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by

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Do outside options matter in matching? A new perspective on the trade-offs in student assignment*

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Abstract

This paper studies a general one-sided matching problem in which outside options do not necessarily exist. An important debate centers around whether it is possible to improve upon the Gale-Shapley student-proposing deferred acceptance mechanism (DA) via alternative strategy-proof mechanisms. In the current paper we introduce a new perspective on this debate by investigating the role of outside options in school choice and college admissions. We show that on a general domain of preferences where all students are able to credibly rank their exogenous or endogenous outside options, no strategy-proof mechanism improving upon DA exists. It is, however, possible to construct natural subdomains allowing for positive results, where some students' preferences are in part induced by an exogenous hierarchy of quality tiers. We then identify maximal domains on which it is possible to improve upon DA without sacrificing strategy-proofness. This result may help better assess the underpinnings of the three-way tension among efficiency, individual rationality/stability, and strategy-proofness in matching.

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Keywords: student-proposing deferred acceptance mechanism, strategy-proofness, Pareto dominance, outside options

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1 Introduction

In many economic contexts, canonical results implied by standard theory are overturned in the shadow of outside options. Examples abound. In auction theory, when buyers have outside options, the seller can exploit these so as to increase either revenue (Jehiel et al., 1996) or both revenue and efficiency (Figueroa and Skreta, 2009). The bargaining literature has long recognized the fundamental roles played by outside options (e.g., Nash, 1950, 1953) as they do not merely represent disagreement points, but can lead to inevitable inefficiencies (Compte and Jehiel, 2009) and constitute an additional source of learning a party's private information (Lee and Liu, 2013). In contract theory, standard properties of the optimal contract are invalidated when agents have type-dependent outside options (see, e.g., Jullien, 2000).[1]

In this paper we explore the role of outside options in one-sided matching problems within the context of school choice and college admissions. The past two decades have witnessed a rapid spread in the implementation of school choice, both within the US and globally, as countless school districts have replaced existing residence-based assignment plans with choice programs. Meanwhile, large centralized college admissions systems have been put in place in many countries including China, Russia, Turkey, Greece, India, and South Korea, where students are prioritized in various categories based on a standardized test.

In the context of school choice, outside options may be limited only to a subset of students. Many if not all families are unable to afford private schools; nor do they have the time or the means for supervising their children's education at home. Therefore, outside options such as private education and home schooling are fundamentally different from standard schooling options, which are accessible to all families within the public education system. Hence, an outside option, if one exists, can be family-specific and can in turn alter the student's strategy set. For example, it is easy to see the effect of outside options when implementing a mechanism, such as the much-debated Boston mechanism,[2] that incentivizes strategic play. A family that is able to afford private education can take a higher risk by aiming at a popular school when choosing its first choice, as opposed to another family without an outside option, which will cautiously stick to its safety school as first choice.[3] In the context of college admissions, exercising the outside option simply implies

[1]In principal-agent models of adverse selection, agents' outside options equip the principal with an additional screening tool, which in turn enhances the efficiency of the optimal contract (Rasul and Sonderegger, 2010). In monopoly pricing theory, the famous Coase conjecture fails when buyers are assumed to have private outside options they can choose to exercise (Board and Pycia, 2014).

[2]The Boston mechanism assigns students lexicographically according to first choice and then priority. Accordingly, the emphasis of the mechanism on first choices incentivizes students to be highly strategic when deciding which school to rank first. For an in-depth game-theoretic and axiomatic analysis of this assignment procedure, see Ergin and Sönmez (2006), Pathak and Sönmez (2008), and Kojima and Ünver (2014).

[3]Indeed, in a recent empirical study of student assignment in Barcelona where the Boston mechanism is in use, Calsamiglia and Güell (2014) report that 14% of students who fail to get into their first-ranked schools go to private school, although only 4% of schools in Barcelona are private. They argue that the availability of outside options for some, but not all, families leads to important inequalities and may significantly diminish any possible gains from school choice. In the US, available data indicate that only a small percentage of participants in NYC and Boston

remaining unassigned to a college (even to a private one) and its cost can be formidably high. In Chinese college admissions, for example, being unassigned often entails repeating the senior year, re-taking the exam the following year, or foregoing higher education altogether (c.f. Chen and Kesten, 2015).

An ongoing debate in the market design literature concerns how to choose the "right" student assignment mechanism based on the three-way tension among incentives, stability, and welfare. In matching problems, if outside options are allowed, the stability requirement encompasses individual rationality along with the no-blocking condition, i.e., that no student would prefer to be assigned to a school where he has higher priority than at least one student currently assigned to that school. In two-sided matching, however, strategy-proofness and stability are incompatible (Roth, 1982). Moreover, there is no strategy-proof, efficient, and individually rational mechanism (Alcalde and Barbera, 1994). School-choice problems, often referred to as one-sided matching problems for lack of schools' preferences,[4] provide a new framework where individual rationality could matter both conceptually and technically.

Dominant-strategy implementation responds to the famous Wilson critique, articulated in Wilson (1987), by imposing the robustness requirement that each agent's strategy be optimal, not only against the actual (or equilibrium) strategies of other agents, but against all possible strategies of other agents. In this sense, a dominant-strategy incentive compatible, or *strategy-proof* mechanism is "detail free" and satisfies this strong form of robustness (Bergemann and Morris, 2005). A strategy-proof mechanism provides strategic simplicity by inducing the straightforward behavior of market participants while also offering a sense of fairness by leveling the playing field among sophisticated and naïve participants (Abdulkadiroğlu et al., 2006; Pathak and Sönmez, 2008). Much in line with these considerations, strategy-proofness has played a critical role in recent school-choice reforms in the US.

In their pioneering work, Abdulkadiroğlu and Sönmez (2003) unearthed an important incentive-related shortcoming of what has come to be known as the Boston mechanism and discussed alternative strategy-proof proposals that make truth-telling a dominant strategy for students. One of these proposals,[5] the Gale-Shapley student-proposing deferred acceptance mechanism (DA), has since become a central student assignment method for school choice both in theory and in practice.[6]

have reported singleton preference lists in recent years.

[4]Mathematically, school priorities are isomorphic to school preferences. However, economically, priorities do not represent schools' preferences over students since they are mandated by law based on families' residence area, sibling status, or other socioeconomic status. Therefore, priorities preclude welfare and incentive analysis on the school side.

[5]Their second proposal, the top trading cycles mechanism, which is also strategy-proof, is currently being implemented in New Orleans. See Kesten (2006) for a theoretical comparison of top trading cycles with respect to deferred acceptance.

[6]DA has long played a prominent role in two-sided matching markets well before it gained much acclaim in school choice. Probably the most well-known of these markets, the U.S. National Residency Matching Program, is generally considered the earliest account of the use of this assignment method in practice (see Roth 1984; Roth and Sotomayor 1990). Balinski and Sönmez (1999) and Guillen and Kesten (2012) show that equivalent versions of DA

In contrast to the Boston mechanism, DA assigns students *tentatively* based on preferences while using priorities to settle all shortages irrespective of preference rankings. This subtle but crucial difference endows DA with the superior incentive property. Thanks to the collaborative efforts of economists and officials, the New York City Department of Education as well as the Boston Public School system transitioned to new designs implementing DA for student assignment beginning in 2004 and 2006, respectively (Abdulkadiroğlu et al., 2005a,b). Abandoning its residence-based system because of major concerns about social diversity, France has also been implementing DA in all of its thirty districts since 2007 (Hiller and Tercieux, 2014). In the context of college admissions, Turkey and China have been implementing variants of DA within the last decade.

DA is not only strategy-proof (Dubins and Freedman, 1981; Roth, 1982) but also implements a *stable* matching. Given a strict priority structure, the outcome of DA is the most favorable to each student within the set of stable matchings, in the sense that each student's assignment at this matching is at least as good as that at any other stable matching (Gale and Shapley, 1962), i.e., it is the student-optimal stable matching. Because of this theoretical appeal, DA has remained as the cornerstone of the important debate concerning the trade-offs faced in school-choice mechanisms.

DA, however, is efficient only when constrained to the stable set. In general, its outcome may be inefficient even when school priorities are strict.[7] When priorities are coarse, it faces an additional welfare loss due to breaking ties in priorities, since DA operates on strict priorities only.[8] To overcome this shortcoming, mechanisms that Pareto improve upon DA have been proposed in the recent literature (see Erdil and Ergin, 2008; Kesten, 2010; Abdulkadiroğlu et al., 2015; Kesten and Ünver, 2015). Nevertheless, none of these mechanisms is strategy-proof. Differently put, these mechanisms Pareto dominate DA at the expense of its fundamental incentive property.[9] Then a natural question is whether or not strategy-proof mechanisms that improve upon DA exist, or, alternatively, whether or not DA is second-best incentive compatible? We show that the answer to this question may depend crucially on the role played by a student's outside option, should he have one.

Departing from the existing literature, where little attention has been paid to the availability of outside options, we argue in this paper that the extent to which students can exercise their outside options may significantly impact the three-way tension among incentives, efficiency, and individual rationality/stability in the school-choice setting. We study a general model of school choice possibly with coarse priority structures where outside options may or may not exist. We show that even when outside options are completely eliminated, if students are unconstrained in

are currently in use respectively for college admissions in Turkey and on-campus dormitory assignments at MIT.

[7]Kesten (2010) provides worst-case scenarios where DA assigns each student to either his worst choice or his second-worst choice in problems of arbitrary size.

[8]Working with the NYC data after the transition to DA, Erdil and Ergin (2008) quantify this second source of welfare loss and show that it is possible to strictly improve the assignment of around 5% of students without hurting any others.

[9]Erdil and Ergin (2008) and Kesten (2010) show that their proposals gain strategic immunity in low-information environments, whereas Kesten and Ünver (2015) establish that their proposal is strategy-proof in a large market.

their strategic rankings of schools, then one must necessarily forgo strategy-proofness to dominate DA (Theorem 1). In establishing this result, we generalize and unify two independent impossibility results by Kesten (2010) and Abdulkadiroğlu et al. (2009), while also correcting an erroneous claim in the latter paper.[10,11]

The rest of the paper is organized as follows: Section 2 gives the formal model and the description of the DA algorithm. Section 3 provides the impossibility result on the full domain and Section 4 the possibility result on a specific subdomain. Section 6 concludes. The technical proofs are relegated to the Appendix.

2 The Model

We consider a general school-choice model with or without outside options. A school-choice problem, or a **problem** for short, is a five-tuple $(I, S, (q_s)_{s\in S}, (P_i)_{i\in I}, (\succeq_s)_{s\in S})$. I is a finite set of students and S is a finite set of schools. Each school $s \in S$ has q_s available seats, or **capacity**, where $q_s \geq 1$. We assume throughout the paper that the total number of seats is no fewer than the number of students, i.e., $|I| \leq \sum_{s\in S} q_s$. If q_s is large enough, say $q_s = |I|$, then school s represents an outside option such as private or home schooling. We call such a school **the outside option** and denote it by o. We do not necessarily assume the existence of the outside option.[12] Each student $i \in I$ has a strict preference relation P_i on S. Let R_i denote the at-least-as-good-as relation associated with P_i. We denote by $\mathcal{P}$ the set of all strict preference relations on S. Given a student $i \in I$, a set $\mathcal{D}_i \subseteq \mathcal{P}$ denotes the set of all admissible preferences on S. A preference profile $P := (P_i)_{i\in I} \in \mathcal{P}^I$ specifies a strict preference for each student $i \in I$. A **(preference) domain** is a set $\mathcal{D} := \times_{i\in I}\mathcal{D}_i \subseteq \mathcal{P}^I$. We say that $\mathcal{D}$ is the **full domain** if $\mathcal{D} = \mathcal{P}^I$. For each $P \in \mathcal{D}$, each $i \in I$, and each $P'_i \in \mathcal{D}_i$, we denote $P_{-i} := (P_j)_{j\in I\setminus\{j\}}$, and by (P'_i, P_{-i}) the profile obtained by replacing

[10] In a model where schools have strict priorities and students do not necessarily have outside options, Kesten (2010) showed that no efficient and strategy-proof mechanism dominates DA. In a two-sided matching model allowing for weak priorities, Abdulkadiroğlu et al. (2009) (henceforth APR) showed that no strategy-proof mechanism dominates DA. APR claimed that the latter result is tighter than the former. We note that this claim does not hold because Kesten's model cannot be embedded into the APR framework, nor is the proof technique used by APR applicable in this setting. More specifically, contrary to Kesten, APR assume that every student has some outside option and is able to submit singleton preference lists that declare only a single school as acceptable.

[11] For an assignment model that also assumes outside options, Erdil (2014) shows that no strategy-proof mechanism dominating a non-wasteful and strategy-proof mechanism exists. We note, however, that that result also no longer holds once outside options are ruled out. Consider, for example, a simple assignment setting with n agents and n objects. Clearly, any constant assignment mechanism is non-wasteful and strategy-proof. However, such a mechanism is dominated by the corresponding core mechanism, which is also strategy-proof. See also Anno and Kurino (2014) for an extension of this impossibility result to multiple-type markets with outside options.

[12] The school-choice model concerns the allocation of indivisible goods and the existence of outside options is not necessarily assumed in this literature (see Sönmez and Ünver, 2011, for a recent survey). On the other hand, the literature on the two-sided matching model typically assumes outside options, which are represented as being assigned with oneself (see Roth and Sotomayor, 1990, for a survey). Because under strict priorities a school-choice problem is isomorphic to a two-sided matching problem, it inherits much in modeling and results. This might be a reason for the modeling differences in outside options.

P_j with P'_j in P. We assume that each school s has a weak priority $\succeq_s$ on I that is a complete and transitive binary relation on I. We say that a priority $\succeq_s$ is strict if it is an antisymmetric weak priority. Let $\succ_s$ represent the asymmetric part of $\succeq_s$. We denote by $\mathcal{A}$ the set of all weak priorities on I. A priority profile $\succeq := (\succeq_s)_{s\in S} \in \mathcal{A}^S$ specifies a weak priority for each school $s \in S$. For convenience we fix $I, S, (q_s)_{s\in S}$ throughout the paper and denote a problem by a pair $(P, \succeq)$.

A **matching** is a correspondence $\mu : I \cup S \to S \cup I$ such that each student is assigned only one school and each school is assigned students up to its capacity, i.e., for each $i \in I$ and each $s \in S$, $\mu(i) \subseteq S$, $\mu(s) \subseteq I$, $|\mu(i)| = 1$, $|\mu(s)| \leq q_s$, and $i \in \mu(s) \Leftrightarrow s \in \mu(i)$. Since $\mu(i)$ is a singleton, we denote $\mu(i) = s$ instead of $\mu(i) = \{s\}$. Let $\mathcal{M}$ be the set of all matchings.

Let a problem $(P, \succeq)$ be given. A matching is **non-wasteful** at P if for each $i \in I$ and each $s \in S$, $s\ P_i\ \mu(i)$ implies $|\mu(s)| = q_s$. A matching μ **dominates** matching ν at P if for each $i \in I$, $\mu(i)\ R_i\ \nu(i)$, and for some $i \in I$, $\mu(i)\ P_i\ \nu(i)$. A matching is **Pareto efficient** at P if it is not dominated by any other matching at P. A pair $(i, s) \in I \times S$ **blocks** a matching μ at $(P, \succeq)$ if $s\ P_i\ \mu(i)$ and $\left[\text{either } |\mu(s)| < q_s \text{ or for some } j \in \mu(s),\ i \succ_s j\ \right]$. A matching is **stable** at $(P, \succeq)$ if it is not blocked by any pair at $(P, \succeq)$. Note that a stable matching at $(P, \succeq)$ is non-wasteful at P. A stable matching is a **student-optimal stable matching** at $(P, \succeq)$ if it is not dominated at P by any other matching that is stable at $(P, \succeq)$.

A **mechanism** is a function $\varphi : \mathcal{D} \times \mathcal{A}^S \to \mathcal{M}$ that maps each problem to a matching. Denote by $\varphi_i(P, \succeq)$ the school that is matched to i by φ at problem $(P, \succeq)$. Similarly, denote by $\varphi_s(P, \succeq)$ the set of students that are matched to s by φ. For notational simplicity, given $j_1, \ldots, j_m \in I$ and $s_1, \ldots, s_m \in S$, we denote $\varphi_{(j_1,\ldots,j_m)}(P, \succeq) = (s_1, \ldots, s_m)$ when $\varphi_{j_1}(P, \succeq) = s_1$, ..., $\varphi_{j_m}(P, \succeq) = s_m$. Similarly, given $J_1, \ldots, J_m \subseteq I$, $\varphi_{J_1,\ldots,J_m}(P, \succeq) = (s_1, \ldots, s_m)$ when $\varphi_{(j_1,\ldots,j_m)}(P, \succeq) = (s_1, \ldots, s_m)$ for all $(j_1, \ldots, j_m) \in J_1 \times \ldots \times J_m$. Mechanism $\varphi : \mathcal{D} \times \mathcal{A}^S \to \mathcal{M}$ is **strategy-proof (on domain $\mathcal{D}$)** if for each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, each $i \in I$, and each $P'_i \in \mathcal{D}_i$, $\varphi_i(P, \succeq)\ R_i\ \varphi_i(P'_i, P_{-i}, \succeq)$. A mechanism φ **dominates** another mechanism ζ if (i) for each problem $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, $\varphi_i(P, \succeq)\ R_i\ \zeta_i(P, \succeq)$, and (ii) for some problem $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, $\varphi(P, \succeq)$ dominates $\zeta(P, \succeq)$ at P. A mechanism φ is **Pareto efficient** (**non-wasteful**) if for each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, matching $\varphi(P, \succeq)$ is Pareto efficient (non-wasteful) at P.

Abdulkadiroğlu and Sönmez (2003) advocated the use of Gale and Shapley's (1962) **student-proposing deferred acceptance (DA) algorithm** as a plausible assignment method in school choice. For a *strict* priority profile $\succeq$ and a preference profile P, the DA algorithm works as follows:

Step 1: Each student applies to her favorite school. Each school s tentatively assigns its seats to its applicants following the priority order $\succeq_s$. Any unassigned student is rejected.

In general,

Step k: Each student who was rejected at the previous step applies to her next favorite school. Each school s considers the students it has been holding together with its new applicants and tentatively assigns its seats following the priority order $\succeq_s$. Any unassigned student is rejected.

The algorithm terminates when no student remains unassigned. At this point all current assignments are final.

A **tie-breaker** for school s is an injective function $\tau_s : I \to \mathbf{N}$ associating $\succeq_s$ with a strict priority $\succeq_s^\tau$ as follows: $i \succ_s^\tau j \Leftrightarrow [(i \succ_s j)$ or $(i \succeq_s j,\ j \succeq_s i,$ and $\tau_s(i) < \tau_s(j))]$. A **tie-breaking rule** is a profile $\tau := (\tau_s)_{s \in S}$ of all schools' tie-breakers.

The **student-proposing deferred acceptance mechanism with a tie-breaking rule** $\boldsymbol{\tau}$, which is denoted by DA^τ, is the mechanism obtained by the student-proposing deferred acceptance algorithm acting on $(P, \succeq_S^\tau) \in \mathcal{D} \times \mathcal{A}^S$, where $\succeq_s^\tau$ is obtained from $\succeq_s$ by breaking ties using τ_s.

It is well known that if $\succeq$ is not strict, there might be multiple student-optimal stable matchings (cf. Erdil and Ergin 2008). But if it is strict, such a matching is unique and dominates any other stable matching (Gale and Shapley, 1962; Balinski and Sönmez, 1999). More precisely, when $\succeq$ is not strict, $DA^\tau(P, \succeq)$ is the unique student-optimal stable matching at $(P, \succeq^\tau)$ and dominates at P any other matching that is stable at $(P, \succeq^\tau)$. Furthermore, DA^τ is strategy-proof (Dubins and Freedman, 1981; Roth, 1982) and non-wasteful on $\mathcal{P}$ and thus on any domain $\mathcal{D}$.[13]

3 Impossibility on the Full Domain

We start our quest of understanding the role of outside options by searching for a strategy-proof mechanism that dominates the student-proposing deferred acceptance (DA) mechanism. The critical question we ask is "Is it possible to improve upon DA without losing its compelling incentive property, if students' outside options taken away?" We give a negative answer to this question and provide an impossibility result on the full preference domain, which generalizes two logically independent impossibilities obtained by Abdulkadiroğlu et al. (2009) [henceforth APR] and Kesten (2010).

Theorem 1. (Impossibility on the full domain) *No strategy-proof mechanism dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule on the full domain $\mathcal{P}^I$ whether an outside option exists or not.*

Corollary 1. (Kesten, 2010) *When priorities are strict, no strategy-proof and efficient mechanism dominates the student-proposing deferred acceptance mechanism on the full domain $\mathcal{P}^I$ whether an outside option exists or not.*

Corollary 2. (Abdulkadiroğlu et al., 2009) *When the outside option exists, no strategy-proof mechanism dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule on the full domain $\mathcal{P}^I$.*

[13] We note, however, that irrespective of the assumption of the existence of the outside option, the strategy-proofness of DA is lost when a quota is imposed on the length of students' preference lists. See Haeringer and Klijn (2009) for an equilibrium analysis of DA when students' choices are constrained in this manner.

Table 1: Preferences and matchings in Example 1

P_{j_1}	P_{j_2}	P_i	P'_{j_1}	P_{j_2}	P_i	$\succeq^{\tau}_{t_1}$	$\succeq^{\tau}_{t_2}$
$\overline{t_2}$	$\overline{t_1}$	t_2	t_2	$\overline{\underline{t_1}}$	$\overline{\underline{t_2}}$	j_1	j_2
$\underline{t_1}$	$\underline{t_2}$	t_1	$\overline{\underline{o}}$	t_1	t_1	i	i
o	o	$\underline{o}$	t_1	o	o	j_2	j_1

Two preference profiles and a priority profile after breaking a tie are shown for Example 1. The underlined (overlined) schools in the preference profiles are assigned to the corresponding students under DA (dominating mechanism φ).

To illustrate the difference between the setups of the two corollaries, consider, for example, a poor neighborhood where the total capacity is sufficient to serve the student body within the neighborhood ($\sum_{s \in S \setminus \{o\}} q_s > |I|$), but students have no outside options.[14] Notice that Corollary 2 is not applicable in this case. Moreover, Theorem 1 implies that the efficiency requirement in Corollary 1 can also be dropped.

It is worth emphasizing that Theorem 1 and Corollary 1 are not implied by the APR impossibility, i.e., Corollary 2. In their proof, APR, whose model assumes the existence of an outside option for each student, critically rely on the assumption that each student is able to submit a singleton preference list where exactly one school can be listed above the outside option. Before proving Theorem 1, we illustrate this point in the following example to help better understand the role of the presence of outside options as well as gain insight into some of the main ideas behind our proof.

Example 1. Let $I = \{i, j_1, j_2\}$ and $S = \{t_1, t_2, o\}$ with $q_{t_1} = q_{t_2} = 1$ where o stands for the outside option. Consider the preference profile P and the priority profile $\succeq$ shown in Table 1. The DA assignment of each student for this problem is indicated as the underlined school in the corresponding preference ranking in Table 1, e.g., student j_1 is assigned to school t_1, etc.

Note that this matching is not Pareto efficient at P, as students j_1 and j_2 can both be better off by swapping their DA assignments.[15] Consider the following mechanism φ which we call a DA mechanism with ex-post swapping:

$$\varphi_{(j_1,j_2,i)}(\hat{P}, \hat{\succeq}) \quad = \quad \begin{cases} (t_2, t_1, o) & \text{if } \hat{P} = P \text{ and } \hat{\succeq} = \succeq, \\ DA^{\tau}_{(j_1,j_2,i)}(\hat{P}, \hat{\succeq}) & \text{otherwise.} \end{cases}$$

[14] Another example of an assignment problem with no outside options is the assignment of senior students to laboratories in engineering faculties in many universities. In this case, while students can declare preference over laboratories, each student must be assigned to work in one of the available labs as a prerequisite for graduation.

[15] As shown in this construction, the DA outcome can be inefficient. Ergin (2002) identifies restrictions on priority structures to ensure the efficiency of DA. See also Kesten (2006) and Ehlers and Erdil (2010).

This exchanges the assignments between students j_1 and j_2 at $(P, \succeq)$ and coincides with DA^τ at any other problem. Each student's assignment under φ is indicated as the overlined school at the corresponding preference ranking in Table 1.

Now consider the student j_1's preference P'_{j_1} that upgrades the outside option o above her DA assignment, t_1, at P. In other words, P'_{j_1} is the singleton preference that declares only t_2 above o (Table 1). At $(P'_{j_1}, P_{-j_1}, \succeq)$, the DA assignment is Pareto efficient and coincides with the assignment by the dominating mechanism φ.

Under φ, student j_1 is assigned the outside option by reporting P'_{j_1}, whereas she is assigned to school t_2 by reporting P_{j_1}. Thus, when she has P'_{j_1} as her true preferences, student j_1 gets a better assignment by misreporting her preference. Thus the dominating mechanism φ is not strategy-proof. $\Diamond$

Although simple, Example 1 gives us much insight into the failure of any dominating mechanism to be strategy-proof. The point is that when a mechanism such as φ dominates DA^τ, a student, such as j_1 in the example, who can get a better school under the dominating mechanism, has the ability to upgrade the outside option above her DA assignment and declare only a single school as acceptable. However, such a strategy is not available in a general setting, which may not allow for the outside option. Recent student assignment data from Boston and New York City, for instance, also highlight that singleton preference lists are generally used by only a small fraction of students (Abdulkadiroğlu et al., 2006).[16]

To prove Theorem 1 we introduce several notions. The first is a notion of a "weakly underdemanded school," which can be seen as a counterpart to the outside option in the general setting where an outside option may not exist.[17]

Definition 1. A school s is **overdemanded at $(P, \succeq)$ under a matching μ** if there is a student $i \in I$ such that $s \; P_i \; \mu(i)$. A school s is **weakly underdemanded at $(P, \succeq)$ under a matching μ** if it is not overdemanded at $(P, \succeq)$ under μ, i.e., for each student $i \in I$, $\mu(i) \; R_i \; s$.

Note that in the DA algorithm, an overdemanded school rejects at least one applicant and thus must be at its full capacity at the DA assignment; on the other hand, a weakly underdemanded school accepts all its applicants and thus need not necessarily be at full capacity.

A set of schools $T^p \subseteq S$ is called **potentially overdemanded** if $\sum_{t \in T^p} q_t < |I|$. Moreover, a set of schools $T^{mp} \subseteq S$ is called **maximally potentially overdemanded with respect to T** if

[16]Since the mechanism in use in both cities is DA, one cannot necessarily infer that these students are being strategic. Rather, we interpret these fractions as lower bound on the proportion of families with outside options.

[17]Put differently, in our general model, a weakly underdemanded school is the analogue of the outside option (or, the option of being unassigned) in the APR model. Therefore, here too a student need not rank any schools below a weakly underdemanded school, as she would not need to rank any schools below the outside option in the APR model.

$$T^{mp} \in \arg\max_{T' \supseteq T} \{ \sum_{t \in T'} q_t \mid \sum_{t \in T'} q_t < |I| \}.$$

Intuitively, for any potentially overdemanded set, it is possible to construct problems where all schools in that set are overdemanded, e.g., when each student specifies only those schools from such a set as acceptable. Note that the outside option is never in T^p or T^{mp} if it exists. We continue with a series of observations.

Lemma 1. *Let τ be a tie-breaking rule, and $(P, \succeq) \in \mathcal{P}^I \times \mathcal{A}^S$.*

1. *When it is available, the outside option is weakly underdemanded at $(P, \succeq)$ under any non-wasteful matching, including $DA^\tau(P, \succeq)$.*
2. *Let T^{mp} be a maximally potentially overdemanded set with respect to any potentiall overdemanded set T, and $\emptyset \neq S' \subseteq S \setminus T^{mp}$. Then, at least one school among those in $T^{mp} \cup S'$ is weakly underdemanded at $(P, \succeq)$ under $DA^\tau(P, \succeq)$.*
3. *There is a weakly underdemanded school at $(P, \succeq)$ under $DA^\tau(P, \succeq)$.*

The proofs of all lemmas in this section are given in Appendix A. There might not be a weakly underdemanded school under a non-wasteful matching at some problem,[18] but there always exists a weakly underdemanded school under the DA matching. Since we focus on over(weakly under)demanded schools under the DA matching, we say that **a school is over(weakly under)demanded at $(P, \succeq; \tau)$** if it is over(weakly under)demanded at $(P, \succeq)$ under $DA^\tau(P, \succeq)$.

The following lemma says that if the DA matching is dominated by another matching and some student i is assigned a weakly underdemanded school under the DA matching, then she is assigned the same school under both of the matchings.

Lemma 2. *Let $(P, \succeq)$ be a problem and τ a tie-breaking rule. Suppose that a matching ν dominates $DA^\tau(P, \succeq)$ at $P \in \mathcal{P}$. For each $i \in I$, if school $DA^\tau_i(P, \succeq)$ is weakly underdemanded at $(P, \succeq; \tau)$ then $\nu(i) = DA^\tau_i(P, \succeq)$.*

In what follows we focus on a specific preference manipulation. To this end, we define some useful notions: For any $S' \subseteq S$, define a strict preference relation $P_i|_{S'}$ on S' if for all $s, s' \in S'$, $s' \, P_i|_{S'} \, s \Leftrightarrow s' \, P_i \, s$. Given preference P_i of student i and schools s^*, s^u with $s^* \; P_i \; s^u$, **preference P'_i upgrades s^u above s^* in P_i** if P'_i ranks s^u right above s^* and the relative ranking of the other schools stays the same, i.e., (i) $s^* \, P_i \, s^u$ and $s^u \, P'_i \, s^*$, (ii) there is no $s \in S$ with $s^u \; P'_i \; s \; P'_i \; s^*$, and (iii) $P_i|_{S \setminus \{s^u, s^*\}} = P'_i|_{S \setminus \{s^u, s^*\}}$.

[18] Consider the following problem where $S = \{s, t\}$, $q_s = q_t = 1$, $I = \{1, 2\}$, and $\succ$ is arbitrarily fixed. Preferences are $s \; P_1 \; t$ and $t \; P_2 \; s$. Then matching $\mu = \begin{pmatrix} 1 & 2 \\ t & s \end{pmatrix}$ is clearly non-wasteful at P. All schools are overdemanded at $(P, \succ)$ under μ.

Lemma 3. *Suppose that under DA, student i is assigned school s^* that is overdemanded at $(P, \succeq; \tau)$. Then there exist $P_i'' \in \mathcal{P}$ and a weakly underdemanded school s^u at $(P, \succeq; \tau)$ such that P_i'' upgrades s^u above s^* in P_i, $DA_i^\tau(P_i'', P_{-i}, \succeq) = s^u$, and s^u is weakly underdemanded at $(P_i'', P_{-i}, \succeq; \tau)$.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. Fix a tie-breaking rule τ. Suppose that a mechanism φ dominates DA^τ on the full domain $\mathcal{P}^I$. Then, there is a problem $(P, \succeq) \in \mathcal{P}^I \times \mathcal{A}^S$ such that for each $i \in I$, $\varphi_i(P, \succeq)\ R_i\ DA_i^\tau(P, \succeq)$ and for some $j \in I$, $\varphi_j(P, \succeq)\ P_j\ DA_j^\tau(P, \succeq)$. As $\varphi_j(P, \succeq) \neq DA_j^\tau(P, \succeq)$, by Lemma 2, school $DA_j^\tau(P, \succeq)$ is overdemanded at $(P, \succeq; \tau)$. Thus, by Lemma 3, there exist $P_j'' \in \mathcal{P}$ and a weakly underdemanded school s^u at $(P, \succeq; \tau)$ such that P_j'' upgrades s^u above $DA_j^\tau(P, \succeq)$ in P_j, and $s^u = DA_j^\tau(P_j'', P_{-j}, \succeq)$ is weakly underdemanded at $(P_j'', P_{-j}, \succeq; \tau)$. Then, since s^u is weakly underdemanded at $(P_j'', P_{-j}, \succeq; \tau)$, Lemma 2 implies $\varphi_j(P_j'', P_{-j}, \succeq) = DA_j^\tau(P_j'', P_{-j}, \succeq) = s^u$. Moreover, since $\varphi_j(P, \succeq)\ P_j\ DA_j^\tau(P, \succeq)$ and P_j'' upgrades s^u above $DA_j^\tau(P, \succeq)$, we have $\varphi_j(P, \succeq)\ P_j''\ s^u$. Therefore, $\varphi_j(P, \succeq)\ P_j''\ \varphi_j(P_j'', P_{-j}, \succeq)$. That is, φ is not strategy-proof. $\square$

Theorem 1 maintains that even if outside options are completely eliminated from the exogenous specification, we still cannot overcome the impossibility to improve upon DA. An important insight that emerges from the proof of Theorem 1 is that although outside options are commonly assumed to be exogenously specified in many matching models, we find that their strategic role may also be overtaken by underdemanded schools that emerge endogenously at a given problem. Given its key role in Theorem 1, however, such an endogenous outside option in this sense,[19] could be more difficult to identify than a readily available exogenous outside option, since it depends on the information about the joint preference profile, i.e., whether a school is weakly underdemanded or not depends on the particular problem. Hence, our analysis indicates that manipulating a Pareto-superior mechanism to DA crucially hinges on the students' ability to identify and upgrade weakly underdemanded schools, and even in cases when profitable manipulation is possible, this may require students to devise rather sophisticated strategies despite holding complete information about the environment. This in turn provides partial and cautious support for the trends in recent theory to try to improve upon DA via nonstrategy-proof mechanisms.

4 Possibility Results on a Restricted Domain

The impossibility result on the full domain follows crucially from the requirement that each student is fully capable of finding and upgrading a weakly underdemanded school anywhere in her preferences. Consider Example 1. We have demonstrated that preference P_{j_1}', which upgrades

[19] Whereas the critical role of endogenous outside options has been much emphasized in mechanism design and search literatures (cf. Lauermann and Virág, 2012 and Atakan and Ekmekci, 2014, for recent applications), we are not aware of any prior exploration of this notion in the matching context.

the outside option above school t_1 — the DA assignment of j_1 under P — causes the dominating mechanism φ to be manipulable. Under the preference profile P, student j_1 is assigned school t_1 but is assigned the outside option by reporting P'_{j_1}. Thus, it may be costly for her to report P'_{j_1} when her true preference is P_{j_1}. In practice, students face similar risks when deciding whether or not to truncate their preferences. After all, an unsuccessful manipulation attempt may cause the student to end up at her outside option. Therefore, a student who actually has no outside option may not be able to credibly misrepresent her preference in this manner. This motivates us to consider restrictions on students' strategy sets that may arise naturally in real-life situations.

Example 2. We revisit Example 1. Suppose that students j_1 and j_2 always have the outside option as the worst. More formally, let $T := \{t_1, t_2\} \subseteq S$ and $\mathcal{P}(T)$ be the set of all preferences under which the first and the second top choices are t_1 or t_2. In the domain $\mathcal{D}$ where $\mathcal{D}_i = \mathcal{P}$ and $\mathcal{D}_{j_1} = \mathcal{D}_{j_2} = \mathcal{P}(T)$, we can show that the DA mechanism with ex-post swapping that we constructed in Example 1 becomes strategy-proof. This is because students j_1 or j_2 can no longer upgrade the outside option above their DA assignments. $\Diamond$

Example 2 shows that under a restricted domain, we might be able to improve on DA while still preserving strategy-proofness. We first develop a general procedure of achieving such a mechanism under an arbitrary preference domain. Then we provide a natural preference domain — tiered preference domain — under which we can improve DA without sacrificing strategy-proofness.

We first introduce notions of cycles: Let φ and ζ be mechanisms. A **cycle for $\boldsymbol{\zeta}$** at a problem $(P, \succeq)$ is a list $C = (i_1, i_2, \ldots, , i_K)$ of distinct students such that $K \geq 1$; and if $K \geq 2$, for each $k \in \{1, \ldots, K\}$, student i_k prefers the assignment of her neighbor i_{k+1} to that of her own under ζ, i.e., $\zeta_{i_{k+1}}(P, \succeq)\ P_{i_k}\ \zeta_{i_k}(P, \succeq)$ where $i_{K+1} = i_1$. We say that such a cycle is **trivial** if $K = 1$. Note that we do not impose any condition for a trivial cycle. We say that a school s^* is the **target school of student i_k in cycle C** if for a nontrivial cycle, s^* is school $\zeta_{i^{k+1}}(P, \succeq)$ — the school assigned to the neighbor i_{k+1}; for a trivial cycle, s^* is the school assigned to herself. Furthermore, a **cycle for $(\boldsymbol{\varphi}, \boldsymbol{\zeta})$** at $(P, \succeq)$ is a cycle for ζ at $(P, \succeq)$, $C = (i_1, i_2, \ldots, i_K)$, such that for each $k \in \{1, \ldots, K\}$, the target school of student i_k under ζ is assigned to her under φ, i.e., for a nontrivial cycle, $\varphi_{i_k}(P, \succeq) = \zeta_{i_{k+1}}(P, \succeq)$; and for a trivial cycle, $\varphi_{i_1}(P, \succeq) = \zeta_{i_1}(P, \succeq)$. We denote by $I(C)$ the set of students that are involved in a cycle C, i..e, $I(C) = \{i_1, \ldots, i_K\}$.

Remark 1. The conclusion of Lemma 2 is equivalent to the following statement: i is in a trivial cycle for (φ, DA^τ) at $(P, \succeq)$.

To construct a dominating mechanism from a given inefficient mechanism, we need additional notions on cycles: We define a **cycle selection for $\boldsymbol{\zeta}$** to be a function $\mathcal{C}$ that maps a problem $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ to a collection $\mathcal{C}(P, \succeq)$ of cycles such that for each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}$, each $C \in \mathcal{C}(P, \succeq)$ is a cycle for ζ at $(P, \succeq)$ and $\{I(C)\}_{C \in \mathcal{C}(P, \succeq)}$ partitions I. We denote $C_i(P, \succeq)$ by the unique cycle C in $\mathcal{C}(P, \succeq)$ such that $i \in I(C)$. Moreover, a **cycle selection for $(\boldsymbol{\varphi}, \boldsymbol{\zeta})$** is a cycle selection $\mathcal{C}$ for ζ such that for each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, each $C \in \mathcal{C}(P, \succeq)$ is a cycle for (φ, ζ) at $(P, \succeq)$. A cycle selection $\mathcal{C}$ for ζ is **trivial** if for each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, each $C \in \mathcal{C}(P, \succeq)$ is trivial.

A mechanism φ is the **ζ-mechanism with ex-post swapping by a cycle selection $\mathcal{C}$** if $\mathcal{C}$ is a cycle selection for ζ; and if for each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ and each $i \in I$, $\varphi_i(P, \succeq)$ is the target school of i in $C_i(P, \succeq)$. Note that when $C_i(P, \succeq)$ is trivial, $\varphi_i(P, \succeq) = \zeta_i(P, \succeq)$. Moreover, we say that a mechanism is the **ζ-mechanism with ex-post swapping** if for some cycle selection $\mathcal{C}$ for ζ, it is the ζ-mechanism with ex-post swapping by $\mathcal{C}$. The next lemma characterizes the mechanism domination by cycle selections.

Lemma 4. *1. If a mechanism φ dominates a non-wasteful mechanism ζ, then there is a non-trivial cycle selection for (φ, ζ) and φ is non-wasteful.*

2. Conversely, if there is a nontrivial cycle selection $\mathcal{C}$ for a mechanism ζ, and a mechanism φ is the ζ-mechanism with ex-post swapping by $\mathcal{C}$, then φ dominates ζ.

The proofs of all lemmas in this section are in Appendix A. Given a strategy-proof but inefficient mechanism ζ, we aim to construct a new strategy-proof mechanism φ that dominates ζ. Lemma 4 is our starting point: a dominating mechanism can be found by identifying a nontrivial cycle selection as the ζ-mechanism with ex-post swapping by it. An important issue, however, is that the ζ-mechanism with ex-post swapping might not be strategy-proof. To maintain strategy-proofness, we will additionally require a cycle selection to be qualified: The qualified cycle selection is the robustness of target schools when an agent manipulates the mechanism such that she values her target school higher in her manipulating preferences. We formalize this notion by using the **upper contour set** of agent i at s, $U(P_i, s) := \{s' \in S \mid s' \, R_i \, s\}$.

Definition 2. 1. A cycle selection $\mathcal{C}$ for a mechanism ζ is **qualified** if it is nontrivial; and for each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, each $i \in I$, and each $P'_i \in \mathcal{D}_i$ if $U(P'_i, s^*) \subseteq U(P_i, s^*)$, where s^* is the target school of i in $C_i(P, \succeq)$, then school s^* is still the target school of i in cycle $C_i(P'_i, P_{-i}, \succeq)$.

2. A mechanism is the **ζ-mechanism with qualified ex-post swapping** if it is the ζ-mechanism with ex-post swapping by some qualified cycle selection for ζ.

The next lemma suggests that the notion of qualified cycle selections is appropriate, as it characterizes the domination of strategy-proof mechanisms by qualified cycle selections.

Lemma 5. *1. If a mechanism φ is strategy-proof and φ dominates a non-wasteful mechanism ζ, then there is a qualified cycle-selection for (φ, ζ) and φ is non-wasteful.*

2. Conversely, if there is a qualified cycle-selection $\mathcal{C}$ for a strategy-proof mechanism ζ, and a mechanism φ is the ζ-mechanism with ex-post swapping by $\mathcal{C}$, then φ is strategy-proof and dominates ζ.

Now we turn to restricted domains in school-choice programs. One possible student-type that may be observed in real-life assignment settings includes those students who either do not have any outside options, or who view their outside options (such as homeschooling) as being inferior to those offered by the public school system. Naturally, these student-types are not likely to truncate their preferences. Indeed, in all the years of Boston and NYC student assignments, for which data exist, thousands of students chose to rank-list as many schools as they were allowed to (see Figure 2).

A generalization of the above type leads us to model correlation among student preferences, a prevalent and salient feature observed in real-life. We do this by assuming student types that divide the set of schools into tiers according to their desirability. In this case, there may be significant welfare differences for these students between a school in a given tier and a school in a lower one. These types, for instance, may then choose to constrain their strategic behavior to within tiers (i.e., may misrepresent their preferences by altering the true rankings of schools that lie within the same tier but not those that belong to different tiers), for otherwise, a miscalculated manipulation may result in an assignment at the lower tier before exhausting all the options in the higher tier. We next formalize these ideas.

Given $T \subseteq S$, a preference $P_i \in \mathcal{P}$ is called a $\boldsymbol{T}$**−tier preference** if under P_i, the top-$|T|$ choices in S are in T.[20] We assume that whether they are truthful or nontruthful, students with $T-$tier preferences always report schools in T as their top-$|T|$ choices, while they may individually differ in their relative rankings of schools within this set.[21] We denote by $\mathcal{P}(T)$ the set of all $T-$tier preferences. Note that a student with T-tier preferences is no longer able to upgrade a non-T school into her top $|T|$ choices. As it turns out, when at least two students' preference reports are restricted to the tiered domain, the impossibility to improve upon DA disappears.

Theorem 2. **(Possibility result)** *Suppose that $T \subseteq S$ is a potentially overdemanded set with $|T| \geq 2$, and $J \subseteq I$ is a set of students with $|J| \geq |T|$, such that each student in J has T-tier preferences. We denote by DA^τ the student-proposing deferred acceptance mechanism with a given tie-breaking rule τ. In the restricted domain $\mathcal{D} = \mathcal{P}(T)^J \times \mathcal{P}^{I \setminus J}$ where any student in J reports a T-tier preference in $\mathcal{P}(T)$ and any student not in J reports a full preference in $\mathcal{P}$, we have the following:*

1. *For some I, S, and q, no strategy-proof and Pareto efficient mechanism dominates DA^τ.*

2. *There is a DA^τ-mechanism with qualified ex-post swapping.*

[20]The set T can be interpreted as a group of schools, or top-tier schools, that this type of student is seriously targeting such that they are not willing to take the risk of ranking a school from set T below a school they actually deem to be inferior. This type of restricted domain has also been considered in the literature (Kesten, 2010; Kandori et al., 2010).

[21]For example, suppose $T = \{x, y\}$. It is possible that a student top-ranks school x, whereas another student top-ranks school y.

3. *There is a maximal DA^τ-mechanism with qualified ex-post swapping, which is a maximal mechanism in the set of strategy-proof mechanisms that dominates DA^τ with respect to the partial order of the dominance of mechanisms.*

4. *In sum, any maximal DA^τ-mechanism with qualified ex-post swapping is strategy-proof and dominates DA^τ. Moreover, it is not dominated by any strategy-proof mechanism.*

The students whose welfare improves are only those with T-tier preferences under any DA^τ-mechanism with qualified ex-post swapping. These students swap their assignments but cannot manipulate this mechanism due to the restriction on their preferences because they cannot upgrade a weakly underdemanded set above their DA assignments. Thus, it follows from Example 1 and Part (2) of Theorem 2 that the more students have tiered preferences, the more students can become better off relative to their DA assignments while preserving strategy-proofness. For example, on the domain of Example 2 where only students j_1 and j_2 have tiered preferences, both can be better off under the priority profile $\succeq^\tau$. But under the priority profile $\tilde{\succeq}^\tau$, the remaining student i has no chance of improvement. However, if all students were to have tiered preferences, it is straightforward to see that student i could also be better off under $\tilde{\succeq}$. Theorem 2 is a rather optimistic possibility result. It shows that once there are at least two students who cannot strategically exploit their exogenous or endogenous outside options, i.e., weakly underdemanded schools, then DA^τ-mechanisms with qualified ex-post swapping can improve upon DA at no incentive cost.

Proof. Part (1) is an extension of Kesten's result – Corollary 1 – to the restricted domain whose proof is in Appendix B. We will show the key in the proof of the other parts – the construction of a qualified cycle selection for DA^τ – by Lemma 5-(2). With this construction, all of statements except Part (1) follows by definitions.

We denote $T := \{t_1, \dots, t_m\}$ where $m \geq 2$. Let T^{mp} be a maximally potentially overdemanded set with respect to T. We denote $T^{mp} \setminus T = \{s_{m+1}, \dots, s_n\}$.[22] Since $|J| \geq |T|$, we arbitrarily choose m students, $j_1, \dots, j_m$, from J. Moreover, let $J_1, \dots, J_n, I_{n+1}$ be a partition of I such that $|J_1| = q_{t_1}$, ..., $|J_m| = q_{t_m}$, $|J_{m+1}| = q_{s_{m+1}}$, ..., $|J_n| = q_{s_n}$, and $j_1 \in J_1$, ..., $j_m \in J_m$. Since $T^{mp} = \{t_1, \dots, t_m, s_{m+1}, \dots, s_n\}$ is potentially overdemanded, $\sum_{s \in T^{mp}} q_s < |I|$. Thus, as $\sum_{s \in S} q_s \geq |I|$ by our assumption, there is $s_{n+1} \in S \setminus T^{mp}$ and I_{n+1} is not empty. We consider the following sets of preferences (see Table 2): For each $\ell \in \{1, \dots, n\}$,

$$\mathcal{P}^\ell = \begin{cases} \{P_i^\ell \in \mathcal{P} \mid t_{\ell+1} \; P_i^\ell \; t_\ell \; P_i^\ell \; u \text{ for each } u \in S \setminus T^{mp}\} & \text{if } \ell \in \{1, \dots, m\}, \\ \{P_i^\ell \in \mathcal{P} \mid s_\ell \text{ is the top choice in } P_i^\ell\} & \text{if } \ell \in \{m+1, \dots, n\}. \end{cases}$$

Let $\tilde{\succeq}^\tau$ be the post-tie-breaking priority profile given by the right table in Table 2. We denote

[22]The subsequent argument assumes $T^{mp} \setminus T \neq \emptyset$, but we can modify it for the other case in a straightforward way.

Table 2: Preferences (left) and priorities (right) in Part 2

$P^1_{i\in J_1}$	$\dots$	$P^m_{i\in J_m}$	$P^{m+1}_{i\in J_{m+1}}$	$\dots$	$P^n_{i\in J_n}$	$P^1_{i\in I_{n+1}}$
$\vdots$		$\vdots$	s_{m+1}	$\dots$	s_n	$\vdots$
t_2		t_{m+1}				t_2
$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
t_1		t_m				t_1
$\vdots$		$\vdots$				$\vdots$

$\bar{\succeq}^\tau_{t_1}$	$\dots$	$\bar{\succeq}^\tau_{t_m}$	$\dots$	$\bar{\succeq}^\tau_{s_n}$	$\bar{\succeq}^\tau_{s\in S\setminus T}$
J_1		J_m		J_n	$\vdots$
I_{n+1}	$\dots$	I_{n+1}	$\dots$	I_{n+1}	
$\vdots$		$\vdots$		$\vdots$	

The left table shows the preference profile. The right table shows the priority profile $\bar{\succeq}^\tau$, where vertical dots represent arbitrary students. For example, the first column means that students in J_1 have higher priority for t_1 than those in I_{n+1}, who have higher priority than the others, and the priorities of all remaining students are arbitrary.

$$\mathcal{D}^* := (\mathcal{P}^1)^{J_1} \times \ldots \times (\mathcal{P}^m)^{J_m} \times (\mathcal{P}^1 \cap \mathcal{P}(T^{mp}))^{I_{m+1}}.$$

Let a matching μ be such that

$$\mu = \begin{pmatrix} J_1 & \dots & J_m & J_{m+1} & \dots & J_n & I_{n+1} \\ t_1 & \dots & t_m & s_{m+1} & \dots & s_n & DA^\tau(P_{I_{n+1}}, \bar{\succeq}_{S\setminus T^p}|_{I_{n+1}}) \end{pmatrix},$$

where $DA^\tau(P_{I_{n+1}}, \bar{\succeq}|_{I_{n+1}})$ is the DA matching for a smaller problem consisting of I_{n+1} and the relative ranking of $\bar{\succeq}|_{I_{n+1}}$ is kept the same as in $\bar{\succeq}$.

<u>Step 1:</u> We show that for each $P \in \mathcal{D}^*$, $DA^\tau_{(J_1,\dots,J_m,J_{m+1},\dots,J_n)}(P, \bar{\succeq}) = (t_1, \dots, t_m, s_{m+1}, \dots, s_n) = \mu_{(J_1,\dots,J_m,J_{m+1},\dots,J_n)}$.

Let $P \in (\mathcal{P}^1_{J_1}, \dots, \mathcal{P}^n_{J_n}, \mathcal{P}^1_{I_{n+1}})$. It is straightforward to see that

$$DA^\tau_{(J_{m+1},\dots,J_n)}(P, \bar{\succeq}) = (s_{m+1}, \dots, s_n). \tag{1}$$

Moreover we can show that μ is stable because under μ, for each $\ell \in \{1, \dots, n\}$, school t_ℓ for $\ell \le m$ (school s_ℓ for $\ell > m$) is matched with its top-priority students in J_ℓ.

Suppose $DA^\tau_{(J_1,\dots,J_m)}(P, \bar{\succeq}) \ne \mu_{(J_1,\dots,J_m)}$. Then, since the DA matching is student-optimal stable, there exist $\ell \in \{1, \dots, m\}$ and $j \in J_\ell$ such that $DA^\tau_j(P, \succeq)\ P_j\ t_\ell$. Thus, as $P_j \in \mathcal{P}^\ell$, $DA^\tau_j(P, \succeq) \in T^{mp} \setminus \{t_\ell\}$. Thus, by (1), $DA^\tau_j(P, \succeq) \in \{t_1, \dots, t_m\} \setminus \{t_\ell\}$. Let $t_k := DA^\tau_j(P, \succeq)$ where $k \ne \ell$. Then, for school t_k, any student $i \in I_{m+1}$ has higher priority than j (statement *). On the other hand, since the DA matching is student-optimal stable and $(P_{J_1}, \dots, P_{J_m}) \in \mathcal{P}^1_{J_1} \times \dots \mathcal{P}^m_{J_m}$, it follows from (1) that under DA, all students in $J_1 \cup \ldots \cup J_m$ are matched with schools in $\{t_1, \dots, t_m\}$, and thus any student i in $I_{m+1}(\ne \emptyset)$ is matched with a school in $S \setminus T^{mp}$. Thus, as $P_i \in \mathcal{P}^1$, i prefers school t_k to school $DA^\tau_i(P, \succeq)$ (statement **). Therefore, by statements (*) and (**), pair (i, t_k) blocks $DA^\tau(P, \succeq)$, which contradicts the stability of DA^τ. This completes Step 1.

<u>Step 2:</u> We show that given $\ell \in \{1, \dots, m\}$ and $P_{j_\ell} \in \mathcal{D}_{j_\ell} = \mathcal{P}(T)$ with $t_{\ell+1}\ P_{j_\ell}\ t_\ell$, we have

$P_{j_\ell} \in \mathcal{P}^\ell$. To this end, we need to show that for each $u \in S \setminus T^{mp}$, $t_\ell \; P_{j_\ell} \; u$. Fix $u \in S \setminus T^{mp}$. Since, $T \subseteq T^{mp}$, $u \in S \setminus T$. Thus, since $P_{j_\ell} \in \mathcal{P}(T)$, we have $t_\ell \; P_{j_\ell} \; u$.

Step 3: We show that the following cycle selection $\mathcal{C}$ for DA^τ is qualified.

$$\mathcal{C}(P, \succeq) = \begin{cases} \{(j_1, \ldots, j_m)\} \cup \{(i) \,|\, i \in I \setminus J\} & \text{if } P \in \mathcal{D}^* \text{ and } \succeq = \bar{\succeq}, \\ \{(i) \,|\, i \in I\} & \text{otherwise.} \end{cases}$$

Let $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, $i \in I$, and $P'_i \in \mathcal{D}^i$ such that $U(P'_i, s^*) \subseteq U(P_i, s^*)$ where s^* is the target school of i in $C_i(P, \succeq)$. Let s^* be the target school of i in $C_i(P'_i, P_{-i}, \succeq)$. We need to show $s^* = s^{**}$.

Case 1: $[i \notin \{j_1, \ldots, j_m\}$ and $\succeq \neq \bar{\succeq}]$ or $[i \in \{j_1, \ldots, j_m\}$, $\succeq = \bar{\succeq}$, $P \notin \mathcal{D}^*$, $(P'_i, P_{-i}) \notin \mathcal{D}^*]$. Then $C_i(P, \succeq) = C_i(P'_i, P_{-i}, \succeq) = \{i\}$ and thus $DA^\tau_i(P, \succeq) = s^*$ and $DA^\tau_i(P'_i, P_{-i}, \succeq) = s^{**}$. By strategy-proofness of DA^τ, $DA^\tau_i(P'_i, P_{-i}, \succeq) = s^*$. Hence $s^* = s^{**}$.

Case 2: $i \in \{j_1, \ldots, j_m\}$, $\succeq = \bar{\succeq}$, and $P \in \mathcal{D}^*$. Let $i = j_\ell$ for some $\ell \in \{1, \ldots, m\}$. As $P \in \mathcal{D}^*$, by Step 1, $s^* = t_{\ell+1}$. Note that $U(P'_i, t_{\ell+1}) \subseteq U(P_i, t_{\ell+1})$, $P_i \in \mathcal{P}^\ell$, and $t_{\ell+1} \; P_i \; t_\ell$. Thus $t_{\ell+1} \; P'_i \; t_\ell$. Hence, by Step 2, $P'_i \in \mathcal{P}^\ell$ and thus $(P'_i, P_{-i}) \in \mathcal{D}^*$. Thus it follows from Step 1 that $s^{**} = t_{\ell+1}$. Hence $s^{**} = t_{\ell+1} = s^*$.

Case 3: $i \in \{j_1, \ldots, j_m\}$, $\succeq = \bar{\succeq}$, and $P \notin \mathcal{D}^*$. It is sufficient to show $(P'_i, P_{-i}) \notin \mathcal{D}^*$, as in this case $s^* = s^{**}$ by Case 1. Suppose $(P'_i, P_{-i}) \in \mathcal{D}^*$. Let $i = j_\ell$ for some $\ell \in \{1, \ldots, m\}$. Note that as $P \notin \mathcal{D}^*$, $C_i(P, \succeq) = \{i\}$ and $DA^\tau_i(P, \succeq) = s^*$. Since $U(P'_i, s^*) \subseteq U(P_i, s^*)$, by strategy-proofness of DA^τ, $DA^\tau_i(P'_i, P_{-i}, \succeq) = s^*$. On the other hand, since $(P'_i, P_{-i}) \in \mathcal{D}^*$, by Step 1, $DA^\tau_i(P'_i, P_{-i}, \succeq) = t_\ell$. Hence $s^* = t_\ell$. Then, since $(P'_i, P_{-i}) \in \mathcal{D}^*$, $t_{\ell+1} \in U(P'_i, s^*) \subseteq U(P_i, s^*)$ and thus $t_{\ell+1} \; P_i \; t_\ell$. Then, by Step 2, $P_i \in \mathcal{P}^\ell$ and thus $P \in \mathcal{D}^*$, a contradiction.

In any case we have $s^* = s^{**}$. Thus cycle selection $\mathcal{C}$ is qualified. □

An extreme case of the tiered domain is when there is a single tier including all schools but the outside option. The following corollary pertains to such cases where students always report the outside option as the least preferred outcome.

Corollary 3. *Suppose that the outside option o exists and the set of schools excluding the null is potentially overdemanded with $|S \setminus \{o\}\,| \geq 2$. If the outside option is the last reported choice of at least $|S \setminus \{o\}|$ students, there is a strategy-proof mechanism that dominates the student-proposing deferred acceptance mechanism with any tie-breaking rule.*

Corollary 3 points to a particular way to improve upon DA in practice in places where the mechanism is currently used. By merely utilizing potential trades among students who declare the outside option as the last choice, it may be possible to make some of the students better off without imposing any welfare cost on any other students. Figure 1 shows the fraction of students who submit full-length preference lists in Boston and NYC, i.e., these are the students who list as many (real) schools as they are allowed. It is not implausible to imagine that a vast majority of

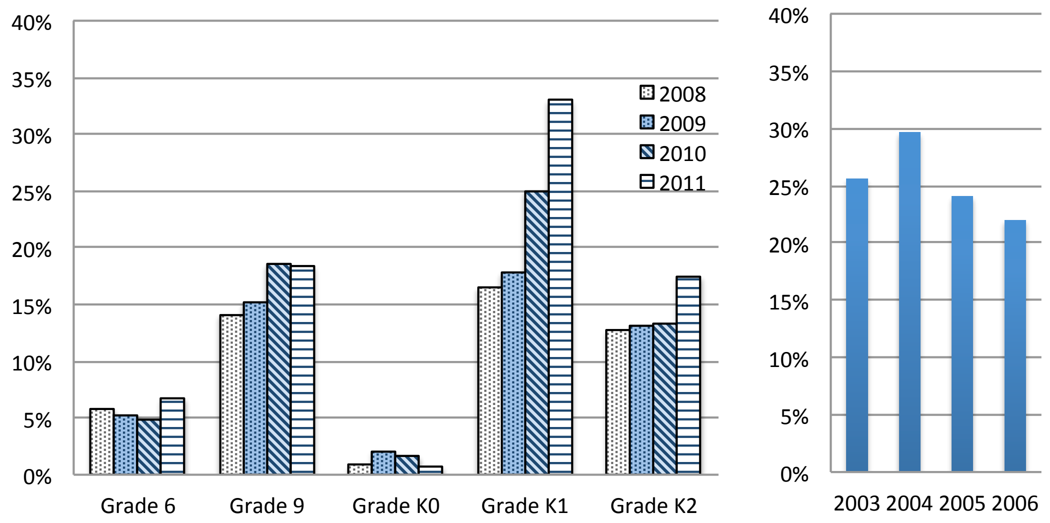

Figure 1: Percentage of students who submitted full-length preference lists in Boston (left) and New York City (right). *Data*: The above plotted statistics for the Boston Public Schools and NYC Public School System are reported in Abdulkadiroğlu et al. (2006) and Abdulkadiroğlu et al. (2009) respectively. In the above calculation of the Boston stats, we consider the percentages of students who submitted six or more schools in their preference lists since the maximum length reported for some grades are six. For the case of NYC, the percentages are for students who submitted the cap of twelve schools in their preference lists.

these students are likely to be without any outside options. Using a DA^τ-mechanism with qualified ex-post swapping to implement trades among them may lead to significant welfare gains of these students without harming others. Most notably, no participant will be given perverse incentives by doing so.

5 A Maximal Domain for Possibility

Thus far, we have seen that while it is impossible to improve upon DA under the full domain of preferences, there exist natural domains under which possibility results may be obtained. This in turn raises a natural question about whether or not we can identify maximal domains where a strategy-proof mechanism dominating DA exists. We next search for such domains.

Definition 3. A domain $\mathcal{D}$ is **maximal** for the possibility result if for each tie-breaking rule τ,

1. there is a strategy-proof mechanism that dominates DA^τ on $\mathcal{D}$;

2. for each $\mathcal{D}'$ with $\mathcal{D} \subsetneq \mathcal{D}' \subseteq \mathcal{P}^I$, no strategy-proof mechanism dominates DA^τ on $\mathcal{D}'$.

Given $T \subseteq S$ and $x, y \in T$, we use the type of preferences, $\mathcal{P} \setminus \mathcal{P}(x, y, T)$, for identifying the maximal domains where

$$\mathcal{P}(x,y,T) \quad = \quad \{\, P_i \in \mathcal{P} \mid \text{for some } u \in S \setminus T,\ x\ P_i\ u\ P_i\ y, \text{ and for each } u' \in S \setminus T,\ x\ P_i\ u' \,\}\,.$$

A student with a preference relation from the set $\mathcal{P}(x,y,T)$ prefers school x to school y, ranks some non-T school between them, and ranks any non-T school below school x.

Example 3. Let $S = \{x, y, o\}$ and $T = \{x, y\}$.

$\mathcal{P}(T)$			$\mathcal{P}(x,y,T)$		$\mathcal{P} \setminus \mathcal{P}(x,y,T)$				
x	y		x		x	o	y	y	o
y	x		o		y	x	x	o	y
o	o		y		o	y	o	x	x

◊

Lemma 6. *For all $T \subseteq T' \subseteq S$ and all $x, y \in T$ with $x \neq y$, $\mathcal{P}(T) \subseteq \mathcal{P}\backslash\mathcal{P}(x,y,T) \subseteq \mathcal{P}\backslash\mathcal{P}(x,y,T')$.*

Theorem 3. (Maximal domain) *Let τ be a tie-breaking rule, $T := \{t_1, \ldots, t_m\}$, and $J := \{j_1, \ldots, j_m\} \subseteq I$ with $|J| = |T| = m \geq 2$. Let T^{mp} a maximally potentially overdemanded set with respect to T. Then define the domain $\mathcal{D}^{\max}$ as follows: for each $i \in I$ and each $\ell \in \{1, \ldots, m\}$,*

$$\mathcal{D}_i^{\max} \quad = \quad \begin{cases} \mathcal{P} \setminus \mathcal{P}(t_{\ell+1}, t_\ell, T^{mp}) & \text{if } i = j_\ell, \\ \mathcal{P} & \text{otherwise,} \end{cases}$$

where $t_{m+1} \equiv t_1$.

1. *Domain $\mathcal{D}^{\max}$ is maximal for the possibility result.*
2. *Suppose that domain $\mathcal{D}$ is such that for each $i \in I$, $\mathcal{P}(T) \subseteq \mathcal{D}_i \subseteq \mathcal{D}_i^{\max}$. Then, there is a strategy-proof mechanism that dominates DA^τ on $\mathcal{D}$.*

Remark 2. By Lemma 6, the restricted domain $\mathcal{P}(T)^J \times \mathcal{P}^{I\setminus J}$ in Theorem 2 is always contained in the maximal domain $\mathcal{D}^{\max}$.

The proof is in Appendix C. Loosely speaking, the maximal domain $\mathcal{D}^{\max}$ rules out preference reversals of students in J with respect to schools in T from the maximally potentially overdemanded set.[23] Theorem 3 sheds light on the logic behind the impossibility theorem on the full domain (Theorem 1) and the possibility result on the tier-preference domain (Theorem 2): Consider a preference profile and a priority structure, where student j_ℓ prefers school $t_{\ell+1}$ to school t_ℓ to

[23] Ergin (2002) has shown that certain reversals in schools' priority orderings (coupled with capacity restrictions) are critical for DA to satisfy Pareto efficiency as well as other desirable properties. Theorem 3 provides another, conceptually related perspective on DA from the point of view of students' preferences.

which she is assigned under DA. On the maximal domain $\mathcal{D}^{\max}$, we can construct a strategy-proof mechanism dominating DA by swapping schools assigned under DA between the students in J at such profiles. The key idea is that once we allow either student to upgrade a non-T^{mp} school above her DA assignment but *below her assignment under the dominating mechanism* (i.e., by reporting a preference outside $\mathcal{D}^{\max}_{j_\ell}$), the dominating mechanism is no longer strategy-proof.

Example 4. Consider the setting in Example 1 where $I = \{i, j_1, j_2\}$ and $S = \{t_1, t_2, o\}$. The unique maximally potentially overdemanded set is $T^{mp} = \{t_1, t_2\}$. Consequently, we have:

$\mathcal{P}(T^{mp})$	
t_1	t_2
t_2	t_1
o	o

$\mathcal{D}^{\max}_{j_1}=\mathcal{P}\setminus\mathcal{P}(t_2,t_1,T^{mp})$				
t_1	t_1	o	t_2	o
t_2	o	t_1	t_1	t_2
o	t_2	t_2	o	t_1

$\mathcal{D}^{\max}_{j_2}=\mathcal{P}\setminus\mathcal{P}(t_1,t_2,T^{mp})$				
t_1	o	t_2	t_2	o
t_2	t_1	t_1	o	t_2
o	t_2	o	t_1	t_1

$\Diamond$

6 Conclusion

The Gale-Shapley deferred acceptance (DA) mechanism has been a focal assignment tool not only in theory but also in practical market design. Recent research has shown a surge of interest in exploring mechanisms that go beyond DA in terms of welfare (either ex-ante or ex-post). We have shown that whether such attempts come at the cost of strategy-proofness may be sensitive to the specifics of the environment. We have argued that weakly underdemanded schools, which we interpret as the endogenously emerging "pseudo outside options," appear to be the primary cause of the general impossibility result shown by Theorem 1. This result suggests that it may be possible to improve upon DA by restricting attention to problems where weakly underdemanded schools do not arise. In circumstances when students cannot credibly use outside options as strategic choices or when it may be difficult to identify weakly underdemanded schools, the scope of manipulation under alternative mechanisms may be diminished. More broadly, our approach puts the three-way tension among efficiency, stability, and strategy-proofness into a new perspective by highlighting the importance of the strategic role of outside options. It remains an interesting future goal to search for new assignment mechanisms in light of this optimistic perspective.

A Appendix

A.1 Proof of Lemma 1

Part (1): Let $\mu \in \mathcal{M}$ be non-wasteful at P. Suppose, on the contrary, that the outside option, o, is overdemanded at $(P, \succeq)$ under μ. Then, for some $i \in I$, $o\, P_i\, \mu(i)$. Thus for some $t \in$

$S \setminus \{o\}$, $\mu(i) = t$, and thus $|\mu(t)| \geq 1$. Moreover, by *non-wastefulness* of μ, $|\mu(o)| = |I|$. Thus, $\sum_{s \in S} |\mu(s)| \geq |I| + 1$, which contradicts the fact that μ is a matching and thus $\sum_{s \in S} |\mu(s)| = |I|$.
Part (2): Let $u \in S' \subseteq S \setminus T^{mp}$. Suppose on the contrary that all schools in $T^{mp} \cup S'$ are overdemanded at $(P, \succeq)$ under $DA^{\tau}(P, \succeq)$. We first show

$$\sum_{s \in T^{mp} \cup \{u\}} |DA^{\tau}_s(P, \succeq)| = |I|. \tag{2}$$

By the definition of T^{mp}, $\sum_{s \in T^{mp} \cup \{u\}} q_s \geq |I|$. Now, for each school $s \in T^{mp} \cup \{u\}$, since s is overdemanded, there is $j \in I$ such that $s \, P_j \, DA^{\tau}_j(P, \succeq)$ and thus $|DA^{\tau}_s(P, \succeq)| = q_s$ by *non-wastefulness* of DA. Thus, as $\sum_{s \in T^{mp} \cup \{u\}} q_s \geq |I|$, $\sum_{s \in T^{mp} \cup \{u\}} |DA^{\tau}_s(P, \succeq)| \geq |I|$. Hence, since $\sum_{s \in S} |DA^{\tau}_s(P, \succeq)| = |I|$ ($\because DA^{\tau}(P, \succeq)$ is a matching), we obtain Equation (2).

By Equation (2), under DA, each application is made only to $T^{mp} \cup \{u\}$. Note that DA ends in at least two steps. Consider the last step $r \geq 2$ of DA at $(P, \succeq^{\tau})$, where some student k who was rejected at step $r-1$ and applies to some school $t \in T^{mp} \cup \{u\}$ at step r. Since t is overdemanded, there is some student $k' \neq k$ such that $t \, P_{k'} \, DA^{\tau}_{k'}(P, \succeq)$. Thus k' must have applied to and been rejected by t at an earlier step than r. Thus, school t has kept q_t applicants at step $r-1$. Hence, school t has at least $(q_t + 1)$ applications at step r and thus rejects one of them. But this contradicts the assumption that r is the last step of DA.
Part (3): Note $\sum_{s \in S} q_s \geq |I|$ by our assumption. Thus, by the definition of T^{mp}, $T^{mp} \neq S$. Then, the set $S' := S \setminus T^{mp}$ is not empty. Part (2) leads to the desired result. □

A.2 Proof of Lemmas 2 and 4

We first prove Lemma 4 and then Lemma 2.

A.2.1 Lemma 4

Part (2) is straightforward by definition. Before proving Part (1), we show two useful claims. Let φ and ζ be mechanisms, and $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ a problem.

Claim 1. If $\varphi(P, \succeq)$ dominates $\zeta(P, \succeq)$ at P and $\zeta(P, \succeq)$ is non-wasteful at P, then there is a nontrivial cycle for (φ, ζ) at $(P, \succeq)$.

Proof. Let $\nu = \varphi(P, \succeq)$ and $\mu = \zeta(P, \succeq)$. We first define a list of students $(i_1, \ldots, i_K)$ to be a **temporary list of size $K \geq 2$** if for each $k \in \{1, \ldots, K-1\}$, $\mu(i_{k+1}) \; P_{i_k} \; \mu(i_k)$ and $\nu(i_k) = \mu(i_{k+1})$; and $\nu(i_K) \; P_{i_K} \; \mu(i_K)$.

We first construct a temporary list of size 2. Since ν dominates μ, there is $i_1 \in I$ such that $\nu(i_1) \; P_{i_1} \; \mu(i_1)$. Let $s_2 := \nu(i_1)$ and $s_1 := \mu(i_1)$. Note $s_1 \neq s_2$. By *non-wastefulness* of μ, as $s_2 \; P_{i_1} \; \mu(i_1)$, we have $|\mu(s_2)| = q_{s_2}$. Then there is $i_2 \in \mu(s_2)$ such that $\nu(i_2) \neq \mu(i_2)$ (otherwise, as $|\mu(s_2)| = q_{s_2}$, $\mu(s_2) = \nu(s_2)$; then $i_1 \in \nu(s_2) = \mu(s_2)$ and thus $s_2 = \mu(i_1) \equiv s_1$, which contradicts

$s_1 \neq s_2$.). Thus, as ν dominates μ, $\nu(i_2)\ P_{i_2}\ \mu(i_2)$. Moreover, as $i_2 \in \mu(s_2)$, $\nu(i_1) \equiv s_2 = \mu(i_2)$; and as $\nu(i_1)\ P_{i_1}\ \mu(i_1)$, we have $\mu(i_2)\ P_{i_1}\ \mu(i_1)$. Thus (i_1, i_2) is a temporary list of size 2. If $\nu(i_1) = \mu(i_1)$, the temporary list is a nontrivial cycle for (φ, ζ) at $(P, \succeq)$. If not, we continue as follows.

Suppose that $(i_1, \ldots, i_{K-1})$ is a temporary list of size $K-1$ where $K \geq 3$. We construct a temporary list of size K. Let $s_k := \mu(i_k)$ for each $k \in \{1, \ldots, K-1\}$. Since $(i_1, \ldots, i_{K-1})$ is a temporary list, $\nu(i_{K-1})\ P_{i_{K-1}}\ \mu(i_{K-1})$. Let $s_K := \nu(i_{K-1})$. Note $s_{K-1} \neq s_K$. By *non-wastefulness* of μ, as $s_K\ P_{i_{K-1}}\ \mu(i_{K-1})$, we have $|\mu(s_K)| = q_{s_K}$. Then there is $i_K \in \mu(s_K)$ such that $\nu(i_K) \neq \mu(i_K)$ (Otherwise, as $|\mu(s_K)| = q_{s_K}$, $\mu(s_K) = \nu(s_K)$; then $i_{K-1} \in \nu(s_K) = \mu(s_K)$ and thus $s_K = \mu(i_{K-1}) \equiv s_{K-1}$, which contradicts $s_{K-1} \neq s_K$). Thus, as ν dominates μ, $\nu(i_K)\ P_{i_K}\ \mu(i_K)$. Moreover, as $i_K \in \mu(s_K)$, $\nu(i_{K-1}) \equiv s_K = \mu(i_K)$; and as $\nu(i_{K-1})\ P_{i_{K-1}}\ \mu(i_{K-1})$, we have $\mu(i_K)\ P_{i_{K-1}}\ \mu(i_{K-1})$. Thus $(i_1, \ldots, i_K)$ is a temporary list of size K. If $\nu(i_K) = \mu(i_k)$ for some $k \in \{1, \ldots, K-1\}$, then the list $(i^k, \ldots, i^K)$ is a nontrivial cycle for (φ, ζ) at $(P, \succeq)$. As the set of students is finite, we eventually obtain a nontrivial cycle for (φ, ζ) at $(P, \succeq)$. □

We say that a **mechanism ζ' is induced by a cycle C for (φ, ζ) at $(P, \succeq)$** if for each $i \in I(C)$, $\zeta'_i(P, \succeq)$ is the target school of i in C; for each $i \in I \setminus I(C)$, $\zeta'(P, \succeq) = \zeta(P, \succeq)$; and for each $(P', \succeq') \neq (P, \succeq)$, $\zeta'(P', \succeq') = \zeta(P', \succeq')$.

Claim 2. Suppose that $\varphi(P, \succeq)$ dominates a non-wasteful matching $\zeta(P, \succeq)$ at P and a mechanism ζ' is induced by a nontrivial cycle C for (φ, ζ) at $(P, \succeq)$. Then, (i) $\varphi(P, \succeq) = \zeta'(P, \succeq)$ or $\varphi(P, \succeq)$ dominates $\zeta'(P, \succeq)$ at P, and (ii) $\zeta'(P, \succeq)$ is non-wasteful at P.

The proof is straightforward and thus is omitted.

Now we prove Part (1). Let φ and ζ be such that φ dominates ζ and ζ is non-wasteful. Define a cycle selection $\mathcal{C}$ for (φ, ζ) such that for each problem $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, if $\varphi(P, \succeq) = \zeta(P, \succeq)$, let $C_i(P, \succeq) = (i)$ for each $i \in I$; if $\varphi(P, \succeq)$ dominates $\zeta(P, \succeq)$ at P (this is true for at least one $(P, \succeq)$ due to the dominance), we find $C(P, \succeq)$ in the following procedure: For $\ell \geq 1$, letting $\zeta^0 = \zeta$,

<u>Step 0:</u> Let $\mathcal{C}^0 = \{(i) |\ \varphi_i(P, \succeq) = \zeta_i(P, \succeq)\}$ be the set of all trivial cycles.

<u>Step ℓ:</u> From the previous step, $\zeta^{\ell-1}(P, \succeq) \neq \varphi(P, \succeq)$ is non-wasteful and $\varphi(P, \succeq)$ dominates $\zeta^{\ell-1}(P, \succeq)$ at P. By Claim 1, there is a nontrivial cycle C^ℓ for $(\varphi, \zeta^{\ell-1})$ at $(P, \succeq)$. Let ζ^ℓ be the mechanism induced by $\mathcal{C}^\ell$ for $(\varphi, \zeta^{\ell-1})$ at $(P, \succeq)$. By Claim 2-(i), $\varphi(P, \succeq) = \zeta^\ell(P, \succeq)$ or $\varphi(P, \succeq)$ dominates $\zeta^\ell(P, \succeq)$ at P. If the former happens, the procedure stops. Otherwise, go to the next step. Note that by Claim 2-(ii), $\varphi(P, \succeq)$ dominates $\zeta^\ell(P, \succeq)$ at P and $\zeta^\ell(P, \succeq)$ is non-wasteful at P.

The above procedure stops in a finite step, L, as the number $|i \in I \,|\, \varphi_i(P, \succeq) \neq \zeta^\ell(P, \succeq)|$ $(\leq |I|)$ is strictly decreasing in ℓ and I is finite.

Let $\mathcal{C} = \mathcal{C}^0 \cup \{C^1, \ldots, C^L\}$. By construction, each C^ℓ is also a cycle for (φ, ζ), and $\{I(C)\}_{c \in \mathcal{C}}$ partitions I. Moreover, $\zeta^L(P, \succeq) = \varphi(P, \succeq)$. Thus, since $\zeta^L(P, \succeq)$ is non-wasteful at P, and thus $\varphi(P, \succeq)$ is non-wasteful at P. Hence, φ is non-wasteful. ■

A.2.2 Lemma 2

We show its contrapositive. Suppose that student i is not in a trivial cycle for (φ, DA^τ) at $(P, \succeq)$. Since φ dominates non-wasteful DA^τ, there is a nontrivial cycle selection $\mathcal{C}$ for (φ, DA^τ). Then, by Lemma 4, i is in a nontrivial cycle $C_i(P, \succeq) := (i_1, \ldots, i_K)$ for (φ, DA^τ) at $(P, \succeq)$ where $K = 2$. Let $i = i_2$ without loss of generality. Then, by definition of cycles, $DA^\tau_{i_2}(P, \succeq)\ P_{i_1}\ DA^\tau_{i_1}(P, \succeq)$. Thus $DA^\tau_{i_2}(P, \succeq)$ is overdemanded at $(P, \succeq; \tau)$. □

A.3 Proof of Lemma 3

We start with the following useful claim whose straightforward proof is omitted.

Claim 3. Let T be the set of all schools that are weakly underdemanded at $(P, \succeq; \tau)$. For all $\succeq'_T \in \mathcal{A}^T$, (i) $DA^\tau(P, \succeq) = DA^\tau(P, \succeq'_T, \succeq_{-T})$, and (ii) if s is weakly underdemanded at $(P, \succeq; \tau)$, then s is also weakly underdemanded at $(P, \succeq'_T, \succeq_{-T}; \tau)$.

Suppose that under DA, student i is assigned school s^*, which is overdemanded at $(P, \succeq; \tau)$. Let S^u be the set of all schools that are weakly underdemanded at $(P, \succeq; \tau)$. Note that $S^u \neq \emptyset$ by Lemma 1-(3), and any weakly underdemanded school $s \in S^u$ is strictly worse than s^* in P_i as s^* is overdemanded. We consider two cases:

P_i	P'_i	P''_i
$P_i\|_{U(P_i,s^*)\setminus\{s^*\}}$	$P_i\|_{U(P_i,s^*)\setminus\{s^*\}}$	$P_i\|_{U(P_i,s^*)\setminus\{s^*\}}$
s^*	all weakly underdemanded schools at $(P, \succeq; \tau)$	s^u
$\vdots$	s^*	s^*
	$\vdots$	$\vdots$

Case 1: for some $s \in S^u$, $|DA^\tau_s(P, \succeq)| < q_s$. Let $s^u \in S^u$ be one such school. Consider the preference P''_i that upgrades s^u above s^* as described in the above table. Let $P'' := (P''_i, P_{-i})$. Let μ'' be a matching such that $\mu''(i) = s^u$ and for all $j \neq i$, $\mu''(j) = DA^\tau_j(P, \succeq)$. Then μ'' is stable at $(P'', \succeq^\tau)$. For problem $(P'', \succeq; \tau)$, DA works in exactly the same way as it does for $(P, \succeq; \tau)$ until right before the step i applies to s^u. Hence $DA^\tau_i(P'', \succeq) \notin U(P_i, s^*) \setminus \{s^*\}$. Note that matching $DA^\tau(P'', \succeq)$ dominates at P'' any matching that is stable at $(P'', \succeq^\tau)$; μ'' is stable at $(P'', \succeq^\tau)$; and $\mu''(i) = s^u$. Thus $DA^\tau_i(P'', \succeq) = s^u$. Also, for each student $j \neq i$, $DA^\tau_j(P'', \succeq)\ R_j\ \mu''(j)$. Thus, for all $j \neq i$, as $s^u \in S^u$ and $\mu''(j) = DA^\tau_j(P, \succeq)$, we have $DA^\tau_j(P'', \succeq)\ R_j\ DA^\tau_j(P, \succeq)\ R_j\ s^u$. Therefore s^u is weakly underdemanded at $(P'', \succeq; \tau)$.

Case 2: for all $s \in S^u$, $|DA^\tau_s(P, \succeq)| = q_s$. As s^* is overdemanded at $(P, \succeq; \tau)$, all schools in $U(P_i, s^*)$ are overdemanded at $(P, \succeq; \tau)$. Thus, since $S^u \neq \emptyset$, all schools in S^u are less desirable than s^* to student i in P_i. We consider the preference P'_i that upgrades all weakly underdemanded schools in S^u above s^* as described in the above table. Moreover, for all $s \in S^u$, let $\succeq'_s$ be the priority such that i has the lowest priority for s, and the relative rankings of all other students are the same as in $\succeq_s$. Let $P' := (P'_i, P_{-i})$ and $\succeq' := (\succeq'_{S^u}, \succeq_{-S^u})$.

We first show that $DA^\tau_i(P',\succeq') \in S^u$. Suppose not. Then, under DA, all schools in S^u reject i at $(P',\succeq';\tau)$. Since i has the lowest priority at all schools in S^u and $DA^\tau(P,\succeq)$ is stable at $(P,\succeq^\tau)$, $DA^\tau(P,\succeq)$ is also stable at $(P',\succeq')$. Then, as the DA matching is student-optimal stable, $DA^\tau(P',\succeq') = DA^\tau(P,\succeq)$. Thus, all overdemanded schools at $(P,\succeq;\tau)$ are still overdemanded at $(P',\succeq';\tau)$, and all weakly underdemanded schools at $(P,\succeq;\tau)$ become overdemanded at $(P',\succeq';\tau)$. Then, all schools are overdemanded at $(P',\succeq';\tau)$, which contradicts Lemma 1-(2).

Now, letting $s^u := DA^\tau_i(P',\succeq') \in S^u$, consider the preference P''_i that upgrades s^u above s^* in P_i as described in the above table. Let $P'' := (P''_i, P_{-i})$. We will show that $DA^\tau_i(P'',\succeq') = s^u$. For $(P'',\succeq';\tau)$ DA works in the same way as for $(P,\succeq;\tau)$ until student i applies to s^u. Thus, $DA^\tau_i(P'',\succeq') \notin U(P''_i, s^u)\setminus\{s^u\}$. Then, by *strategy-proofness* of DA^τ, $DA^\tau_i(P'',\succeq')\ R''_i\ DA^\tau_i(P',\succeq')$, i.e., $DA^\tau_i(P'',\succeq')\ R''_i\ s^u$ and thus $DA^\tau_i(P'',\succeq') = s^u$.

It remains to show that $DA^\tau_i(P'',\succeq) = s^u$ and s^u is weakly underdemanded at $(P'',\succeq;\tau)$. Note that $DA^\tau(P,\succeq)$ is stable at $(P'',\succeq')$, since $DA^\tau(P,\succeq)$ is stable at $(P,\succeq^\tau)$ and i has the lowest priority for s^u with $|DA^\tau_{s^u}(P,\succeq)| = q_{s^u}$. Since matching $DA^\tau(P'',\succeq')$ is student-optimal stable, we have for all $j \neq i$ and all $s \in S^u$, $DA^\tau_j(P'',\succeq')\ R''_j\ DA^\tau_j(P,\succeq)\ R_j\ s$ and $DA^\tau_i(P'',\succeq')\ R''_i\ s^u\ R''_i\ s$. Since $P''_{-i} = P_{-i}$, for all $j \in I$ and all $s \in S^u$, $DA^\tau_j(P'',\succeq')\ R''_j\ s$. That is, all schools in S^u, including s^u, are weakly underdemanded at $(P'',\succeq';\tau)$. Hence, by Claim 3, $DA^\tau_i(P'',\succeq) = DA^\tau_i(P'',\succeq') = s^u$, which is weakly underdemanded at $(P'',\succeq;\tau)$. □

A.4 Proof of Lemma 5

Part (1). Suppose that a strategy-proof mechanism φ dominates a non-wasteful mechanism ζ. Then, by Lemma 4 (1), there is a nontrivial cycle selection $\mathcal{C}$ for (φ,ζ). We show that $\mathcal{C}$ is qualified for (φ,ζ). Let $(P,\succeq) \in \mathcal{D}\times\mathcal{A}^S$, $i \in I$, and $P'_i \in \mathcal{D}_i$ such that $U(P'_i,s^*) \subseteq U(P_i,s^*)$, where s^* is the target school of i in $C_i(P,\succeq)$. Since cycle selection $\mathcal{C}$ is for (φ,ζ), $\varphi_i(P,\succeq) = s^*$. Now, since φ is strategy-proof, $\varphi_i(P'_i,P_{-i},\succeq)\ R'_i\ \varphi_i(P,\succeq) = s^*$. Thus, since $U(P'_i,s^*) \subseteq U(P_i,s^*)$, $\varphi_i(P'_i,P_{-i},\succeq)\ R_i\ \varphi_i(P,\succeq)$. On the other hand, since φ is strategy-proof, $\varphi_i(P,\succeq)\ R_i\ \varphi_i(P'_i,P_{-i},\succeq)$. Therefore, by strictness of preferences, $\varphi_i(P'_i,P_{-i},\succeq) = \varphi_i(P,\succeq) = s^*$, which shows that s^* is the target school of i in $C_i(P'_i,P_{-i},\succeq)$. Hence, $\mathcal{C}$ is qualified for (φ,ζ). □

Part (2). Suppose that there is a qualified cycle-selection $\mathcal{C}$ for a strategy-proof mechanism ζ. Let φ be the mechanism induced by $\mathcal{C}$. By Lemma 4 (2), φ dominates ζ.

It remains to show that φ is strategy-proof. Suppose, for a contradiction, that there exist $\succeq\in\mathcal{A}^S$, $i \in I$, $P \in \mathcal{D}$, and $P'_i \in \mathcal{D}_i$ such that $\varphi_i(P'_i,P_{-i},\succeq)\ P_i\ \varphi_i(P,\succeq)$. First, since φ dominates ζ, $\varphi_i(P,\succeq)\ R_i\ \zeta_i(P,\succeq)$. Moreover, since ζ is strategy-proof, $\zeta_i(P,\succeq)\ R_i\ \zeta_i(P'_i,P_{-i},\succeq)$. Hence, by transitivity of preferences, $\varphi_i(P'_i,P_{-i},\succeq)\ P_i\ \zeta_i(P'_i,P_{-i},\succeq)$.

Let $s^* = \varphi_i(P,\succeq)$ and $s^{**} = \varphi_i(P'_i,P_{-i},\succeq)$. Then s^* is the i's target school in $C_i(P,\succeq)$ and s^{**} is the i's target school in $C(P'_i,P_{-i},\succeq)$. Note that $s^{**}\ P_i\ s^*$ and thus $s^* \neq s^{**}$.

Now consider the preference P''_i of student i where s^{**} is her top choice and s^* is her second top

Table 3: Preferences and matchings in Part 1

P_{j_1}	P_{j_2}	P_i	P_{j_1}	P_{j_2}	P'_i	$\tilde{\succsim}^{\tau}_{t_1}$	$\tilde{\succsim}^{\tau}_{t_2}$
t_2	$\overline{t_1}$	$\overline{t_2}$	$\overline{\underline{t_2}}$	$\overline{\underline{t_1}}$	t_2	i	j_2
t_1	$\underline{t_2}$	$\underline{t_1}$	t_1	t_2	$\overline{\underline{o}}$	j_1	j_1
$\underline{o}$	o	o	o	o	t_1	j_2	i

Two preference profiles and a priority profile after breaking a tie are shown for the proof of Part 1 in Theorem 2. The underlined (overlined) schools in the preference profiles are assigned to the corresponding students under DA (dominating mechanism φ).

choice. Then, since $s^{**}\ P_i\ s^*$, we have $\{s^*, s^{**}\} = U(P''_i, s^*) \subseteq U(P_i, s^*)$. Note that $\mathcal{C}$ is qualified and s^* is the i's target school in $C_i(P, \succeq)$. Thus, s^* is the i's target school in $C_i(P''_i, P_{-i}, \succeq)$ and thus $\varphi_i(P''_i, P_{-i}, \succeq) = s^*$. On the other hand, $\{s^{**}\} = U(P''_i, s^{**}) \subseteq U(P'_i, s^{**})$ and s^{**} is the i's target school in $C_i(P'_i, P_{-i}, \succeq)$. Thus, since $\mathcal{C}$ is qualified, s^{**} is the i's target school in $C_i(P''_i, P_{-i}, \succeq)$ and thus $\varphi_i(P''_i, P_{-i}, \succeq) = s^{**}$. In sum, we have $\varphi_i(P''_i, P_{-i}, \succeq) = s^* = s^{**}$, which contradicts $s^* \neq s^{**}$. □

A.5 Proof of Lemma 6

Let $T \subseteq T' \subseteq S$ and $x, y \in T$ be distinct. It is clear that $\mathcal{P}(T) \subseteq \mathcal{P} \setminus \mathcal{P}(x, y, T)$. To show $\mathcal{P} \setminus \mathcal{P}(x, y, T) \subseteq \mathcal{P} \setminus \mathcal{P}(x, y, T')$, we prove its equivalent statement where $\mathcal{P}(x, y, T') \subseteq \mathcal{P}(x, y, T)$. Let $P_i \in \mathcal{P}(x, y, T')$. Then, by definition, for some $u \in S \setminus T'$, $x\ P_i\ u\ P_i\ y$ and for each $\bar{u} \in S \setminus T'$, $x\, P_i\, \bar{u}$. Since $S \setminus T' \subseteq S \setminus T$, we have $u \in S \setminus T$, $x\ P_i\ u\ P_i\ y$ and for each $\bar{u} \in S \setminus T$, $x\ P_i\ \bar{u}$. Thus, $P_i \in \mathcal{P}(x, y, T)$. □

B Proof of Theorem 2 - Part (1)

To facilitate the exposition, we take the same setting (I, S, q, P) as that in Example 1. Let $I = \{i, j_1, j_2\}$, $S = \{t_1, t_2, o\}$, $J = \{j_1, j_2\}$, $T = \{t_1, t_2\}$, and $q_{t_1} = q_{t_2} = 1$. The only difference from Example 1 is the priority profile $\tilde{\succsim}^{\tau}$ where the roles of students i and j_1 are switched. See Table 3.

Suppose that there is a Pareto efficient mechanism $\tilde{\varphi}$ that dominates DA^{τ}. We show that student i can manipulate $\tilde{\varphi}$, whose proof is very similar to that of Example 1, where student j_1 manipulates the dominating mechanism φ.

At problem $(P, \tilde{\succsim})$, $DA^{\tau}_{(j_1, j_2, i)}(P, \tilde{\succsim}) = (o, t_2, t_1)$, and thus there is a unique Pareto efficient matching $\nu_{(j_1, j_2, i)} = (o, t_1, t_2)$ that dominates $DA^{\tau}(P, \tilde{\succsim})$ at P (Table 3). Thus, since $\tilde{\varphi}$ is Pareto efficient and dominates DA^{τ}, $\tilde{\varphi}(P, \tilde{\succsim}) = \nu$. Consider the singleton preference P'_i of student i

that declares only t_2 above o. Then $DA^{\tau}_{(j_1,j_2,i)}(P'_i, P_{-i}, \tilde{\succeq}) = (t_2, t_1, o)$, which is Pareto efficient at $(P'_i, P_{-i}, \tilde{\succeq})$. Thus, since $\tilde{\varphi}$ is Pareto efficient and dominates DA^{τ}, $\tilde{\varphi}(P'_i, P_{-i}, \tilde{\succeq}) = DA^{\tau}(P'_i, P_{-i}, \tilde{\succeq})$. Now $\tilde{\varphi}_i(P, \tilde{\succeq}) = t_2 \; P'_i \; o = \tilde{\varphi}_i(P'_i, P_{-i}, \tilde{\succeq})$. Hence, $\tilde{\varphi}$ is not strategy-proof. □

C Proof of Theorem 3

For the proof of Theorem 3, it suffices to show:

- (Possibility result — Part 2 in Theorem 3) On any domain $\mathcal{D}$ where for each $i \in I$, $\mathcal{P}(T) \subseteq \mathcal{D}_i \subseteq \mathcal{D}_i^{\max}$, there is a strategy-proof mechanism that dominates DA^{τ}.
- (Impossibility result) On any domain $\mathcal{D}$ where for each $\mathcal{D}$ where $\mathcal{D}^{\max} \subsetneq \mathcal{D} \subseteq \mathcal{P}^I$, no strategy-proof mechanism dominates DA^{τ}.

C.1 Proof of the possibility result

We follow the same proof strategy as the one in the proof of Theorem 2-(2). The only difference is that our domain varies from $\mathcal{P}(T)$ to $\mathcal{D}_i^{\max}$. As in the proof of Theorem 2-(2), we use the same notations and argument up to Step 1, and will modify Steps 2 and 3 due to the domain change.

Step 2: We show

$$\forall \ell \in \{1, \dots, m\} \, \forall P_{j_\ell} \in \mathcal{D}_{j_\ell}[t_{\ell+1} \; P_{j_\ell} \; t_\ell \text{ and } \forall u \in S \setminus T^{mp}, \, t_{\ell+1} \; P_{j_\ell} \; u] \Rightarrow P_{j_\ell} \in \mathcal{P}^{\ell}.$$

Let $\ell \in \{1, \dots, m\}$ and $P_{j_\ell} \in \mathcal{D}_{j_\ell}$ such that the hypothesis in the above is satisfied. Since $P_{j_\ell} \in \mathcal{D}_{j_\ell}^{\max} = \mathcal{P} \setminus \mathcal{P}(t_{\ell+1}, t_\ell, T^{mp})$, we have

$$\begin{aligned} & [\forall u \in S \setminus T^{mp}, \, u \, P_{j_\ell} \, t_{\ell+1} \text{ or } t_\ell \, P_{j_\ell} \, u] \text{ or } \exists u' \in S \setminus T^{mp}, \, u' \, P_{j_\ell} \, t_{\ell+1} \\ \Rightarrow \;& \forall u \in S \setminus T^{mp}, \, t_\ell \, P_{j_\ell} u \qquad (\because \text{the hypothesis}) \\ \Rightarrow \;& P_{j_\ell} \in \mathcal{P}^{\ell}. \end{aligned}$$

Step 3: We show that the following cycle selection $\mathcal{C}$ for DA^{τ} is qualified.

$$\mathcal{C}(P, \succeq) = \begin{cases} \{(j_1, \dots, j_m)\} \cup \{(i) \,|\, i \in I \setminus J\} & \text{if } P \in \mathcal{D}^* \text{ and } \succeq = \tilde{\succeq}, \\ \{(i) \,|\, i \in I\} & \text{otherwise.} \end{cases}$$

Let $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$, $i \in I$, and $P'_i \in \mathcal{D}^i$ such that $U(P'_i, s^*) \subseteq U(P_i, s^*)$ where s^* is the target school of i in $C_i(P, \succeq)$. Let s^* be the target school of i in $C_i(P'_i, P_{-i}, \succeq)$. We need to show $s^* = s^{**}$.

Case 1: $[i \notin \{j_1, \ldots, j_m\}$ and $\succeq \neq \bar{\succeq}]$ or $[i \in \{j_1, \ldots, j_m\}, \succeq = \bar{\succeq}, P \notin \mathcal{D}^*, (P'_i, P_{-i}) \notin \mathcal{D}^*]$. We can conclude that $s^* = s^{**}$ using the corresponding argument in the proof of Theorem 2-(2).

Case 2: $i \in \{j_1, \ldots, j_m\}$, $\succeq = \bar{\succeq}$, and $P \in \mathcal{D}^*$. Let $i = j_\ell$ for some $\ell \in \{1, \ldots, m\}$. As $P \in \mathcal{D}^*$, by Step 1, $s^* = t_{\ell+1}$.

We show $P'_i \in \mathcal{P}^\ell$ and thus $(P'_i, P_{-i}) \in \mathcal{D}^*$. By Step 2, it is sufficient to show that $t_{\ell+1}\ P'_i\ t_\ell$ and for each $u \in S \setminus T^{mp}$, $t_{\ell+1}\ P'_i\ u$. Fix $u \in S \setminus T^{mp}$. Note that $U(P'_i, t_{\ell+1}) \subseteq U(P_i, t_{\ell+1})$, $P_i \in \mathcal{P}^\ell$, and $t_{\ell+1}\ P_i\ t_\ell$. Thus $t_{\ell+1}\ P'_i\ t_\ell$ and $t_{\ell+1}\ P'_i\ u$. Hence $P'_i \in \mathcal{P}^\ell$.

Then, since $(P'_i, P_{-i}) \in \mathcal{D}^*$, it follows from Step 1 that $s^{**} = t_{\ell+1}$. Hence $s^{**} = t_{\ell+1} = s^*$.

Case 3: $i \in \{j_1, \ldots, j_m\}$, $\succeq = \bar{\succeq}$, and $P \notin \mathcal{D}^*$. It is sufficient to show $(P'_i, P_{-i}) \notin \mathcal{D}^*$, as in this case $s^* = s^{**}$ by Case 1. Suppose $(P'_i, P_{-i}) \in \mathcal{D}^*$. Let $i = j_\ell$ for some $\ell \in \{1, \ldots, m\}$.

We first show $s^* = t_\ell$. Note that as $P \notin \mathcal{D}^*$, $C_i(P, \succeq) = \{i\}$ and $DA^\tau_i(P, \succeq) = s^*$. Since $U(P'_i, s^*) \subseteq U(P_i, s^*)$, by strategy-proofness of DA^τ, $DA^\tau_i(P'_i, P_{-i}, \succeq) = s^*$. On the other hand, since $(P'_i, P_{-i}) \in \mathcal{D}^*$, by Step 1, $DA^\tau_i(P'_i, P_{-i}, \succeq) = t_\ell$. Hence $s^* = t_\ell$.

We finally obtain $P_i \in \mathcal{P}^\ell$, a contradiction. By Step 2, it is sufficient to show that $t_{\ell+1}\ P_i\ t_\ell$ and for each $u \in S \setminus T^{mp}$, $t_{\ell+1}\ P_i\ u$. Fix $u \in S \setminus T^{mp}$. Note that $t_{\ell+1} \in U(P'_i, s^*) \subseteq U(P_i, s^*)$ and $s^* = t_\ell$. Thus $t_{\ell+1}\, P_i\, t_\ell$. Moreover, $t_{\ell+1}\, P_i\, u$ by the following reasoning. Suppose $u\ P_i\ t_{\ell+1}$. Since the DA matching is student-optimal stable, we would have $DA^\tau(P, \succ) = DA^\tau(P'_i, P_{-i}, \succeq)$ and thus for some $i_{n+1} \in I_{n+1}$, $DA^\tau_{i_{n+1}}(P'_i, P_{-i}, \succeq) = u$. Hence, as $u\ P_i\ t_{\ell+1}\ P_i\ t_\ell$ and $DA^\tau_i(P'_i, P_{-i}, \succeq) = t_\ell$, u is weakly underdemanded at $(P'_i, P_{-i}, \succeq; \tau)$. Moreover, as $P_{i_{n+1}} \in \mathcal{P}^1 \cap \mathcal{P}(T^{mp})$, we have for each $k \in \{1, \ldots, m\}$, $t_k\ P_{i_{n+1}}\ u = DA^\tau_{i_{n+1}}(P'_i, P_{-i}, \succeq)$. This implies that all schools in T^{mp} are weakly underdemanded at $(P'_i, P_{-i}, \succeq; \tau)$. Hence all schools in $T^{mp} \cup \{u\}$ are weakly underdemanded at $(P'_i, P_{-i}, \succeq; \tau)$, which violates Lemma 1-(2). This completes Step 3. □

C.2 Proof of the impossibility result

We show the impossibility result. Let $\mathcal{D}^{\max} \subsetneq \mathcal{D} \subseteq \mathcal{P}^I$. Suppose that a mechanism φ dominates DA^τ on $\mathcal{D}$. We start with several claims. To this end, we introduce a notation $\mathcal{Q}_i(P, \succeq)$: For each $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ and each $i \in I$ with $\varphi_i(P, \succeq) \neq DA^\tau_i(P, \succeq)$, let $\mathcal{Q}_i(P, \succeq)$ be the set of preferences $P'_i \in \mathcal{P}$ such that for some weakly underdemanded school s^u at $(P, \succeq; \tau)$, P'_i upgrades s^u above $DA^\tau_i(P, \succeq)$ in P_i, and $s^u = DA^\tau_i(P'_i, P_{-i}, \succeq)$ is weakly underdemanded at $(P'_i, P_{-i}, \succeq; \tau)$.

Claim 4. Let $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ and $i \in I$ with $\varphi_i(P, \succeq) \neq DA^\tau_i(P, \succeq)$. Then $\mathcal{Q}_i(P, \succeq) \neq \emptyset$.

Proof. By Lemma 2, school $DA^\tau_i(P, \succeq)$ is overdemanded at $(P, \succeq; \tau)$. Then Claim 4 follows from Lemma 3. □

Claim 5. Let $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ and $i \in I$ with $\varphi_i(P, \succeq) \neq DA^\tau_i(P, \succeq)$. If $\mathcal{Q}_i(P, \succeq) \subseteq \mathcal{D}_i$, then φ is not strategy-proof.

Proof. Suppose $\mathcal{Q}_i(P, \succeq) \subseteq \mathcal{D}_i$. By Claim 4 there is $P'_i \in \mathcal{Q}_i(P, \succeq)$. Then some school $s^u = DA^\tau_i(P'_i, P_{-i}, \succeq)$ is weakly underdemanded at $(P'_i, P_{-i}, \succeq; \tau)$. By Lemma 2, $\varphi_i(P'_i, P_{-i}, \succeq) = DA^\tau_i(P'_i, P_{-i}, \succeq) = s^u$. Moreover, since $\varphi_i(P, \succeq)\ P_i\ DA^\tau_i(P, \succeq)$ and P'_i upgrades s^u above $DA^\tau_i(P, \succeq)$, we have $\varphi_i(P, \succeq)\ P'_i\ s^u$. Thus $\varphi_i(P, \succeq)\ P'_i\ \varphi_i(P'_i, P_{-i}, \succeq)$. Hence, since $P \in \mathcal{D}$ and $P'_i \in \mathcal{Q}_i(P, \succeq) \subseteq \mathcal{D}_i$, φ is not strategy-proof on $\mathcal{D}$. □

Claim 6. Let $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ and $i \in I$ with $\varphi_i(P, \succeq) \neq DA^\tau_i(P, \succeq)$. If $i \in I \setminus J$, then φ is not strategy-proof.

Proof. Let $i \in I \setminus J$. By Claim 5, it suffices to show $\mathcal{Q}_i(P, \succeq) \subseteq \mathcal{D}_i$. Since $i \in I \setminus J$, $\mathcal{D}_i = \mathcal{P}$. Thus, since $\mathcal{Q}_i(P, \succeq) \subseteq \mathcal{P}$, we have $\mathcal{Q}_i(P, \succeq) \subseteq \mathcal{D}_i$. □

Claim 7. Let $(P, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ and $\ell \in \{1, \ldots, m\}$ with $\varphi_{j_\ell}(P, \succeq) \neq DA^\tau_{j_\ell}(P, \succeq)$. If $DA^\tau_{j_\ell}(P, \succeq) \neq t_\ell$ or $\varphi_{j_\ell}(P, \succeq) \neq t_{\ell+1}$, then φ is not strategy-proof.

Proof. Suppose $DA^\tau_{j_\ell}(P, \succeq) \neq t_\ell$. By Claim 5 it suffices to show $\mathcal{Q}_{j_\ell}(P, \succeq) \subseteq \mathcal{D}_{j_\ell}$. Let $P'_i \in \mathcal{Q}_i(P, \succeq)$. Then for some weakly underdemanded school s^u at $(P, \succeq; \tau)$, P'_i upgrades s^u above $DA^\tau_i(P, \succeq)$ in P_i. Note that $\mathcal{D}^{\max}_{j_\ell} = \mathcal{P} \setminus \mathcal{P}(t_{\ell+1}, t_\ell, T^{mp}) \subseteq \mathcal{D}_{j_\ell}$ and $s^u \in S$. Since $DA^\tau_{j_\ell}(P, \succeq) \neq t_\ell$ and P'_{j_ℓ} upgrades s^u above $DA^\tau_{j_\ell}(P, \succeq)$, $P'_{j_\ell} \in \mathcal{P} \setminus \mathcal{P}(t_{\ell+1}, t_\ell, T^{mp}) \subseteq \mathcal{D}_{j_\ell}$. Hence $\mathcal{Q}_{j_\ell}(P, \succeq) \subseteq \mathcal{D}_{j_\ell}$.

Suppose on the contrary that $\varphi_{j_\ell}(P, \succeq) \neq t_{\ell+1}$ but φ is strategy-proof. Then it follows from what we just proved that $DA^\tau_{j_\ell}(P, \succeq) = t_\ell$. Since $t_\ell \neq t_{\ell+1}$, we have two cases: $t_\ell\ P_{j_\ell}\ t_{\ell+1}$ and $t_{\ell+1}\ P_{j_\ell}\ t_\ell$.

<u>Case 1:</u> $t_\ell\ P_{j_\ell}\ t_{\ell+1}$. Let $P'_{j_\ell} \in \mathcal{Q}_{j_\ell}(P, \succeq)$. Then $DA^\tau_{j_\ell}(P, \succeq) = t_\ell\ P_{j_\ell}\ t_{\ell+1}$. Thus, since P'_{j_ℓ} upgrades some s^u above $DA^\tau_{j_\ell}(P, \succeq) = t_\ell$, we have $P'_{j_\ell} \in \mathcal{P} \setminus \mathcal{P}(t_{\ell+1}, t_\ell, T^{mp}) = \mathcal{D}^{\max}_{j_\ell} \subseteq \mathcal{D}_{j_\ell}$. Thus $\mathcal{Q}_{j_\ell}(P, \succeq) \subseteq \mathcal{D}_{j_\ell}$. By Claim 5, φ is not strategy-proof. This is a contradiction.

<u>Case 2:</u> $t_{\ell+1}\ P_{j_\ell}\ t_\ell$. Denote $\tilde{t} = \varphi_{j_\ell}(P, \succeq)$. Let $\tilde{P}_{j_\ell} \in \mathcal{P}$ such that $t_{\ell+1}$ is put below t_ℓ and the relative ranking over the other schools in $\tilde{P}_{j_\ell}$ is kept the same as in P_{j_ℓ}. Then, since $\tilde{t} = \varphi_{j_\ell}(P, \succeq)\ P_{j_\ell}\ DA^\tau_{j_\ell}(P, \succeq) = t_\ell$ by the domination of φ over DA^τ, we have $U(\tilde{P}_{j_\ell}, \tilde{t}) \subseteq U(P_{j_\ell}, \tilde{t})$. By strategy-proofness of φ, $\varphi_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq)\ \tilde{R}_{j_\ell}\ \varphi_{j_\ell}(P, \succeq) \equiv \tilde{t}$. Thus, since $U(\tilde{P}_{j_\ell}, \tilde{t}) \subseteq U(P_{j_\ell}, \tilde{t})$, $\varphi_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq)\ R_{j_\ell}\ \tilde{t}$. Moreover, by strategy-proofness of φ, $\tilde{t} \equiv \varphi_{j_\ell}(P, \succeq)\ R_{j_\ell}\ \varphi_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq)$. Therefore, since preferences are strict, $\varphi_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq) = \tilde{t}$. Similarly, since $U(\tilde{P}_{j_\ell}, t_\ell) \subseteq U(P_{j_\ell}, t_\ell)$ and $DA^\tau_{j_\ell}(P, \succeq) = t_\ell$, the strategy-proofness of DA^τ implies $DA^\tau_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq) = t_\ell$.

Now, since $\varphi_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq) = \tilde{t} \neq t_\ell = DA^\tau_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq)$, $\mathcal{Q}_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq)$ is not empty by Claim 4. Let $P'_{j_\ell} \in \mathcal{Q}_{j_\ell}(\tilde{P}_{j_\ell}, P_{-j_\ell}, \succeq)$. Then P'_{j_ℓ} updates some school s^u above t_ℓ in $\tilde{P}_{j_\ell}$. Thus, as $t_\ell\ \tilde{P}_{j_\ell}\ t_{\ell+1}$, we have $t_\ell\ P'_{j_\ell}\ t_{\ell+1}$. Thus $P'_{j_\ell} \in \mathcal{P} \setminus \mathcal{P}(t_{\ell+1}, t_\ell, T^{mp}) = \mathcal{D}^{\max}_{j_\ell} \subseteq \mathcal{D}_{j_\ell}$. Thus, by Claim 5, φ is not strategy-proof. This is a contradiction. □

Finally we prove the impossibility result by contradiction. Suppose that mechanism φ is strategy-proof on $\mathcal{D}$. Then, by Lemma 5, there is a qualified cycle-selection $\mathcal{C}$ for (φ, DA^τ). Since it is qualified, $\mathcal{C}$ is nontrivial. Thus there is $(\hat{P}, \succeq) \in \mathcal{D} \times \mathcal{A}^S$ such that some cycle in $\mathcal{C}(\hat{P}, \succeq)$ is not trivial. For notational simplicity, as we keep $\succeq$ fixed, we omit $\succeq$ in the expression of mechanisms throughout the proof.

Step 1: We show that for each $P \in \mathcal{D}$ with $\varphi(P) \neq DA^\tau(P)$, $DA^\tau_{(j_1,\ldots,j_m)}(P) = (t_1, \ldots, t_m)$ and $\varphi_{(j_1,\ldots,j_m)}(P) = (t_2, \ldots, t_{m+1})$. Let $P \in \mathcal{D}$ with $\varphi(P) \neq DA^\tau(P)$. Then $\mathcal{C}(P, \succ)$ has a nontrivial cycle. By Claims 6 and 7, $\mathcal{C}$ has the unique cycle $(j_1, \ldots, j_m)$ for (φ, DA^τ), which leads to the desired result.

Step 2: For each $\ell \in \{1, \ldots, m\}$, let $P''_{j_\ell} \in \mathcal{P}(T^{mp})$ such that school $t_{\ell+1}$ is the top choice in S and school t_ℓ is the top-$|T^{mp}|$ choice in S. For each $\ell \in \{0, \ldots, m\}$, let $J_\ell := \emptyset$ for $\ell = 0$; and $J_\ell := \{j_1, \ldots, j_\ell\}$ otherwise. We show by induction on $\ell \in \{0, \ldots, m\}$ that $DA^\tau_{(j_1,\ldots,j_m)}(P''_{J_\ell}, \hat{P}_{-J_\ell}) = (t_1, \ldots, t_m)$ and $\varphi_{(j_1,\ldots,j_m)}(P''_{J_\ell}, \hat{P}_{-J_\ell}) = (t_2, \ldots, t_{m+1})$.

Since the claim is trivial for $\ell = 0$, suppose that the claim is true up to $\ell - 1$, that is, $DA^\tau_{(j_1,\ldots,j_m)}(P''_{J_{\ell-1}}, \hat{P}_{-J_{\ell-1}}) = (t_1, \ldots, t_m)$ and $\varphi_{(j_1,\ldots,j_m)}(P''_{J_{\ell-1}}, \hat{P}_{-J_{\ell-1}}) = (t_2, \ldots, t_{m+1})$. By strategy-proofness of φ, $\varphi_{j_\ell}(P''_{J_\ell}, \hat{P}_{-J_\ell}) \ R''_{j_\ell} \ \varphi_{j_\ell}(P''_{J_{\ell-1}}, \hat{P}_{-J_{\ell-1}}) = t_{\ell+1}$. Thus, as $t_{\ell+1}$ is the top choice in S under P''_{j_ℓ}, $\varphi_{j_\ell}(P''_{J_\ell}, \hat{P}_{-J_\ell}) = t_{\ell+1}$. Also, by strategy-proofness of DA^τ, $DA^\tau_{j_\ell}(P''_{J_{\ell-1}}, \hat{P}_{-J_{\ell-1}}) \ \hat{R}_{j_\ell} \ DA^\tau_{j_\ell}(P''_{J_\ell}, \hat{P}_{-J_\ell})$. Thus, since $t_{\ell+1} \ \hat{P}_{j_1} \ t_\ell = DA^\tau_{j_\ell}(P''_{J_{\ell-1}}, \hat{P}_{-J_{\ell-1}})$, we have $DA^\tau_{j_\ell}(P''_{J_\ell}, \hat{P}_{-J_\ell}) \neq t_{\ell+1}$. Therefore, $\varphi_{j_\ell}(P''_{J_\ell}, \hat{P}_{-J_\ell}) \neq DA^\tau_{j_\ell}(P''_{J_\ell}, \hat{P}_{-J_\ell})$. By Step 1, we have the desired result.

Step 3: We finally show a contradiction. Since $\mathcal{D}^{\max} \subsetneq \mathcal{D} \subseteq \mathcal{P}^I$, for each $i \in I \setminus J$, $\mathcal{D}_i = \mathcal{P}$ and thus we may assume without loss of generality that $\mathcal{D}^{\max}_{j_1} \subsetneq \mathcal{D}_{j_1}$. Then there is $P'_{j_1} \in \mathcal{D}_{j_1} \setminus \mathcal{D}^{\max}_{j_1} \subseteq \mathcal{P}(t_2, t_1, T^{mp})$. Thus for some $u \in S \setminus T^{mp}$, $t_2 \ P'_{j_1} \ u \ P'_{j_1} \ t_1$. There are two cases:
<u>Case 1:</u> $\varphi(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) \neq DA^\tau(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J})$. Then, by Step 1, $DA^\tau_{(j_1,\ldots,j_m)}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) = (t_1, \ldots, t_m)$ and $\varphi_{(j_1,\ldots,j_m)}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) = (t_2, \ldots, t_{m+1})$. Thus j_2 is assigned the last school t_2 in T^{mp} under DA, and thus each school in $T^{mp} \setminus \{t_2\}$ is overdemanded at $(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}, \succeq; \tau)$. Moreover, by our assumption, $t_2 \ P'_{j_1} \ u \ P'_{j_1} \ t_1 = DA^\tau_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J})$, and thus schools t_2 and u are overdemanded at $(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}, \succeq; \tau)$. Hence all schools in T^{mp} and school $u \in S \setminus T^{mp}$ are overdemanded at the same problem, which contradicts Lemma 1-(2).
<u>Case 2:</u> $\varphi(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) = DA^\tau(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J})$. By strategy-proofness of DA^τ, $DA^\tau_{j_1}(P''_J, \hat{P}_{-J}) \ R''_{j_1} \ DA^\tau_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J})$. Thus, since school $DA^\tau_{j_1}(P''_J, \hat{P}_{-J}) = t_1$ (the equality follows from Step 2) is the last choice in T^{mp} under P''_{j_1}, we have $DA^\tau_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) \in \{t_1\} \cup (S \setminus T^{mp})$. Thus, since $U(P'_{j_1}, t_2) \subseteq T^{mp} \setminus \{t_1\}$ ($\because$ as $P'_{j_1} \in \mathcal{P}(t_2, t_1, T^{mp})$, $t_2 \ P'_{j_1} \ t_1$ and for each $u' \in S \setminus T^{mp}$, $t_2 \ P'_{j_1} \ u'$), we have

$$DA^\tau_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) \notin U(P'_{j_1}, t_2). \tag{3}$$

On the other hand, by strategy-proofness of φ, $\varphi_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) \ R'_{j_1} \ \varphi_{j_1}(P''_J, \hat{P}_{-J})$. Thus, since $\varphi_{j_1}(P''_J, \hat{P}_{-J}) = t_2$ by Step 2, we have $\varphi_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) \in U(P'_{j_1}, t_2)$. Hence, by (3), $\varphi_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J}) \neq DA^\tau_{j_1}(P'_{j_1}, P''_{J\setminus\{j_1\}}, \hat{P}_{-J})$, which is a contradiction. ■

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