaces* $\{(T_k, \trianglerighteq_k)\}_{k=1}^{K}$ *is simplified to the one with the length two, i.e.,* $K = 2$*, such that* $(T_1, \trianglerighteq_1)$ *is the blood type space* $(T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$*;* $(T_2, \trianglerighteq_2)$ *is the other factor space that needs coincidence, i.e., for all* $t_2, t_2' \in T_2$*,* $t_2 \trianglerighteq_2 t_2' \Leftrightarrow t_2 = t_2'$*;* T_2 *contains at least four elements. We denote it as* $T_2 = \{I, II, III, IV, \ldots\}$*.*

This assumption seems too specific, but can reasonably accommodate the current practice. For example, consider the simplest space $T_{\mathcal{B}} \times T_2 = T_{\mathcal{B}} \times \{l, s\} \times \{c_1, c_2\}$ where the component space $\{l, s\}$ of T_2 expresses the sizes of grafts, large (l) or small (s); moreover, the component space $\{c_1, c_2\}$ of T_2 does the types of leucocyte. There are many types of leucocyte which are an important compatibility condition for lung transplantation. However, these types cannot be classified with a clear formula.[36] For this reason, the cross-match test is used and there would be at least two types which are incompatible with each other.[37] What is common within a T_2 space is that it needs a coincidence of types for a donor and a recipient. With these two component spaces of T_2, it is reasonable to assume at least four elements in T_2. Hence, our example space satisfies Assumption 3. Note that T_2 can be any larger cardinality as long as it contains four elements.

Assumption 4 (On the number of living donors of each patient)**.** *The following three conditions hold.*[38]

1. *Each patient has at most two living donors, i.e., for each* $i \in N$*,* $|D_i^L| \leq 2$*.*
2. *At least four patients have multiple living donors, i.e., there exist distinct* $i, j, k, \ell \in N$ *such that for each* $m \in \{i, j, k, \ell\}$*,* $|D_m^L| = 2$*.*
3. *The highest-priority patient* $\sigma(1)$ *has multiple living donors, i.e.,* $|D_{\sigma(1)}^L| \geq 2$*.*

[36]We would like to thank Prof. Takahiro Oto for numerous useful comments from his expertise.

[37]Although the red blood cell has four types of A, B, O, AB, leucocyte also has many types. The human leucocyte is first classified into three types of A, B, DR, and then each of A, B, DR is classified into dozens of antigen types. Moreover, there are unknown types, i.e., the antigens have not yet been exhausted. It is known that every human being has two of the HLA (human leucocyte antigen)s. Thus, for example, when each of A, B, DR types are assumed to contain 20 antigens, human HLA types are $\binom{60}{2} = 1770$. The extreme diversity and the existence of unknown types make it hard to specify which type is compatible with the given type. For this reason, in practice, the cross-match test is carried out to experiment whether the patient's and donor's blood have the immunological rejection in HLA type. In our example, for simplicity, the HLA type is described by the two types. Note that our general setup, especially the fact that T_2 can be large as long as it contains four types, allows more complex type spaces.

[38]A special case of the model satisfying this assumption is a market formed by four or more patients with exactly two living donors for each (Ergin, Sönmez, and Ünver, 2017).

The first condition is weak compared with the corresponding one in Ergin, Sönmez, and Ünver (2017) that requires each patient to have exactly two living donors. The second condition is also weak because hundreds of patients are in line for deceased donors in the Japanese lung market. The last condition seems strong but is actually weak, because our notion of donors include potential donors who are incompatible and usually do not show up in hospitals. Theoretically speaking, this assumption is for simplification. That is, to maintain our main result of Theorem 3, we can drop Assumption 4 when there are at least four patients and each of them faces further uncertainty over the number of the other patients' living donors.

Assumption 5 (Type space restriction). *For each $i \in N$, we redefine the type space Θ_i to slightly restrict a feasible type profile.*[39]

$$\Theta_i := \left\{ \theta_i \in T^{\{i\} \cup D_i^L} \;\middle|\; \begin{array}{l} i)\ \forall d, d' \in D_i^L, \theta_i(d) \trianglerighteq \theta_i(i) \text{ and } \theta_i(d') \trianglerighteq \theta_i(i) \Rightarrow d = d', \\ ii)\ \forall d \in D_i^L, \theta_i(d) \neq \theta_i(i) \end{array} \right\}.$$

The new definition of the type space excludes that a patient has a living donor who is not only compatible with the own patient but also has the identical type with the patient. Let us emphasize that the new definition does not necessarily exclude a compatible living donor. It only excludes the complete coincidence between the type of a patient and her own donor. Since T_2 can be any large set, the restriction of the type space leaves almost no loss of generality.[40]

We are now ready to state our main result of this subsection which asserts that each priority mechanism is robust against strategic behavior under incomplete information even when the living-donor exchange is allowed.

Theorem 3. *The truth-telling strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in \prod_{i \in N} \mathcal{S}_i$ is a Bayesian Nash equilibrium in G.*

Theorem 3 shows that priority mechanisms satisfy a version of *the ordinal Bayesian incentive compatibility* discussed in D'Aspremont and Peleg (1988) and Majumdar and Sen (2004). Note that we adopt Assumption 1 that ensures the truth-telling about patients' medical type throughout this paper, and Assumption 2 that might be a plausible restriction along with the problem pertaining to patients' death throughout this section. Thus, the appropriately adjusted version of the ordinal Bayesian incentive compatibility with respect

[39] The role of Assumption 5 is critical only when $Y = HE$. Theorem 3 can be proved without it if $Y \in \{O, E, H\}$.

[40] For example, a cross-match type space based on individual tissue type can be large.

to the given prior belief $\{p_i\}_{i\in N}$ is the following: given a regime $Y \in \{O, E, H, HE\}$, a mechanism φ is **ordinally Bayesian incentive-compatible with respect to** $\{p_i\}_{i\in N}$ *if* for each $i \in N$, each $(R_i, \theta_i) \in \mathcal{R} \times \Theta_i$, each $R'_i \in \mathcal{R}$, and each $u_i(\cdot;\theta_i) : \{20, 10, 11, 02, 00\} \to \mathbb{R}$ satisfying Assumption 2,

$$\sum_{(R_{-i};\theta_{-i})\in\mathcal{R}^{N\setminus\{i\}}\times\Theta_{-i}} p_i(R_{-i};\theta_{-i} \mid R_i,\theta_i)u_i\left(\varphi_i(\theta_{d_c};(R_i,\theta_i);(R_j,\theta_j)_{j\neq i}) \,\Big|\, R_i,\theta_i\right)$$

$$\geq \sum_{(R_{-i};\theta_{-i})\in\mathcal{R}^{N\setminus\{i\}}\times\Theta_{-i}} p_i(R_{-i};\theta_{-i} \mid R_i,\theta_i)u_i\left(\varphi_i(\theta_{d_c};(R'_i,\theta_i);(R_j,\theta_j)_{j\neq i}) \,\Big|\, R_i,\theta_i\right).$$

Now we are ready to restate the incentive-compatible nature of priority mechanisms.

Corollary 1. *Under any regime $Y \in \{O, E, H, HE\}$, any priority mechanisms are ordinally Bayesian incentive-compatible.*

5 Conclusion

We introduce a dual-organ market where patients are in the two markets for deceased donors and living donors. We investigated the properties of the priority mechanism. Without donor exchange, the priority mechanism is shown to be individually rational, Pareto efficient, fair, and strategy-proof. However, once we allow for donor exchange, we lose its strategy-proofness. Because patients' manipulation is risky, we show that the priority mechanism is robust against any manipulation by showing that the truth-telling strategy profile is a Bayesian Nash equilibrium under the uncertainty over other patients' type and preference, i.e., priority mechanisms are ordinally Bayesian incentive-compatible.

For countries with many deceased donors, it will be more appropriate to consider a static model with multiple deceased donors or a dynamic model developed by Ünver (2010) for a kidney exchange. We believe that our model of a static model with a single deceased donor is important in the light of the current status of many countries, including Japan, and is also a benchmark for extended models.

A Appendix : The Japanese mechanism in lung market

In this appendix, we provide a formal description of the Japanese mechanism introduced in Subsection 4.1. To describe the Japanese mechanism formally, we need some notations. For each $(R;\theta) \in \mathcal{R}^N \times \Theta$,

- $N^{20}(R;\theta) = \{i \in N(R;\theta) \mid 20\ P_i\ 00\ P_i\ 10\}$ is the set of all patients who think the only the dual-graft transplant as acceptable;

- $N^{21}(R;\theta) = \{i \in N(R;\theta) \mid 20\ P_i\ 10\ P_i\ 00\}$ is the set of all patients who prefer the dual-graft transplant to single-graft ones with no unacceptable surgery types;

- $N^{10}(R;\theta) = \{i \in N(R;\theta) \mid 10\ P_i\ 00\ P_i\ 20\}$ is the set of all patients who think the only single-graft transplant as acceptable;

- $N^{12}(R;\theta) = \{i \in N(R;\theta) \mid 10\ P_i\ 20\ P_i\ 00\}$ is the set of all patients who prefer the single-graft transplant to dual-graft ones with no unacceptable surgery types.

Note that the sets $N^{20}(R;\theta)$, $N^{21}(R;\theta)$, $N^{10}(R;\theta)$, and $N^{12}(R;\theta)$ are disjoint.

Definition 8 (Japanese mechanism)**.** Under the regime O, the **Japanese mechanism**, φ^J, selects an allocation for each $(R;\theta) \in \mathcal{R}^N \times \Theta$ as follows.

First, we consider the case with $\theta_{d_cq} = 2$. If $N(R;\theta) = \emptyset$, then let $\varphi^J(R;\theta) = \big((0,\emptyset), \cdots, (0,\emptyset)\big)$. Otherwise, let $i_1 := Top(\succeq; N(R;\theta))$.

Case 1: $i_1 \in N^{20}(R;\theta) \cup N^{21}(R;\theta)$.
For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (2,\emptyset) \text{ if } i = i_1, \\ (0,\emptyset) \text{ if } i \neq i_1. \end{cases}$$

Case 2: $i_1 \in N^{10}(R;\theta) \cup N^{12}(R;\theta)$ and $\big(N^{10}(R;\theta) \cup N^{12}(R;\theta)\big)\backslash\{i_1\} \neq \emptyset$.
Let $i_2 := Top(\succeq; \big(N^{10}(R;\theta) \cup N^{12}(R;\theta)\big)\backslash\{i_1\})$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1,\emptyset) \text{ if } i = i_1 \text{ or } i = i_2, \\ (0,\emptyset) \text{ if } i \in N\backslash\{i_1, i_2\}. \end{cases}$$

Case 3: $N^{10}(R;\theta) \cup N^{12}(R;\theta) = \{i_1\}$.

Case 3.1: $N^{21}(R;\theta) \neq \emptyset$.
Let $i_3 := Top(\succeq; N^{21}(R;\theta))$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1,\emptyset) \text{ if } i = i_1 \text{ or } i = i_3, \\ (0,\emptyset) \text{ if } i \in N\backslash\{i_1, i_3\}. \end{cases}$$

Case 3.2: $N^{21}(R;\theta) = \emptyset$.

Case 3.2.1: $i_1 \in N^{10}(R;\theta)$ and $N^{20}(R;\theta) \neq \emptyset$.
Let $i_4 := Top(\succeq; N^{20}(R;\theta))$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (2,\emptyset) \text{ if } i = i_4, \\ (0,\emptyset) \text{ if } i \neq i_4. \end{cases}$$

Case 3.2.2: $i_1 \in N^{12}(R;\theta)$ or $N^{20}(R;\theta) = \emptyset$.
For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1,\emptyset) \text{ if } i = i_1, \\ (0,\emptyset) \text{ if } i \neq i_1. \end{cases}$$

Next, we consider the case with $\theta_{d_{cq}} = 1$. If $N^{21}(R;\theta) \cup N^{12}(R;\theta) \cup N^{10}(R;\theta) = \emptyset$, then let $\varphi^J(R;\theta) = \big((0,\emptyset), \cdots, (0,\emptyset)\big)$. Otherwise,

Case 4: $N^{12}(R;\theta) \cup N^{10}(R;\theta) \neq \emptyset$.

Let $i_5 := Top(\succeq; N^{12}(R;\theta) \cup N^{10}(R;\theta))$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1,\emptyset) \text{ if } i = i_5, \\ (0,\emptyset) \text{ if } i \neq i_5. \end{cases}$$

Case 5: $N^{12}(R;\theta) \cup N^{10}(R;\theta) = \emptyset$.

Let $i_6 := Top(\succeq; \big(N^{21}(R;\theta))$. For each $i \in N$,

$$\varphi_i^J(R;\theta) = \begin{cases} (1,\emptyset) \text{ if } i = i_6, \\ (0,\emptyset) \text{ if } i \neq i_6. \end{cases}$$

B Appendix: Proofs

In this appendix, we provide the omitted proofs in the main text.

Proof of Proposition 1. Let $a^\theta \in \mathcal{A}(\theta)$ be arbitrary. First, we show two claims.

Claim 1. $\displaystyle\bigcup_{i\in N} \left(a_i^{\theta L} \backslash D_i^L\right) = \bigcup_{i\in N} \left[D_i^L \cap \left(\bigcup_{j\neq i} a_j^{\theta L}\right)\right]$.

To show that the left hand side of the equality is a subset of the right hand side, let $d \in \cup_{i\in N} \left(a_i^{\theta L} \backslash D_i^L\right)$. Then there is $i \in N$ such that $d \in a_i^{\theta L} \backslash D_i^L$. Since $d \in D^L = \cup_{j\in N} D_j^L$ and $d \notin D_i^L$, there is $j_0 \in N\backslash\{i\}$ such that $d \in D_{j_0}^L$. Moreover, $d \in a_i^{\theta L} \subseteq \cup_{j\neq j_0} a_j^{\theta L}$. Thus $d \in D_{j_0}^L \cap \left(\cup_{j\neq j_0} a_j^{\theta L}\right)$.

To show the converse, let $d \in \cup_{i\in N} \left[D_i^L \cap \left(\cup_{j\neq i} a_j^{\theta L}\right)\right]$. Then there is $i \in N$ such that $d \in D_i^L \cap \left(\cup_{j\neq i} a_j^{\theta L}\right)$. Thus, there is $j_0 \in N\backslash\{i\}$ such that $d \in a_{j_0}^{\theta L}$. Since $D_i^L \cap D_{j_0}^L = \emptyset$, $d \in D_i^L$ implies $d \notin D_{j_0}^L$. Thus $d \in a_{j_0}^{\theta L} \backslash D_{j_0}^L$. □

Claim 2. Both $\displaystyle\bigcup_{i\in N} \left(a_i^{\theta L} \backslash D_i^L\right)$ and $\displaystyle\bigcup_{i\in N} \left[D_i^L \cap \left(\bigcup_{j\neq i} a_j^{\theta L}\right)\right]$ are a direct union.

Let $i, i' \in N$ be distinct. If $d \in \left(a_i^{\theta L} \backslash D_i^L\right) \cap \left(a_{i'}^{\theta L} \backslash D_{i'}^L\right)$, then $d \in a_i^{\theta L} \cap a_{i'}^{\theta L}$. This violates the second condition of an allocation (2). Thus $\cup_{i\in N} \left(a_i^{\theta L} \backslash D_i^L\right)$ is direct a union.

If $d \in \left[D_i^L \cap \left(\bigcup_{j \neq i} a_j^{\theta L}\right)\right] \cap \left[D_{i'}^L \cap \left(\bigcup_{j \neq i'} a_j^{\theta L}\right)\right]$, then $d \in D_i^L \cap D_{i'}^L$, a contradiction. Thus, $\cup_{i \in N} \left[D_i^L \cap \left(\cup_{j \neq i} a_j^{\theta L}\right)\right]$ is also a direct union. □

Now we turn back to the proof of Proposition 1. We have

$$\sum_{i \in N} \left|a_i^{\theta L} \backslash D_i^L\right| = \left|\bigcup_{i \in N} \left(a_i^{\theta L} \backslash D_i^L\right)\right| = \left|\bigcup_{i \in N} \left[D_i^L \cap \left(\bigcup_{j \neq i} a_j^{\theta L}\right)\right]\right| = \sum_{i \in N} \left|D_i^L \cap \left(\bigcup_{j \neq i} a_j^{\theta L}\right)\right|. \qquad (*)$$

Note that the first and the third equalities follow from Claim 2, while the second one follows from Claim 1. On the other hand, the balanced condition requires

$$\forall i \in N, \left|a_i^{\theta L} \backslash D_i^L\right| \leq \left|D_i^L \cap \left(\cup_{j \neq i} a_j^{\theta L}\right)\right|.$$

For the equality $(*)$ to be true, the balanced condition must hold with equality for each $i \in N$. □

To show Remark 3, we need the following lemma.

Lemma 1. *Under any regime $Y \in \{O, H\}$, non-wastefulness implies individual rationality.*

Proof. Let $(R; \theta) \in \mathcal{R}^N \times \Theta$. We show the contrapositive. Suppose that $a^\theta \in \mathcal{A}^Y(\theta)$ is not individually rational at $(R; \theta)$. Then we have a patient $i \in N$ such that $(0, \emptyset)\ P_i\ a_i^\theta$. Since $Y \in \{O, H\}$, no patient $j \in N \backslash \{i\}$ uses a graft from a living donor in D_i^L at a^θ. That is, for each $j \in N \backslash \{i\}$, $a_j^{\theta L} \cap D_i^L = \emptyset$. Thus, letting $b_i^\theta := (0, \emptyset)$, we have (i) $b_i^\theta\ P_i\ a_i^\theta$, (ii) $(b_i^\theta; a_{-i}^\theta) \in \mathcal{A}^Y(\theta)$ and (iii) $b_i^{\theta C} = 0 < \overline{a}_i^{\theta C}$ and $b_i^{\theta L} = \emptyset \subseteq \overline{a}_i^{\theta L}$. This means that a^θ is wasteful at $(R; \theta)$. □

Proof of Remark 3. (Item 1) The first part is trivial. The second one is already shown in Example 1.

(Item 2) Since the first part is trivial, we only show the second part. Let $Y \in \{O, H\}$ and $(R; \theta) \in \mathcal{R}^N \times \Theta$. Suppose that $a^\theta \in \mathcal{A}^Y(\theta)$ is non-wasteful at $(R; \theta)$. Let $b^\theta \in \mathcal{A}^Y(\theta)$ be such that for each $i \in N$, $b_i^\theta\ R_i\ a_i^\theta$. It is sufficient to show that there is no $i \in N$ such that $b_i^\theta\ P_i\ a_i^\theta$. We consider four cases separately according to the distribution of grafts from d_c. Note that a^θ is individually rational at $(R; \theta)$ by Lemma 1. Consequently, b^θ is also individually rational at $(R; \theta)$.

Case 1: $\exists i \in N$ s.t. $a_i^\theta \in X_i^{20}(\theta)$. Note that no patient, except for patient i, receives a non-null transplant at a^θ, since all grafts from d_c are used by i and the regime under consideration is $Y \in \{O, H\}$. Thus, at the induced allocation, $\overline{a}_i^\theta = (2, D_i^L)$. If patient i's assignment b_i^θ is not $(2,\emptyset)$, i.e., $b_i^\theta \in X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$, then $b_i^\theta \; P_i \; a_i^\theta$, $b_i^{\theta C} \leq \overline{a}_i^{\theta C}$, and $b_i^{\theta L} \subseteq \overline{a}_i^{\theta L}$. This violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_i^\theta = (2,\emptyset)$. Note that as $b_i^{\theta C} = 2$ and $Y \in \{O,H\}$, no patient, except for patient i, receives a non-null transplant at b^θ. Thus $b^\theta = a^\theta$.

Case 2: $\sum_{i\in N} a_i^{\theta C} = 2$ and $\nexists i \in N$ s.t. $a_i^\theta \in X_i^{20}(\theta)$. Then there exist $i,j \in N$ such that $i \neq j$ and $a_i^{\theta C} = a_j^{\theta C} = 1$. Note that no patient, except for patients i and j, receives a non-null transplant at a^θ, since all grafts from d_c are used by i and j, and the regime under consideration is $Y \in \{O,H\}$. Thus, at the induced allocation, $\overline{a}_i^\theta = (1, D_i^L)$ and $\overline{a}_j^\theta = (1, D_j^L)$.

If $b_i^\theta = (2,\emptyset)$, then $b_j^{\theta C} = 0$. Since $Y \in \{O,H\}$, $b_j^\theta = (0,\emptyset)$. However, this violates the fact that $b_j^\theta \; R_j \; a_j^\theta \; P_j \; (0,\emptyset)$, as $b_j^\theta \; R_j \; a_j^\theta$, a_j^θ $(\neq (0,\emptyset))$ is individually rational at $(R;\theta)$. Thus $b_i^\theta \neq (2,\emptyset)$. Thus, as $Y \in \{O,H\}$, $b_i^\theta \in X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$. Note that $b_i^{\theta C} \leq \overline{a}_i^{\theta C}$ and $b_i^{\theta L} \subseteq \overline{a}_i^{\theta L}$. Therefore, $b_i^\theta \; P_i \; a_i^\theta$ is impossible as it violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_i^\theta \; I_i \; a_i^\theta$. Since $Y \in \{O,H\}$, $b_i^\theta = a_i^\theta$. By the identical argument, we have $b_j^\theta = a_j^\theta$. Note that, as $Y \in \{O,H\}$, no patient, except for patients i and j, receives a non-null transplant at b^θ. Thus $b^\theta = a^\theta$.

Case 3: $\sum_{i\in N} a_i^{\theta C} = 1$. We treat two cases separately according to the number of available grafts from d_c.

Case 3.1: $\theta_{d_c q} = 1$. We can show that $b^\theta = a^\theta$ by the same argument as the proof for Case 1.

Case 3.2: $\theta_{d_c q} = 2$. Let $i \in N$ be the patient such that $a_i^{\theta C} = 1$. Note that, since $Y \in \{O,H\}$, no patient, except for patient i, receives a non-null transplant at a^θ. Note also that one graft from d_c is disposed at a^θ. Thus, the induced allocation is as follows: For each $j \in N$,

$$\overline{a}_j^\theta = \begin{cases} (2, D_j^L) \text{ if } j = i, \\ (1, D_j^L) \text{ if } j \neq i. \end{cases}$$

Note that, since $Y \in \{O,H\}$, b_i^θ satisfies that $b_i^{\theta C} \leq \overline{a}_i^{\theta C}$ and $b_i^{\theta L} \subseteq \overline{a}_i^{\theta L}$. Therefore, $b_i^\theta \; P_i \; a_i^\theta$ is impossible as it violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_i^\theta \; I_i \; a_i^\theta$. Since $Y \in \{O,H\}$, $b_i^\theta = a_i^\theta$. Thus $b_i^{\theta C} = 1$.

Since patient i uses one graft from d_c at b^θ, for each $j \in N \backslash \{i\}$, $b_j^{\theta C} \leq 1$. Thus $b_j^\theta \in X_j^{10}(\theta) \cup \tilde{X}_j^{11}(\theta) \cup X_j^{00}(\theta)$. Note that b_j^θ satisfies that $b_j^{\theta C} \leq \overline{a}_j^{\theta C}$ and $b_j^{\theta L} \subseteq \overline{a}_j^{\theta L}$, since $Y \in \{O,H\}$. Therefore, $b_j^\theta \; P_j \; a_j^\theta$ is impossible as it violates the non-wastefulness of a^θ at $(R;\theta)$. Thus $b_j^\theta \; I_j \; a_j^\theta$. Since $Y \in \{O,H\}$, $b_j^\theta = a_j^\theta = (0,\emptyset)$. Summing up with $a_i^\theta = b_i^\theta$, we obtain $b^\theta = a^\theta$.

Case 4: $\sum_{i \in N} a_i^{\theta C} = 0$. Note that no patient receives a non-null transplant at a^θ, since $Y \in \{O, H\}$. Note also that all grafts from d_c are disposed at a^θ. Thus the induced allocation is: for each $i \in N$, $\overline{a}_i^\theta = (\theta_{d_c q}, D_i^L)$. If patient i's assignment b_i^θ is in $X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$ and $b_i^\theta \; P_i \; a_i^\theta$, then a^θ is wasteful at $(R; \theta)$. Thus, $b_i^\theta \notin X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$ or $a_i^\theta \; R_i \; b_i^\theta$. Note that since $Y \in \{O, H\}$, $b_i^\theta \notin X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$ is impossible. Thus $a_i^\theta \; R_i \; b_i^\theta$. Because we assume that $b_i^\theta \; R_i \; a_i^\theta$, this implies $a_i^\theta \; I_i \; b_i^\theta$. Since $Y \in \{O, H\}$, $b_i^\theta = a_i^\theta$. Thus $b^\theta = a^\theta$.

As $b^\theta = a^\theta$ for all of the four cases, no patient prefers b^θ to a^θ. This completes the proof of Item 2.

(Item 3) We show the statement by two examples. First, we show that Pareto efficiency does not imply strong non-wastefulness. Obviously, allocation a^θ described in Example 1 is Pareto efficient, while it is strongly wasteful. Next, we show that strong non-wastefulness does not imply Pareto efficiency under any regime $Y \in \{E, HE\}$. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors. Let $d_{11}, d_{12} \in D_1^L$ and $d_{21}, d_{22} \in D_2^L$. Let $\theta \in \Theta$ be such that $\theta_{d_c T} = AB$, $\theta(1) = \theta(d_{21}) = \theta(d_{22}) = A$ and $\theta(2) = \theta(d_{11}) = \theta(d_{12}) = B$. Suppose that for each $i \in N \backslash \{1, 2\}$ and each $d \in D^L \backslash \{d_{11}, d_{12}, d_{21}, d_{22}\}$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

R_1	02	00	$\cdots$
R_2	02	00	$\cdots$

Consider the allocation $a^\theta := ((0, \emptyset), (0, \emptyset))$. The allocation is trivially non-wasteful at $(R; \theta)$. Note that no feasible allocation uses a graft from d_c at all, since the deceased donor is not compatible with any patients. Thus a^θ is strongly non-wasteful at $(R; \theta)$. On the other hand, since it is Pareto dominated by allocation $((0, \{d_{21}, d_{22}\}), (0, \{d_{11}, d_{12}\}))$, a^θ is not Pareto efficient at $(R; \theta)$. □

Proof of Remark 4. (First half of Item 1: $\succeq$-fairness $\not\Rightarrow$ individual rationality) We prove it by an example. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Let i be the highest-priority patient. Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, A)$ and $\theta(i) = A$. Let $R \in \mathcal{R}^N$ be such that $R_i \mid \cdots \;\; 00 \;\; 20 \;\; \cdots$.
An allocation where i receives $(2, \emptyset)$ is trivially $\succeq$-fair because the highest-priority patient receives all grafts from d_c. However, $(2, \emptyset)$ is not acceptable for patient i, i.e., the allocation is not in $\mathcal{I}^Y(R; \theta)$.

(The latter half of Item 1: Pareto efficiency and $\succeq$-fairness $\not\Rightarrow$ individual rationality) We prove it by an example. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors. Let $d_{11}, d_{12} \in D_1^L$ and $d_{21}, d_{22} \in D_2^L$. Let $\theta \in \Theta$ be such that $\theta_{d_c T} = AB$, $\theta(1) = \theta(d_{21}) = \theta(d_{22}) = A$ and $\theta(2) = \theta(d_{11}) = \theta(d_{12}) = B$. Suppose that for each $i \in N \backslash \{1,2\}$ and each $d \in D^L \backslash \{d_{11}, d_{12}, d_{21}, d_{22}\}$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_2 & 00 & 02 & \cdots \end{array}$$

Consider allocation $a^\theta := ((0, \{d_{21}, d_{22}\}), (0, \{d_{11}, d_{12}\})) \in \mathcal{A}^Y(\theta)$. Note that a^θ is not individually rational at $(R;\theta)$ because a_2^θ is not acceptable for patient 2.

We claim that a^θ is Pareto efficient and $\succeq$-fair at $(R;\theta)$. Since no patient is compatible with the grafts from d_c, a^θ is trivially $\succeq$-fair at $(R;\theta)$. Next we show the Pareto efficiency. Suppose that $b^\theta \in \mathcal{A}^Y(\theta)$ is such that $b_1^\theta \; R_1 \; a_1^\theta$ and $b_2^\theta \; R_2 \; a_2^\theta$. Note that $X_1^{02} = \{(0, \{d_{21}, d_{22}\})\}$. Thus, since the living-donor dual-graft transplantation, 02, is the top choice at R_1, $b_1^\theta = a_1^\theta$. By Proposition 1, patient 2 receives two grafts from other patients' living-donors because her own donors (d_{21} and d_{22}) donate two grafts to other patient in total. Thus $b_2^\theta = (0, \{d_{11}, d_{12}\})$. Thus $b^\theta = a^\theta$. This completes the proof of Pareto efficiency.

(Item 2) We show it by an example. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that $1 \succ 2$. Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, A)$ and $\theta(1) = \theta(2) = A$. Suppose that for each $i \in N \backslash \{1,2\}$ and each $d \in D^L$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

$$\begin{array}{c|cccc} R_1 & 10 & 20 & 00 & \cdots \\ R_2 & 20 & 00 & \cdots & \end{array}$$

First, consider allocation $a^\theta := ((0, \emptyset), (2, \emptyset)) \in \mathcal{A}^Y(\theta)$. This allocation is non-wasteful, strongly non-wasteful, and Pareto efficient at $(R;\theta)$, because the two grafts from d_c go to patient 2 whose top choice is $(2, \emptyset)$ and the market has no living donor compatible with a patient. However, a^θ is not $\succeq$-fair at $(R;\theta)$, because patient 1 can be better off by using a graft from d_c assigned to lower-priority patient (patient 2). Thus, neither non-wastefulness, strong non-wastefulness, nor Pareto efficiency implies $\succeq$-fairness.

Next, consider allocation $b^\theta := ((2, \emptyset), (0, \emptyset))$. Since the highest-priority patient 1 uses all grafts from d_c, b^θ is trivially $\succeq$-fair at $(R;\theta)$. However, b^θ is wasteful at $(R;\theta)$, because patient 1 can be better off by disposing of one graft from d_c without affecting patient 2's assignment. Formally, letting $c_1^\theta := (1, \emptyset)$, (i) $c_1^\theta \; P_1 \; b_1^\theta$, (ii) $(c_1^\theta, b_2^\theta) \in \mathcal{A}^Y(\theta)$ and (iii) $c_1^{\theta C} = 1 \leq \bar{b}_1^{\theta C}$ and $c_1^{\theta L} = \emptyset \subseteq \bar{b}_1^{\theta L}$. This means that b^θ is wasteful at $(R;\theta)$. Thanks to Remark 3, b^θ is not strongly non-wasteful at $(R;\theta)$, and not Pareto efficient at $(R;\theta)$. Thus

$\succeq$-fairness does not imply any one of non-wastefulness, strong non-wastefulness and Pareto efficiency. $\square$

Proof of Proposition 2. We prove it by an example. We use the same example as in Example 1. Assume, without loss of generality, that the priority is given as $1 \succ 2$. Obviously, $\mathcal{I}^Y(R;\theta)$ consists of just three allocations: $a^\theta := \big((1,\emptyset),(0,\emptyset)\big)$, $b^\theta := \big((0,\emptyset),(2,\emptyset)\big)$, and $c^\theta := \big((0,\emptyset),(0,\emptyset)\big)$. Allocation a^θ is $\succeq$-fair, but is not strongly non-wasteful, since the two grafts from d_c can be used at allocation b^θ. Next, allocation b^θ is strongly non-wasteful, but is not $\succeq$-fair, since agent 1 can be better off by using one graft from d_c assigned to agent 2 who is of lower priority than agent 1. Finally, allocation c^θ is wasteful, since one of the agents 1 and 2 can be better off by using the grafts from d_c that are disposed of at c^θ. Thus, no individually rational allocation satisfies $\succeq$-fairness and strong non-wastefulness at $(R;\theta)$. $\square$

Proof of Proposition 3. Item (i) is trivial. Items (iii) to (v) are shown in Example 2. So it remains to show (ii). Since Pareto efficiency is equivalent to non-wastefulness under regime O by Remark 3, we show that φ^J is non-wasteful. Note that under regime O, non-wastefulness only requires that no patient be better off by using a disposed graft from d_c. This is straightforward. $\square$

Proof of Remark 5. (Item 1) Trivial.

(Item 2) Let $Y \in \{O, H\}$. We show that for each $(R;\theta) \in \mathcal{R}^N \times \Theta$, $\Phi^Y(R;\theta)$ is a singleton. Let $a^\theta, b^\theta \in \Phi^Y(R;\theta)$. By Item 1, for each $i \in N$, $a^\theta_i \; I_i \; b^\theta_i$. Note that regimes O and H allow only transplants in $X^{20}_i(\theta), X^{10}_i(\theta), \tilde{X}^{11}_i(\theta)$, and $X^{00}_i(\theta)$. Thus, $a^\theta_i \; I_i \; b^\theta_i$ implies $a^\theta_i = b^\theta_i$. Thus $a^\theta = b^\theta$.

(Item 3) Let $Y \in \{E, HE\}$. We show that for some $(R;\theta) \in \mathcal{R}^N \times \Theta$, $\Phi^Y(R;\theta)$ contains at least two allocations. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors. Let $d_{11}, d_{12} \in D^L_1$ and $d_{21}, d_{22} \in D^L_2$. Let $\theta \in \Theta$ be such that $\theta_{d_cT} = AB$, $\theta(1) = \theta(d_{21}) = A$, $\theta(2) = \theta(d_{11}) = B$ and $\theta(d_{12}) = \theta(d_{22}) = O$. Suppose that for each $i \in N\backslash\{1,2\}$ and each $d \in D^L\backslash\{d_{11}, d_{12}, d_{21}, d_{22}\}$, $\theta(i) = O$ and $\theta(d) = AB$. Let $R \in \mathcal{R}^N$ be such that

R_1	02	00	$\cdots$
R_2	02	00	$\cdots$

Note that the only compatible donors for patients in $N\backslash\{1,2\}$ are d_{12} and d_{22}. Thus, if a patient $i \in N\backslash\{1,2\}$ receives a non-null transplant at an allocation, it must be $(0,\{d_{12},d_{22}\})$. However, in that case, patients 1 and 2 also receive a non-null transplant with receiving at least one graft from other patient's living donor (Proposition 1). Because there are not enough compatible donors for patients 1 and 2 to receive a non-null transplant without d_{12} and d_{22}, every feasible allocation assigns the null transplant $(0,\emptyset)$ to patients in $N\backslash\{1,2\}$.

For any priority order $\succeq$, $\Phi^Y(R;\theta)$ contains all allocations where patients 1 and 2 receive living-donor transplants. Note that both $((0,\{d_{12},d_{21}\}),(0,\{d_{11},d_{22}\}))$ and $\big((0,D_2^L),(0,D_1^L)\big)$ are in $\Phi^Y(R;\theta)$. □

Note that in the following proof, there is no specification of regime except for strategy-proofness.

Proof of Theorem 1. We assume, without loss of generality, that $1 \succ 2 \succ \ldots \succ n$.

(Individual rationality) Trivial.

(Pareto efficiency) Let $(R;\theta) \in \mathcal{R}^N \times \Theta$. Suppose to the contrary that there is an allocation $a^\theta \in \mathcal{A}^Y(\theta)$ such that for each $i \in N$, $a_i^\theta \ R_i \ \varphi_i^P(R;\theta)$ and for some $i \in N$, $a_i^\theta \ P_i \ \varphi_i^P(R;\theta)$. Note that $a^\theta \in \mathcal{I}^Y(R;\theta) = \Phi_0^Y(R;\theta)$, since a^θ Pareto-dominates an individually rational allocation $\varphi^P(R;\theta)$. Let $i \in N$ be the highest-priority patient among those who prefer a^θ to $\varphi^P(R;\theta)$. Note that this implies that for each $j \in N$ with $j \succ i$, $a_j^\theta \ I_j \ \varphi_j^P(R;\theta)$. Thus $a^\theta \in \Phi_{i-1}^Y(R;\theta)$. Since $a_i^\theta \ P_i \ \varphi_i^P(R;\theta)$, $\varphi^P(R;\theta) \notin \Phi_i^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$, a contradiction.

($\succeq$-fairness) Let $(R;\theta) \in \mathcal{R}^N \times \Theta$. Let $a^\theta := \varphi^P(R;\theta)$. Suppose to the contrary that there is an allocation $b^\theta \in \mathcal{I}^Y(R;\theta)$ such that for some $i \in N$, (i) $b_i^\theta \ P_i \ a_i^\theta$, (ii) $b_i^{\theta C} > a_i^{\theta C}$ and for each $j \in N$ with $j \succ i$, $b_j^{\theta C} = a_j^{\theta C}$ and (iii) for each $j \in N$, $a_j^\theta \ P_j \ b_j^\theta$ implies $a_j^{\theta C} > b_j^{\theta C}$.

If for each $j \in N$ with $j \succ i$, $b_j^\theta \ R_j \ a_j^\theta$, then allocation a^θ is excluded from the priority correspondence at some step, the latest step i, of the algorithm. That is, there is $j \in \{1,\ldots,i\}$ such that $a^\theta \notin \Phi_j^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$, a contradiction. Thus there is $j \in N$ with $j \succ i$ such that $a_j^\theta \ P_j \ b_j^\theta$. By Item (iii), $a_j^{\theta C} > b_j^{\theta C}$. This contradicts the second part of Item (ii).

(Strategy-proofness) Let $Y \in \{O,H\}$. Suppose to the contrary that there are $(R;\theta) \in \mathcal{R}^N \times \Theta, i \in N$ and $R_i' \in \mathcal{R}$ such that $\varphi_i^P(R_i',R_{-i};\theta) \ P_i \ \varphi_i^P(R;\theta)$. For notational simplicity, let $R' := (R_i',R_{-i})$.

Since regime Y does not allow for donor exchange, any preference misreporting cannot affect the individual assignment of higher-priority patients in the priority mechanism. That is, for each $j \in N$ with $j \succ i$, $\varphi_j^P(R;\theta) \ I_j \ \varphi_j^P(R';\theta)$. Thus, since indifferent individual assignments are identical in regime Y, for each $j \in N$ with $j \succ i$, $\varphi_j^P(R;\theta) = \varphi_j^P(R';\theta)$. Note that $\varphi^P(R';\theta)$ is individually rational at $(R;\theta)$ because $\varphi_i^P(R';\theta)\ P_i\ \varphi_i^P(R;\theta)\ R_i\ (0,\emptyset)$ and $R'_{-i} = R_{-i}$. Thus $\varphi^P(R';\theta) \in \mathcal{I}^Y(R;\theta) = \Phi_0^Y(R;\theta)$. Thus we obtain $\varphi^P(R';\theta) \in \Phi_{i-1}^Y(R;\theta)$. Hence $\varphi^P(R;\theta) \notin \Phi_i^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$, a contradiction. □

Proof of Proposition 4. The proof of individual rationality, Pareto efficiency, and $\succeq$-fairness is identical with that of Theorem 1. Non-strategy-proofness is by Example 4. □

Proof of Theorem 2. Suppose to the contrary that a mechanism under $Y \in \{E, HE\}$, φ, satisfies all axioms stated in Theorem 2. Let $(T, \trianglerighteq) = (T_{\mathcal{B}}, \trianglerighteq_{\mathcal{B}})$. Assume, without loss of generality, that patients 1 and 2 have multiple living donors and that $1 \succ 2$. Let $d_{11}, d_{12} \in D_1^L$ and $d_{21}, d_{22} \in D_2^L$. Let $\theta \in \Theta$ be such that $\theta_{d_c} = (2, O)$, $\theta(1) = \theta(d_{21}) = \theta(d_{22}) = A$ and $\theta(2) = \theta(d_{11}) = \theta(d_{12}) = B$. Let $R_1, R'_1, R_2, R'_2 \in \mathcal{R}$ be preferences of patients 1 and 2 described by the following table.

R_1	02	10	00	$\cdots$	
R'_1	02	20	10	00	$\cdots$
R_2	10	02	00	$\cdots$	
R'_2	10	00	$\cdots$		

Let $R_{-\{1,2\}} \in \mathcal{R}^{N\setminus\{1,2\}}$ be a preference profile such that each patient $j \in N\setminus\{1,2\}$ has $(0,\emptyset)$ as the most preferred in R_j. Note that each patient in $N\setminus\{1,2\}$ receives $(0,\emptyset)$, no matter when patients 1 and 2 submit any preference because φ is individually rational. In the subsequent part of the proof, we omit their assignments in the description of an allocation. First, we show the following claim.

Claim. $\varphi(R_1, R_2, R_{-\{1,2\}};\theta) = ((1,\emptyset),(1,\emptyset))$.

Suppose to the contrary that $\varphi(R_1, R_2, R_{-\{1,2\}};\theta) \neq ((1,\emptyset),(1,\emptyset))$. Note that the only individually rational and Pareto efficient allocations at $(R_1, R_2, R_{-\{1,2\}};\theta)$ are $a^\theta := ((0,\{d_{21}, d_{22}\}),(0,\{d_{11}, d_{12}\}))$ and $b^\theta := ((1,\emptyset),(1,\emptyset))$. Thus, by the contradiction hypothesis, $\varphi(R_1, R_2, R_{-\{1,2\}};\theta) = a^\theta$. On the other hand, b^θ is the only individually rational and Pareto efficient allocation at $(R_1, R'_2, R_{-\{1,2\}};\theta)$. Thus $\varphi(R_1, R'_2, R_{-\{1,2\}};\theta) = b^\theta$. Therefore, $\varphi_2(R_1, R'_2, R_{-\{1,2\}};\theta)\ P_2\ \varphi_2(R_1, R_2, R_{-\{1,2\}};\theta)$, a violation of strategy-proofness of φ. This completes the proof of Claim.

Next, we consider patient 1's assignment at $\varphi(R'_1, R_2, R_{-\{1,2\}};\theta)$. Note that since φ is individually rational, $\varphi_1(R'_1, R_2, R_{-\{1,2\}};\theta)$ is one of the following assignments: $(0,\emptyset), (1,\emptyset), (2,\emptyset), (0,\{d_{21}, d_{22}$ We separately derive a contradiction for each case.

Case 1: $\varphi_1(R'_1, R_2, R_{-\{1,2\}}; \theta) = (0, \emptyset)$. Then $\varphi_1(R_1, R_2, R_{-\{1,2\}}; \theta) = (1, \emptyset)\ P'_1\ (0, \emptyset) = \varphi_1(R'_1, R_2, R_{-\{1,2\}}; \theta)$, a violation of strategy-proofness of φ.
Case 2: $\varphi_1(R'_1, R_2, R_{-\{1,2\}}; \theta) = (1, \emptyset)$. Since $\varphi(R'_1, R_2, R_{-\{1,2\}}; \theta)$ is Pareto efficient at $(R'_1, R_2, R_{-\{1,2\}}; \theta)$, the assignment of patient 2 is $\varphi_2(R'_1, R_2, R_{-\{1,2\}}; \theta) = (1, \emptyset)$, i.e., $\varphi(R'_1, R_2, R_{-\{1,2\}}; \theta) = b^\theta$. However, b^θ is not $\succeq$-fair at $(R'_1, R_2, R_{-\{1,2\}}; \theta)$, because $c^\theta := ((2, \emptyset), (0, \emptyset)) \in \mathcal{I}^Y(R'_1, R_2, R_{-\{1,2\}}; \theta)$. This violates the $\succeq$-fairness of φ.
Case 3: $\varphi_1(R'_1, R_2, R_{-\{1,2\}}; \theta) = (2, \emptyset)$. Note that no patient in $N \backslash \{2\}$ uses patient 2's living donor. By Proposition 1, patient 2's assignment does not use any graft from other's living donor. Moreover, patient 2 cannot use a graft from d_c, since all grafts from d_c are assigned to patient 1. Thus the assignment of patient 2 is $\varphi_2(R'_1, R_2, R_{-\{1,2\}}; \theta) = (0, \emptyset)$, i.e., $\varphi(R'_1, R_2, R_{-\{1,2\}}; \theta) = c^\theta$. However, c^θ is Pareto dominated by a^θ at $(R'_1, R_2, R_{-\{1,2\}}; \theta)$. This violates Pareto efficiency of φ.
Case 4: $\varphi_1(R'_1, R_2, R_{-\{1,2\}}; \theta) = (0, \{d_{21}, d_{22}\})$. We have $\varphi_1(R'_1, R_2, R_{-\{1,2\}}; \theta) = (0, \{d_{21}, d_{22}\})\ P_1\ (1, \emptyset) = \varphi_1(R_1, R_2, R_{-\{1,2\}}; \theta)$, a violation of strategy-proofness of φ.

Since the above four cases exhaust all possibilities of $\varphi_1(R'_1, R_2, R_{-\{1,2\}}; \theta)$, we conclude that $\varphi(R'_1, R_2, R_{-\{1,2\}}; \theta)$ is not well-defined, a contradiction. □

To prove Theorem 3, we need two lemmas. Lemma 2 says that if a patient can successfully manipulate a priority mechanism, then the assignment under the true preference necessarily contains a graft from other patient's living donor. Consequently, the transplant is an acceptable one at the true preference. Moreover, at the false preference, the assignment under the true preference is evaluated as unacceptable. That is, successful manipulation forces the patient to pretend that she cannot accept a transplantation type which is actually an acceptable one.

Lemma 2. *Under any regime $Y \in \{E, HE\}$, for each $(R; \theta) \in \mathcal{R}^N \times \Theta$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $\varphi^P_i(R'_i, R_{-i}; \theta)\ P_i\ \varphi^P_i(R_i, R_{-i}; \theta)$, then*

(i) $\varphi^P_i(R_i, R_{-i}; \theta) \in \left(X^{11}_i(\theta) \backslash \tilde{X}^{11}_i(\theta)\right) \cup X^{02}_i(\theta)$. Consequently, $\varphi^P_i(R_i, R_{-i}; \theta) \neq (0, \emptyset)$.

(ii) $\varphi^P_i(R'_i, R_{-i}; \theta)\ P'_i\ (0, \emptyset)\ P'_i\ \varphi^P_i(R_i, R_{-i}; \theta)$.

Proof. Let $(R; \theta) \in \mathcal{R}^N \times \Theta$, $i \in N$ and $R'_i \in \mathcal{R}$. Suppose $\varphi^P_i(R'_i, R_{-i}; \theta)\ P_i\ \varphi^P_i(R_i, R_{-i}; \theta)$. For notational simplicity, let $b^\theta := \varphi^P(R'_i, R_{-i}; \theta)$ and $a^\theta := \varphi^P(R_i, R_{-i}; \theta)$. Assume, without loss of generality, that $1 \succ 2 \succ \ldots \succ n$.

Proof of Item *(i)*. Suppose to the contrary that $a^\theta_i \notin \left(X^{11}_i(\theta) \backslash \tilde{X}^{11}_i(\theta)\right) \cup X^{02}_i(\theta)$. Namely, a^θ_i is a deceased-donor dual-graft, deceased-donor single-graft, hybrid with own donor, or null transplant. We claim

$$\forall j \in N \text{ with } j \succ i, b^\theta_j\ I_j\ a^\theta_j. \tag{4}$$

To show (4), suppose to the contrary that at least one patient $j \in N$ with $j \succ i$ is not indifferent between b_j^θ and a_j^θ, i.e., $b_j^\theta \; P_j \; a_j^\theta$ or $a_j^\theta \; P_j \; b_j^\theta$. We derive a contradiction for each case separately. Without loss of generality, suppose that j is the highest-priority patient who has such a preference.

Case 1: $a_j^\theta \; P_j \; b_j^\theta$. Let c^θ be such that

$$c_k^\theta = \begin{cases} (0, \emptyset) \text{ if } k = i, \\ a_k^\theta \text{ if } k \neq i. \end{cases}$$

Note that $c^\theta \in \mathcal{A}^Y(\theta)$, since a_i^θ does not use other's donor. Moreover, $c^\theta \in \mathcal{I}^Y(R_i', R_{-i}; \theta)$, since the only difference between $(R_i, R_{-i}; \theta)$ and $(R_i', R_{-i}; \theta)$ is patient i's preference. Thus $c^\theta \in \Phi_0^Y(R_i', R_{-i}; \theta)$. By the definition of j, for each $k < j$, $c_k^\theta = a_k^\theta \; I_k \; b_k^\theta$. Thus $c^\theta \in \Phi_{j-1}^Y(R_i', R_{-i}; \theta)$. Since $a_j^\theta \; P_j \; b_j^\theta$, $b^\theta \notin \Phi_j^Y(R_i', R_{-i}; \theta) \supseteq \Phi^Y(R_i', R_{-i}; \theta)$. However, $b^\theta = \varphi^P(R_i', R_{-i}; \theta) \in \Phi^Y(R_i', R_{-i}; \theta)$, a contradiction.

Case 2: $b_j^\theta \; P_j \; a_j^\theta$. Since $b^\theta \in \mathcal{I}^Y(R_i', R_{-i}; \theta)$ and $b_i^\theta \; P_i \; a_i^\theta \; R_i \; (0, \emptyset)$, $b^\theta \in \mathcal{I}^Y(R_i, R_{-i}; \theta)$. Thus $b^\theta \in \Phi_0^Y(R_i, R_{-i}; \theta)$. By the definition of j, for each $k < j$, $a_k^\theta \; I_k \; b_k^\theta$. Thus $b^\theta \in \Phi_{j-1}^Y(R_i, R_{-i}; \theta)$. Thus, $a^\theta \notin \Phi_j^Y(R_i, R_{-i}; \theta) \supseteq \Phi^Y(R_i, R_{-i}; \theta)$. However, $a^\theta = \varphi^P(R_i, R_{-i}; \theta) \in \Phi^Y(R_i, R_{-i}; \theta)$, a contradiction.

Summing up Cases 1 and 2, we get (4).

Now we complete the proof of Item *(i)*. Note that as we have seen in Case 2, $b^\theta \in \mathcal{I}^Y(R_i, R_{-i}; \theta)$. Thus, by (4), $b^\theta \in \Phi_{i-1}^Y(R_i, R_{-i}; \theta)$. Since $b_i^\theta \; P_i \; a_i^\theta$, $a^\theta \notin \Phi_i^Y(R_i, R_{-i}; \theta) \supseteq \Phi^Y(R_i, R_{-i}; \theta)$. However, $a^\theta = \varphi^P(R_i, R_{-i}; \theta) \in \Phi^Y(R_i, R_{-i}; \theta)$, a contradiction. □

Proof of Item *(ii)*: We show the first part of Item *(ii)*. Since φ^P is individually rational, it is obvious that $b_i^\theta = \varphi_i^P(R_i', R_{-i}; \theta) \; R_i' \; (0, \emptyset)$. If $b_i^\theta = (0, \emptyset)$, then $(0, \emptyset) \; P_i \; a_i^\theta = \varphi_i^P(R_i, R_{-i}; \theta)$, a violation to individual rationality of φ^P. Thus we obtain $b_i^\theta \; P_i' \; (0, \emptyset)$.

Next, we show the second part of Item *(ii)*, i.e., $(0, \emptyset) \; P_i' \; a_i^\theta$. Suppose to the contrary that $a_i^\theta \; R_i' \; (0, \emptyset)$. Then, both a^θ and b^θ are individually rational at both $(R_i, R_{-i}; \theta)$ and $(R_i', R_{-i}; \theta)$, since the only difference between $(R_i, R_{-i}; \theta)$ and $(R_i', R_{-i}; \theta)$ is patient i's preference. We consider the following two cases separately, and derive a contradiction for each.

Case 1: $\forall j \in N$ with $j \succ i$, $a_j^\theta \; I_j \; b_j^\theta$. By $b^\theta \in \mathcal{I}^Y(R_i, R_{-i}; \theta)$ and the assumption for Case 1, $b^\theta \in \Phi_{i-1}^Y(R_i, R_{-i}; \theta)$. Since $b_i^\theta \; P_i \; a_i^\theta$, $a^\theta \notin \Phi_i^Y(R_i, R_{-i}; \theta) \supseteq \Phi^Y(R_i, R_{-i}; \theta)$. Thus, $a^\theta = \varphi^P(R_i, R_{-i}; \theta) \notin \Phi^Y(R_i, R_{-i}; \theta)$, a contradiction.

Case 2: Let $j \in N$ be the highest-priority patient with $j \succ i$ and not $a_j^\theta \; I_j \; b_j^\theta$.

Case 2.1: $a_j^\theta \; P_j \; b_j^\theta$. By $a^\theta \in \mathcal{I}^Y(R_i', R_{-i}; \theta)$ and the assumption for Case 2.1, $a^\theta \in$

$\Phi^Y_{j-1}(R'_i, R_{-i};\theta)$. Since $a^\theta_j \; P_j \; b^\theta_j$, $b^\theta \notin \Phi^Y_j(R'_i, R_{-i};\theta) \supseteq \Phi^Y(R'_i, R_{-i};\theta)$. Thus, $b^\theta = \varphi^P(R'_i, R_{-i};\theta) \notin \Phi^Y(R'_i, R_{-i};\theta)$, a contradiction.

Case 2.2: $b^\theta_j \; P_j \; a^\theta_j$. By $b^\theta \in \mathcal{I}^Y(R_i, R_{-i};\theta)$ and the assumption for Case 2.2, $b^\theta \in \Phi^Y_{j-1}(R_i, R_{-i};\theta)$. Since $b^\theta_j \; P_j \; a^\theta_j$, $a^\theta \notin \Phi^Y_j(R_i, R_{-i};\theta) \supseteq \Phi^Y(R_i, R_{-i};\theta)$. Thus, $a^\theta = \varphi^P(R_i, R_{-i};\theta) \notin \Phi^Y(R_i, R_{-i};\theta)$, a contradiction. □

Lemma 3 says that a patient i has a profitable deviation only if she has a donor d_i with a special medical type: a graft with patient i's medical type is not compatible with a patient with d_i's medical type. To see why this is true, let us take a look at Figure 8. Let i be a patient who has donors with identical type except for the blood type. In that figure, x, y, and z represent the blood types of patient i, 1st donor of i, and 2nd donor of i, respectively. Suppose that patient i has a profitable deviation. That is, she has a false preference with which she can get a more preferable transplant to the one with the true preference. In figure 8, the solid arrows represent the flow of living donors at the allocation under patient i's true preference. Let us call the allocation a. By Lemma 2, patient i gets a hybrid transplant with other's donor or living-donor dual-graft transplant at a. In either case, she uses a graft from other patient's living donor, say d_k. By Proposition 1, one of the patient i's donors donates to a patient, say j.

We claim that group i's contribution at a must be critical to maintain the welfare level of other patients. If $x \trianglerighteq_1 z$ holds, then the donor d_k can donate to patient j directly, because the blood type compatibility relation $\trianglerighteq_1 = \trianglerighteq_{\mathcal{B}}$ is transitive. This means that the patients in $N\setminus\{i\}$ can attain the welfare level of a without the contribution of group i. Consequently, even if patient i reports a preference which states that a_i is unacceptable, other patients can keep consuming transplants indifferent with a_{-i}. Thus patient i's deviation has no effect on the priority mechanism. This contradicts that patient i has a profitable deviation. Thus, x cannot be blood-type-compatible with z, i.e., $x \not\trianglerighteq_1 z$.

Lemma 3. *Let $Y \in \{E, HE\}$, $(R;\theta) \in \mathcal{R}^N \times \Theta$ and $i \in N$. Suppose that for each $d \in D^L_i$, $\theta(d)$ is identical with $\theta(i)$ except for the blood type, i.e., $\theta_2(d) = \theta_2(i)$. Then,*

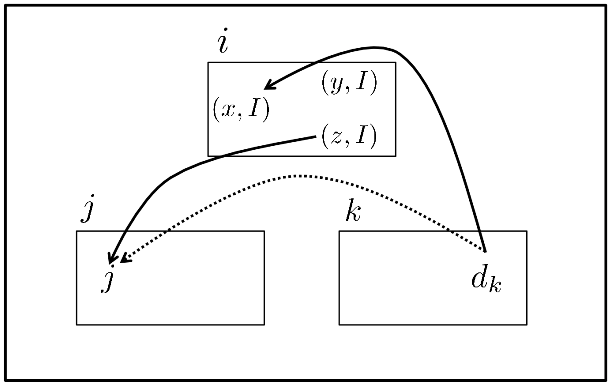
$$\left[\exists R'_i \in \mathcal{R} \text{ s.t. } \varphi^P_i(R'_i, R_{-i};\theta) \; P_i \; \varphi^P_i(R_i, R_{-i};\theta)\right] \Rightarrow \left[\exists d \in D^L_i \text{ s.t. } \theta_1(i) \not\trianglerighteq_1 \theta_1(d)\right].$$

Proof. Assume, without loss of generality, $1 \succ 2 \succ \ldots \succ n$. Suppose to the contrary that

$$\forall d \in D^L_i, \theta_1(i) \trianglerighteq_1 \theta_1(d). \tag{5}$$

Fix patient i's profitable deviation $R'_i \in \mathcal{R}$. For notational simplicity, let $b^\theta := \varphi^P(R'_i, R_{-i};\theta)$

Figure 8: Patient with a profitable deviation necessarily plays a critical role in the donor exchange.



and $a^\theta := \varphi^P(R;\theta)$. The proof consists of three steps.

Step 1: Defining new allocations b'^θ and a'^θ. We define b'^θ and a'^θ in $\mathcal{A}^Y(\theta)$ based on b^θ and a^θ. Since the way to generate a'^θ from a^θ is the same as the one for b'^θ from b^θ, we only describe the construction of b'^θ in detail. The definition of b'^θ varies according to the number of other patient's living donors in $b_i^{\theta L}$. Each case below corresponds to the case where the number is 0, 1 and 2, respectively.

Case 1: $b_i^\theta \in X_i^{20}(\theta) \cup X_i^{10}(\theta) \cup \tilde{X}_i^{11}(\theta) \cup X_i^{00}(\theta)$. For each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ b_m^\theta \text{ if } m \neq i. \end{cases}$$

Obviously, $b'^\theta \in \mathcal{A}^Y(\theta)$.

Case 2: $b_i^\theta \in X_i^{11}(\theta) \setminus \tilde{X}_i^{11}(\theta)$ or $\left[b_i^\theta \in X_i^{02}(\theta) \text{ and } b_i^{\theta L} \cap D_i^L \neq \emptyset\right]$. Let $d_i \in D_i^L$ be i's donor who donates to other patient, say $k \in N\setminus\{i\}$, at b^θ, i.e., $d_i \in b_k^{\theta L}$. Let $d_\ell \in D_\ell^L$ be other patient's donor who donates to i at b^θ, i.e., $\ell \neq i$ and $d_\ell \in b_i^{\theta L}$. For each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ (b_k^{\theta C}, (b_k^{\theta L} \setminus \{d_i\}) \cup \{d_\ell\}) \text{ if } m = k, \\ b_m^\theta \text{ if } m \notin \{i, k\}. \end{cases}$$

We claim that $b'^\theta \in \mathcal{A}^Y(\theta)$. To show this, it is sufficient to prove that d_ℓ is compatible with

k, i.e., $\theta(d_\ell) \trianglerighteq \theta(k)$. First, we show

$$\begin{aligned}
\theta_2(d_\ell) &\trianglerighteq_2 \theta_2(i) \quad (\because d_\ell \in b_i^{\theta L}) \\
&\trianglerighteq_2 \theta_2(d_i) \quad (\because \text{Assumption of Lemma 3}) \\
&\trianglerighteq_2 \theta_2(k). \quad (\because d_i \in b_k^{\theta L})
\end{aligned}$$

Since the binary relation $\trianglerighteq_2$ is the equality "=", it is transitive. Thus $\theta_2(d_\ell) \trianglerighteq_2 \theta_2(k)$. Similarly,

$$\begin{aligned}
\theta_1(d_\ell) &\trianglerighteq_1 \theta_1(i) \quad (\because d_\ell \in b_i^{\theta L}) \\
&\trianglerighteq_1 \theta_1(d_i) \quad (\because \text{The contradiction hypothesis (5)}) \\
&\trianglerighteq_1 \theta_1(k). \quad (\because d_i \in b_k^{\theta L})
\end{aligned}$$

Since the binary relation $\trianglerighteq_1$ is transitive, $\theta_1(d_\ell) \trianglerighteq_1 \theta_1(k)$. In sum, $\theta(d_\ell) \trianglerighteq \theta(k)$. Thus $b'^\theta \in \mathcal{A}^Y(\theta)$.

Case 3: $b_i^\theta \in X_i^{02}(\theta)$ and $b_i^{\theta L} \cap D_i^L = \emptyset$. Let $d_{i1}, d_{i2} \in D_i^L$ be i's donor who donate to other patient(s), say $k, \ell \in N \backslash \{i\}$, at b^θ, i.e., $d_{i1} \in b_k^{\theta L}$ and $d_{i2} \in b_\ell^{\theta L}$. Let $d_p, d_q \in D^L \backslash D_i^L$ be other patient's donors who donate to i at b^θ, i.e., $b_i^{\theta L} = \{d_p, d_q\}$. If $k = \ell$, then let b'^θ be such that for each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ (0, \{d_p, d_q\}) \text{ if } m = k, \\ b_m^\theta \text{ if } m \notin \{i, k\}. \end{cases}$$

If $k \neq \ell$, then let b'^θ be such that for each $m \in N$,

$$b'^\theta_m := \begin{cases} (0, \emptyset) \text{ if } m = i, \\ (b_k^{\theta C}, (b_k^{\theta L} \backslash \{d_{i1}\}) \cup \{d_p\}) \text{ if } m = k, \\ (b_\ell^{\theta C}, (b_\ell^{\theta L} \backslash \{d_{i2}\}) \cup \{d_q\}) \text{ if } m = \ell, \\ b_m^\theta \text{ if } m \notin \{i, k, \ell\}. \end{cases}$$

In either case, the proof for $b'^\theta \in \mathcal{A}^Y(\theta)$ is the same as the one given in Case 2. Thus we omit it.

In the same manner, we define a'^θ based on a^θ.[41] Note that every patient, except for i, receives the same transplantation type at b'^θ and b^θ in each case. This is true at a'^θ and a^θ.

[41] By Lemma 2, a_i^θ is a hybrid transplant with other's donor or a living-donor dual-graft transplant. Thus Case 1 is redundant to define a'^θ.

Thus we have

$$\forall k \in N\backslash\{i\}, b_k'^\theta \; I_k \; b_k^\theta \text{ and } a_k'^\theta \; I_k \; a_k^\theta. \tag{6}$$

Step 2: We show that for each $j \in N$ with $j \succ i$, $b_j^\theta \; I_j \; a_j^\theta$. Suppose to the contrary that for some $j \in N$ with $j \succ i$, $b_j^\theta \; P_j \; a_j^\theta$ or $a_j^\theta \; P_j \; b_j^\theta$. Let j be the highest-priority patient among the patients who are not indifferent between b^θ and a^θ.

First, suppose $b_j^\theta \; P_j \; a_j^\theta$. Note that $b'^\theta \in \mathcal{I}^Y(R;\theta)$ by $b_i'^\theta = (0,\emptyset)$ and (6). Thus $b'^\theta \in \Phi_0^Y(R;\theta)$. By the definition of j and (6), for each $k \in N$ with $k \succ j$, $a_k^\theta \; I_k \; b_k^\theta \; I_k \; b_k'^\theta$. Thus $b'^\theta \in \Phi_{j-1}^Y(R;\theta)$. By (6), $b_j'^\theta \; I_j \; b_j^\theta \; P_j \; a_j^\theta$. Thus we conclude $a^\theta \notin \Phi_j^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$, contradicting $a^\theta = \varphi^P(R;\theta) \in \Phi^Y(R;\theta)$.

Next, suppose that $a_j^\theta \; P_j \; b_j^\theta$. Note that $a'^\theta \in \mathcal{I}^Y(R_i', R_{-i};\theta)$ by $a_i'^\theta = (0,\emptyset)$ and (6). Thus $a'^\theta \in \Phi_0^Y(R_i', R_{-i};\theta)$. By the definition of j and (6), for each $k \in N$ with $k \succ j$, $b_k^\theta \; I_k \; a_k^\theta \; I_k \; a_k'^\theta$. Thus $a'^\theta \in \Phi_{j-1}^Y(R_i', R_{-i};\theta)$. By (6), $a_j'^\theta \; I_j \; a_j^\theta \; P_j \; b_j^\theta$. Thus we conclude $b^\theta \notin \Phi_j^Y(R_i', R_{-i};\theta) \supseteq \Phi^Y(R_i', R_{-i};\theta)$. This contradicts $b^\theta = \varphi^P(R_i', R_{-i};\theta) \in \Phi^Y(R_i', R_{-i};\theta)$.

In either case, we obtain a contradiction. This completes the proof of Step 2.

Step 3: We complete the proof. Since b^θ satisfies that $b_i^\theta \; P_i \; a_i^\theta \; R_i \; (0,\emptyset)$ and $b_k^\theta = \varphi_k^P(R_i', R_{-i};\theta) \; R_k \; (0,\emptyset)$ for all $k \in N\backslash\{i\}$, we have $b^\theta \in \mathcal{I}^Y(R;\theta) = \Phi_0^Y(R;\theta)$. Thus, by Step 2, $b^\theta \in \Phi_{i-1}^Y(R;\theta)$. Since $b_i^\theta \; P_i \; a_i^\theta$, we conclude $a^\theta \notin \Phi_i^Y(R;\theta) \supseteq \Phi^Y(R;\theta)$. This contradicts $a^\theta = \varphi^P(R;\theta) \in \Phi^Y(R;\theta)$. □

Proof of Theorem 3. Without loss of generality, assume that $1 \succ 2 \succ \ldots \succ n$. Suppose to the contrary that for some $i \in N$, $(R_i^*, \theta_i^*) \in \mathcal{R} \times \Theta_i$, and $R_i \in \mathcal{R}$,

$$\begin{aligned} &\sum_{(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}} p_i(R_{-i};\theta_{-i} \mid R_i^*, \theta_i^*) u_i^* \left(\varphi_i^P(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right) \\ < &\sum_{(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}} p_i(R_{-i};\theta_{-i} \mid R_i^*, \theta_i^*) u_i^* \left(\varphi_i^P(\theta_{d_c}; (R_i, \theta_i^*); (R_j, \theta_j)_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right). \end{aligned} \tag{7}$$

A direct consequence of the hypothesis (7) is that at least one $(R_{-i}'; \theta_{-i}') \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, we have

$$u_i^* \left(\varphi_i^P(\theta_{d_c}'; (R_i^*, \theta_i^*); (R_j', \theta_j')_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right) < u_i^* \left(\varphi_i^P(\theta_{d_c}'; (R_i, \theta_i^*); (R_j', \theta_j')_{j \neq i}) \,\middle|\, R_i^*, \theta_i^* \right) \tag{8}$$

Thus, patient i is not the highest-priority patient, i.e., $i \neq 1$. Since $\varphi_i^P(\theta_{d_c}'; (R_i^*, \theta_i^*); (R_j', \theta_j')_{j \neq i})$ is at least as good as $(0,\emptyset)$ at R_i^*, patient i's true preference R_i^* has at least one acceptable transplantation type, i.e., $Ac_i(R_i^*) \neq \emptyset$. Thus we can apply Assumption 2 to patient i. Moreover, by Lemma 2, $\varphi_i^P(\theta'_{d_c}; (R_i^*, \theta_i^*); (R'_j, \theta'_j)_{j\neq i}) \in \left(X_i^{11}(\theta_i^*, \theta'_{-i}) \backslash \tilde{X}_i^{11}(\theta_i^*, \theta'_{-i})\right) \cup X_i^{02}(\theta_i^*, \theta'_{-i})$ and $(0, \emptyset)\ P_i\ \varphi_i^P(\theta'_{d_c}; (R_i^*, \theta_i^*); (R'_j, \theta'_j)_{j\neq i})$. Consequently, at least one of the following two statements holds:

$$02 \text{ is acceptable at } R_i^*, \text{ but not acceptable at } R_i. \tag{9}$$

$$11 \text{ is acceptable at } R_i^*, \text{ but not acceptable at } R_i. \tag{10}$$

In the subsequent part of the proof, we show

$$\exists (R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i} \text{ s.t. } \begin{bmatrix} \varphi_i^P(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}) \in Ac_i(R_i^*) \\ \text{and} \\ \varphi_i^P(\theta_{d_c}; (R_i, \theta_i^*); (R_j, \theta_j)_{j\neq i}) = (0, \emptyset) \end{bmatrix}. \tag{11}$$

Note that (11) contradicts inequality (7) because Assumption 2 states that even if misreporting R_i is successful for every $(\tilde{R}_{-i}; \tilde{\theta}_{-i}) \in \left(\mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}\right) \backslash \{(R_{-i}; \theta_{-i})\}$, i.e., $\varphi_i^P(R_i, \tilde{R}_{-i}; \theta_i^*, \tilde{\theta}_{-i})\ P_i^*\ \varphi_i^P(R_i^*, \tilde{R}_{-i}; \theta_i^*, \tilde{\theta}_{-i})$, the expected utility gain from misreporting is canceled out by the failure of misreporting at $(R_{-i}; \theta_{-i})$ (See statement (11)).

In the following, we construct $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ to show (11) for each case separately. For notational simplicity, we will use the following notation in every case.

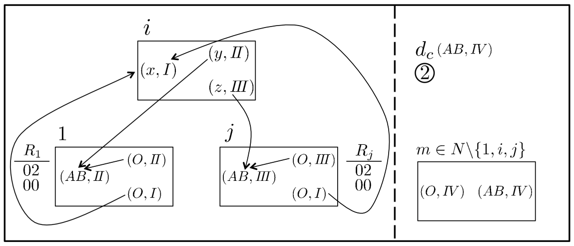
$$a := \varphi^P(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}) \text{ and } b := \varphi^P(\theta_{d_c}; (R_i, \theta_i^*); (R_j, \theta_j)_{j\neq i}).$$

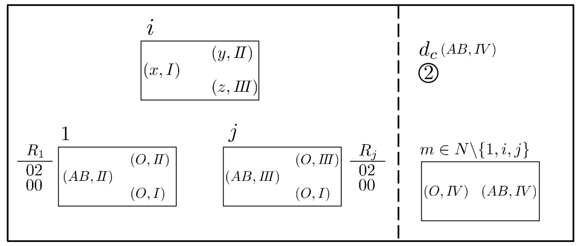
That is, a denotes the allocation under i's truth-telling at $(R_{-i}; \theta_{-i})$ constructed in the case under consideration, and b does the allocation under i's misreporting at $(R_{-i}; \theta_{-i})$ constructed in the case under consideration.

By Assumption 4, let $j, k \in N\backslash\{1, i\}$ be distinct patients who have two living donors. Let $D_1^L = \{d_{11}, d_{12}\}$, $D_j^L = \{d_{j1}, d_{j2}\}$, and $D_k^L = \{d_{k1}, d_{k2}\}$.

Case 1: Patient i has two living donors, i.e., $|D_i^L| = 2$. Let $D_i^L = \{d_{i1}, d_{i2}\}$.
Case 1.1: (9) holds.
Case 1.1.1: $|\{\theta_{i2}^*(i), \theta_{i2}^*(d_{i1}), \theta_{i2}^*(d_{i2})\}| = 3$. Without loss of generality, let $\theta_i^*(i) = (x, I), \theta_i^*(d_{i1}) = (y, II), \theta_i^*(d_{i2}) = (z, III)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

Figure 9: Allocations a and b in Case 1.1.1.





- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D_m^L$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, IV)\Big) \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, I) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, III) \\ \theta_j(d_{j1}) = (O, III) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.[42]

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

Claim 1.1.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, \{d_{i1}, d_{11}\}) \text{ if } m = 1, \\ (0, \{d_{12}, d_{j2}\}) \text{ if } m = i, \\ (0, \{d_{i2}, d_{j1}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j \neq i})$, φ^P selects it. □

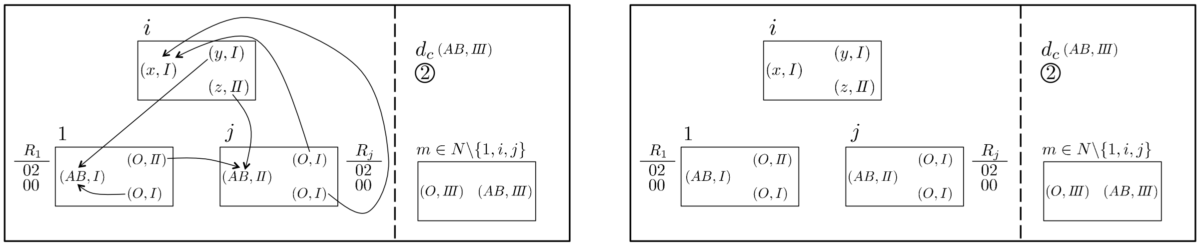
Claim 1.1.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$, $b_m = (0, \emptyset)$.

Proof. Since 02 is unacceptable at R_i, patients 1 and j cannot receive living-donor dual-graft transplants. Moreover the grafts from the deceased donor are not compatible with any patient. □

Case 1.1.2: $|\{\theta_{i2}^*(i), \theta_{i2}^*(d_{i1}), \theta_{i2}^*(d_{i2})\}| = 2$.

[42] The preferences of patients in $N\setminus\{1, i, j\}$ are omitted because they are free. In the later cases, the omitted preferences are free, too.

Figure 10: Allocations a and b in Case 1.1.2.1.



Case 1.1.2.1: $\theta^*_{i2}(d_{i1}) \neq \theta^*_{i2}(d_{i2})$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, I), \theta^*_i(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, III)\Big) \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, I) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

Claim 1.1.2.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, i, j\}, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

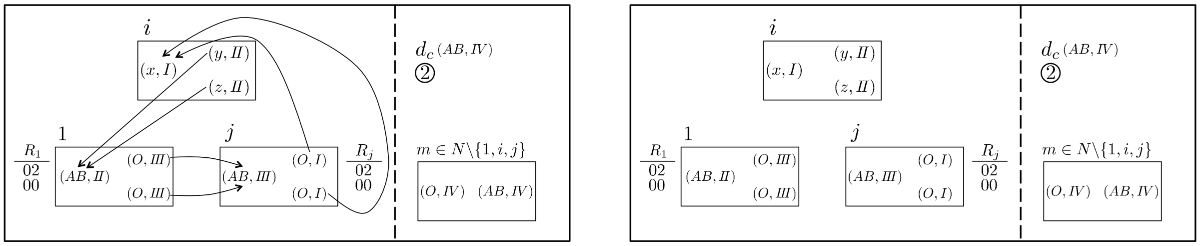
Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □

Claim 1.1.2.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$, $b_m = (0, \emptyset)$.

Proof. Since 02 is unacceptable at R_i, b_i cannot be 02. Moreover the grafts from the deceased donor are not compatible with patient i. Thus $b_i = (0, \emptyset)$. This implies that d_{i2} does not donate to any patient (Proposition 1). Thus, $b_j = (0, \emptyset)$, since 02 is the only acceptable transplantation type for patient j. This implies that d_{j1} and d_{j2} do not donate to any patient. Thus $b_1 = (0, \emptyset)$. □

Case 1.1.2.2: $\theta^*_{i2}(d_{i1}) = \theta^*_{i2}(d_{i2})$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, II), \theta^*_i(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

Figure 11: Allocations a and b in Case 1.1.2.2.



- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, IIV)\Big) \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, III) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, III) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

<u>Claim 1.1.2.2a:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, D^L_i) \text{ if } m = 1, \\ (0, D^L_j) \text{ if } m = i, \\ (0, D^L_1) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

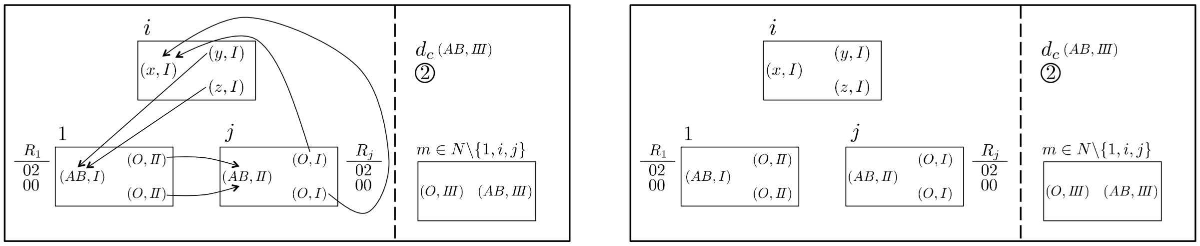
Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □

<u>Claim 1.1.2.2b:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$, $b_m = (0, \emptyset)$.

Proof. Since 02 is unacceptable at R_i, b_i cannot be 02. Moreover the grafts from the deceased donor are not compatible with patient i. Thus, $b_i = (0, \emptyset)$. This implies that d_{i1} and d_{i2} do not donate to any patient (Proposition 1). Thus $b_1 = (0, \emptyset)$ since 02 is the only acceptable transplantation type for patient 1. This implies that d_{11} and d_{12} do not donate to any patient. Thus $b_j = (0, \emptyset)$. □

<u>Case 1.1.3:</u> $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 1$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, I), \theta^*_i(d_{i2}) = (z, I)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

Figure 12: Allocations a and b in Case 1.1.3.



- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (AB, III)\Big) \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, II) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|ccc} R_1 & 02 & 00 & \cdots \\ R_j & 02 & 00 & \cdots \end{array}$$

<u>Claim 1.1.3a:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, i, j\}, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

Proof. Since the above allocation is the only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □

<u>Claim 1.1.3b:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (0, D^L_j) \text{ if } m = 1, \\ (0, D^L_1) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, j\}. \end{cases}$$

Proof. Since 02 is unacceptable at R_i, b_i cannot be 02. Moreover the grafts from the deceased donor are not compatible with patient i. Thus $b_i = (0, \emptyset)$. The only individually rational and Pareto efficient allocation at $(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, φ^P selects it. □

<u>**Case 1.2:**</u> (10) holds.

Case 1.2.1: $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 3$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, II), \theta^*_i(d_{i2}) = (z, III)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: For $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$,

$$\Big\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, I) \\ \theta_1(d_{12}) = (O, II) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{c|cccc} R_1 & 02 & 20 & 00 & \cdots \\ R_j & 02 & 00 & \cdots & \end{array}$$

Claim 1.2.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, j\}, \\ X^{11}_m(\theta^*_i, \theta_{-i}) \text{ if } m = i, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

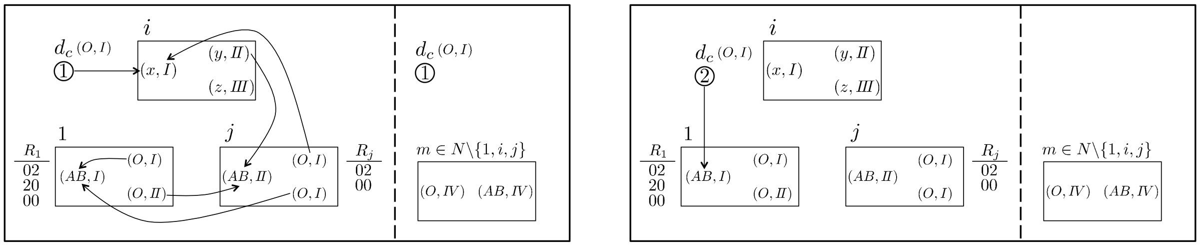
Proof. Since the allocation in the left hand side of Figure 13 is in $\mathcal{I}^Y(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}) = \Phi^Y_0(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since a_1 is 02, d_{12} donates to a patient (Proposition 1). Since only patient j can receive d_{12}'s donation, $d_{12} \in a^L_j$. Since the only acceptable transplantation type for j is 02, $a_j \in X^{02}_j(\theta^*_i, \theta_{-i})$. Since a_1 and a_j is 02, d_{11}, d_{j1} and d_{j2} donate to a patient respectively (Proposition 1). Since two of them donate to patient 1, the remaining one donates to patient i. Since no other living donor is compatible with patient i, $a_i \in X^{11}_i(\theta^*_i, \theta_{-i})$. □

Claim 1.2.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.1a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Thus each patient in $N\setminus\{1\}$ cannot use a graft from d_c. Thus b_m is not 20, 10 or 11 for each $m \in N\setminus\{1\}$. Since patient j cannot find two compatible living donors in $D^L\setminus D^L_1$, $b_j \notin X^{02}_j(\theta^*_i, \theta_{-i})$. Thus $b_j = (0, \emptyset)$. Since patient $m \in N\setminus\{1, j\}$ cannot find two compatible living donors in $D^L\setminus(D^L_1 \cup D^L_j)$, $b_m \notin X^{02}_m(\theta^*_i, \theta_{-i})$. Thus $b_m = (0, \emptyset)$. □

Figure 13: Allocations a and b in Case 1.2.1.



Case 1.2.2: $|\{\theta^*_{i2}(i), \theta^*_{i2}(d_{i1}), \theta^*_{i2}(d_{i2})\}| = 2$.
Case 1.2.2.1: $\theta^*_{i2}(d_{i1}) \neq \theta^*_{i2}(d_{i2})$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, I), \theta^*_i(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. We consider the following three cases separately.
Case 1.2.2.1.1: $y \not\trianglerighteq_1 x$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (x, I) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, III) \end{cases} \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, III) \end{cases} \begin{cases} \theta_k(k) = (AB, III) \\ \theta_k(d_{k1}) = (O, I) \\ \theta_k(d_{k2}) = (O, I) \end{cases}$$

$$\begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

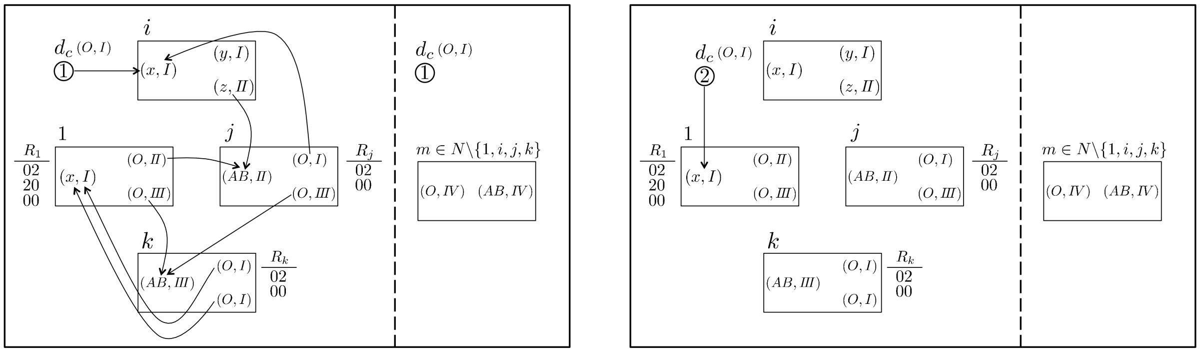
R_1	02	20	00	$\cdots$
R_j	02	00	$\cdots$	
R_k	02	00	$\cdots$	

Claim 1.2.2.1.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, j, k\}, \\ X^{11}_m(\theta^*_i, \theta_{-i}) \text{ if } m = i, \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j, k\}. \end{cases}$$

Proof. Since the allocation in the left hand side of Figure 14 is in $\mathcal{I}^Y(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}) = \Phi^Y_0(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since a_1 is 02, d_{11} and d_{12} donate to a patient respectively (Proposition 1). Since only patients j and k can receive d_{11}'s and d_{12}'s donation respectively, $d_{11} \in a^L_j$ and $d_{12} \in a^L_k$. Since the only acceptable transplantation

Figure 14: Allocations a and b in Case 1.2.2.1.1.



type for j and k is 02, $a_j \in X_j^{02}(\theta_i^*, \theta_{-i})$ and $a_k \in X_k^{02}(\theta_i^*, \theta_{-i})$. Thus $a_j^L = \{d_{11}, d_{i2}\}$ and $a_k^L = \{d_{12}, d_{j2}\}$. Since a_j and a_k are 02, d_{j1}, d_{k1} and d_{k2} donate to a patient respectively (Proposition 1). Since two of them donate to patient 1, the remaining one donates to patient i (Recall that $y \not\trianglerighteq_1 x$). Since no other living donor is compatible with patient i, $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. $\square$

Claim 1.2.2.1.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.2.1.1a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Thus each patient in $N\setminus\{1\}$ cannot use a graft from d_c. Thus b_m is not $20, 10$ or 11 for each $m \in N\setminus\{1\}$. Since patient k cannot find two compatible living donors in $D^L\setminus D_1^L$, $b_k \notin X_k^{02}(\theta_i^*, \theta_{-i})$. Thus $b_k = (0, \emptyset)$. Since patient $m \in N\setminus\{1, k\}$ cannot find two compatible living donors in $D^L\setminus(D_1^L \cup D_k^L)$, $b_m \notin X_m^{02}(\theta_i^*, \theta_{-i})$. Thus $b_m = (0, \emptyset)$. $\square$

Case 1.2.2.1.2: $y = x$. Note that $y = x$ implies that $\theta_i^*(i) = (x, I) = (y, I) = \theta_i^*(d_{i1})$. Thus, by Assumption 5, this case is excluded.

Case 1.2.2.1.3: $y \trianglerighteq_1 x$ and $y \neq x$. Note that $y \trianglerighteq_1 x$ and $y \neq x$ imply that the combination of x and y is one of the following five: $(O, A), (O, B), (O, AB), (A, AB), (B, AB)$. Note also that each of them satisfies $x \not\trianglerighteq_1 y$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\left\{\theta_{d_c} = \Big(2, (O, I)\Big)\right. \quad \begin{cases} \theta_1(1) = (y, I) \\ \theta_1(d_{11}) = (O, III) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, III) \\ \theta_j(d_{j1}) = (x, I) \\ \theta_j(d_{j2}) = (y, I) \end{cases} \quad \begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

$$\begin{array}{r|llll} R_1 & 02 & 20 & 00 & \cdots \\ R_j & 02 & 00 & \cdots & \end{array}$$

Claim 1.2.2.1.3a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, \{d_{i1}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{j1}\}) \text{ if } m = i, \\ (0, \{d_{11}, d_{12}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

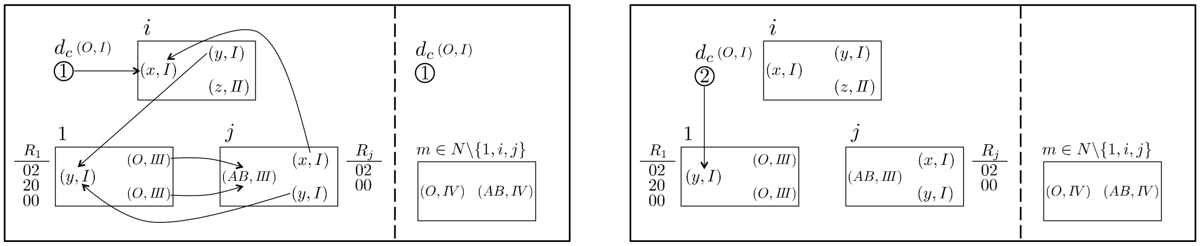
Proof. Since the above allocation is in $\mathcal{I}^Y\left(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}\right) = \Phi^Y_0\left(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}\right)$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since only d_{i1} and d_{j2} are compatible with patient 1, $a_1 = (0, \{d_{i1}, d_{j2}\})$ (Recall that $x \ntrianglerighteq_1 y$). Since a_1 is 02, d_{11} and d_{12} donate to a patient respectively (Proposition 1). Since only patients j can receive d_{11}'s and d_{12}'s donation, $a_j = (0, \{d_{11}, d_{12}\})$. Since a_j is 02, d_{j1} donates to a patient (Proposition 1). Since patient 1 cannot receive d_{j1}'s donation, it goes to i, i.e., $d_{j1} \in a^L_i$. Since no other living donor is compatible with patient i, $a_i = (1, \{d_{j1}\})$. □

Claim 1.2.2.1.3b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.2.1.3a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Thus each patient in $N\setminus\{1\}$ cannot use a graft from d_c. Thus b_m is not $20, 10$ or 11 for each $m \in N\setminus\{1\}$. Since patient j cannot find two compatible living donors in $D^L\setminus D^L_1$, $b_j \notin X^{02}_j(\theta^*_i, \theta_{-i})$. Thus $b_j = (0, \emptyset)$. Since patient $m \in N\setminus\{1, j\}$ cannot find two compatible living donors in $D^L\setminus(D^L_1 \cup D^L_j)$, $b_m \notin X^{02}_m(\theta^*_i, \theta_{-i})$. Thus $b_m = (0, \emptyset)$. □

Figure 15: Allocations a and b in Case 1.2.2.1.3.



Case 1.2.2.2: $\theta^*_{i2}(d_{i1}) = \theta^*_{i2}(d_{i2})$. Without loss of generality, let $\theta^*_i(i) = (x, I), \theta^*_i(d_{i1}) = (y, II), \theta^*_i(d_{i2}) = (z, II)$, where $x, y, z \in \mathcal{B}$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\left\{\theta_{d_c} = \Big(2, (O, I)\Big)\right. \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, I) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, III) \end{cases} \quad \begin{cases} \theta_k(k) = (AB, III) \\ \theta_k(d_{k1}) = (O, II) \\ \theta_k(d_{k2}) = (O, I) \end{cases}$$

$$\begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

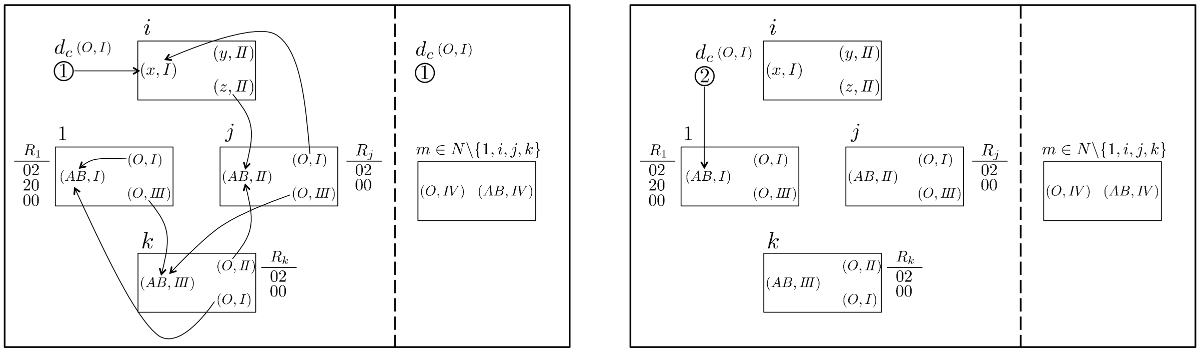
R_1	02	20	00	$\cdots$
R_j	02	00	$\cdots$	
R_k	02	00	$\cdots$	

Claim 1.2.2.2a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X^{02}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in \{1, j, k\} \\ X^{11}_m(\theta^*_i, \theta_{-i}) \text{ if } m = i \\ X^{00}_m(\theta^*_i, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j, k\}. \end{cases}$$

Proof. Since the allocation in the left hand side of Figure 16 is in $\mathcal{I}^Y(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}) = \Phi^Y_0(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since a_1 is 02, d_{12} donates to a patient (Proposition 1). Since only patients k can receive d_{12}'s donation, $d_{12} \in a^L_k$. Since the only acceptable transplantation type for patient k is 02, $a_k \in X^{02}_k(\theta^*_i, \theta_{-i})$. Thus $a^L_k = \{d_{12}, d_{j2}\}$. Since d_{j2} donates to patient k, patient j receives an acceptable transplant, i.e. $a_j \in X^{02}_j(\theta^*_i, \theta_{-i})$. Thus a^L_j consists of d_{k1} and one of d_{i1} and d_{i2} (Note that if $a^L_j = D^L_i$,

Figure 16: Allocations a and b in Case 1.2.2.2.



then d_{k1} cannot donate to any patient). Since a_1, a_j and a_k are 02, d_{11}, d_{j1} and d_{k2} donate to a patient respectively (Proposition 1). Since two of them donate to patient 1, the remaining one donates to patient i, i.e., a_i uses a living donor. Since no other living donor is compatible with patient i, $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. □

Claim 1.2.2.2b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.2.2a, we can show that b_1 is 02 only if b_i is 11. However, since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Thus each patient in $N\backslash\{1\}$ cannot use a graft from d_c. Thus b_m is not 20, 10 or 11 for each $m \in N\backslash\{1\}$. Since patient k cannot find two compatible living donors in $D^L\backslash D_1^L$, $b_k \notin X_k^{02}(\theta_i^*, \theta_{-i})$. Thus $b_k = (0, \emptyset)$. Since patient j cannot find two compatible living donors in $D^L\backslash(D_1^L \cup D_k^L)$, $b_j \notin X_j^{02}(\theta_i^*, \theta_{-i})$. Thus $b_j = (0, \emptyset)$. Since patient $m \in N\backslash\{1, j, k\}$ cannot find two compatible living donors in $D^L\backslash(D_1^L \cup D_j^L \cup D_k^L)$, $b_m \notin X_m^{02}(\theta_i^*, \theta_{-i})$. Thus $b_m = (0, \emptyset)$. □

Case 1.2.3: $|\{\theta_{i2}^*(i), \theta_{i2}^*(d_{i1}), \theta_{i2}^*(d_{i2})\}| = 1$. Without loss of generality, let $\theta_i^*(i) = (x, I), \theta_i^*(d_{i1}) = (y, I), \theta_i^*(d_{i2}) = (z, I)$, where $x, y, z \in \mathcal{B}$. By the definition of Θ_i, at least one of d_{i1} and d_{i2} is not compatible with patient i, i.e., $y \ntrianglerighteq_1 x$ or $z \ntrianglerighteq_1 x$. Without loss of generality, assume that $y \ntrianglerighteq_1 x$. Thus $x \neq AB$. By Lemma 3, $x \ntrianglerighteq_1 y$ or $x \ntrianglerighteq_1 z$. Thus $x \neq O$. Summing up, we have $x \in \{A, B\}$. Without loss of generality, we assume $x = A$ till the end of Case 1.2.3.[43]

Note that since $y \ntrianglerighteq_1 x$, $y \in \{B, AB\}$. Moreover, we have the following two claims that

[43]The same argument works for the case with $x = B$ by replacing A with B and B with A in the proof given here.

narrow down the combination of x, y and z.

Claim 1.2.3: The combination of x, y and z, written as (x, y, z), is one of the following five: (i) (A, B, O), (ii) (A, B, B), (iii) (A, B, AB), (iv) (A, AB, O), and (v) (A, AB, B).

Proof. First, we show that $y = B$ or $z \in \{O, B\}$ by contradiction. Suppose to the contrary that $y \neq B$ and $z \notin \{O, B\}$. Since $y \in \{B, AB\}$, $y = AB$. Since $z \in \mathcal{B}\backslash\{O, B\} = \{A, AB\}$, we have $x = A \trianglerighteq_1 AB = y$ and $x = A \trianglerighteq_1 z$, contradicting Lemma 3.

Now we complete the proof of Claim 1.2.3. Note that $x = A$ and $y \in \{B, AB\}$. First consider the case with $y = B$. Since $z = A$ is impossible by Assumption 5, we have (i), (ii), and (iii). Next consider the case with $y = AB$. By the fact shown in the previous paragraph, we have $z \in \{O, B\}$. Thus we have (iv) and (v). □

We omit the proof for the case (v) because it is same as the one for case (iii). Let us consider the following two cases of 1.2.3.1 and 1.2.3.2 separately.

Case 1.2.3.1: (x, y, z) is $(i)(A, B, O)$ or $(iv)(A, AB, O)$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D^L_m$.

$$\Big\{\theta_{d_c} = \Big(1, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (O, I) \end{cases} \quad \begin{cases} \theta_j(j) = (O, I) \\ \theta_j(d_{j1}) = (A, I) \\ \theta_j(d_{j2}) = (O, II) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}$$

- The definition of R_{-i}.

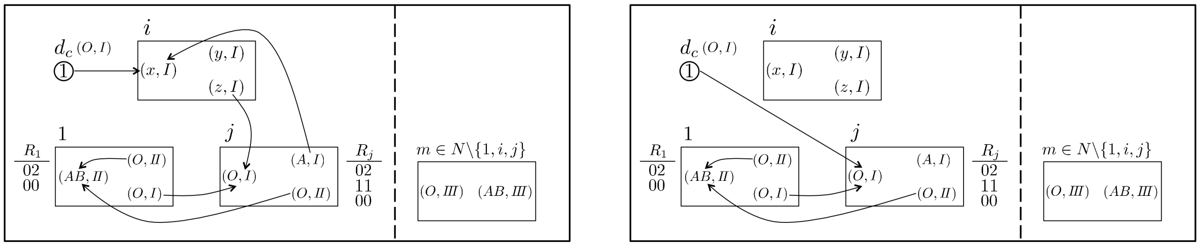
$$\begin{array}{c|cccc} R_1 & 02 & 00 & \cdots & \\ R_j & 02 & 11 & 00 & \cdots \end{array}$$

Claim 1.2.3.1a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{j1}\}) \text{ if } m = i, \\ (0, \{d_{12}, d_{i2}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1, i, j\}. \end{cases}$$

Proof. Since the allocation in the left hand side of Figure 17 is in $\mathcal{I}^Y(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i}) = \Phi^Y_0(\theta_{d_c}; (R^*_i, \theta^*_i); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X^{02}_1(\theta^*_i, \theta_{-i})$. Since only d_{11} and d_{j2} are compatible with patient 1, $a_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that a^L_j contains at least one living donor, i.e., a_j is 11 or 02.

Figure 17: Allocations a and b in Case 1.2.3.1.



Since $\theta_{d_c q} = 1$, a_i cannot be 20. Moreover, a_i cannot be 10 since it implies that a_j is not 11 ($\because$ patient j cannot use the graft from d_c) and not 02 ($\because$ patient j cannot receive a donation from d_{i2} by Proposition 1). Moreover, a_i cannot be 02 since it implies that d_{i1} who has no compatible patient donates to a patient. Moreover, a_i cannot be 00 ($\because$ Since 11 is acceptable at R_i^*, the allocation described in the left hand side of Figure 17 excludes the allocations that assign $(0, \emptyset)$ to patient i). Summing up, $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. The allocation described in the left hand side of Figure 17 enable patient j to receive 02 under the condition that $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$ and $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. Thus a_j is 02. Thus $a_j = (0, \{d_{12}, d_{i2}\})$ ($\because$ Only d_{12} and d_{i2} are compatible living donors with patient j). Thus $a_i = (1, \{d_{j1}\})$. □

Claim 1.2.3.1b: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{12}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\backslash\{1, j\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.3.1a, we can show that $b_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that b_j^L contains at least one living donor, i.e., b_j is 11 or 02.

In a similar manner to Claim 1.2.3.1a, we can show that b_i is not $20, 10$ or 02. Moreover, since 11 is unacceptable at R_i, b_i is not 11. Thus $b_i = (0, \emptyset)$. Since patient j cannot find compatible living donors in $D^L \backslash D_1^L$, b_j is not 02. Thus b_j is 11 since the allocation described in the right hand side of Figure 17 is available. Since it is the only allocation that assigns 02 to patient 1 and 11 to patient j, we are done. □

Case 1.2.3.2: (x, y, z) is $(ii)(A, B, B)$ or $(iii)(A, B, AB)$. Define $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\backslash\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D_m^L$.

$$\left\{\theta_{d_c} = \Big(1, (O, I)\Big)\right. \quad \begin{cases} \theta_1(1) = (AB, II) \\ \theta_1(d_{11}) = (O, II) \\ \theta_1(d_{12}) = (B, I) \end{cases} \quad \begin{cases} \theta_j(j) = (B, I) \\ \theta_j(d_{j1}) = (A, I) \\ \theta_j(d_{j2}) = (O, II) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}$$

- The definition of R_{-i}.

R_1	02	00	$\cdots$	
R_j	02	11	00	$\cdots$

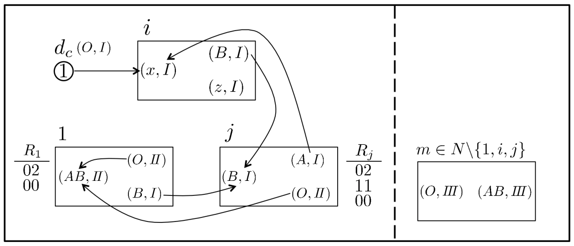
Claim 1.2.3.2a: For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

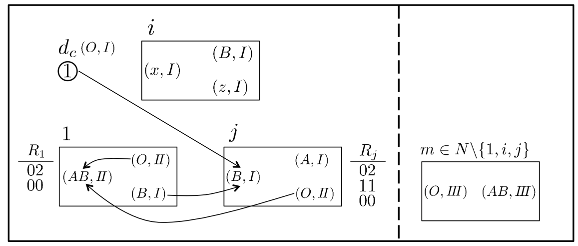
$$a_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{j1}\}) \text{ if } m = i, \\ (0, \{d_{12}, d_{i1}\}) \text{ if } m = j \text{ and } (iii) \text{ holds}, \\ (0, \{d_{12}, d_{i1}\}) \text{ or } (0, \{d_{12}, d_{i2}\}) \text{ if } m = j \text{ and } (ii) \text{ holds}, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, i, j\}. \end{cases}$$

Proof. Since the allocation in the left hand side of Figure 18 is in $\mathcal{I}^Y\left(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}\right) = \Phi_0^Y\left(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}\right)$, $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$. Since only d_{11} and d_{j2} are compatible with patient 1, $a_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that a_j^L contains at least one living donor, i.e., a_j is 11 or 02.

Since $\theta_{d_cq} = 1$, a_i cannot be 20. Moreover, a_i cannot be 10 since it implies that a_j is not 11 ($\because$ patient j cannot use the graft from d_c) and not 02 ($\because$ patient j cannot receive a donation from a patient in D_i^L by Proposition 1). Moreover, a_i cannot be 02 since it implies that both d_{i1} and d_{i2} donate to a patient respectively. Note that d_{12} also donates to a patient since a_1 is 02. However, since the economy can receive donation from at most two of d_{i1}, d_{i2} and d_{12}, one of d_{i1} and d_{i2} cannot donate any patient. Thus a_i is not 02. Moreover, a_i cannot be 00 ($\because$ Since 11 is acceptable at R_i^*, the allocation described in the left hand side of Figure 18 excludes the allocations that assign $(0, \emptyset)$ to patient i). In sum, $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. The allocation described in the left hand side of Figure 18 enable patient j to receive 02 under the condition that $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$ and $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. Thus a_j is 02. Thus a_j^L consists of d_{12} and a donor in D_i^L ($\because$ Donors in D_j^L are not compatible with patient j). Thus $a_i = (1, \{d_{j1}\})$. $\square$

Figure 18: Allocations a and b in Case 1.2.3.2.





Claim 1.2.3.2b: For the above $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (0, \{d_{11}, d_{j2}\}) \text{ if } m = 1, \\ (1, \{d_{12}\}) \text{ if } m = j, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, j\}. \end{cases}$$

Proof. In a similar manner to Claim 1.2.3.2a, we can show that $b_1 = (0, \{d_{11}, d_{j2}\})$. Note that this implies that b_j^L contains at least one living donor, i.e., b_j is 11 or 02.

In a similar manner to Claim 1.2.3.2a, we can show that b_i is not $20, 10$ or 02. Moreover, since 11 is unacceptable at R_i, b_i is not 11. Thus $b_i = (0, \emptyset)$. Since patient j cannot find compatible living donors in $D^L \setminus D_1^L$, b_j is not 02. Thus b_j is 11 since the allocation described in the right hand side of Figure 18 is available. Since it is the only allocation that assigns 02 to patient 1 and 11 to patient j, we are done. □

Case 2: Patient i has one living donors, i.e., $|D_i^L| = 1$. Let $D_i^L = \{d_i\}$. Note that, by Proposition 1, patient i never receives a living-donor dual-graft transplant at any profile in $\mathcal{R}^N \times \Theta$. Thus, (10) holds.

Case 2.1: $|\{\theta_{i2}^*(i), \theta_{i2}^*(d_i)\}| = 2$. Without loss of generality, let $\theta_i^*(i) = (x, I)$ and $\theta_i^*(d_i) = (y, II)$, where $x, y \in \mathcal{B}$. Define $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

- The definition of θ_{-i}: Let $m \in N \setminus \{1, i, j\}$ and $d \in D_m^L$.

$$\Big\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (AB, I) \\ \theta_1(d_{11}) = (O, I) \\ \theta_1(d_{12}) = (O, III) \end{cases} \quad \begin{cases} \theta_j(j) = (AB, II) \\ \theta_j(d_{j1}) = (O, I) \\ \theta_j(d_{j2}) = (O, III) \end{cases} \quad \begin{cases} \theta_k(k) = (AB, III) \\ \theta_k(d_{k1}) = (O, II) \\ \theta_k(d_{k2}) = (O, I) \end{cases}$$

$$\begin{cases} \theta_m(m) = (O, IV) \\ \theta_m(d) = (AB, IV) \end{cases}$$

- The definition of R_{-i}.

R_1	02	20	00	$\cdots$
R_j	02	00	$\cdots$	
R_k	02	00	$\cdots$	

<u>Claim 2.1a:</u> For the above $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m \in \begin{cases} X_m^{02}(\theta_i^*, \theta_{-i}) \text{ if } m \in \{1, j, k\}, \\ X_m^{11}(\theta_i^*, \theta_{-i}) \text{ if } m = i, \\ X_m^{00}(\theta_i^*, \theta_{-i}) \text{ if } m \in N\setminus\{1, i, j, k\}. \end{cases}$$

Proof. Since the allocation in the left hand side of Figure 19 is in $\mathcal{I}^Y(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}) = \Phi_0^Y(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i})$, $a_1 \in X_1^{02}(\theta_i^*, \theta_{-i})$. Since a_1 is 02, d_{12} donates to a patient (Proposition 1). Since only patients k can receive d_{12}'s donation, $d_{12} \in a_k^L$. Since the only acceptable transplantation type for patient k is 02, $a_k \in X_k^{02}(\theta_i^*, \theta_{-i})$. Thus $a_k^L = \{d_{12}, d_{j2}\}$. Since d_{j2} donates to patient k, patient j receives an acceptable transplant, i.e. $a_j \in X_j^{02}(\theta_i^*, \theta_{-i})$. Thus $a_j^L = \{d_i, d_{k1}\}$ ($\because$ Only d_i and d_{k1} are compatible with patient j). Thus patient i receives a donation from a living donor, i.e., $a_i \in X_i^{11}(\theta_i^*, \theta_{-i})$. $\square$

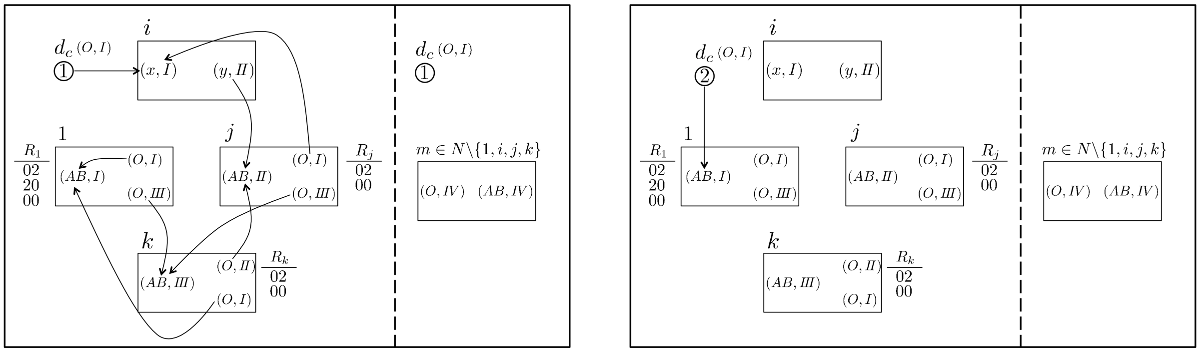
<u>Claim 2.1b:</u> For the above $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1\}. \end{cases}$$

Proof. In a similar manner to Case 2.1a, we can show that b_1 is 02 only if b_i is 11. Since 11 is unacceptable at R_i, b_1 is not 02. Thus $b_1 = (2, \emptyset)$. Since b_1 does not use a living donor, d_{11} and d_{12} do not donate to any patient. Since patient k cannot find two compatible living donors in $D^L\setminus D_1^L$, b_k is not 02. Thus $b_k = (0, \emptyset)$. Since b_k does not use a living donor, d_{k1} and d_{k2} do not donate to any patient. Since patient j cannot find two compatible living donors in $D^L\setminus(D_1^L \cup D_k^L)$, b_j is not 02. Thus $b_j = (0, \emptyset)$. Since patient i cannot use a graft from d_c, b_i is not 20, 10 or 11. $\square$

Case 2.2: $|\{\theta_{i2}^*(i), \theta_{i2}^*(d_i)\}| = 1$. Without loss of generality, let $\theta_i^*(i) = (x, I)$ and $\theta_i^*(d_i) = (y, I)$, where $x, y \in \mathcal{B}$. Note that $x \not\trianglerighteq_1 y$ by Lemma 3. Define $(R_{-i};\theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$ as follows:

Figure 19: Allocations a and b in Case 2.1.



- The definition of θ_{-i}: For $m \in N \setminus \{1, i, j\}$ and $d \in D_m^L$,

$$\left\{\theta_{d_c} = \Big(2, (O, I)\Big) \quad \begin{cases} \theta_1(1) = (y, I) \\ \theta_1(d_{11}) = (x, I) \\ \theta_1(d_{12}) = (O, II) \end{cases} \quad \begin{cases} \theta_m(m) = (O, III) \\ \theta_m(d) = (AB, III) \end{cases}\right.$$

- The definition of R_{-i}.

$$R_1 \;\Big|\; 11 \quad 20 \quad 00 \quad \cdots$$

<u>Claim 2.2a:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation a is, for each $m \in N$,

$$a_m = \begin{cases} (1, \{d_i\}) \text{ if } m = 1, \\ (1, \{d_{11}\}) \text{ if } m = i, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1, i\}. \end{cases}$$

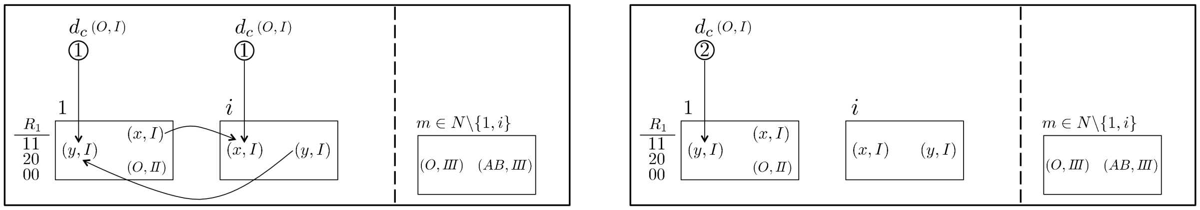
Proof. Since the above allocation is in $\mathcal{I}^Y\left(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}\right) = \Phi_0^Y\left(\theta_{d_c}; (R_i^*, \theta_i^*); (R_j, \theta_j)_{j\neq i}\right)$, $a_1 \in X_1^{11}(\theta_i^*, \theta_{-i})$. Since d_i is the only living donor compatible with patient 1, $a_1 = (1, \{d_i\})$. By Proposition 1, one of d_{11} and d_{12} donates to a patient. Since donor d_{12} has no compatible patient, d_{11} donates to patient i. Thus $a_i = (1, \{d_{11}\})$. □

<u>Claim 2.2b:</u> For the above $(R_{-i}; \theta_{-i}) \in \mathcal{R}^{N\setminus\{i\}} \times \Theta_{-i}$, allocation b is, for each $m \in N$,

$$b_m = \begin{cases} (2, \emptyset) \text{ if } m = 1, \\ (0, \emptyset) \text{ if } m \in N\setminus\{1\}. \end{cases}$$

Proof. Note that patient 1 cannot receive a hybrid transplant with own donor since $x \ntrianglerighteq_1 y$. Since 11 is unacceptable at R_i, b_1 is not 11. Thus $b_1 = (2, \emptyset)$. Since patient 1 uses two grafts

Figure 20: Allocations a and b in Case 2.2.



from d_c, patient i cannot use a graft from d_c. Thus b_i is not $20, 10$ or 11. Thus $b_i = (0, \emptyset)$.
□

Case 3: D_i^L contains no living donor. Note that, by Proposition 1, patient i never receives a living-donor dual-graft transplant or a hybrid transplant at any profile in $\mathcal{R}^N \times \Theta$. Thus, by Lemma 2, patient i cannot manipulate φ^P. □

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