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The Impact of Rising Income Inequality on Rent Distribution in Housing Markets with Indivisibilities

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The Impact of Rising Income Inequality on Rent Distribution in Housing Markets with Indivisibilities

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Abstract

This paper studies the impact of rising income inequality on a rent distribution of rental housings. The model is based on the assignment market model with a finite number of households and apartments. We show that when household income inequality increases, then (i) a rent rises at every apartment, (ii) rise at apartments of higher quality but declines at lower quality or (iii) decline at every apartment. These cases are characterized by the location of the specific household who divides households into income-increased/declined groups. This characterization implies (i) is a special case. Numerical examples confirm our results. We also discuss equitability of competitive allocations in our market model.

1 Introduction

We present the impact of an increase in household income inequality on apartment rents. The market model we adopt is the rental housing market model by Kaneko, Ito and Osawa (2006).

The model by Kaneko et al. is an application of the assignment market without the assumption of quasi-linear utility functions. In the model, the market participants are divided into households and landlords. Each household demands at most one apartment unit and each landlord provides some apartment units. The apartments as indivisible commodities are classified into finite categories $1, ..., T$ based on their qualities. The goods other than apartments are aggregated and consumed as composite good (money). Household utility function is assumed to be homogeneous, and allows income effect on housing qualities.

It is known that this model guarantees the existence of a competitive equilibrium (Kaneko, 1982; Kaneko and Yamamoto, 1986). In particular, under our assumptions on the utility functions, a household with a higher income rents an apartment of a better category than a household with lower income at any equilibrium (Proposition 2.1). We can then represents the maximum competitive rent vector by a solution of a certain system of equations. This system

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of equations and its solution are called the *rent equation* and the *differential rent vector*. In our analysis, we directly consider a differential rent vector rather than the competitive rent vector.

Here, we briefly introduce our comparative statics result. The effect of an increase in income inequality is divided by three cases: (i) rise in rent at every category, (ii) decline at every category or (iii) rise at higher categories and decline at lower categories. Cases (i) and (ii) are counterintuitive because rising income inequality seems to cause decline in rents at worse apartments and rise in rents at better apartments. Indeed, we show that (i) is said to be an extreme case, while (ii) [and also (iii)] may possible. Three cases (i)-(iii) are associated with the location of household who divides the households into the income-increased group and the decreased group.

Here, we introduce related literatures. Kaneko et al. (2006) studied effects of changes in incomes of boundary households on a competitive rent vector. The boundary household is defined for each category of apartments, and play a crucial role in the model. The authors showed that when the boundary income difference is larger (smaller) for a better category of apartments, the rent difference forms convex (concave) shape.

Ito (2007) presented the effects of rise in only the boundary household income of category k on competitive rents, under a more restricted assumption on a utility function. The author showed that rents are unchanged at $k+1, ..., T$, increase at $1, ..., k$ and a rent difference of each category $1, ..., k-2$ is smaller for a better category of apartments.

Määttänen and Terviö (2014) studied the effect of rising income inequality on house prices in the one-sided assignment model. One-sided means that the agents are potentially seller and buyer. The authors assume a continuum of agents and housing types (thus, an analytical method is calculus), and the homogeneity and normality on the utility functions. The authors presented a similar result to our main result with the exclusion of the case (i). Braid (1981) also studied the effects of parameter changes on rent distributions under the two-sided version of Määttänen and Terviö's framework.

This paper is organized as follows. Section 2 formulates our rental housing market model and gives our definition of competitive equilibrium. Next we introduce the rent equation and differential rent vector. Section 3 examines the impact of rising income inequality on a rent distribution. Numerical examples confirm our theorems. Section 4 studies equitability property of competitive allocations in our model. Section 5 presents our conclusions and some remarks.

2 The market model

The *rental housing market model* (Kaneko et al., 2006) is denoted by (M, N), where the symbol $M = \{1, \ldots, m\}$ denotes the set of *households*, and $N = \{1, \ldots, T\}$ denotes the set of *landlords*. The objects of trade are apartments (indivisible) and money (perfectly divisible).

The apartments are classified into finite T number of categories by their housing attributes (e.g., a housing size and a commuting time). Each landlord $k \in N$ supplies units of apartments of the k-th category (thus k is the only landlord providing the k-th apartments).[1]

Each household $i \in M$ initially has an income $I_i > 0$ but no dwelling. He wants to live in some apartment by paying rent from his income. Without loss of generality, we can assume that the households are ordered in their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. The *consumption set* is written by $X := \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$, where $\mathbf{e}^k$ is the T-dimensional unit vector with k-th component is 1 ($\mathbf{e}^0 = \mathbf{0}$), and $\mathbb{R}_+$ is the set of nonnegative real numbers. A *consumption bundle* $(\mathbf{e}^k, c) \in X$ with $k \neq 0$ means that household i rents one apartment unit of category k and enjoys the consumption $c = I_i - p_k$ paying rent p_k of category k. For $k = 0$, no apartment is consumed. An initial endowment of $i \in M$ is given as $(\mathbf{e}^0, I_i)$ with $I_i > 0$. Each household has an *identical utility function* $u : X \to \mathbb{R}$ satisfying the following assumption:

Assumption A. For each $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$, $u(x, c)$ is a continuous and strictly monotone function of c, and $u(\mathbf{e}^0, I_i) > u(\mathbf{e}^k, 0)$ for all $k = 1, ..., T$.

The identical utility function implies that a housing market (M, N) represents a monocentric city, and every households commute to the same business district. In Assumption A, continuity and monotonicity on money are standards; the latter inequality means the indispensability of money. We also assume the following: B-D on $u(\cdot, \cdot)$.

Assumption B. If $u(x_i, c) = u(x_i', c')$, and $c < c'$, then $u(x_i, c + \delta) > u(x_i', c' + \delta)$ for any $\delta > 0$.

Assumption C. If $u(x_i, c) > u(x_i', c')$, then $u(x_i, c) = u(x_i', c' + \delta)$ for some $\delta > 0$.

Assumption D. $u(\mathbf{e}^1, 0) > u(\mathbf{e}^2, 0) > \cdots > u(\mathbf{e}^T, 0)$.

Assumption B is the *normality* assumption on the quality of apartments in the following sense. In B, the k-th apartment has a better quality than k', since living in k with smaller consumption c is indifferent to living in k' with larger c'. When an income is increased by the same magnitude $\delta > 0$, the household strictly demands better apartment. The normality implies that even if we assume the identical utility function, households having different incomes demand different qualities of apartments. Assumption C means that housing quality of an apartment is substitutable for money. Assumption D means that the apartment qualities are strictly ordered by the numerical order.[2]

We next define the seller side. Each landlord $k \in N = \{1, ..., T\}$ provides apartments of k-th category. The landlord has a *cost function* $C_k(y_k) : \mathbb{Z}_+ \to \mathbb{R}_+$, where $\mathbb{Z}_+$ is the set of nonnegative integers. For each $y_k \in \mathbb{Z}_+$, $C_k(y_k)$ represents the cost (in terms of money) of

[1] The original model of Kaneko et al (2006) assume that $|N| \geq T$ and there are more than one seller providing apartments of type k $(= 1, ..., T)$. As far as competitive equilibrium is concerned, we can assume *without of generality* that only one seller provides apartments of type $k (= 1, ..., T)$ (thus the set N becomes $N = \{1, ..., T\}$). See Section 5 of Sai (2014).

[2] Assumption D together with Assumptions A, B and C imply that $u(\mathbf{e}^1, c) > u(\mathbf{e}^2, c) > \cdots > u(\mathbf{e}^T, c)$ for all $c \in \mathbb{R}_+$.

supplying y_k units of apartments of k-th category. In this study, we employ the following simple form of $C_k(\cdot)$.

Assumption E. For each $k \in N$, $C_k(y_k)$ is expressed as

$$C_k(y_k) = \begin{cases} a_k y_k & \text{if } y_k \leq w_k, \\ \text{“large”} & \text{if } y_k \geq w_k + 1, \end{cases}$$

In Assumption E, the constant $a_k > 0$ is the marginal cost of providing additional unit. The “large” is a sufficiently large number. The remaining constant w_k is the number of all apartment units owned by landlord k. This cost function means that landlord k supplies units up to w_k with the constant marginal cost a_k, while he never supplies more than w_k units since the cost to build a new one is very large relative to the market.[3].

We define a competitive equilibrium in (M, N). Let $p \in \mathbb{R}^T_+$ be the price vector, $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}^m$ be the demand vector and $y \in \mathbb{Z}^T_+$ be the supply vector. A triple (p, x, y) is a *competitive equilibrium* iff

(UM): for all $i \in M$, (i) $I_i - px_i \geq 0$, where $px_i = \sum_{k=1}^T p_k x_{ik}$;
(ii) $u(x_i, I_i - px_i) \geq u(x'_i, I_i - px'_i)$ for all $x'_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$ with $I_i - px_i \geq 0$.

(PM): for all $k \in N$, $p_k y_k - C_k(y_k) \geq p_k y'_k - C_k(y'_k)$ for all $y'_k \in \mathbb{Z}_+$.

(BDS): $\sum_{i \in M} x_i = \sum_{k=1}^T y_k \mathbf{e}^k$.

There exists a competitive equilibrium (p, x, y) in (M, N) (Kaneko and Yamamoto, 1986), the maximum and minimum competitive rent vectors (Kaneko et al., 2006; Sai, 2015).[4,5] In our analysis, we focus on the maximum competitive rent vector. This rent vector is calculated by the solution of a certain system of equations called the *rent equation*.[6] The following proposition is necessary to define the rent equation.

Proposition 2.1 (Kaneko et al., 2006). *Let (p, x, y) be a competitive equilibrium. Then,*
(1) If $k < k'$ and $x_i = \mathbf{e}^{k'}$ for some i, then $p_k > p_{k'}$.
(2) If $x_i = \mathbf{e}^k$, $x_j = \mathbf{e}^{k'}$ and $I_i > I_j$ for some i, j, then $k \leq k'$.

This states that in any competitive equilibrium, (1) the price of a better apartment is higher than a worse one, and (2) a household with a higher income rents a better apartment. Note that Proposition 2.1.(1) does not exclude the case of $y_k = 0$. The following assumption eliminates such a case.

Assumption F. Let (p, x, y) be a competitive equilibrium. Then there exists some category f such that $y_k > 0$ for $k = 1, ..., f$ and $y_k = 0$ for $k = f + 1, ..., T$.

[3] In this sense, our approach is short-run analysis.

[4] A vector $p \in \mathbb{R}^T_+$ is a *competitive price vector* iff (p, x, y) is a competitive equilibrium, and p is the *maximum* (*minimum*) *competitive price vector* iff $p \geq p'$ ($p \leq p'$) for any competitive price vector p'.

[5] Indeed, these existence theorems are guaranteed only under Assumptions A and E.

[6] Instead of the maximum one, we may focus on the minimum competitive rent vector. It follows from Sai (2014) and/or Sai (2015) that the difference between $p^{\max}$ and $p^{\min}$ is rather small when a market is thick with landlords and/or households.

We call this f the *marginal category.* By Proposition 2.1.(1) and Assumption F, we have $p_1 > p_2 > \cdots > p_f$.

Recall that the households $1, ..., m$ are ordered by their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. We define the household with the lowest income in each active category. Let (p, x, y) be a maximum competitive equilibrium. For each category $k \leq f$, we define the household $G(k)$ with the lowest income in the k-th category as:

$$G(k) := \sum_{t=1}^{k} y_t.$$

For each k, we call $G(k)$ the *boundary household* of the k-th category.

The rent equation (Kaneko et al., 2006) is defined as the system of equations with unknowns $r_1, ..., r_f$:

$$\left.\begin{array}{r} u(\mathbf{e}^{f-1}, I_{G(f-1)} - r_{f-1}) = u(\mathbf{e}^{f}, I_{G(f-1)} - r_f), \\ u(\mathbf{e}^{f-2}, I_{G(f-2)} - r_{f-2}) = u(\mathbf{e}^{f-1}, I_{G(f-2)} - r_{f-1}), \\ \vdots \qquad\qquad \\ u(\mathbf{e}^{1}, I_{G(1)} - r_1) = u(\mathbf{e}^{2}, I_{G(1)} - r_2). \end{array}\right\} \tag{3.1}$$

Note that the rent equation (3.1) has f unknowns, while which is constituted by $f - 1$ equations. Eq. (3.1) states that a household $G(k)$ is indifferent between renting the $k+1$-th apartment at rent r_{k+1} and renting the k-th category at r_k. In Eq. (3.1), if the rent of marginal category r_f is given, the first equation of Eqs. (3.1) determines r_{f-1}. In the same manner, the remaining rents $r_{f-2}, ..., r_1$ are recursively determined. We call a solution $(r_1, ..., r_f)$ of Eq. (3.1) a *differential rent vector.* Under our assumptions, if r_f is given with $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)} - r_f)$, then a differential rent vector is uniquely determined and satisfies $r_1 > \cdots > r_{f-1} > r_f$.

We conclude this section by noting the relation between a differential rent vector and a competitive rent vector. Let $p = (p_1, ..., p_T)$ be the maximum competitive rent vector and $(r_1, ..., r_f)$ is a differential rent vector given by $r_f \geq p_f$. Then, it holds that $r_k \geq p_k$ for all $k = 1, ..., f$ (Theorem 3.1 by Sai, 2015). In particular, if $r_f = p_f$ and some condition holds, then $r_k = p_k$ for all $k = 1, ..., f$.[7] Hereafter, we use a differential rent vector for a comparative statics.

[7] They are two conditions by Kaneko et al. (2006), Theorem 2.6: (1) $I_{G(k)} = I_{G(k)+1}$ *for each* $k = 1, ..., f-1$; (2) $p_k < C_k(y_k + 1) - C_k(y_k)$ *for each* $k = 1, ..., f-1$.

Under our assumption E on the cost function, the condition (2) holds because the cost of additional unit from w_k, $C_k(w_k + 1) =$"large". Even when neither conditions hold, a differential rent vector can be an approximation of the maximum competitive rent vector. See Sai (2015), Section 3.1.

3 The Impact of an increase in Income Inequalities on Competitive Rents

3.1 Comparative statics

In this section, we study the relation between household income distribution and competitive equilibria. The main purpose is to explain how rising income inequality affects a competitive rent distribution. Recall that the households M are ordered by their income levels as $I_1 \geq \cdots \geq I_m$. Here, we consider a new market where only the household incomes change. Precisely, $\{I_1, \cdots, I_m\}$ changes to $\{\widehat{I}_1, \cdots, \widehat{I}_m\}$, but the remaining parameters, the sets M, N, utility and cost functions $u(\cdot, \cdot)$, $c_k(\cdot)$, the marginal category f and the marginal rent r_f are unchanged.[8] By assumption E, the supply amount of each category $1, ..., f$ is also unchanged in the market. Therefore, the boundary household $G(k) = \sum_{t=1}^{k} w_t$ $(k = 1, ..., f)$ remains the same. We consider the following condition on the household incomes.

Condition InE (*Increase in Income Inequality*). There exists a household $i^* \in M \backslash \{m\}$ such that $I_i < \widehat{I}_i$ for $i \in \{1, \cdots, i^*\}$ and $I_i > \widehat{I}_i$ for $i \in \{i^* + 1, \cdots, m\}$, and $\sum_{i \in M}(I_i - \widehat{I}_i) = 0$.

This condition states that in the new market, an income increases at upper households than $i^* + 1$ and declines at lower households than i^*, preserving the gross income.

Let $(r_1, \cdots, r_{f-1}, r_f)$ and $(\widehat{r}_1, \cdots, \widehat{r}_{f-1}, r_f)$ be differential rent vectors in the original and new markets determined by r_f with $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, \widehat{I}_{G(f-1)} - r_f)$. In the next theorem, we examine how the new rent vector $(\widehat{r}_1, \cdots, \widehat{r}_{f-1})$ changes under Condition InE (the proof will be given in Section 3.2).

Theorem 3.1 (*The Possible Cases of Rent Change*). *Under Condition InE, either* (1), (2) *or* (3) *holds*:

(1) $r_k < \widehat{r}_k$ *for* $k = 1, ..., f - 1$.

(2) *There exist a category* $k^* (\leq f - 2)$ *such that* $\begin{cases} r_k < \widehat{r}_k \text{ for } k = 1, ..., k^* - 1, \\ r_{k^*} \leq \widehat{r}_{k^*}, \\ r_k > \widehat{r}_k \text{ for } k = k^* + 1, ..., f - 1. \end{cases}$

(3) $r_k > \widehat{r}_k$ *for* $k = 1, ..., f - 1$.

This theorem shows three possibilities of rent change when income inequality increases. Theorems 3.1.(1) and (3) are straightforward: (1) [(3), respectively] states that a rent rises (declines) at every category $1, ..., f - 1$ in the new market. Thus, an average rent rises (falls). The remaining (2) states that a rent increases at upper categories $1, ..., k^*$, declines at lower categories $k^* + 1, ..., f - 1$. The illustration of (2) is depicted in Fig. 1.

One may think Theorem 3.1.(1) and (3) are counterintuitive: it is natural that rising income inequality causes decline in rents at lower categories and rise in rents at upper categories [case (2)]. Indeed, in the next theorem we show (1) is an extreme case; on the other hand, we

[8] We assume $\{\widehat{I}_1, \cdots, \widehat{I}_m\}$ also satisfies $\widehat{I}_1 \geq \cdots \geq \widehat{I}_m$

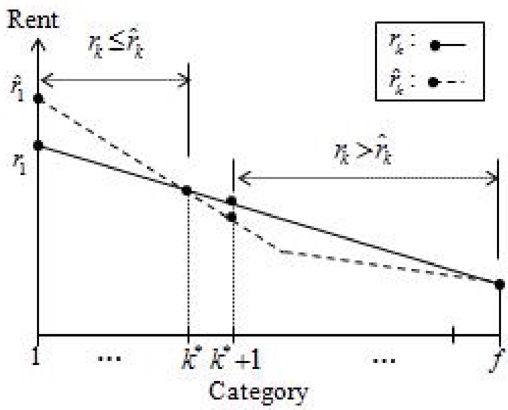

Figure 1: An illustration of Theorem 3.1.(2).

also show (3) is a common case.

Theorem 3.2 (*Location of Household i^* and Rent Change*). *Under Condition InE, the following holds*:

(1) $G(f-1) \leq i^*$ *implies Theorem 3.1.(1).*

(2) $G(1) \leq i^* < G(f-1)$ *implies Theorem 3.1.(2) or (3).*

(3) $i^* < G(1)$ *implies Theorem 3.1.(3).*

This theorem characterizes three cases of Theorem 3.1 by the location of household i^* of Condition InE. In Theorem 3.2.(1), the inequality $G(f-1) \leq i^*$ implies that a boundary income of every category rises, i.e., $I_{G(k)} < \widehat{I}_{G(k)}$ for every $k = 1, ..., f-1$. In this case, a differential rent $\widehat{r}_k$ rises for every $k = f-1, ..., 1$. (3) is also understood in a similar manner. The remaining (2) is the case that household i^* is located below $G(1)$ and above $G(f-1)$. In this case, there are two possibilities: (i) decline in rents at every category or (ii) rise in rents above some category k^* and decline below k^*.

The condition $G(f-1) \leq i^*$ of Theorem 3.2.(1) is an extreme case in that every *income declined* household is assigned to the marginal category f and the income declined segment $\{\widehat{I}_{l+1}, ..., \widehat{I}_m\}$ is irrelevant to the determination of rents $\widehat{r}_1, ..., \widehat{r}_{f-1}$. The condition $i^* < G(1)$ of Theorem 3.2.(3) is another extreme case in that every *income increased* household is assigned to the first category 1. Theorem 3.2 states that even if we eliminate the case $i^* < G(1)$, Theorem 3.1.(2) still possible.

Now, we compare our results and other related studies. Kaneko et al. (2006) studied effects of changes in boundary incomes on a differential rent vector. In particular, they considered the case: $\widehat{I}_{G(f-1)} - I_{G(f-1)} \leq \widehat{I}_{G(f-2)} - I_{G(f-2)} \leq \cdots \leq \widehat{I}_{G(1)} - I_{G(1)}$, i.e., the boundary income increment is larger for a better category of apartments.[9] We can apply their condition to our

[9]They also considered the opposite case: $\widehat{I}_{G(f-1)} - I_{G(f-1)} \geq \widehat{I}_{G(f-2)} - I_{G(f-2)} \geq \cdots \geq \widehat{I}_{G(1)} - I_{G(1)}$.

Condition InE as follows:

$$\widehat{I}_{G(f-1)} - I_{G(f-1)} \leq \cdots \leq \widehat{I}_{G(\overline{k})} - I_{G(\overline{k})} < 0 < \widehat{I}_{G(\overline{k}-1)} - I_{G(\overline{k}-1)} \leq \cdots \leq \widehat{I}_{G(1)} - I_{G(1)}$$
$$\text{for some } \overline{k} \in \{2, ..., f-1\}.$$

This could be understood as the income inequality significantly increases. Then, by Theorem 5.2.(1) and Corollary 6.2.(1) (Kaneko et al., p.160 and p.162), the rent differences form convex shape

$$0 < \widehat{r}_{f-1} - r_{f-1} < \cdots < \widehat{r}_{k_1} - r_{k_1} = \cdots = \widehat{r}_{k_2} - r_{k_2} < \cdots < \widehat{r}_1 - r_1,$$
$$\text{where } \overline{k} \leq k_2 \leq k_1 \leq f-1,$$

that is, the decrement of $\widehat{r}_k$ to r_k gradually gets large from category $f-1$ to k_1 (and takes maximal from k_1 to k_2, and gradually decrease as

Määttänen and Terviö (2014) also studied the effect of rising income inequality on house prices by using the one-sided assignment model. Their model assume a continuum of agents and housing types, and the homogeneity and normality on the utility functions. Their main result (Proposition 4, p.391) is essentially the same as our Theorem 3.1 with the exclusion of the case (1).[10] Nevertheless, their analytical method is different from ours in that they uses a calculus for analyses, while our model is based on finiteness.

3.2 Proofs of Theorem 3.1 and Theorem 3.2

It suffices to prove Theorem 3.2.

Proof of Theorem 3.2.(1). Suppose $G(f-1) \leq i^*$, i.e., $I_{G(k)} < \widehat{I}_{G(k)}$ for every $k = 1, ..., f-1$. We prove this by mathematical induction over $f-1, ..., 1$. Let $\delta = \widehat{I}_{G(f-1)} - I_{G(f-1)} > 0$. the rent equation (3.1) and the normality assumption (Assumption B) imply

$$u(\mathbf{e}^{f-1}, I_{G(f-1)} - r_{f-1} + \delta) > u(\mathbf{e}^{f}, I_{G(f-1)} - r_f + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{f-1}, \widehat{I}_{G(f-1)} - r_{f-1}) &> u(\mathbf{e}^{f}, \widehat{I}_{G(f-1)} - r_f) \\ &= u(\mathbf{e}^{f-1}, \widehat{I}_{G(f-1)} - \widehat{r}_{f-1}) \text{ by Eqs. (3.1).} \end{aligned}$$

This inequality and the monotonicity (Assumption A) imply $\widehat{I}_{G(f-1)} - r_{f-1} > \widehat{I}_{G(f-1)} - \widehat{r}_{f-1}$, that is, $r_{f-1} < \widehat{r}_{f-1}$.

Suppose $r_k < \widehat{r}_k$ for k with $1 < k \leq f-1$. Then we show this relation also holds for $k-1$. Let $\delta = \widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k)$ and suppose $\delta > 0$. Then, Eqs. (3.1) and Assumption

[10] Their condition on the income distribution excludes the occurrence of antecedents of Theorem 3.2.(1) and (3).

B imply

$$u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1} + \delta) > u(\mathbf{e}^{k}, I_{G(k-1)} - r_k + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k) &> u(\mathbf{e}^{k}, \widehat{I}_{G(k-1)} - \widehat{r}_k) \\ &= u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}) \text{ by Eqs. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $\widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k > \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}$, that is, $\widehat{r}_{k-1} - r_{k-1} > \widehat{r}_k - r_k > 0$.

Suppose the other case $\delta \leq 0$. Then, Assumption A imply $u(\mathbf{e}^{k}, I_{G(k-1)} - r_k) \geq u(\mathbf{e}^{k}, \widehat{I}_{G(k-1)} - \widehat{r}_k)$. Since the left hand side equals $u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$ and the right hand side equals $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$ by Eqs. (3.1), we have $u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1}) \geq u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$. Again, by Assumption A, we have $I_{G(k-1)} - r_{k-1} \geq \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}$, that is, $\widehat{r}_{k-1} - r_{k-1} \geq \widehat{I}_{G(k-1)} - I_{G(k-1)} > 0$. Hence we obtain $r_{k-1} < \widehat{r}_{k-1}$. ∎

Proof of (2). Suppose $G(1) \leq i^* < G(f-1)$ and let $k^\circ = \min[k : i^* < G(k)]$. We first prove the inequality $r_k > \widehat{r}_k$ holds for $k = k^\circ, ..., f-1$ by mathematical induction. Let $\delta = I_{G(f-1)} - \widehat{I}_{G(f-1)} > 0$. The rent equation (3.1) and Assumption B imply

$$u(\mathbf{e}^{f-1}, \widehat{I}_{G(f-1)} - \widehat{r}_{f-1} + \delta) > u(\mathbf{e}^{f}, \widehat{I}_{G(f-1)} - r_f + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{f-1}, I_{G(f-1)} - \widehat{r}_{f-1}) &> u(\mathbf{e}^{f}, I_{G(f-1)} - r_f) \\ &= u(\mathbf{e}^{f-1}, I_{G(f-1)} - r_{f-1}) \text{ by Eqs. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $I_{G(f-1)} - \widehat{r}_{f-1} > I_{G(f-1)} - r_{f-1}$, that is, $r_{f-1} > \widehat{r}_{f-1}$.

Suppose the inequality $r_k > \widehat{r}_k$ holds for k with $k^\circ < k \leq f-1$. We show this also holds for $k-1$. Let $\delta = I_{G(k-1)} - r_k - (\widehat{I}_{G(k-1)} - \widehat{r}_k)$ and suppose $\delta > 0$. Eqs. (3.1) and Assumption B imply

$$u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1} + \delta) > u(\mathbf{e}^{k}, \widehat{I}_{G(k-1)} - \widehat{r}_k + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{k-1}, I_{G(k-1)} - \widehat{r}_{k-1} - r_k + \widehat{r}_k) &> u(\mathbf{e}^{k}, I_{G(k-1)} - r_k) \\ &= u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1}) \text{ by Eqs. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $I_{G(k-1)} - \widehat{r}_{k-1} - r_k + \widehat{r}_k > I_{G(k-1)} - r_{k-1}$, that is, $r_{k-1} - \widehat{r}_{k-1} > r_k - \widehat{r}_k > 0$. Hence we obtain $r_{k-1} > \widehat{r}_{k-1}$.

Suppose the other case $\delta \leq 0$. Then, Assumption A imply $u(\mathbf{e}^{k}, \widehat{I}_{G(k-1)} - \widehat{r}_k) \geq u(\mathbf{e}^{k}, I_{G(k-1)} - r_k)$. Since the left hand side equals $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$ and the right hand side equals

$u(\mathbf{e}^{k-1}, I_{G(k-1)}-r_{k-1})$ by Eqs. (3.1), we have $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)}-\widehat{r}_{k-1}) \geq u(\mathbf{e}^{k-1}, I_{G(k-1)}-r_{k-1})$. Again, by Assumption A, we have $\widehat{I}_{G(k-1)} - \widehat{r}_{k-1} \geq I_{G(k-1)} - r_{k-1}$, that is, $r_{k-1} - \widehat{r}_{k-1} \geq I_{G(k-1)} - \widehat{I}_{G(k-1)} > 0$. Hence we obtain $r_{k-1} > \widehat{r}_{k-1}$.

From the above discussion, we have $r_k > \widehat{r}_k$ holds for $k = k^\circ, ..., f-1$. We next show either $r_k > \widehat{r}_k$ or $r_k \leq \widehat{r}_k$ holds for $k = 1, ..., k^\circ - 1$. Furthermore, we show that once $r_k \leq \widehat{r}_k$ appears for some $k^* \leq k^\circ - 1$, then it holds that $r_k < \widehat{r}_k$ for $k = 1, ..., k^* - 1$.

Let $\delta = \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ} - (I_{G(k^\circ-1)} - r_{k^\circ})$. By condition InE, we have

$$\widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ} - (I_{G(k^\circ-1)} - r_{k^\circ}) > 0. \tag{3.2}$$

Eqs. (3.1) and Assumption B imply

$$u(\mathbf{e}^{k^\circ-1}, I_{G(k^\circ-1)} - r_{k^\circ-1} + \delta) > u(\mathbf{e}^{k^\circ}, I_{G(k^\circ-1)} - r_{k^\circ} + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - r_{k^\circ-1} - \widehat{r}_{k^\circ} + r_{k^\circ}) &> u(\mathbf{e}^{k^\circ}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ}) \\ &= u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1}) \text{ by Eqs. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $\widehat{I}_{G(k^\circ-1)} - r_{k^\circ-1} - \widehat{r}_{k^\circ} + r_{k^\circ} > \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1}$, that is,

$$r_{k^\circ} - \widehat{r}_{k^\circ} > r_{k^\circ-1} - \widehat{r}_{k^\circ-1}. \tag{3.3}$$

On the other hand, Eq. (3.2) and Assumption A imply $u(\mathbf{e}^{k^\circ}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ}) > u(\mathbf{e}^{k^\circ}, I_{G(k^\circ-1)} - r_{k^\circ})$. Since the left hand side equals $u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1})$ and the right hand side equals $u(\mathbf{e}^{k^\circ-1}, I_{G(k^\circ-1)} - r_{k^\circ-1})$ by Eqs. (3.1), we have $u(\mathbf{e}^{k^\circ-1}, \widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1}) > u(\mathbf{e}^{k^\circ-1}, I_{G(k^\circ-1)} - r_{k^\circ-1})$. Again, by assumption A, we have $\widehat{I}_{G(k^\circ-1)} - \widehat{r}_{k^\circ-1} > I_{G(k^\circ-1)} - r_{k^\circ-1}$, that is, $r_{k^\circ-1} - \widehat{r}_{k^\circ-1} > I_{G(k^\circ-1)} - \widehat{I}_{G(k^\circ-1)}$. By this and Eq. (3.3), we have

$$r_{k^\circ} - \widehat{r}_{k^\circ} > r_{k^\circ-1} - \widehat{r}_{k^\circ-1} > I_{G(k^\circ-1)} - \widehat{I}_{G(k^\circ-1)}.$$

Since $r_{k^\circ} > \widehat{r}_{k^\circ}$ and $I_{G(k^\circ-1)} < \widehat{I}_{G(k^\circ-1)}$, there are two cases: $r_{k^\circ-1} > \widehat{r}_{k^\circ-1}$ or $r_{k^\circ-1} \leq \widehat{r}_{k^\circ-1}$. If the latter case, the category k^* of Theorem 3.1.(2) is $k^* = k^\circ - 1$.

Let k with $1 < k \leq k^\circ - 1$.

(Case $r_k > \widehat{r}_k$): By Condition InE, $\widehat{I}_{G(k-1)} > I_{G(k-1)}$. Thus, we have $\widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k) > 0$. In the same manner with the above discussion, we have

$$r_k - \widehat{r}_k > r_{k-1} - \widehat{r}_{k-1} > I_{G(k-1)} - \widehat{I}_{G(k-1)},$$

and there may be two cases $r_{k-1} > \widehat{r}_{k-1}$ or $r_{k-1} \leq \widehat{r}_{k-1}$. If the latter case, the category k^* of Theorem 3.1.(2) is $k^* = k - 1$.

(Case $r_k \leq \widehat{r}_k$): Suppose that $\delta = \widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k) > 0$. Eqs. (3.1) and Assumption

B imply

$$u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1} + \delta) > u(\mathbf{e}^k, I_{G(k-1)} - r_k + \delta),$$

that is,

$$\begin{aligned} u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k) &> u(\mathbf{e}^k, \widehat{I}_{G(k-1)} - \widehat{r}_k) \\ &= u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}) \text{ by Eqs. (3.1).} \end{aligned}$$

This inequality and Assumption A imply $\widehat{I}_{G(k-1)} - r_{k-1} - \widehat{r}_k + r_k > \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}$, that is, $r_{k-1} < \widehat{r}_{k-1}$.

Suppose the other case $\widehat{I}_{G(k-1)} - \widehat{r}_k - (I_{G(k-1)} - r_k) \leq 0$. This inequality and Assumption A imply $u(\mathbf{e}^k, \widehat{I}_{G(k-1)} - \widehat{r}_k) \leq u(\mathbf{e}^k, I_{G(k-1)} - r_k)$. Since the left hand side equals $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1})$ and the right hand side equals $u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$ we have $u(\mathbf{e}^{k-1}, \widehat{I}_{G(k-1)} - \widehat{r}_{k-1}) \leq u(\mathbf{e}^{k-1}, I_{G(k-1)} - r_{k-1})$. Again, by Assumption A, $\widehat{I}_{G(k-1)} - \widehat{r}_{k-1} \leq I_{G(k-1)} - r_{k-1}$. Since $\widehat{I}_{G(k-1)} > I_{G(k-1)}$, we obtain $r_{k-1} < \widehat{r}_{k-1}$. ▮

Proof of (3). The proof is the same as the early part of the proof of (2). ▮

3.3 Numerical Examples

In this section, we confirm our comparative statics results by numerical examples. In examples, we find that rising income inequality possibly causes (i) decline in rents at every category or (ii) rise in rents above some category k^* and decline below k^* [the statement of Theorem 3.2.(2)].

Suppose that there are 6 categories of apartments ($T = 6$) and $T = f$. For each category $k = 1, ..., 6$, w_k amount of apartments are already built and owned by landlord k for sale. Here, suppose that $w_1 = w_2 = 200$, $w_3 = w_4 = 300$ and $w_5 = w_6 = 500$. We assume the same number of households are coming to the market to seek the dwelling, and that all the apartment units are traded in the end. Therefore, $m = \sum_{t=1}^{k} w_t$ and $G(k) = \sum_{t=1}^{k} w_t$ for $k = 1, ..., 6$.

Each household has the following utility function:

$$u(\mathbf{e}^k, c) = h_k + \sqrt{c} \ (k = 0, ..., 6),$$

where $h_1 = 5.1, h_2 = 4.4, h_3 = 3.7, h_4 = 3, h_5 = 2, h_6 = 1$ and $h_0 = 0$. We assume that a household (monthly) income is lognormally distributed.[11] In this example, we adopt the

[11] We say that a (positive) random variable X is lognormally distributed with parameters μ and σ^2 iff $Y = \ln X$ is normally distributed with mean μ and variance σ^2. The lognormal distribution is denoted by $\Lambda(\mu, \sigma^2)$. The probability density function of $X \sim \Lambda(\mu, \sigma^2)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right] \quad (x > 0).$$

The mean E, variance V, median M and mode D of $\Lambda(\mu, \sigma^2)$ are given by $E = \exp(\mu + \frac{1}{2}\sigma^2)$, $V =$

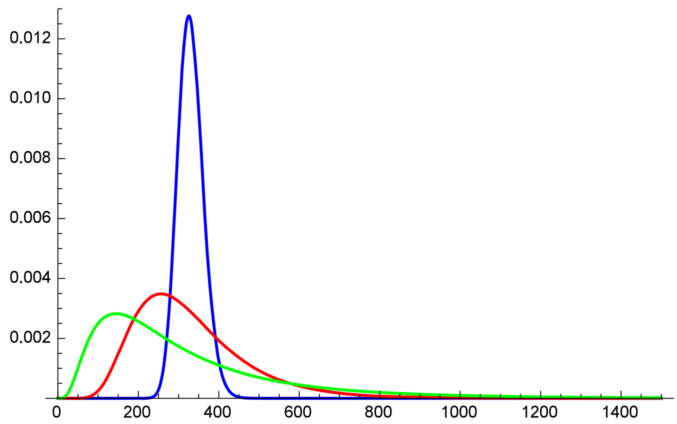

Figure 2: Probability density distributions of lognormal distributions.

		Changes in $I_{G(k)}$			Differences in $I_{G(k)}$	
k	$G(k)$	$I_{G(k)}$	$\widehat{I}_{G(k)}$	$\widehat{\widehat{I}}_{G(k)}$	$\widehat{I}_{G(k)} - I_{G(k)}$	$\widehat{\widehat{I}}_{G(k)} - \widehat{I}_{G(k)}$
1	200	371.6	514.7	660.5	143.1	145.8
2	400	356.7	433.8	470.1	77.1	36.3
3	700	341.8	356.1	346.0	14.3	−10.1
4	1000	329.3	305.2	264.3	−24.1	−40.9
5	1500	310.1	231.4	167.5	−78.6	−63.9
Gini		0.05	0.22	0.36		

Table 1: Boundary incomes in the example

following three lognormal distributions: the mean of lognormal distribution is fixed as $E = 330$, and variances are $V_1 = 1000$, $V_2 = 20000$ and $V_3 = 80000$. Fig. 2 depicts probability density distributions.

In Fig. 2, the most highest graph corresponds to the mean $E = 330$ and the variance $V_1 = 1000$, the second highest one corresponds to $E = 330$ and $V_2 = 20000$, and the remaining one is $E = 330$ and $V_2 = 80000$. We generate three sets of 2000 random numbers following each distribution. We suppose the initial income distribution is $V_1 = 1000$, and it changes into $V_1 = 20000$ (denote the new incomes by hats); and it also changes into $V_1 = 80000$ (by double hats). Table 1 gives boundary incomes and Gini coefficients of each generated incomes.

Table 1 shows that income inequality increases as the variance increases. The table also shows the magnitude of income difference is monotonically increasing. Locations of household i^* of Condition InE is as follows: (i) $G(3) \leq \widehat{i^*} < G(4)$ and (ii) $G(2) \leq \widehat{\widehat{i^*}} < G(3)$. Both satisfy Condition of Theorem 3.2.(2).

Let the marginal rent $r_6 = 50$. We then calculates differential rent vectors $(r_1, ..., r_6)$,

$\exp(2\mu + \sigma^2)\left[\exp(\sigma^2) - 1\right]$, $M = \exp(\mu)$ and $D = \exp(\mu - \sigma^2)$. By them, we have $D < M < E$, and thus, $\Lambda(\mu, \sigma^2)$ has a long-tail form. These difinitions and properties are due to Crow and Shimizu (1988). The lognormal distribution is aften used as an approximation of an income distribution.

	Changes in r_k			Differences in r_k	
k	r_k	$\widehat{r}_k$	$\widehat{\widehat{r}}_k$	$\widehat{r}_k - r_k$	$\widehat{\widehat{r}}_k - \widehat{r}_k$
1	173.2	177.2	176.1	4.0	−1.0
2	153.0	150.9	114.8	−2.0	−6.1
3	132.5	126.9	119.1	−5.6	−7.8
4	111.8	105.2	97.5	−6.5	−7.7
5	81.3	75.9	70.7	−5.3	−5.3
6	50	50	50	0	0

Table 2: Differential rents in the example

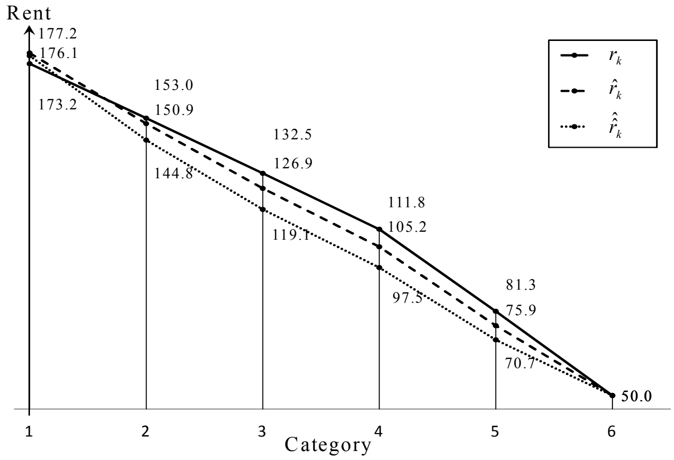

Figure 3: Rent changes in the example.

$(\widehat{r}_1, ..., \widehat{r}_5, r_6)$ and $(\widehat{\widehat{r}}_1, ..., \widehat{\widehat{r}}_5, r_6)$ by Eq. (3.1). The calculation result and its illustration are given in Table 2 and Fig. 3.

As seen from Table 2, the first income change causes decline in rents except the first category and rise in rent at the first category, that is, Theorem 3.1.(2). On the other hand, the second change causes decline in rents at every category [Theorem 3.1.(3)]. These results are consistent with Theorem 3.2. In sum, it can be said that increase of an income inequality often lowers a rent at every category (as well as average rent).

4 Income inequality and equitability of competitive allocations

Here, we briefly mention the equitability property of competitive allocations in our market model. Foley (1967) and Varian (1974) developed the theory of equitability (or fairness) in markets with perfectly divisible goods. Svensson (1983), Alkan, Demange and Gale (1991) and Sakai (2007) also studied equitability/fairness in the indivisibility framework. However, their models are different to ours in that their models consist of (i) only buyers (ii) the same number of buyers and indivisible units (iii) no initial endowments and (iiii) without homogeneous preference assumption. Here, we focus on competitive equilibria, and consider equitability of competitive allocations in our model.

We first give some notations (definitions are due to Foley, 1967). Recall that the consumption set of households are given by $X = \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$. Let an m-tuple $a = (a_1, ..., a_m) \in X^m$ be a consumption allocation. We say that i *envies* j at $a \in X^m$ iff $u(a_j) > u(a_i)$. We say that $a \in X^m$ is the *equitable* (*envy-free*) *allocation* iff $u(a_i) \geq u(a_j)$ for every $i, j \in M$. Note that in our framework, this condition can be translated by $u(a_i) = u(a_j)$ for every $i, j \in M$.

The following proposition holds in our market model.

Proposition 4.1. *Let (p, x, y) be a competitive equilibrium and let $i, j \in M$. Then, $I_i \geq I_j$ if and only if $u(x_i, I_i - px_i) \geq u(x_j, I_j - px_j)$ (note that $\geq$ is replaced by $\leq, >, <$, or $=$).*

Proof. (*Only If*) By the antecedent $I_i > I_j$ and utility maximization condition, we have $u(x_i, I_i - px_i) \geq u(x_j, I_i - px_j) > u(x_j, I_j - px_j)$. (*If*) Suppose, on the contrary, $I_i \leq I_j$. Then, we obtain the contradictory inequality by utility maximization condition: $u(x_j, I_j - px_j) \geq u(x_i, I_j - px_i) \geq u(x_i, I_i - px_i)$. ∎

This proposition means that if there exist two households having different incomes, then the lower-income household envies the higher-income household in any competitive allocations; conversely, if some household envies the other in a competitive allocation, then the income of the envied household is higher. Furthermore, if incomes of some two households are the same, then their utility levels also the same in any competitive allocations; conversely, if utility levels of some two households are the same in a competitive allocation, then their incomes also the same.

The following corollary follows from the proposition.

Corollary 4.2. *Let (p, x, y) be a competitive equilibrium. Then every household has the same income if and only if an m-tuple $((x_1, I_1 - px_1), ..., (x_m, I_m - px_m))$ is an equitable allocation.*

Thus, when the household income distribution has even a little inequality, any competitive allocation does not satisfies equitability (conversely, if a competitive allocation does not satisfies equitability, then the income distribution has an inequality). Theorems 3.1,2 and Corollary 4.2 imply that rising income inequality tends to cause both dampening the equitability on household allocations and a decline in landlord revenues. Note that since any competitive equilibrium is Pareto efficient in our market model, an equitable competitive allocation is a fair allocation.[12] Note also that the only-if part of the corollary holds without identical utility function assumption, whereas the if part does not holds without this assumption. The

[12] Svensson (1983) and Sakai (2007) gave a result related to Corollary 4.2. According to them, a consumption allocation $((x_1, c_1), ..., (x_m, c_m)) \in X^m$ is a *Walrasian allocation from equal income* iff there exist $p \in \mathbb{R}^T_+$ and $I \in \mathbb{R}_+$ such that $c_i = I - px_i$ for all $i \in M$ and every household maximizes his utility, where I is the implicit imcome. They showed that the set of equitable allocations coincides with the set of Walrasian allocations from equal income.

next example shows a case that income inequality exists but a competitive allocation satisfies equitability.

Example 4.3 (*Equitable competitive equilibrium with income inequality exists*). Suppose that there are two households 1 and 2 with incomes $I_1 = 150$ and $I_2 = 100$, two different apartments 1 and 2 (with reservation prices 50 and 36). Suppose that their utility functions are given as

$$u_1(\mathbf{e}^k, c) = \begin{cases} 0 + \sqrt{c} \text{ for } k = 0, \\ 4 + \sqrt{c} \text{ for } k = 1, \\ 1 + \sqrt{c} \text{ for } k = 2, \end{cases} \quad u_2(\mathbf{e}^k, c) = \begin{cases} 0 + \sqrt{c} \text{ for } k = 0, \\ 1 + \sqrt{c} \text{ for } k = 1, \\ 4 + \sqrt{c} \text{ for } k = 2. \end{cases}$$

This setting explains, for example, the following situation: the apartment 1 is a relatively large one located in a suburban area and the apartment 2 is a small one located in a central city. Household 1 with higher income prefers the apartment 1 to 2, while the household 2 prefers the apartment 2 to 1.

Let $p = (p_1, p_2) = (50, 36)$. Then, $u_1(\mathbf{e}^1, I_1 - p_1) = 14 > u_1(\mathbf{e}^0, I_1) > u_1(\mathbf{e}^2, I_1 - p_2)$ and $u_2(\mathbf{e}^2, I_2 - p_2) = 12 > u_2(\mathbf{e}^0, I_1) > u_2(\mathbf{e}^1, I_1 - p_1)$. Hence, a triple $(p, (\mathbf{e}^1, \mathbf{e}^2), (1, 1))$ is a competitive equilibrium. On the other hand, $u_1(\mathbf{e}^1, I_1 - p_1) = 14 > u_1(\mathbf{e}^2, I_2 - p_2) = 9$ and $u_2(\mathbf{e}^2, I_2 - p_2) = 12 > u_2(\mathbf{e}^1, I_1 - p_1) = 11$. Hence, this equilibrium satisfies equitability but income inequality exists.

5 Conclusions

We have studied the comparative statics analysis based on the assignment market model. In particular, we present how rising income inequality affects a competitive rent distribution. The key assumptions of the model are homogeneous and normality assumptions on the household utility functions. A competitive rent vector can be then calculated by a system of equations.

Our main comparative statics result is Theorem 3.1, stating that an increase in income inequality effects three cases on the competitive rent vector: (i) rise at every category, (ii) rise at higher categories and decline at lower categories or (iii) decline at every category. Another Theorem 3.2 implies that (i) is a special case, while (iii) [as well as (ii)] is possible in a general situation. Numerical examples facilitated our comparative statics results. We also mentioned the equitability property of competitive allocations in our market model.

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