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On the evaluation of a differential rent vector in housing markets with indivisibilities

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On the evaluation of a differential rent vector in housing markets with indivisibilities

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Abstract

We study the difference between two differential rent vectors in a rental housing market model, where the apartments are classified into a finite number of categories and each household demands (at most) one apartment unit. A differential rent vector is the representative solution of the maximum (or minimum) competitive rent vector. We show that the difference between the maximum and minimum differential rent vectors is bounded by the income difference of specific two neighboring households. This implies that the difference is rather small, and thus, the gap between two analyses using the maximum or minimum rent vector is also small. As an application of results, we show that the differential rent difference shrinks to zero as the market size gets large and the household income distribution becomes dense.

Keywords: Rental housing market; Indivisibilities; Differential rent; Multiplicity of equilibrium prices

JEL classification: R31; R30; D58

1 Introduction

This paper investigates a certain property of a differential rent vector in a rental housing market model by Kaneko, Ito and Osawa [9]. In particular, we evaluate the difference between the maximum and minimum values of the differential rent vectors.

This model is an application of the generalized assignment market by Kaneko [7],[1] where the market participants are divided into households and landlords, the rental housings to be traded are treated as indivisible goods and classified into finite categories, and each household

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[1]Kaneko (1982) studied a generalized model of Shapley and Shubik's [6] assignment game, where the quasi-linear assumption on the buyers' utility functions is removed. Kaneko showed the existence of a competitive equilibrium in the model. Furthermore, the existence of the maximum and minimum competitive price vectors is guaranteed.

can consume (at most) one housing unit. A differential rent vector (which plays a key role in their model) is a solution of a certain system of equations called "the rent equation". This vector coincides with the maximum competitive rent vector under some reasonable conditions. Thus, a differential rent vector is regarded as a good approximation of the maximum competitive rent vector. Kaneko et al. and their following papers (e.g., Ito [4]) presented numerous comparative statics results using a differential rent vector.

On the other hand, there is a concern regarding the adequacy of adopting the maximum competitive rent vector to represent the market equilibrium. While there are multiple equilibrium candidates in the market model, we cannot observe which one is actually selected in the real housing market. There may be a gap between the comparative statics results of the model the and real markets. Regarding this problem, Kaneko et al. noted the availability of the minimum instead of the maximum competitive rent vector to represent the market equilibrium. However, a similar concern still remains, even when we adopt the minimum vector. There has been little investigation concerning this problem.

One reasonable solution to the above problem is to study the difference between the maximum and minimum differential rent vectors. In this manner, we can identify the gap between the two comparative statics results. Here, we introduce a summary of one of our results, Theorem 3.5 in Section 3.2. Theorem 3.5 states that the difference between the maximum and minimum differential rent vectors is bounded by the largest income difference between two specific neighboring households. This implies that the rent difference is not large, and that it tends to smaller as the distribution of household income gets dense. Thus, we could argue that, when we target a considerably large housing market, the difference of the comparative statics results is small whether we use the maximum or minimum vectors. The numerical examples in Section 4.1 confirm our results. We present a more precise discussion on the shrinkage of the difference of the differential rent vectors in Section 4.2.

Our results are related to a recent paper by Sai [11], who studied the difference of the maximum and minimum competitive price vectors under weaker conditions. His approach is different from ours in that he characterizes the price difference by the difference in the seller's marginal costs. On the other hand, the cost structure is not easily observable from the real data in comparison to the household income data. Furthermore, the households (buyers) rather than the landlords (sellers) play the main role in our market model. Therefore, we could be say that, for studying the housing market model, our characterization is more helpful in a housing market analysis.

We should note that the following two assumptions are important in our market model: the identical utility function, and the normality of the quality of housing. The former assumption is necessary for the derivation of the differential rent vectors, and the latter assumption plays a crucial role in the results. These assumptions imply that the difference in housing preferences (i.e., the difference between households) is only characterized by the difference in incomes. These assumptions are salient features of our model because the quasi-linear utility assumption is still the main-stream view of markets with indivisibilities. Recently, Määttänen and Terviö [5] studied the housing market using a one-sided assignment model that assumed the identical utility function and the normality of the quality of housing, as outlined above.[2]

[2] In their model, the set of buyers (who are, at the same time, sellers) is assumed to be continuum and the

They presented comparative statics results on the relation between income distribution and housing prices.

As viewed from the perspective of the urban economics literature, we can find the source of our approach from Alonso's [1] bid rent theory. He studied a mechanism of land pricing in urban area with a central business district, where land sizes are treated as continuous variables. Our differential rent approach is based on his bid rent theory.[3]

This paper is organized as follows. Section 2 formulates our rental housing market model and gives our definition of competitive equilibrium. Section 3 first introduces two recursive equation systems, *the maximum and minimum rent equations*, from which we can derive the maximum and minimum differential rent vectors. This section then outlines the main results of our study. Section 4 provides some numerical examples and an application of our theorems. Section 5 presents our conclusions and further remarks.

2 The market model

This section introduces the rental housing market model of Kaneko et al. [9]. In Section 2.1, we give our basic assumptions and the definition of competitive equilibrium. In Section 2.2, we introduce additional assumptions that facilitate our study.

2.1 General formulation

The *rental housing market* is denoted by (M, N), where $M = \{1, \ldots, m\}$ denotes the set of households, and $N = \{1', \ldots, n'\}$ denotes the set of landlords. The apartments are classified into finite *categories* $1, ..., T$, and are assumed to be already built.

Each household $i \in M$ initially has an income $I_i > 0$ but no dwelling. The household wants to rent at most one apartment unit paying rent from his income. Without loss of generality, we can assume that the households are ordered in their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. The *consumption set* is written by $X := \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbf{R}_+$, where $\mathbf{e}^k$ is the T-dimensional unit vector with $\mathbf{e}^k_k = 1$ ($\mathbf{e}^0 = \mathbf{0}$), and $\mathbf{R}_+$ is the set of nonnegative real numbers. A consumption vector $(\mathbf{e}^k, c) \in X$ with $k \neq 0$ means that household i rents one unit of the k-th category of an apartment and enjoys the consumption $c = I_i - p_k$, where p_k is the rent of the k-th apartment. For $k = 0$, no apartment is consumed. An *initial endowment* of $i \in M$ is given as $(\mathbf{e}^0, I_i)$ with $I_i > 0$.

A *utility function* of household i is given by $u_i : X \to \mathbf{R}$. We make the following assumption.

Assumption A. For each $i \in M$ and $x \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$, $u_i(x_i, c)$ is a continuous and strictly monotone function of c, and $u_i(\mathbf{e}^0, I_i) > u_i(\mathbf{e}^k, 0)$ for all k with $1 \leq k \leq T$.

The first part of Assumption A allows a utility function to have an "income effect". An inequality in the last part is a boundary condition, meaning that a household prefers keeping his income to rent any apartment by paying all his income.

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housing attributes are also continuous variables.

[3]We can find a survey of the urban economics literatures in the textbook by Fujita [2].

Each landlord $j \in N$ provides apartments of exactly one category (say k), but may provide more than one unit. The landlord has a cost function $C_j(y_j) : \mathbf{Z}_+ \to \mathbf{R}_+$, where $\mathbf{Z}_+$ is the set of nonnegative integers. For each $y_j \in Z_+$, $C_j(y_j)$ represents the cost (in terms of money) of supplying y_j units of the k-th category. We make the following assumption for $C_j(\cdot)$.

Assumption B. For each $j \in N$, $C_j(0) = 0$ and $C_j(y_j+2) - C_j(y_j+1) \geq C_j(y_j+1) - C_j(y_j)$ for all $y_j \in \mathbf{Z}_+$.

The first part of Assumption B means that no fixed cost is required for no production. The last part is a discrete version of the standard convexity assumption on a cost function, meaning that the marginal cost is increasing. As mentioned above, we assume that the apartments in the market are already built, and thus only the operating costs need to be covered by the landlords.[4]

For notational simplicity, we assume that *only one* landlord k provides apartments in the k-th category. Thus, the set N becomes $\{1, \ldots, T\}$, and landlord $k \in N$ is the only landlord providing the k-th apartments. As far as the competitive equilibrium is concerned, this can be assumed *without loss of generality*.[5]

Let $(p, x, y) = ((p_1, \ldots, p_T),\ (x_1, \ldots, x_m),\ (y_1, \ldots, y_T))$ be a triple of $p \in \mathbf{R}^T_+$, $x \in \left\{\mathbf{e}^0,\ \mathbf{e}^1,\ \ldots,\ \mathbf{e}^T\right\}^m$ and $y \in \mathbf{Z}^T_+$. The competitive equilibrium is defined by the following:

Definition 2.1. We say that a triple (p, x, y) is a *competitive equilibrium* iff

(1) Utility Maximization under the Budget Constraint (UMC): for all $i \in M$,

(i) $I_i - px_i \geq 0$, where $px_i = \sum_{k=1}^T p_k x_{ik}$;
(ii) $u_i(x_i, I_i - rx_i) \geq u_i(x'_i, I_i - rx'_i)$ for all $x'_i \in \left\{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\right\}$ with $I_i - rx_i \geq 0$.

(2) Profit Maximization (PM): For all k with $1 \leq k \leq T$,

$p_k y_k - C_k(y_k) \geq p_k y'_k - C_k(y'_k)$ for all $y'_k \in \mathbf{Z}_+$.

(3) Balance of the Total Demand and Supply (BDS): $\sum_{i \in M} x_i = \sum_{k=1}^T y_k \mathbf{e}^k$.

Under Assumptions A and B, we have a competitive equilibrium in (M, N).

Theorem 2.1 (Kaneko, 1982; Kaneko and Yamamoto, 1986)**.** *There exists a competitive equilibrium* (p, x, y) *in a rental housing market* (M, N).

We say that $p = (p_1, \ldots, p_T)$ is a *competitive rent vector* iff (p, x, y) is a competitive equilibrium for some $x \in \{\mathbf{0}, \mathbf{e}^1, \ldots, \mathbf{e}^T\}^m$ and $y \in \mathbf{Z}^T_+$. Note that there may be multiple competitive equilibria. In particular, the maximum and minimum competitive rent vectors exist and play an important role in our analysis.[6]

Theorem 2.2. *There exist the maximum and minimum competitive rent vectors in* (M, N).

[4] The operating costs contain, for example, the maintenance and operation costs and real estate taxes. Kaneko et al. [9], Section 2.1 presents a detailed discussion.

[5] Under this simplification, each seller is interpreted as an "aggregated seller". Sai [9], Section 5.1 presents a detailed discussion.

[6] A competitive rent vector p is the maximum (minimum) iff for any competitive rent vector p', $p_t \geq p'_t$ ($p_t \leq p'_t$) for all t with $1 \leq t \leq T$.

Kaneko [8] and Kaneko et al. [9] proved the existence of the maximum competitive rent vector. A complete proof is in Appendix A. We say that (p, x, y) is a *maximum* (*minimum*) *competitive equilibrium* iff p is the maximum (minimum). By definition, the maximum (minimum) competitive rent vector is uniquely determined; however, multiple maximum (minimum) competitive equilibria may exist.

Kaneko et al. and their following subsequent papers (e.g. Ito [4]) adopted the maximum competitive rent vector to represent the market equilibrium rent of their comparative statics. In their analysis, the maximum competitive rent vector is calculated according to a certain system of equations called "the rent equation." On the other hand, it is also possible to use the minimum competitive rent vector in their analysis. As stated in the introduction, we cannot observe what (competitive) equilibrium rent is selected when the real market achieves equilibrium. Thus, there is room to consider the adequacy of using the maximum competitive rent vector as the market equilibrium.

2.2 Specific assumptions for (M, N)

In addition to Assumptions A and B, we assume that every household has identical utility function; that is,

Assumption C. $u_i(\cdot, \cdot) = u_j(\cdot, \cdot)$ for all $i, j \in M$.

By Assumption C, we can simplify the utility function u_i as u. In an urban economics context, Assumption C implies that the housing market (M, N) represents a mono-centric city, and all the households commute to an identical business district. Thus, under C, each household only is characterized by its initial income level. Some readers may be concerned that C implies an identical apartment preference for each household. However, this concern will be eliminated by the next assumption.

Assumption D. If $u(x_i, c) = u(x_i', c')$, and $c < c'$, then $u(x_i, c + \delta) > u(x_i', c' + \delta)$ for any $\delta > 0$.

Assumption D is the normality assumption on the quality of apartments. In D, apartment x_i is better than x_i' because a household living in x_i with a smaller consumption c is indifferent to living in x_i' with a larger consumption c'. This implies that, for each household, the demand shifts to a better apartment or remains the same if their income increases.

The next assumption is regarding the quality of apartments.

Assumption E. $u(\mathbf{e}^1, 0) > u(\mathbf{e}^2, 0) > \cdots > u(\mathbf{e}^T, 0)$.[7]

Thus, the apartments are numbered according to their quality level. The first category is the best one and the T-th category is the worst. These assumptions facilitate our study of using the "differential rent approach".

[7] This assumption together with Assumptions A and D imply that $u(\mathbf{e}^1, c) > u(\mathbf{e}^2, c) > \cdots > u(\mathbf{e}^T, c)$ for all $c \in \mathbb{R}_+$.

3 The rent equation and the evaluation of the differential rent vector

In Section 3.1, we introduce two systems of equations. One was formulated by Kaneko et al., which we call the *maximum rent equation.* Another is newly formulated in a parallel manner, which we call the *minimum rent equation.* Each solution of the equations is called the *maximum/minimum differential rent vector*, corresponding to the maximum and minimum competitive rent vectors, respectively. Using both differential rent vectors, we present two theorems on the evaluation of the difference between the maximum and minimum differential rent vectors in Section 3.2. Section 3.3 gives proofs of two theorems.

3.1 The rent equation

To introduce the rent equation, we give more detailed assumptions and some lemmas. The purpose of this paper is to evaluate the difference between the maximum and minimum competitive rent vectors. Therefore, hereafter, we consider the case where the maximum competitive rent vector (denoted by $p^{\max}$) and the minimum competitive rent vector (denoted by $p^{\min}$) satisfy $p_k^{\min} < p_k^{\max}$ for all k with $1 \leq k \leq T$. This assumption together with Theorem 3.1 by Sai (2014) (also Lemma A.2 in the Appendix) imply that for any competitive equilibria (p, x, y) and (p', x', y'),

$$y_k = y'_k \text{ for all } k \text{ with } 1 \leq k \leq T, \tag{3.1}$$

that is, for any category k, the equilibrium supply for k is uniquely determined.

The following lemma has an important role.

Lemma 3.1 (Kaneko et al., 2006). *Let (p, x, y) be a competitive equilibrium. Then,*
(1) If $k' < k$ and $x_i = e^k$ for some i, then $p_k < p_{k'}$;
(2) If $x_i = e^k$, $x_{i'} = e^{k'}$ and $I_i > I_{i'}$ for some i, i', then $k \leq k'$.

This lemma states that in any competitive equilibrium, (1) the price of a better apartment is higher, and (2) a household with a higher income rents a better apartment. Note that in (1), it may be possible that no one rents an apartment in the k'-th category, while the k-th apartment is rented by someone. To eliminate this case, we assume that there is a category f dividing the apartments into active categories and inactive categories. That is,

Assumption F. There exists some category f such that for any competitive equilibrium (p, x, y), $y_k > 0$ for k with $1 \leq k \leq f$ and $y_k = 0$ for k with $f \leq k \leq T$.

We call this f the *marginal category.* Note that Assumption F includes the same f for each competitive equilibria. The condition of equation (3.1) guarantees such a treatment. In the literature of urban economics, the marginal category corresponds to the marginal land in Ricardo's differential rent theory.

Recall that all the households $M = \{1, ..., m\}$ are ordered by their incomes as $I_1 \geq I_2 \geq \cdots \geq I_m$. We next define the household with the lowest income in each active category. This household plays a crucial role in the rent equation. Let (p, x, y) be a competitive equilibrium. For each category $k(\leq f)$, we define the household $G(k)$ with the lowest income in the k-th category as:

$$G(k) := \sum_{t=1}^{k} y_t.$$

For each k, we call $G(k)$ the *boundary household* of the k-th category. Note that by equation (3.1), $G(k)$ is uniquely determined for each k.

We now introduce the rent equation. The maximum rent equation is defined as the system of equations with the unknowns $(\overline{r}_1, ..., \overline{r}_f)$. The formulation is from Kaneko et al. In a parallel manner, the minimum rent equation is defined with the unknowns $(\underline{r}_1, ..., \underline{r}_f)$.

Definition 3.1. (1) (Kaneko et al, 2006): We call the following system of equations the *maximum rent equation*:

$$\left.\begin{array}{c} u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1}) = u(\mathbf{e}^{f}, I_{G(f-1)} - \overline{r}_{f}), \\ u(\mathbf{e}^{f-2}, I_{G(f-2)} - \overline{r}_{f-2}) = u(\mathbf{e}^{f-1}, I_{G(f-2)} - \overline{r}_{f-1}), \\ \vdots \\ u(\mathbf{e}^{1}, I_{G(1)} - \overline{r}_{1}) = u(\mathbf{e}^{2}, I_{G(1)} - \overline{r}_{2}). \end{array}\right\} \tag{3.2}$$

(2) We call the following system of equations the *minimum rent equation*:

$$\left.\begin{array}{c} u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1}) = u(\mathbf{e}^{f}, I_{G(f-1)+1} - \underline{r}_{f}), \\ u(\mathbf{e}^{f-2}, I_{G(f-2)+1} - \underline{r}_{f-2}) = u(\mathbf{e}^{f-1}, I_{G(f-2)+1} - \underline{r}_{f-1}), \\ \vdots \\ u(\mathbf{e}^{1}, I_{G(1)+1} - \underline{r}_{1}) = u(\mathbf{e}^{2}, I_{G(1)+1} - \underline{r}_{2}). \end{array}\right\} \tag{3.3}$$

The difference between equations (3.2) and (3.3) is the replacement of the boundary income $I_{G(k)}$ by $I_{G(k)+1}$ for $k = 1, ..., f-1$. In equation (3.2), if $\overline{r}_f$ is given with $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f)$, then the unknown $\overline{r}_{f-1}$ is uniquely determined by the first equation. In the same manner, the remaining unknowns $\overline{r}_1, ..., \overline{r}_{f-2}$ are recursively and uniquely determined, and it holds that $\overline{r}_1 > \cdots > \overline{r}_{f-1} > \overline{r}_f$.[8] Similarly, a solution $(\underline{r}_1, ..., \underline{r}_{f-1})$ is uniquely determined if $\underline{r}_f$ is given with $u(\mathbf{e}^1, 0) < u(\mathbf{e}^f, I_{G(f-1)} - \underline{r}_f)$ and it holds that $\underline{r}_1 > \cdots > \underline{r}_{f-1} > \underline{r}_f$. We say that $(\overline{r}_1, ..., \overline{r}_f)$ is a *maximum differential rent vector* iff it is a solution of the rent equation (3.2) and $(\underline{r}_1, ..., \underline{r}_f)$ is a *minimum differential rent vector* iff it it is a solution of the rent equation (3.3).

We then have the following relationships for the vectors $(\overline{r}_1, ..., \overline{r}_f)$, $(\underline{r}_1, ..., \underline{r}_f)$ and the maximum and minimum competitive rent vectors.

Theorem 3.1. (1) *Let (p, x, y) be a maximum competitive equilibrium and $(\overline{r}_1, ..., \overline{r}_f)$ with $\overline{r}_f = p_f$ be a solution of equation (3.2). Then $\overline{r}_k \geq p_k$ for all k* with $1 \leq k \leq f-1$.
(2) Let (p, x, y) be a minimum competitive equilibrium and $(\underline{r}_1, ..., \underline{r}_f)$ with $\underline{r}_f = p_f$ be a solution of equation (3.3). Then $\underline{r}_k \leq p_k$ for all k with $1 \leq k \leq f-1$.

Proof is in Appendix B. In our study, although the competitive rent p_f of the marginal category is endogenously determined, the rent p_f of the marginal category f is exogenously

[8] See Kaneko et al., Lemma 2.5.

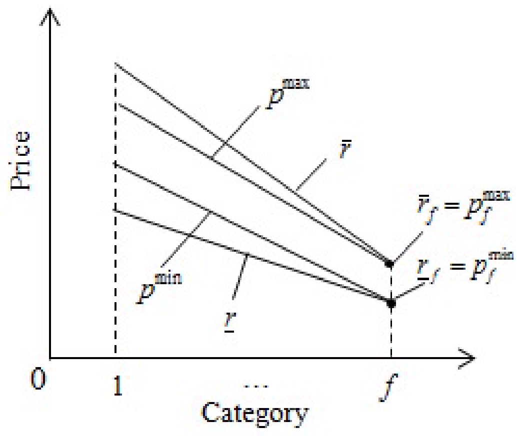

Figure 1: An illustration of Theorem 3.1.

given (e.g., Kaneko et al. adopted an estimated rent $\widetilde{p}_f$ from the real data). The meaning of Theorem 3.1.(2) is that if such a p_f is close to the maximum competitive rent, the vector $(\overline{r}_1, ..., \overline{r}_f)$ derived from equation (3.2) corresponds to an upper bound of the set of competitive rent vectors. Similarly, (2) means that if p_f is close to the minimum competitive rent, the vector $(\underline{r}_1, ..., \underline{r}_f)$ derived from equation (3.3) corresponds to an lower bound of the set of competitive rent vectors. An illustration of Theorem 3.1 is given by Fig. 1, which depict a positional relationship of a differential and a competitive rent vectors.

Kaneko et al. provided two sufficient conditions for the maximum differential rent vector to coincide with the maximum competitive rent vector (Theorem 3.2). We can also expect a similar condition for the minimum competitive rent vector to coincide with the minimum differential rent vector (Theorem 3.3).

Theorem 3.2. (Kaneko et al., 2006) *Let (p, x, y) be a maximum competitive equilibrium. If at least one of the following holds:*

(1) $I_{G(k)} = I_{G(k)+1}$ *for each* k with $1 \leq k \leq f-1$;

(2) $p_k < C_k(y_k + 1) - C_k(y_k)$ *for each* k with $1 \leq k \leq f-1$,

then the maximum differential rent vector $(\overline{r}_1, ..., \overline{r}_f)$ determined by $\overline{r}_f = p_f$ coincides with $(p_1, ..., p_f)$.

Theorem 3.3. *Let (p, x, y) be a minimum competitive equilibrium. If at least one of the following holds:*

(1) $I_{G(k)} = I_{G(k)+1}$ *for each* k with $1 \leq k \leq f-1$;

(2) $p_k > C_k(y_k) - C_k(y_k - 1)$ *for each* k with $1 \leq k \leq f-1$,

then the minimum differential rent vector $(\underline{r}_1, ..., \underline{r}_f)$ determined by $\underline{r}_f = p_f$ coincides with $(p_1, ..., p_f)$.

The proof is in Appendix C. Each theorem has the same Condition (1), which states that the income of the last household in the k-th category coincides with the income of the first household in the $k+1$-th category. This implies that, when the number of households is large and the income distribution is more or less dense (i.e., Condition (1) holds approximately), then the maximum and minimum differential rent vectors can be regarded as approximations of the maximum and minimum competitive rent vectors, respectively. Condition (2) of each theorem states that a profit maximization condition strictly holds for the k-th category, in other words, a competitive price p_k does not coincide with a marginal cost for the k-th category.

Note that under condition (1) of Theorems 3.2 and 3.3, if the competitive rent p_f of the marginal category is determined uniquely, then by the rent equations, the maximum differential rent vector (determined by $\overline{r}_f = p_f$) coincides with the minimum differential rent vector (determined by $\underline{r}_f = p_f$). When conditions (1) and (2) fail, however, the difference between the maximum and minimum differential rent vectors is an open question.

3.2 The difference between the maximum and the minimum differential rent vectors

As presented in the previous section, the maximum/minimum differential rent vectors coincides with the maximum/minimum competitive rent vectors under a certain condition. We may use either of the maximum or minimum differential rent vectors for comparative statics; however, our results may differ, depending on which one we use.

This section provides an answer to this question. In particular, we evaluate the difference between the maximum and minimum differential rent vectors. The proofs of theorems are in Section 3.3.

The following theorem concerns the relationship between the income difference and the rent difference.

Theorem 3.4. *Let* $(\overline{r}_1, ..., \overline{r}_f)$ *and* $(\underline{r}_1, ..., \underline{r}_f)$ *be the maximum and minimum differential rent vectors determined by* $\overline{r}_f = \overline{p}_f$ *and* $\underline{r}_f = \underline{p}_f$ *with* $\underline{p}_f \leq \overline{p}_f$, and k with $1 \leq k \leq f$. *Then,*

$$\overline{r}_k - \underline{r}_k \leq I_{G(k-1)} - I_{G(k-1)+1} \text{ if and only if } \overline{r}_k - \underline{r}_k \leq \overline{r}_{k-1} - \underline{r}_{k-1} \quad (3.4)$$

Note that $\leq$ *of equation (3.4) can be replaced by* $\geq$, $>$, $<$ *or* $=$.

The form of Theorem 3.4 is similar to the "Basic comparative statics theorem" of Kaneko [8] and Kaneko et al. [9]. Nevertheless, the meaning is quite different. Theorem 3.4 states that the rent difference of the k-th category is smaller than the income difference of two neighboring households numbered $G(k-1)$ and $G(k-1)+1$ if and only if the rent difference of $k-1$ is greater than the rent difference of k. This implies that we can reduce the comparison of the differences $\overline{r}_k - \underline{r}_k$ and $\overline{r}_{k-1} - \underline{r}_{k-1}$ to the comparison of the differences $\overline{r}_k - \underline{r}_k$ and $I_{G(k-1)} - I_{G(k-1)+1}$.

Fig. 2 depicts three examples of a shape of rent differences $\overline{r}_k - \underline{r}_k$ $(1 \leq k \leq f)$. Fig.2.(1) explains the case of (3.4) holds for each k. In this case, the difference $\overline{r}_k - \underline{r}_k$ gradually increases as k reaches 1. Fig.2.(2) explains the case where the opposite inequality of (3.4) holds for each k. In this case, the difference $\overline{r}_k - \underline{r}_k$ gradually decreases as k reaches 1. The

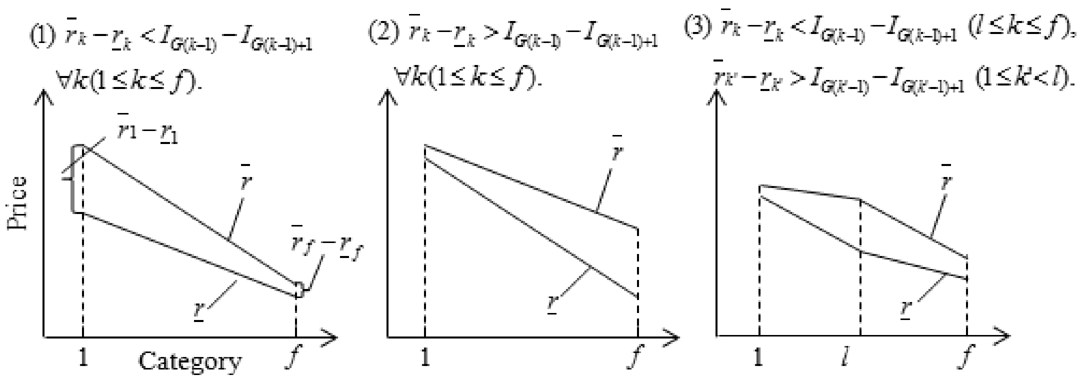

Figure 2: Shapes of the rent difference

remaining Fig.2.(3) explains the case where there is a category l $(1 \le l \le f)$ such that an inequality of (3.4) switches at l: the difference $\overline{r}_k - \underline{r}_k$ gradually increases for k with $l \le k \le f$ and decreases for k with $1 \le k < l$. Numerical examples are given in Section 4.1.

The next theorem evaluates the rent difference by the income difference.

Theorem 3.5. *Let* $(\overline{r}_1, ..., \overline{r}_f)$ *and* $(\underline{r}_1, ..., \underline{r}_f)$ *be the maximum and minimum differential rent vectors determined by* $\overline{r}_f = \overline{p}_f$ *and* $\underline{r}_f = \underline{p}_f$ *with* $\underline{p}_f \le \overline{p}_f$. *Suppose that* $\overline{r}_f - \underline{r}_f \le I_{G(f-1)} - I_{G(f-1)+1}$. *Then,*

$$0 \le \overline{r}_k - \underline{r}_k \le \max_{l \in \{k,...,f-1\}} \{I_{G(l)} - I_{G(l)+1}\} \text{ for all } k \text{ with } 1 \le k \le f-1.$$

Theorem 3.5 states that if the rent difference $\overline{r}_f - \underline{r}_f$ of the marginal category f is less than the income difference $I_{G(f-1)} - I_{G(f-1)+1}$ of two neighboring households, then the rent difference of the k-th $(k < f)$ category is bounded by at most the largest income difference $I_{G(l)} - I_{G(l)+1}$ $(k \le l < f)$.

In our differential rent approach, the rent of the marginal category f is regarded as a uniquely determined value.[9] This implies that the maximum and minimum differential rent vectors are determined by the same $p^*_f (= \overline{r}_f = \underline{r}_f)$; thus, the supposition of Theorem 3.5 holds. Under this situation, the theorem implies that the rent differences $\overline{r}_k - \underline{r}_k$ for each k are rather small. In particular, when we target a considerably large housing market with a dense household income distribution (i.e., the equality $I_{G(k)} = I_{G(k)+1}$ approximately holds), the difference can be approximated by zero. Consequently, the comparative statics results are not very different, whether or not we use the maximum or minimum differential rent vectors.

Theorem 3.5, together with Theorem 3.1 in Section 3.1, imply the following:

$$\begin{aligned} p_k^{\max} - p_k^{\min} &\le \max_{l \in \{k,...,f-1\}} \{I_{G(l)} - I_{G(l)+1}\} \text{ for all } k \text{ with } 1 \le k \le f-1 \qquad (3.5) \\ \textit{if} \quad \underline{r}_f &\le p_f^{\min} \le p_f^{\max} \le \overline{r}_f, \end{aligned}$$[10]

[9] For instance, Kaneko et al. adopted the estimated rent $\widetilde{p}_f$ from the real rent data as the differential rent r_f, and Ito adopted the (constant) marginal cost of the marginal category a_f as r_f.

that is, the difference of the maximum and minimum competitive rents of k-th category is also bounded by the largest income difference $I_{G(l)} - I_{G(l)+1}$ with $k \le l < f$. This implies the shrinkage result on the competitive rent vector set, which will be presented in Section 4.2.

3.3 Proofs of Theorems 3.4 and 3.5

Here, we prove Theorems 3.4 and 3.5.

Proof of Theorem 3.4. (*Only if*) By (3.3), we have $u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \underline{r}_{k-1}) = u(\mathbf{e}^k, I_{G(k-1)+1} - \underline{r}_k)$. Let $\delta = I_{G(k-1)} - \overline{r}_k - (I_{G(k-1)+1} - \underline{r}_k) \ge 0$. By Assumption D, $u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \underline{r}_{k-1} + \delta) \ge u(\mathbf{e}^k, I_{G(k-1)+1} - \underline{r}_k + \delta)$, that is,

$$\begin{aligned} u\left(\mathbf{e}^{k-1}, I_{G(k-1)} - \underline{r}_{k-1} - \overline{r}_k + \underline{r}_k\right) &\ge u(\mathbf{e}^k, I_{G(k-1)} - \overline{r}_k) \\ &= u(\mathbf{e}^{k-1}, I_{G(k-1)} - \overline{r}_{k-1}) \text{ by (3.2).} \end{aligned}$$

This inequation together with Assumption A imply $I_{G(k-1)} - \underline{r}_{k-1} - \overline{r}_k + \underline{r}_k \ge I_{G(k-1)} - \overline{r}_{k-1}$, that is, $\overline{r}_k - \underline{r}_k \le \overline{r}_{k-1} - \underline{r}_{k-1}$. □

(*If*) We prove the contraposition of the claim. Suppose that $\overline{r}_k - \underline{r}_k > I_{G(k-1)} - I_{G(k-1)+1}$. By (3.2), we have $u(\mathbf{e}^{k-1}, I_{G(k-1)} - \overline{r}_{k-1}) = u(\mathbf{e}^k, I_{G(k-1)} - \overline{r}_k)$. Let $\delta = I_{G(k-1)+1} - \underline{r}_k - (I_{G(k-1)} - \overline{r}_k) \ge 0$. By Assumption D, $u(\mathbf{e}^{k-1}, I_{G(k-1)} - \overline{r}_{k-1} + \delta) > u(\mathbf{e}^k, I_{G(k-1)} - \overline{r}_k + \delta)$, that is,

$$\begin{aligned} u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \overline{r}_{k-1} - \underline{r}_k + \overline{r}_k) &> u\left(\mathbf{e}^k, I_{G(k-1)+1} - \underline{r}_k\right) \\ &= u(\mathbf{e}^{k-1}, I_{G(k-1)+1} - \underline{r}_{k-1}) \text{ by (3.3).} \end{aligned}$$

This inequation together with Assumption A imply $I_{G(k-1)+1} - \overline{r}_{k-1} - \underline{r}_k + \overline{r}_k > I_{G(k-1)+1} - \underline{r}_{k-1}$, that is, $\overline{r}_k - \underline{r}_k > \overline{r}_{k-1} - \underline{r}_{k-1}$. □

Proof of Theorem 3.5. We proof this by mathematical induction over $k = f-1, f-2$,..., 1. Let $k = f-1$. By the hypothesis and monotonicity (Assumption A), we have $u(\mathbf{e}^f, I_{G(f-1)+1} - \underline{p}_f) \le u(\mathbf{e}^f, I_{G(f-1)} - \overline{p}_f)$. The left hand side is equal to $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1})$ by (3.3), and the right hand side is equal to $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1})$ by (**3.2**). Hence, we have $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1}) \le u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1})$. This and Assumption A imply

$$\overline{r}_{f-1} - \underline{r}_{f-1} \le I_{G(f-1)} - I_{G(f-1)+1}. \tag{3.6}$$

Let $\delta = I_{G(f-1)} - I_{G(f-1)+1} \ge 0$. Since $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1}) = u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f)$ by (**3.2**), we have, by Assumption D, $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1} - \delta) \le u(\mathbf{e}^f, I_{G(f-1)} - \overline{r}_f - \delta)$. This inequation is restated as

$$\begin{aligned} u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \overline{r}_{f-1}) &\le u\left(\mathbf{e}^f, I_{G(f-1)+1} - \overline{r}_f\right) \\ &\le u\left(\mathbf{e}^f, I_{G(f-1)+1} - \underline{r}_f\right) \\ &= u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1}) \text{ by (3.3).} \end{aligned}$$

This and Assumption A imply $I_{G(f-1)+1} - \overline{r}_{f-1} \leq I_{G(f-1)+1} - \underline{r}_{f-1}$, that is, $\overline{r}_{f-1} \geq \underline{r}_{f-1}$. By this and (**3.6**), we have the relation $0 \leq \overline{r}_{f-1} - \underline{r}_{f-1} \leq I_{G(f-1)} - I_{G(f-1)+1}$.

Suppose that for $k = j$ with $1 < j \leq f-1$,

$$0 \leq \overline{r}_j - \underline{r}_j \leq \max_{l \in \{j,\dots,f-1\}} \{I_{G(l)} - I_{G(l)+1}\}. \tag{3.7}$$

Then, for $k = j-1$,
(i) Suppose $\overline{r}_j - \underline{r}_j \leq I_{G(j-1)} - I_{G(j-1)+1}$. Then, $u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) \leq u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j)$. The left hand side is equal to $u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1})$ by (3.3), and the right hand side is equal to $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1})$ by (3.2). Hence, $u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) \leq u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1})$. This and Assumption A imply $I_{G(j-1)+1} - \underline{r}_{j-1} \leq I_{G(j-1)} - \overline{r}_{j-1}$, that is,

$$\overline{r}_{j-1} - \underline{r}_{j-1} \leq I_{G(j-1)} - I_{G(j-1)+1}. \tag{3.8}$$

Let $\delta = I_{G(j-1)} - I_{G(j-1)+1} \geq 0$. Since $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1}) = u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j)$ by (3.2), we have, by Assumption D, $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1} - \delta) \leq u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j - \delta)$. This inequation is restated as

$$\begin{aligned} u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \overline{r}_{j-1}) &\leq u(\mathbf{e}^j, I_{G(j-1)+1} - \overline{r}_j) \\ &\leq u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) \text{ by } (3.7) \\ &= u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) \quad \text{by } (3.3). \end{aligned}$$

This and Assumption A imply $I_{G(j-1)+1} - \overline{r}_{j-1} \leq I_{G(j-1)+1} - \underline{r}_{j-1}$, that is, $\overline{r}_{j-1} \geq \underline{r}_{j-1}$. By this and (3.8), we get

$$0 \leq \overline{r}_{j-1} - \underline{r}_{j-1} \leq I_{G(j-1)} - I_{G(j-1)+1}. \tag{3.9}$$

(ii) Suppose $\overline{r}_j - \underline{r}_j > I_{G(j-1)} - I_{G(j-1)+1}$. Then, $u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) > u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j)$. This together with (3.2) and (3.3) we have $u(\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) > u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1})$. This and Assumption A imply $I_{G(j-1)+1} - \underline{r}_{j-1} > I_{G(j-1)} - \overline{r}_{j-1}$, that is,

$$\overline{r}_{j-1} - \underline{r}_{j-1} > I_{G(j-1)} - I_{G(j-1)+1}. \tag{3.10}$$

Let $\delta = \overline{r}_{j-1} - \underline{r}_{j-1} - (I_{G(j-1)} - I_{G(j-1)+1}) \geq 0$. Since $u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1}) = u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j)$ by (3.2), we have, by Assumption D, $u(\mathbf{e}^j, I_{G(j-1)} - \overline{r}_j + \delta) \leq u(\mathbf{e}^{j-1}, I_{G(j-1)} - \overline{r}_{j-1} + \delta)$. This is restated as

$$\begin{aligned} u(\mathbf{e}^j, I_{G(j-1)+1} - \overline{r}_j + \overline{r}_{j-1} - \underline{r}_{j-1}) &\leq (\mathbf{e}^{j-1}, I_{G(j-1)+1} - \underline{r}_{j-1}) \\ &= u(\mathbf{e}^j, I_{G(j-1)+1} - \underline{r}_j) \quad \text{by } (3.3). \end{aligned}$$

This and Assumption A imply $I_{G(j-1)+1} - \overline{r}_j + \overline{r}_{j-1} - \underline{r}_{j-1} < I_{G(j-1)+1} - \underline{r}_j$, that is, $\overline{r}_{j-1} - \underline{r}_{j-1} < \overline{r}_j - \underline{r}_j$. By this and (3.10), we get $I_{G(j-1)} - I_{G(j-1)+1} < \overline{r}_{j-1} - \underline{r}_{j-1} < \overline{r}_j - \underline{r}_j$. This together with (3.7) implies

$$0 \leq I_{G(j-1)} - I_{G(j-1)+1} < \overline{r}_{j-1} - \underline{r}_{j-1} \leq \max_{l \in \{j,\dots,f-1\}} \{I_{G(l)} - I_{G(l)+1}\}.$$

This inequation together with (3.9), we have

$$0 \leq \overline{r}_{j-1} - \underline{r}_{j-1} \leq \max_{k \in \{j-1,\dots,f-1\}} \{I_{G(k)} - I_{G(k)+1}\}.$$

Hence, for all k with $1 \leq k \leq f-1$, we have $0 \leq \overline{r}_k - \underline{r}_k \leq \max_{l \in \{k,\dots,f-1\}} \{I_{G(l)} - I_{G(l)+1}\}$. □

4 Numerical examples and the application

In this section, we give three examples of the rental housing market model. The examples confirm our results in Section 3.2. As an application of Theorem 3.5, we also present the shrinkage result on the difference between the maximum and minimum differential rent vectors.

4.1 Numerical examples: Calculation of maximum and minimum differential rent vector

First, we give common settings for our examples. Suppose that there are six categories of apartments ($T = 6$). Let w_k ($k = 1, ..., 6$) be the number of apartment units for rent in the k-th category. We assume that the number of households and apartments for rent are the same, and that all the apartments are ultimately rented. That is, the marginal category is $f = 6$ and the number of households is $m = \sum_{k=1}^{6} w_k$. Assume that each household has the following utility function:

$$u(x, c) = h_k + \sqrt{c} \text{ for } k = 0, 1, ..., 6,$$

where $h_1 = 9$, $h_2 = 7$, $h_3 = 5$, $h_4 = 4$, $h_5 = 3$, $h_6 = 2$ and $h_0 = 0$. Assume that the income of each household is uniformly distributed over the interval $[100, 500]$. (Then this utility function satisfies Assumption A, C,D and E).

The settings for the following Examples 4.1 and 4.2 are the same, except for the number of apartment units and households. These examples show the difference between the maximum and minimum differential rents, as well as the smaller difference in a market with a large number of households compared to a market with a small number of households.

Example 4.1. Let $w_k = 5$ for each $k = 1, ..., 6$. This implies $I_{G(k)} - I_{G(k)+1} \simeq 13.8$ for each k. Let $\overline{r}_6 = \underline{r}_6 = 20$. Then, we can calculate the maximum and minimum differential rent vectors by the rent equations (3.2) and (3.3). Table 4.1 shows the calculation results of $\overline{r}_k$, $\underline{r}_k$ and $\overline{r}_k - \underline{r}_k$.

Table 4.1.

k	1	2	3	4	5	6
$\overline{r}_k$	225.7	162.5	100.1	70.3	43.4	20
$\underline{r}_k$	220.4	158.3	97.2	68.2	42.3	20
$\overline{r}_k - \underline{r}_k$	5.3	4.2	2.8	2.1	1.2	0

By Table 4.1, an inequality $\overline{r}_k - \underline{r}_k \leq \max_{j \in \{k,...,f-1\}} \{I_{G(j)} - I_{G(j)+1}\}$ of Theorem 3.5 holds for each k. Furthermore, we observe $\overline{r}_k - \underline{r}_k < \overline{r}_{k-1} - \underline{r}_{k-1}$ for each k. This is consistent with Theorem 3.4 because $\overline{r}_k - \underline{r}_k < I_{G(k-1)} - I_{G(k-1)+1}$ for each k (which correspond to Fig. 2.(1) in Section 3.2). To sum up, the difference $\overline{r}_k - \underline{r}_k$ is significantly smaller than $I_{G(k)} - I_{G(k)+1}$ for each k; however, the difference tends to increase as k reaches 1.

Example 4.2. Let $w_k = 20$ for each $k = 1, ..., 6$. This implies $I_{G(k)} - I_{G(k)+1} \simeq 3.4$ for each k. Let $\overline{r}_6 = \underline{r}_6 = 20$. Table 4.2 shows the calculation results of $\overline{r}_k$, $\underline{r}_k$ and $\overline{r}_k - \underline{r}_k$.

Table 4.2.

	$k=1$	2	3	4	5	6
$\overline{r}_k$	223.4	161	99.4	69.9	43.3	20
$\underline{r}_k$	222.1	160	98.7	69.4	43	20
$\overline{r}_k - \underline{r}_k$	1.3	1.0	0.7	0.5	0.3	0

As with Example 4.1, the difference $\overline{r}_k - \underline{r}_k$ is smaller than $I_{G(k)} - I_{G(k)+1}$ for each k; however,the difference gradually becomes larger as k reaches 1. We can also observe that for each k, the difference $\overline{r}_k - \underline{r}_k$ is significantly smaller than Example 4.1.

Next, we give another example where the hypothesis of Theorem 3.5 fails. This example shows that whereas the rent difference $\overline{r}_k - \underline{r}_k$ exceeds the income difference $I_{G(k)} - I_{G(k)+1}$, the rent difference tends to decrease as k goes to 1.

Example 4.3. Let $w_k = 20$ for each k. This implies $I_{G(k)} - I_{G(k)+1} \simeq 3.4$. Let $\overline{r}_6 = 20$ and $\underline{r}_6 = 15$. Then, we have $\overline{r}_6 - \underline{r}_6 = 5 > 3.4 \simeq I_{G(5)} - I_{G(5)+1}$; that is, the hypothesis of Theorem 3.5 fails. Table 4.3 shows the calculation results of $\overline{r}_k$, $\underline{r}_k$ and, $\overline{r}_k - \underline{r}_k$.

Table 4.3.

k	1	2	3	4	5	6
$\overline{r}_k$	223.4	161	99.4	69.9	43.3	20
$\underline{r}_k$	219	156.5	94.7	65.1	38.4	15
$\overline{r}_k - \underline{r}_k$	4.4	4.5	4.7	4.8	4.9	5

From Table 4.3, we have $\overline{r}_k - \underline{r}_k > \max_{j \in \{k,\ldots,f-1\}} \{I_{G(j)} - I_{G(j)+1}\}$ for each k (Theorem 3.5 fails). On the other hand, the difference $\overline{r}_k - \underline{r}_k$ tends to decrease as k reaches 1. This is consistent with Theorem 3.4 because $\overline{r}_k - \underline{r}_k > I_{G(k-1)} - I_{G(k-1)+1}$ for each k (corresponding to Fig . 2.(2) in Section 3.2). Note that this example does not explain the necessity of the condition $\overline{r}_f - \underline{r}_f \leq I_{G(f-1)} - I_{G(f-1)+1}$ for Theorem 3.5. It may be possible that Theorem 3.5 holds but $\overline{r}_f - \underline{r}_f > I_{G(f-1)} - I_{G(f-1)+1}$.

4.2 Application: Shrinkage of differential rent vectors with a large number of households

In their pioneering work of assignment games, Shapley and Shubik [6] conjectured that a set of competitive price vectors tends to shrink as the number of traders becomes large. However, they did not suggest a formulation for a suitable limit procedure. For an answer to their argument, Kamecke [12] and Gretsky, Ostroy and, Zame [3] presented a limit model of the linear assignment model with continuum of traders. Recently, Sai [11] showed that without the quasi-linear utility assumption, the set of competitive price vectors shrinks to a unique point as the number of sellers become large (though they did not deal with a limit model). The point is that Sai's result required no condition for the buyers' set.[11] According to our Theorem 3.5, we also obtain the same shrinkage result when only the buyers' set is large.

[11] Nevertheless, Sai stated that it would be natural to require the number of buyers also become large proportional to the sellers.

Here, we consider a sequence of rental housing markets $\{(M^\nu, N^\nu)\}_{\nu=1}^{\infty}$. We consider the situation where for a large ν, the market has many households and their income distribution gets dense. This is formalized by the following condition.

Condition 4.1. There is some constant $\alpha > 0$ such that for any ν, $\{(M^\nu, N^\nu)\}_{\nu=1}^{\infty}$ satisfies
(1) $|M^\nu| \to \infty$ as $\nu \to \infty$;
(2) Let $I_i^\nu \leq \alpha$ for all $i \in M^\nu$;
(3) $\max_{1\leq i\leq m-1}[I_i^\nu - I_{i+1}^\nu] \leq \alpha/|M^\nu|$.

Condition 4.1.(2) together with (1) implies that, although the number of households becomes large, the income of each household is bounded; while (3) implies that an interval of two adjacent incomes tends to be small as the number of households becomes large.

For $\nu \geq 1$, let f^ν be a marginal category in the market (M^ν, N^ν) and $(\overline{r}_1, ..., \overline{r}_{f^\nu})$, $(\underline{r}_1, ..., \underline{r}_{f^\nu})$ be the maximum and minimum differential rent vectors determined by $\overline{r}_{f^\nu} = \overline{p}_{f^\nu}$ and $\underline{r}_{f^\nu} = \underline{p}_{f^\nu}$ with $0 \leq \overline{p}_{f^\nu} - \underline{p}_{f^\nu} \leq I^\nu_{G(f^\nu-1)} - I^\nu_{G(f^\nu-1)+1}$.

Theorem 4.1. Suppose that $\{(M^\nu, N^\nu)\}_{\nu=1}^{\infty}$ satisfies Condition 4.1. Then, $\sum_{k=1}^{f^\nu} \overline{r}_k - \underline{r}_k \to 0$ as $\nu \to \infty$.

Proof. By Theorem 3.2, we have $0 \leq \sum_{k=1}^{f^\nu-1} \overline{r}_k - \underline{r}_k \leq \sum_{k=1}^{f^\nu-1} \max_{l\in\{k,...,f^\nu-1\}}\{I^\nu_{G(l)} - I^\nu_{G(l)+1}\}$. This inequality and Condition 4.1.(3) imply the consequence. □

Theorem 4.1 together with equation (**3.5**) in Section 3.2 imply that a competitive rent vector $(p_1, ..., p_f)$ of a relevant part also shrinks to a unique point ($\sum_{k=1}^{f^\nu} p_k^{\max} - p_k^{\min} \to 0$ as $\nu \to \infty$).[12] Thus, we obtain the same shrinkage result for the competitive rent vectors only under the condition for the household income distribution.

5 Conclusion

We have evaluated the difference between the maximum and minimum differential rent vectors in a rental housing market model by Kaneko et al., where the identical utility function for household and the normality for apartment quality are assumed. The differential rent vector is a solution of a certain system of equations called "the rent equation," and it is regarded as a representative of the competitive rent vector in the model.

Our main result (Theorem 3.5) is that the rent difference of the k-th category is smaller than the largest income difference between specific neighboring households numbered $G(l)$ and $G(l)+1$ ($k \leq l \leq f-1$). This implies that the rent difference can be regarded as rather small and consequently, the difference between the maximum and minimum competitive rent vectors is also small. Furthermore, the difference shrinks to zero as the market becomes larger and the household income distribution becomes denser. A precise discussion of the shrinkage result is presented in Theorem 4.2 of Section 4.2. Another result (Theorem 3.4) indicates that we can reduce the comparison of two rent differences of the k-th and $k-1$-th categories into a comparison of the rent differences of the k-th category and the income differences of

[12] Indeed, the remaining categories $f+1, ..., T$ are inessential in our market model since no units of the k-th category ($f < k \leq T$) are traded (though a competitive rent p_k ($k > f$) is determined with $p_k \leq C_k(1)$).

neighboring households numbered $G(k-1)$ and $G(k-1)+1$. Our results argue that, when we study considerably large housing markets, the difference in the comparative statics results is small, whether or not we use the maximum or minimum differential rent vectors.

We conclude with a remark on a future empirical study. It may be possible to study whether our shrinkage result on the rent difference is actually viewed from a real housing market by comparing real apartment rent data of a densely populated area and a sparsely populated area.

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Appendix

A. Proof of Theorem 2.2.

To prove Theorem 2.2, we need the following lemma.

Lemma A.1. *Let (p, x, y) and (p', x', y') be any competitive equilibria and suppose that there is $i \in M$ such that $x_i = e^k$ and $x'_i = e^l$, $k \neq l$. Then,*
(1): $p_k < p'_k$ if and only if $p_l < p'_l$;
(2): $p_k = p'_k$ if and only if $p_l = p'_l$;
(3): $p_k > p'_k$ if and only if $p_l > p'_l$.

The proof of Lemma A.1 needs the following lemma from Sai [11].

Lemma A.2 (Sai, 2014). *Let (p, x, y) and (p', x', y') be any competitive equilibria and let k be an integer with $1 \leq k \leq T$. Then, $p_k < p'_k$ implies $y_k = y'_k$.*

Proof of Lemma A.1. (*If part of (1)*): Assume $p_l < p'_l$. It follows from the assumption and household i's UMC, $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^l, I_i - p_l) > u_i(\mathbf{e}^l, I_i - p'_l) \geq u_i(\mathbf{e}^k, I_i - p'_k)$. Thus we have $u_i(\mathbf{e}^k, I_i - p_k) > u_i(\mathbf{e}^k, I_i - p'_k)$, which implies $p_k < p'_k$.
(*Only if of (1)*): Suppose $p_k < p'_k$. Suppose, on the contrary, $p_l \geq p'_l$. By lemma A.1, it holds that $y_k = y'_k$. On the other hand, in equilibrium (p', x', y'), one household i switches his housing choice from k to l. This implies that at least one household $j (\neq i)$ switches his housing choice from $m (\neq k)$ to k. By the assumption and household j's UMC, $u_j(\mathbf{e}^m, I_j - p_m) \geq u_j(\mathbf{e}^k, I_j - p_k) > u_j(\mathbf{e}^k, I_j - p'_k) \geq u_j(\mathbf{e}^m, I_j - p'_m)$. This inequality derives $p_m < p'_m$, which implies $m \neq l$. In the same manner with the above discussion, $p_m < p'_m$ implies $y_m = y'_m$, and in equilibrium (p', x', y'), at least one household switches his choice from $n (\neq m)$ to m. This also derives $p_n < p'_n$, $n \neq l$ and $y_n = y'_n$, so this process continues. Since M is finite, this process does not finish even with all the possible household switched. This implies the hypothesis $p_l \geq p'_l$ is false. Thus, we obtain $p_l < p'_l$.
(*If of (2)*): Suppose $p_l = p'_l$. By only if part of (1), it is enough to show that $p_k \leq p'_k$. It follows from the supposition and buyer i's UMC, $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^l, I_i - p_l) = u_i(\mathbf{e}^l, I_i - p'_l) \geq u_i(\mathbf{e}^k, I_i - p'_k)$. Thus we have $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^k, I_i - p'_k)$, which implies $p_k \leq p'_k$.
(*Only if of (2)*): Suppose $p_k = p'_k$. By if part of (1), it is enough to show that $p_l \leq p'_l$. Suppose, on the contrary, $p_l > p'_l$. By lemma A.1, it holds that $y_l = y'_l$. On the other

hand, in equilibrium (p', x', y'), one household i switches his housing choice from k to l. This implies that at least one household $j(\neq i)$ switches his housing choice from l to $m(\neq l)$. By the supposition and household j's UMC,$u_j(\mathbf{e}^m, I_j - p'_m) \geq u_j(\mathbf{e}^l, I_j - p'_l) > u_j(\mathbf{e}^l, I_j - p_l) \geq u_j(\mathbf{e}^m, I_j - p_m)$. This inequality derives $p_m > p'_m$, which implies $m \neq k$. In the same manner with the above discussion, $p_m > p'_m$ implies $y_m = y'_m$, and in equilibrium (p', x', y'), at least one household switches his choice from m to $n(\neq m)$. This also derives $p_n > p'_n$, $n \neq k$ and $y_n = y'_n$, so this process continues. Since M is finite, the process does not finish even with all the possible household switched. This implies the hypothesis $p_l > p'_l$ is false. Thus, we obtain $p_l \leq p'_l$.
(*(3)*): It is immediately proved by using Lemma A.1.(1) and (2). □

Proof of Theorem 2.2. Let (p, x, y) and (p', x', y') be any competitive equilibria and suppose that $p'_k < p_k$ and $p'_l > p_l$ for some k, l. Then we construct a tuple $(\underline{p}, \underline{x}, \underline{y})$ such that
(m-1): $\underline{p}_k = \min\{p_k, p'_k\}$ for k with $1 \leq k \leq T$;

(m-2): for each $i \in M$, $$\underline{x}_i = \begin{cases} x_i \text{ if } x_i = \mathbf{e}^k \text{ and } p_k \leq p'_k \text{ for some } k \text{ with } 1 \leq k \leq T, \\ x'_i \text{ if } x'_i = \mathbf{e}^k \text{ and } p_k > p'_k \text{ for some } k \text{ with } 1 \leq k \leq T, \\ \mathbf{0} \quad \text{otherwise;} \end{cases}$$

(m-3): for $k = 1, \cdots, T$, $\underline{y}_k = y_k$.

Note that the above $\underline{x}$ is well defined: Indeed, by Lemma A.1, each $i \in M$ chooses at most one category k in $\underline{x}$. In the following, we show that a tuple $(\underline{p}, \underline{x}, \underline{y})$ satisfies competitive equilibrium conditions UMC, PM and BDS.

UMC: Let $i \in M$. We have three cases.
(Case 1): $\underline{x}_i = x_i = \mathbf{e}^k$. By (m-1), $\underline{p}_k = p_k$. It is obvious that $u_i(\mathbf{e}^k, I_i - \underline{p}_k) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p_m$. Let l be the category which household i chooses in (p', x', y'). By Lemma A.1, we have $p_l \leq p'_l$. By this inequality and the UMC, $u_i(\mathbf{e}^k, I_i - p_k) \geq u_i(\mathbf{e}^l, I_i - p_l) \geq u_i(\mathbf{e}^l, I_i - p'_l) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p'_m$. Thus i satisfies UMC in $(\underline{p}, \underline{x}, \underline{y})$.
(Case 2): $\underline{x}_i = x'_i = \mathbf{e}^k$. By Then $\underline{p}_k = p'_k$. It is obvious that $u_i(\mathbf{e}^k, I_i - \underline{p}_k) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p'_m$. Let l be the category which household i chooses in (p, x, y). By Lemma A.1, we have $p_l > p'_l$. By this inequality and the UMC, $u_i(\mathbf{e}^k, I_i - p'_k) \geq u_i(\mathbf{e}^l, I_i - p'_l) > u_i(\mathbf{e}^l, I_i - p_l) \geq u_i(\mathbf{e}^m, I_i - \underline{p}_m)$ for all m with $\underline{p}_m = p_m$. Thus i satisfies UMC in $(\underline{p}, \underline{x}, \underline{y})$.
(Case 3): $\underline{x}_i = \mathbf{0}$. by (m-2), we have $x_i = x'_i = \mathbf{0}$. Hence i satisfies UMC in $(\underline{p}, \underline{x}, \underline{y})$.

PM and BDS: Let k with $1 \leq k \leq T$. If $\underline{p}_k = p_k$, landlord k maximizes his profit with production $\underline{y}_k = y_k$. By (m-2), $\underline{x}_i = x_i = \mathbf{e}^k$ for all $i \in M_k$. This implies $\sum_{i \in M_k} \underline{x}_i = \sum_{i \in M_k} x_i = y_k \mathbf{e}^k = \underline{y}_k \mathbf{e}^k$, that is, BDS holds for category k. Otherwise ($\underline{p}_k = p'_k$), the landlord k maximizes his profit with production $\underline{y}_k = y_k = y'_k$ (by Lemma A.2). The balance of total demand and supply is inherited from the equilibrium (p', x', y').

The vector $\underline{p}$ satisfies $\underline{p} \leq p$ and $\underline{p} \leq p'$. Since the set of competitive rent vectors is a compact set, there is the minimum competitive rent vector in the market (M, N). In the dual manner, we can also prove the existence of the maximum competitive rent vector. □

B. Proof of Theorem 3.1.

Proof. of (1). We proof this by mathematical induction over $k = f-1,\ f-2\ ,...,\ 1$. Let $k = f-1$. By utility maximization condition and (3.2), we have $u(\mathbf{e}^{f-1}, I_{G(f-1)} - p_{f-1}) \geq u(\mathbf{e}^{f}, I_{G(f-1)} - p_f)$ and $u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1}) = u(\mathbf{e}^{f}, I_{G(f-1)} - \overline{r}_f)$. These together with $\overline{r}_f = p_f$ imply $u(\mathbf{e}^{f-1}, I_{G(f-1)} - p_{f-1}) \geq u(\mathbf{e}^{f-1}, I_{G(f-1)} - \overline{r}_{f-1})$. This inequality and Assumption A imply $I_{G(f-1)} - p_{f-1} \geq I_{G(f-1)} - \overline{r}_{f-1}$, that is, $\overline{r}_{f-1} \geq p_{f-1}$.

Suppose that for $k = l, 1 < l \leq f-1, \overline{r}_l \geq p_l$ and let $k = l-1$. By utility maximization condition and (3.2), we have $u(\mathbf{e}^{l-1}, I_{G(l-1)} - p_{l-1}) \geq u(\mathbf{e}^{l}, I_{G(l-1)} - p_l)$ and $u(\mathbf{e}^{l-1}, I_{G(l-1)} - \overline{r}_{l-1}) = u(\mathbf{e}^{l}, I_{G(l-1)} - \overline{r}_l)$. On the other hand, $\overline{r}_l \geq p_l$ and Assumption A imply $u(\mathbf{e}^{l}, I_{G(l-1)} - p_l) \geq u(\mathbf{e}^{l}, I_{G(l-1)} - \overline{r}_l)$. This inequality together with previous inequalities imply $u(\mathbf{e}^{l-1}, I_{G(l-1)} - p_{l-1}) \geq u(\mathbf{e}^{l-1}, I_{G(l-1)} - \overline{r}_{l-1})$. This and Assumption A imply $I_{G(l-1)} - p_{l-1} \geq I_{G(l-1)} - \overline{r}_{l-1}$, that is, $\overline{r}_{l-1} \geq p_{l-1}$. Therefore we have $\overline{r}_k \geq p_k$ for all k with $1 \leq k \leq f-1$. □

Proof of (2). It is proved by the dual manner with (1). Let $k = f-1$. By utility maximization condition and the first equation of (3.3), we have $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - p_{f-1}) \leq u(\mathbf{e}^{f}, I_{G(f-1)+1} - p_f)$ and $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1}) = u(\mathbf{e}^{f}, I_{G(f-1)+1} - \underline{r}_f)$. These together with $\underline{r}_f = p_f$ imply $u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - p_{f-1}) \leq u(\mathbf{e}^{f-1}, I_{G(f-1)+1} - \underline{r}_{f-1})$. Thus, we have $\underline{r}_{f-1} \leq p_{f-1}$.

Suppose that for $k = l, 1 < l \leq f-1$, $\underline{r}_l \geq p_l$ and let $k = l-1$. By utility maximization condition and (3.3), we have $u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - p_{l-1}) \leq u(\mathbf{e}^{l}, I_{G(l-1)+1} - p_l)$ and $u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - \underline{r}_{l-1}) = u(\mathbf{e}^{l}, I_{G(l-1)+1} - \underline{r}_l)$. On the other hand, $\underline{r}_l \leq p_l$ and Assumption A imply $u(\mathbf{e}^{l}, I_{G(l-1)+1} - p_l) \leq u(\mathbf{e}^{l}, I_{G(l-1)+1} - \underline{r}_l)$. This inequality together with previous inequalities imply $u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - p_{l-1}) \leq u(\mathbf{e}^{l-1}, I_{G(l-1)+1} - \underline{r}_{l-1})$. Thus, we have $\underline{r}_{l-1} \leq p_{l-1}$. Therefore we have $\underline{r}_k \leq p_k$ for all k with $1 \leq k \leq f-1$. □

C. Proof of Theorem 3.3.

By the definition of $G(k)$ and the utility maximization condition for households $G(k)$ and $G(k+1)$, it holds that

$$\begin{aligned} u(\mathbf{e}^{k}, I_{G(k)} - p_k) &\geq u(\mathbf{e}^{k+1}, I_{G(k)} - p_{k+1}) \text{ and} \\ u(\mathbf{e}^{k+1}, I_{G(k)+1} - p_{k+1}) &\geq u(\mathbf{e}^{k}, I_{G(k)+1} - p_k). \end{aligned}$$

Suppose that condition (1) of Theorem 3.3 holds. Then, By the above inequalities, we have

$$u(\mathbf{e}^{k}, I_{G(k)+1} - p_k) = u(\mathbf{e}^{k+1}, I_{G(k)+1} - p_{k+1}),$$

that is, the rent equation holds.

Suppose that condition (2) of Theorem 3.3 holds. We prove by contradiction. Suppose that there is a category t with $1 \leq t \leq f-1$ such that

$$u(\mathbf{e}^{k}, I_{G(k)+1} - p_k) = u(\mathbf{e}^{k+1}, I_{G(k)+1} - p_{k+1}) \text{ for } k \text{ with } 1 \leq k \leq t-1;$$
$$u(\mathbf{e}^{t}, I_{G(t)+1} - p_t) < u(\mathbf{e}^{t+1}, I_{G(t)+1} - p_{t+1}).$$

Then, we can decrease p_t and p_{t-1}, ..., p_1 slightly into p'_t and p'_{t-1}, ..., p'_1 such that

$$\begin{aligned} u(\mathbf{e}^{t}, I_{G(t)+1} - p'_t) &< u(\mathbf{e}^{t+1}, I_{G(t)+1} - p_{t+1}); \\ p'_t &> C_t(y_t) - C_t(y_t - 1). \end{aligned} \tag{C.1}$$

$$u(\mathbf{e}^k, I_{G(k)+1} - p'_k) = u(\mathbf{e}^{k+1}, I_{G(k)+1} - p'_{k+1}) \text{ and}$$
$$p'_k > C_k(y_k) - C_k(y_k - 1) \text{ for } k \text{ with } 1 \leq k \leq t-1. \tag{C.2}$$

We now let the new rent vector p^* as

$$p^*_k = \begin{cases} p_k \text{ for } k \geq t+1; \\ p'_k \text{ for } k \leq t. \end{cases}$$

In the following, we show a tuple (p^*, x, y) is also a competitive rent vector: this is a contradictory claim since p is the minimum competitive rent vector. Since (x, y) is a competitive allocation, the balance of total supply and demand condition is satisfied. Furthermore, by the bottom of (C.1) and (C.2), each landlord's profit maximization condition holds with (p^*, y). The utility maximization condition of households is checked by as follows. Let $i \in M$ with $x_i = \mathbf{e}^k$. We easily find $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^{k'}, I_i - p^*_{k'})$ for price unchanged categories $k' = t+1, ..., T$. The remaining part is shown by the following case analysis:
(i) The case of $k \geq t+1$. By the definition of $G(k)$, we have $I_i \leq I_{G(t)+1}$. This together with the top of (C.1) and Assumption E imply $u(\mathbf{e}^k, I_i - p^*_k) > u(\mathbf{e}^t, I_i - p^*_t)$. Furthermore, this inequality together with the top of (C.2) and Assumption E imply $u(\mathbf{e}^k, I_i - p^*_k) > u(\mathbf{e}^t, I_i - p^*_t) \geq u(\mathbf{e}^{t-1}, I_i - p^*_{t-1}) \geq \cdots \geq u(\mathbf{e}^1, I_i - p^*_1)$.
(ii) The case of $k < t+1$. Let k' with $k < k' < t+1$. By the definition of $G(k)$, we have $I_i \geq I_{G(k)} \geq I_{G(k)+1}$. This together with (C.2) and Assumption E imply $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^{k'}, I_i - p^*_{k'})$. Furthermore let k'' with $1 < k'' < k$. By the definition of $G(k)$, we have $I_i \leq I_{G(k-1)+1}$. This together with (C.2) and Assumption E imply $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^{k''}, I_i - p^*_{k''})$. Combining them, we have $u(\mathbf{e}^k, I_i - p^*_k) \geq u(\mathbf{e}^l, I_i - p^*_l)$ for all $l = 1, ..., t$. □

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