Institute of Socio-Economic Planning
Discussion Paper Series

No.129(81-30)

A NEW VARIABLE DIMENSION ALGORITHM : EXTENSION FOR SEPARABLE MAPPINGS, GEOMETRIC INTERPRETATION AND SOME APPLICATIONS

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Oct. 1981

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Title: A New Variable Dimension Algorithm: Extension for Separable Mappings, Geometric Interpretation and Some Applications

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Abstract: We consider the 2-ray method, a new variable dimension fixed-point algorithm proposed by one of the authors. We give a computational procedure of the method for separable mappings which will save a certain amount of pivot operations. We also give a geometric interpretation of the method. Applications to two-point boundary value problems and variational problems are also discussed.

Key Words: fixed point, variable dimension algorithm, separable mapping, piecewise linear equations, two-point boundary value problem, variational problem

Abbreviated Title: A New Variable Dimension Algorithm

1. INTRODUCTION

Since van der Laan and Talman [8-12,17] developed a new fixed-point algorithm, variable dimension algorithm, several new algorithms and various interpretations have been proposed (Freund [3], Kojima and Yamamoto [6,7], Reiser [14], Saigal [15], Todd [18,19], Todd and Wright [20], and Wright [21]). We have also proposed a new variable dimension algorithm in [24] (see also [23]) for solving the system of n equations in n variables

(1.1) $\quad f(x) = 0, \qquad x \in R^n,$

where $f\colon R^n \rightarrow R^n$ is a continuous mapping. This new algorithm always moves either in the direction of the first coordinate axis or in the opposite direction, then we refer to this algorithm as "2-ray method". The purpose of this paper is to give an improved computational procedure, a geometric interpretation and some applications of the 2-ray method.

In Section 2 we shall briefly review the basic system and the computational procedure of the 2-ray method. In Section 3 we shall consider the case where f is a separable mapping and propose a procedure which improves the computational efficiency of the method. In Section 4 we shall show that the 2-ray method can be considered as a method for tracing a solution path of a piecewise linear mapping from a certain subset of $R^n \times [0,1]$ to R^n. This interpretation makes the index theorem applicable to the 2-ray method. In Section 5 we shall show how the 2-ray method is applied to the two-point boundary value problem and the variational problem.

2. PRELIMINARIES

A convex polyhedral set C in some Euclidean space R^n is called a cell and its dimension, denoted by dim C, is defined to be the dimension of the affine subspace aff(C) spanned by C. We shall abbreviate m-dimensional cells by m-cells. $B \subseteq R^n$ is said to be a face of a cell C if B is a convex subset of C such that $x,y \in C$, $\lambda \in (0,1)$ and $\lambda x+(1-\lambda) y \in B$ always imply that $x,y \in B$. We write $B < C$ if B is a face of C.

Let L be a finite or countable collection of m-cells in R^n. We denote by $\bar{L}$ the collection $\{B: B < C \text{ for some } C \in L\}$. L is called a subdivided m-manifold if the following conditions are satisfied:

(2.1) For $B,C \in L$ $B \cap C$ is a face of B and C.

(2.2) Each (m-1)-cell of $\bar{L}$ lies in at most two m-cells of L.

We call the collection of all (m-1)-cells of $\bar{L}$ each of which lies in exactly one m-cell of L the boundary of L and denote by ∂L.

By an interval we mean a convex subset of R^1 which contains more than one point. It is known that a connected subdivided 1- manifold is homeomorphic to either an interval or the unit circle $\{x: x \in R^2, \|x\|=1\}$. We call a connected subdivided 1-manifold a path if it is homeomorphic to an interval, and a loop otherwise.

Let L and L' be subdivided m-manifolds with $|L| = |L'|$, where $|L| = \bigcup\{C : C \in L\}$. If each m-cell $C' \in L'$ is contained in some m-cell $C \in L$, L' is said to be a refinement of L. If, in addition, all cells of L' are simplicial, L' is said to be a simplicial refinement or a triangulation of L.

Now we shall briefly review the new variable dimension algorithm 2-ray method. The proofs of the following lemmas are found in [24] and omitted here. Let $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^t \in R^n$ be an arbitrary point. For $\alpha \in \{+1, -1\}$ and $k \in N = \{1, 2, \ldots, n\}$ define

$$X(k,\alpha) = \{ x \in R^n : \alpha(x_k - x_k^0) \geq 0, \quad x_j - x_j^0 = 0 \text{ for } j > k \}$$

$$Y(k,\alpha) = \{ y \in R^n : \alpha y_k \geq 0, \qquad y_j = 0 \quad \text{for } j < k \},$$

with the convention that

$$X(0,\alpha) = \{x^0\}, \qquad Y(n+1,\alpha) = \{0\}$$

for any $\alpha \in \{+1,-1\}$. Let T be a triangulation of R^n such that

(2.3) mesh size $\delta_0 = \sup_{\sigma \in T} \sup\{ \|u-v\| : u, v \in \sigma \}$ is bounded,

(2.4) each point $x \in |T|$ has a neighborhood which intersects finitely many simplices of T, and

(2.5) $T(k,\alpha) = \{ \sigma \in \overline{T} : \sigma \subseteq X(k,\alpha), \dim \sigma = \dim X(k,\alpha) \}$ is a simplicial refinement of $X(k,\alpha)$ for any $\alpha \in \{+1,-1\}$ and any $k \in N^* = \{0,1,\ldots, n\}$.

Let

$$M = \{ \sigma \times Y(k,\alpha) : \sigma \in T(k,\alpha), \alpha \in \{+1,-1\}, k \in N \},$$

then M is a countable collection of (n+1)-cells in R^{2n}. For $\tau, \sigma \in \overline{T}$ if $\tau < \sigma$ and $Y(j,\beta) \subseteq Y(k,\alpha)$, we say that $\tau \times Y(j,\beta)$ is a pseudo face of $\sigma \times Y(k,\alpha)$ and write $\tau \times Y(j,\beta) \prec \sigma \times Y(k,\alpha)$. In the same way as in defining $\overline{L}$ and ∂L we define

$$\overline{M} = \{ B : B \prec C \text{ for some } C \in M \}$$

$$\partial M = \{ B : B \in \overline{M}, \dim B = n, B \text{ lies in exactly one } (n+1)\text{-cell of } M \}$$

and call each cell of $\overline{M}$ a pseudo face of M. Then M closely resembles subdivided manifolds in structure as follows.

<u>Lemma 2.1.</u> Let B and C be (n+1)-cells of M with $B \cap C \neq \emptyset$. Then $B \cap C$ is either an n-pseudo face of M or the union of several pseudo faces of M whose dimensions are less than n.

<u>Lemma 2.2.</u>

(2.6) Let $\tau \times Y(k,\alpha)$ be an n-pseudo face of M. If $2 \leq k \leq n$, then $\tau \times Y(k,\alpha)$ is a pseudo face of exactly two (n+1)-cells of M.

(2.7) The n-pseudo face $\{x^0\} \times Y(1,\alpha)$ of M lies in exactly one (n+1)-cell $\sigma \times Y(1,\alpha)$ of M, where σ is a 1-simplex such that $\{x^0\} < \sigma \in T(1,\alpha)$.

(2.8) Let σ be an n-simplex of T. Then the n-pseudo face $\sigma \times \{0\}$ lies in exactly one (n+1)-cell $\sigma \times Y(n,\alpha)$ of M, where α is such that $\sigma \in T(n,\alpha)$.

The next lemma is immediate from Lemma 2.2.

Lemma 2.3.

(2.9) $|\partial M| = (\{x^0\} \times R^n) \cup (R^n \times \{0\})$.

Let $F : |T| \longrightarrow R^n$ be a piecewise linear (abbreviated by PL) approximation of $f: R^n \longrightarrow R^n$ with respect to the triangulation T, i.e., for each $x \in \sigma \in T$ with

$$x = \sum_{i=0}^{n} \lambda_i v^i, \quad \sum_{i=0}^{n} \lambda_i = 1, \quad \lambda_i \geqq 0 \quad \text{for } i = 0,1,\ldots,n,$$

where $v^0, v^1, \ldots, v^n$ are the vertices of σ,

$$F(x) = \sum_{i=0}^{n} \lambda_i f(v^i).$$

Since F closely approximates f if the mesh size δ_0 of T is small enough, we can locate an approximate solution of the system (1.1) by solving the system of PL equations

(2.10) $F(x) = 0, \quad x \in |T|$.

Let $H: |M| \longrightarrow R^n$ be a PL mapping such that

(2.11) $H(x,y) = F(x) + y$

and consider the following basic system of PL equations

(2.12) $H(x,y) = 0, \quad (x, y) \in |M|$.

Since n-1 variables out of 2n variables of (2.12) are fixed to some constant values, (2.12) is a system of n equations in, virtually, n+1 variables. Hence under an appropriate regularity condition it is shown that

(2.13) the solution set $S = \{ (x, y) \in |M| : H(x, y)=0 \}$ of (2.12) is the disjoint union of paths and loops,

(2.14) each loop of S does not intersect with $|\partial M|$, and

(2.15) if a path P of S is compact, then ∂P consists of two distinct points in $|\partial M|$.

By the definitions of M and H $(x^0, y^0) = (x^0, -F(x^0))$ is a point in $S \cap |\partial M|$. Let S^0 be a connected component of S having (x^0, y^0). Then by (2.14) we see that S^0 is a path. If S^0 is compact, S^0 has another endpoint, say (x^1, y^1), in $|\partial M|$. Hence by Lemma 2.3 we have either

$$(x^1, y^1) \in \{x^0\} \times R^n \quad \text{or} \quad (x^1, y^1) \in R^n \times \{0\} .$$

If the former case occurs two endpoints of S^0 coincide with each other, which is contrary to (2.15). Hence the latter case occurs, i.e. $y^1 = 0$ and thus $F(x^1) = 0$. Therefore by tracing the path S^0 from the starting point (x^0, y^0) we can locate an approximate solution x^1 of (1.1).

Now we shall explain the procedure to trace the solution path S^0. Suppose that $(x, y) \in S^0 \cap (\sigma \times Y(k, \alpha))$ for a k-simplex $\sigma \in T(k, \alpha)$. Then there exists a solution $(\lambda, y) \in R^{(k+1)+n}$ of the following system (2.16)

$$\text{(2.16)} \qquad \begin{bmatrix} e^t \\ \hline f(v^0) \cdots\cdots f(v^k) \end{bmatrix} \begin{bmatrix} \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ \hline I_n \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix}$$

$$\lambda \geqq 0, \quad y \in Y(k, \alpha),$$

where $e \in R^{k+1}$ is the vector of one's and I_n is the $n \times n$ indentity matrix. Since $y \in Y(k, \alpha)$ implies that $y_j = 0$ for $j < k$, $\alpha y_k \geqq 0$ and y_j is not restricted at all for $j > k$, (2.16) has a solution if and only if the following smaller system (2.17) has a solution $(\lambda, z_k) \in R^{(k+1)+1}$.

$$(2.17)_k \quad \begin{bmatrix} e^t \\ \hdashline f^{(k)}(v^0) \dots f^{(k)}(v^k) \end{bmatrix} \begin{bmatrix} \\ \lambda \\ \\ \end{bmatrix} + \begin{bmatrix} \\ 0 \\ \hdashline \alpha \end{bmatrix} z_k = \begin{bmatrix} 1 \\ \hdashline \\ 0 \\ \\ \end{bmatrix}$$

$$\lambda \geqq 0, \quad z_k \geqq 0,$$

where $f^{(k)}$ is the k-vector consisting of the first k components of f. Thus we can trace the solution path S^0 by handling the system (2.17) of k+1 equations in k+2 variables.

The 2-ray method is formally stated as follows. Here for a simplex $\tau = \mathrm{co}\{v^0, v^1, \dots, v^k\}$ and a point $v \in R^n$ we denote by $\tau + v$ (resp. $\tau - v$) the simplex consising of the vertex set $\{v^0, v^1, \dots, v^k\} \cup \{v\}$ (resp. $\{v^0, v^1, \dots, v^k\} \setminus \{v\}$), where co V means the convex hull of V. For a real value y sign y is defined such that

$$\text{sign } y = \begin{cases} +1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0. \end{cases}$$

Step 0 (initialization): $\tau := \{x^0\}$, $\lambda_0 := 1$, $y_1 := -f(x^0)$, $z_1 := |y_1|$, $\alpha :=$ sign y_1, $k := 1$.

Step 1 : Find a vertex v^+ of T such that $\sigma := \tau + v^+$ is a k-simplex of $T(k, \alpha)$.

Step 2 : In the system $(2.17)_k$ increase λ^+, the variable corresponding to the column $f^{(k)}(v^+)$. If z_k vanishes, go to Step 3. If some λ_j vanishes go to Step 4.

Step 3 (dimension increasing) : If $k = n$, stop. Otherwise, $\tau := \sigma$,

$y_{k+1} := -\left\{ \sum_{i=0}^{k-1} \lambda_i f_{k+1}(v^i) + \lambda^+ f_{k+1}(v^+) \right\}$, $z_{k+1} := |y_{k+1}|$, $\alpha :=$ sign y_{k+1}, $k := k+1$ and go to Step 1.

Step 4 : $\tau := \sigma - v_j$. If $\tau \in T(k-1, \beta)$ for some $\beta \in \{+1,-1\}$, then $\sigma := \tau$, $\alpha := \beta$ and go to Step 5. Otherwise, $v^- := v^j$ and go to Step 6.

Step 5 (dimension decreasing) : In the system $(2.17)_{k-1}$ increase z_{k-1} until some λ_i vanishes. $\tau := \sigma - v^i$, $v^- := v^i$, $k := k-1$ and go to Step 6.

Step 6 (replacing a vertex) : Find a vertex v^+ of T which can be replaced with v^- in order to obtain another k-simplex of $T(k, \alpha)$ having τ as a face. $\sigma := \tau + v^+$ and go to Step 2.

3. THE 2-RAY METHOD FOR SEPARABLE MAPPINGS

In this section we suppose that $f: R^n \rightarrow R^n$ is a separable mapping, i.e., $f(x) = \sum_{j=1}^{n} g_j(x_j)$ for some $g_j: R^1 \rightarrow R^n$. Kojima [5] has developed two different separable homotopies, H^2 and H^3, to increase the computational efficiency of the homotopic approach to solving the system of equations (1.1). Theoretical arguments in terms of the directional density and preparatory computaional experiments have proved the proposed homotopies to be more efficient than the ordinary ones. It should be noted, however, that

(i) one 2-dimensional triangulation is needed to make a PL approximation of the proposed homotopies, and

(ii) the system for tracing the solution path consists of 3n or 2n equations in 3n+1 or 2n+1 variables.

We shall propose a separable version of the 2-ray method which requires no triangulations with higher dimension than unity and no more than n variables (the reader might refer to Saigal [16] which provides a separable version of the variable dimension algorithm proposed by himself). For j = 1,2,...,n let T_j be a 1-dimensional triangulation of the real line R^1 and $\bar{T}_j$ be the collection of all faces of T_j. We assume that T_j subdivides the j-th coordinate axis of R^n. Let $G_j(x_j)$ be a PL approximation of $g_j(x_j)$ on T_j. Then the separable mapping $f(x) = \sum_{j=1}^{n} g_j(x_j)$ is approximated by PL mapping

$$F(x) = \sum_{j=1}^{n} G_j(x_j).$$

Each piece of linearity of F is of the form $\prod_{j=1}^{n} [l_j, u_j]$, where for j = 1,2,...,n the closed interval $[l_j, u_j]$ is a 1-dimensional simplex of T_j. Here we further assume that each component x_j^0 of the initial point $x^0 \in R^n$ is a vertex of T_j. Then the basic system (2.12) is equivalently recast as

$$(3.1)\quad \begin{aligned} &\sum_{j=1}^{n} G_j(x_j) + y = 0 \\ &x_j \in [l_j,\ u_j] \in \bar{T}_j\ , \quad j = 1,2,\ldots,n \\ &\prod_{j=1}^{n} [l_j,\ u_j] \subseteq X(k,\alpha) \\ &y \in Y(k,\alpha). \end{aligned}$$

By the definitions $X(k,\alpha)$ and $Y(k,\alpha)$ the last two conditions of (3.1) are equivalent to

$$\begin{aligned} &\alpha\,[l_k,\ u_k] \geqq 0 \\ &l_j = u_j = x_j^0, \qquad j > k \\ &\alpha\, y_k \geqq 0 \\ &y_j = 0, \qquad j < k, \end{aligned}$$

where $\alpha\,[l_k,\ u_k] \geqq 0$ means that $\alpha\, x \geqq 0$ for any $x \in [l_k,\ u_k]$. Therefore (3.1) has a solution $(x,\ y) \in R^{2n}$ if and only if the following smaller system (3.2) has a solution $(x_1, x_2, \ldots, x_k, z_k)^t \in R^{k+1}$:

$$(3.2)_k\quad \begin{aligned} &\sum_{j=1}^{k} G_j^{(k)}(x_j) + \sum_{j=k+1}^{n} G_j^{(k)}(x_j^0) + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} z_k = 0 \\ &l_j \leqq x_j \leqq u_j\ , \qquad j \leqq k \\ &\alpha\,[l_k,\ u_k] \geqq 0 \\ &z_k \geqq 0. \end{aligned}$$

Since $G_j(x_j)$ is affine on each interval $[l_j,\ u_j]$, i.e., $G_j(x_j) = a_j x_j + b_j$ for some $a_j,\ b_j \in R^n$, the system of equations in (3.2) is further rewritten as

$$(3.3)_k\quad \sum_{j=1}^{k} a_j^{(k)} x_j + \sum_{j=1}^{n} b_j^{(k)} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} z_k = 0.$$

The above system has the following advantage:

(i) the system $(3.2)_k$ or $(3.3)_k$ consists of k equations and k pairs of bounded variable constraints in k+1 variables,

(ii) k is initially unity and never exceeds n,

(iii) the bounded variable constraints are easily handled by the bounded variable technique of the linear programming, and

(iv) a_j and b_j are directly calculated from the function values g_j at two end points of the interval $[l_j, u_j]$ as follows:

$$a_j = (g_j(u_j) - g_j(l_j)) / (u_j - l_j)$$

$$b_j = (u_j g_j(l_j) - l_j g_j(u_j)) / (u_j - l_j).$$

4. GEOMETRIC INTERPRETATION OF THE 2-RAY METHOD

Some of the variable dimension algorithms were originally proposed as a method for tracing a solution path of a PL mapping on a special subdivision of a subset of $R^n \times [0, 1]$, and some were given such geometric interpretations afterwards. The purpose of this section is to give a geometric interpretation to the 2-ray method and cast it in the basic framework of Eaves [1], and Eaves and Scarf [2].

4.1. A SUBDIVISION OF A CERTAIN SUBSET OF $R^n \times [0, 1]$

We begin with some definitions. We shall call a vector of R^n each component of which is -1, 0 or 1 a sign vector and usually denote it by s. For $k \in N$ and $\alpha \in \{+1, -1\}$ let

$$S(k, \alpha) = \{ s \in R^n: \quad s_j = 0, \quad \forall j < k$$
$$s_k = \alpha$$
$$s_j \in \{+1, -1\}, \quad \forall j > k \}$$

and for notational convenience let $S(0, \beta) = \{0\}$ for any $\beta \in \{+1, -1\}$.

Let

$$B = \{ y \in R^n: \ \| y \|_1 = 1 \},$$

where $\| y \|_1 = \sum_{j=1}^{n} | y_j |$. For a nonzero sign vector s define

$$B(s) = \{ y \in B: \ s_j y_j \geqq 0, \quad \forall j \in N$$
$$y_j = 0 \ \text{ if } s_j = 0 \ \}$$

and for s = 0

$$B(0) = \{ y \in R^n: \ \| y \|_1 \leqq 1 \}.$$

We denote the collection $\{ B(s): s \in S(k, \alpha) \}$ by $B(k, \alpha)$.

For a pair of subsets P and Q of R^n we shall denote by Q # P the join of $Q \times \{1\}$ and $P \times \{0\}$, i.e.,

$$Q \# P = \{ z \in R^{n+1}: z = \lambda (y, 1) + (1 - \lambda)(x, 0)$$
$$\text{for some } y \in Q,\ x \in P \text{ and } \lambda \in [0, 1] \}.$$

Define

$$L(k, \alpha) = \{ C \# \tau \ : C \in B(k, \alpha), \quad \tau \in T(k, \alpha) \}$$

and

$$L = \{ L(k, \alpha): k \in N^*, \quad \alpha \in \{+1, -1\} \},$$

where $N^* = \{0, 1, \ldots, n\}$.

In the sequel we shall prove several preparatory lemmas to show that L is a subdivision, not necessarily simplicial, of a certain subset of $R^n \times [0, 1]$ after the discussion in Yamamoto [22].

<u>Lemma 4.1.</u> $|B(k, \alpha)| = B \cap Y(k, \alpha)$ for $k \in N$ and $\alpha \in \{+1, -1\}$.

proof. Let $y \in C$ for some $C \in B(k, \alpha)$. Then $y_j = 0$ for any $j < k$ and $\alpha\ y_k \geqq 0$. This implies that $y \in Y(k, \alpha)$. Hence we have seen that

$$|B(k, \alpha)| \subseteq B \cap Y(k, \alpha).$$

To see the reverse relation let y be a point in $B \cap Y(k, \alpha)$ and define $s \in R^n$ such that

$$s_j = \begin{cases} 0 & \text{if } j < k \\ \alpha & \text{if } j = k \\ +1 & \text{if } j > k \text{ and } y_j \geqq 0 \\ -1 & \text{if } j > k \text{ and } y_j < 0 \end{cases}$$

Then it is clear that $s \in S(k, \alpha)$ and $y \in B(s) \in B(k, \alpha)$. Q.E.D.

For a convex set $X \subseteq R^n$ define $tng(X) = aff(X) - X = \{y - x: y \in aff(X), x \in X\}$ and call it the tagential space of X.

<u>Lemma 4.2.</u> $tng(C) \cap tng(X(k, \alpha)) = \{0\}$ for $C \in B(k, \alpha)$ and $k \in N^* = \{0, 1, \ldots, n\}$.

proof. If $k = 0$, then the lemma is trivial because $X(k, \alpha) = \{0\}$. Hence we shall assume that $k \neq 0$. It can be checked easily that

$$tng(C) = \{ y \in R^n: \ y_j = 0, \ \forall j < k \\ \sum_{j=1}^{n} s_j y_j = 0 \}$$

for some $s \in S(k, \alpha)$ and

$$tng(X(k, \alpha)) = \{ x \in R^n: \ x_j = 0, \ \forall j > k \}.$$

Accordingly, if $z \in tng(C) \cap tng(X(k, \alpha))$, then $z_j = 0$ for any $j \neq k$. Moreover

$$0 = \sum_{j=1}^{n} s_j z_j = s_k z_k = \alpha\ z_k,$$

implies that $z_k = 0$. Q.E.D.

Lemma 4.3. Let $k \in N$. If $C \in B(k, \alpha)$ and $\tau \in T(k, \alpha)$ for some $\alpha \in \{+1, -1\}$, then $C \# \tau$ is an $(n+1)$-dimensional simplex.

proof. We can readily see that

$$C \# \tau = \text{co}\{(u^0, 1), \ldots, (u^{n-k}, 1), (v^0, 0), \ldots, (v^k, 0)\},$$

where $u^0, \ldots, u^{n-k}$ are the vertices of $(n-k)$-simplex C and $v^0, \ldots, v^k$ are the vertices of k-simplex τ. Hence it is sufficient to show that the $n+2$ vectors above are affinely independent. Suppose that

$$\sum_{j=0}^{n-k} \alpha_j \{(u^j, 1) - (v^k, 0)\} + \sum_{j=0}^{k-1} \beta_j \{(v^j, 0) - (v^k, 0)\} = 0.$$

Then

$$\sum_{j=0}^{n-k} \alpha_j = 0 \tag{4.1}$$

and thus

$$\sum_{j=0}^{n-k-1} \alpha_j \{(u^j, 1) - (u^{n-k}, 1)\} = - \sum_{j=0}^{k-1} \beta_j \{(v^j, 0) - (v^k, 0)\}.$$

Since the left hand side of the above system lies in $\text{tng}(C)$ and the right hand side in $\text{tng}\,\tau \subseteq \text{tng}(X(k, \alpha))$, we obtain by Lemma 4.2 that both sides are zero. Therefore $\alpha_j = 0$ for $j = 0, 1, \ldots, n-k-1$ and $\beta_j = 0$ for $j = 0, 1, \ldots, k-1$ since $\{u^0, \ldots, u^{n-k}\}$ and $\{v^0, \ldots, v^k\}$ are the sets of affinely independent vectors. Furthermore (4.1) implies that $\alpha_{n-k} = 0$. This completes the proof. Q.E.D.

Lemma 4.4. For an arbitrary point $(z, t) \in |L|$ with $t \in (0, 1)$ x and $y \in R^n$ such that

(4.2) $x \in X(k, \alpha)$ for some $k \in N^*$ and some $\alpha \in \{+1, -1\}$,

(4.3) $y \in C$ for some $C \in B(k, \alpha)$, and

(4.4) $(z, t) = \lambda (y, 1) + (1 - \lambda)(x, 0)$ for some $\lambda \in [0, 1]$

are unique.

proof. For a given point $(z, t) \in |L|$ suppose that there exist two pairs (x, y) and (x′, y′) satisfying the three conditions (4.2) - (4.4) for the triplets (k, α, λ) and (k', α', λ'), respectively. It is trivial that $\lambda = \lambda' = t$.

Case 1: $k = k'$ and $\alpha = \alpha'$.

From the condition (4.4)

$$(1 - t)(x - x') = -t(y - y').$$

Hence by Lemma 4.2 and $t \in (0, 1)$, $x = x'$ and $y = y'$.

Case 2: $k = k'$ and $\alpha \neq \alpha'$.

If $k = k' = 0$, this case is reduced to Case 1. Then we shall suppose $k = k' \neq 0$. Since the condition (4.4) is

$$\begin{aligned} & t(0, \ldots, 0, y_k, \ldots, y_n) + (1 - t)(x_1, \ldots, x_k, 0, \ldots, 0) \\ = \; & t(0, \ldots, 0, y'_k, \ldots, y'_n) + (1 - t)(x'_1, \ldots, x'_k, 0, \ldots, 0), \end{aligned}$$

we immediately have

$$x_j = x'_j, \quad \forall j \neq k$$
$$y_j = y'_j, \quad \forall j \neq k$$

and

$$ty_k + (1 - t)x_k = ty'_k + (1 - t)x'_k. \tag{4.5}$$

Since $\alpha x_k, \; \alpha y_k, \; \alpha' x'_k, \; \alpha' y'_k \geq 0$,

$$\alpha(ty_k + (1 - t)x_k) \geqq 0, \quad \alpha'(ty'_k + (1 - t)x'_k) \geqq 0.$$

Hence applying the assumption that $\alpha \neq \alpha'$ we see that both sides of (4.5) are zero. This together with the assumption that $t \in (0, 1)$ implies that $x_k = y_k = x'_k = y'_k = 0$. Thus we have shown that $x = x'$ and $y = y'$.

Case 3: $k \neq k'$

Without loss of generality we here assume that $k' < k$. Then from the condition (4.4) we have

$$(4.6) \qquad y'_j = \begin{cases} ((1-t)/t)(x_j - x'_j) & \text{if } j = k' \\ ((1-t)/t)x_j & \text{if } k' < j < k \\ ((1-t)/t)x_k + y_k & \text{if } j = k \\ y_j & \text{if } j > k . \end{cases}$$

Since $k \geqq 1$, $y \in B$, i.e., $\| y \|_1 = 1$. Hence

$$\sum_{j > k} | y'_j | = \sum_{j > k} | y_j | = 1 - | y_k | .$$

Since the sign of x_k coincides with that of y_k,

$$| y'_k | = ((1-t)/t) \; | x_k | + | y_k | .$$

Therefore

$$(4.7) \qquad \sum_{j \geqq k} | y'_j | = 1 - | y_k | + ((1-t)/t) \; | x_k | + | y_k |$$
$$= 1 + ((1-t)/t) \; | x_k | .$$

Since $\| y' \|_1 \leqq 1$, (4.7) implies that $x_k = 0$ and

$$(4.8) \qquad y'_j = 0 \quad \text{for } \forall j < k .$$

Consequently

$$(4.9) \qquad x_j = 0 \quad \text{for } \forall j > k' .$$

The assertion follows from (4.6)(4.8) and (4.9). Q.E.D.

Now we are ready to prove the following lemma.

Lemma 4.5. Let $C_1 \# \tau_1$ and $C_2 \# \tau_2$ be cells of L. If the intersection $(C_1 \# \tau_1) \cap (C_2 \# \tau_2)$ is not empty, it is a face of both $C_1 \# \tau_1$ and $C_2 \# \tau_2$.

proof. It is sufficient to show that

$$(C_1 \# \tau_1) \cap (C_2 \# \tau_2) = (C_1 \cap C_2) \# (\tau_1 \cap \tau_2) .$$

Since the left hand side clearly includes the right hand side, we shall only prove the reverse relation. Let $(z, t) \in (C_1 \# \tau_1) \cap (C_2 \# \tau_2)$. Then for $i = 1, 2$ there exist $x^i \in \tau_i$, $y^i \in C_i$ such that

$$(z, t) = t(y^i, 1) + (1-t)(x^i, 0).$$

By Lemma 4.4 we immediately conclude that $x^1 = x^2$ and $y^1 = y^2$. Hence

$x^1 = x^2 \in \tau_1 \cap \tau_2$ and $y^1 = y^2 \in C_1 \cap C_2$. This completes the proof. Q.E.D.

Theorem 4.6. L is a subdivided (n + 1)-manifold whose cells are simplicial with one exception $B(0) \# \{0\}$.

proof. Straightforward from the fact that $L(k, \alpha)$ is a class of (n + 1)-dimensional simplices and Lemma 4.5. Q.E.D.

Thus we have proved that L subdivides a certain subset of $R^n \times [0, 1]$.

In what follows we shall show that $\partial L = (T \times \{0\}) \cup \{B(0) \times \{1\}\}$, where $T \times \{0\} = \{\tau \times \{0\} : \tau \in T\}$. The next lemma follows directly from the definitions of $T(k, \alpha)$ and $B(k, \alpha)$.

Lemma 4.7.

$$\partial T(k, \alpha) = T(k - 1, + 1) \cup T(k - 1, - 1)$$

$$\partial B(k, \alpha) = B(k + 1, + 1) \cup B(k + 1, - 1).$$

Lemma 4.8. For any n-dimensional cell $E \in \overline{L}$ such that $E \nsubseteq R^n \times \{0, 1\}$ there exist exactly two (n + 1)-dimensional cells $D_1, D_2 \in L$ having E as a face.

proof. Since E is a face of some (n + 1)-dimensional cell of L, we assume that $E < D_1 = C_1 \# \tau_1$ for some $C_1 \in B(k, \alpha)$ and $\tau_1 \in T(k, \alpha)$. One of the following two cases occurs:

Case 1: $E = C' \# \tau_1$ for some $C' < C_1$ with $\dim C' = \dim C_1 - 1$,

case 2: $E = C_1 \# \eta$ for some $\eta < \tau_1$ with $\dim \eta = \dim \tau_1 - 1$.

We shall show that there is an (n+1)-dimensional cell $D_2 \neq D_1$ of L in either case.

Case 1: If $C' \notin \partial B(k, \alpha)$, then there exists a cell $C_2 \in B(k, \alpha)$ such that $C' < C_2$ and $C_1 \neq C_2$. Hence $E < C_2 \# \tau_1 \in L$. If $C' \in \partial B(k, \alpha)$, then by Lemma 4.7 $C' \in C(k + 1, \beta)$ for some $\beta \in \{+1, -1\}$. Again by Lemma 3.7 there exists a simplex $\tau_2 \in T(k + 1, \beta)$ such that $\tau_1 < \tau_2$. Therefore

$E < C' \# \tau_2 \in L$.

Case 2: If $\eta \notin \partial T(k, \alpha)$, then there exists a simplex $\tau_2 \in T(k, \alpha)$ such that $\eta < \tau_2$ and $\tau_1 \neq \tau_2$. Hence $E < C_1 \# \tau_2 \in L$. If $\eta \in \partial T(k,\alpha)$, then by Lemma 4.7 $\eta \in T(k - 1, \beta)$ for some $\beta \in \{+1, -1\}$. Again by Lemma 4.7 there exists a cell $C_2 \in C(k - 1, \beta)$ such that $C_1 < C_2$. Therefore $E < C_2 \# \eta \in L$. Q.E.D.

Theorem 4.9. $\partial L = (T \times \{0\}) \cup \{B(0) \times \{1\}\}$.

proof. Note that Lemma 4.8 implies that

$$|\partial L| \subseteq R^n \times \{0, 1\}.$$

Then

$$\begin{aligned} |\partial L| &= |\partial L| \cap (R^n \times \{0, 1\}) \\ &= |\partial L| \cap (R^n \times \{0\}) \cup |\partial L| \cap (R^n \times \{1\}) \\ &= (R^n \times \{0\}) \cup (B(0) \times \{1\}), \end{aligned}$$

and thus the theorem follows. Q.E.D.

4.2. GEOMETRIC INTERPRETATION

Now let us show that the 2-ray method can be viewed as a method for tracing a solution path of a PL mapping from $|L|$ into R^n. Let K be a PL mapping such that for every vertex (z, t) of L

$$K(z, t) = \begin{cases} f(z) & \text{if } t = 0 \\ z & \text{if } t = 1 \end{cases}$$

and K is affine on each cell of L (note that K is also well defined on non-simplicial cell $B(0) \neq \{0\}$). Then K is written as

$$(4.10) \qquad K(z, t) = (1 - t)F(x) + ty ,$$

where (x, y) is the unique pair satisfying the three conditions (4.2) - (4.4) for (z, t) in Lemma 4.4 and F is a PL approximation of f on the triangulation T. We here assume that zero is a regular value of K: $|L| \longrightarrow R^n$. Then the solution set of K(z, t) = 0 turns out to be a disjoint union of paths and loops (see Eaves [1] and Eaves and Scarf [2]). Let S_K^0 be a connected component of the solution set which has a trivial solution $(x^0, t^0) = (0, 1) \in R^n \times [0, 1]$. Since $(0, 1) \in |\partial L|$, S_K^0 forms a path. Suppose S_K^0 connects (x^0, t^0) with another boundary point (x^1, t^1) of L. As we have seen in Lemma 4.9, L has no boundary point in $R^n \times (0, 1)$, so that t^1 is either 0 or 1. If $t^1 = 1$, x^1 must be zero, which is contrary to the regularity assumption. Hence $t^1 = 0$ and $F(x^1) = 0$. Thus by tracing the solution path leading from $(x^0, t^0) = (0, 1)$ to another boundary point we can locate an approximate solution of the system of equations f(x) = 0.

Suppose z and t satisfy K(z, t) = 0 and $t \in [0, 1)$. Then there exist x and y such that

$$(1 - t)F(x) + ty = 0$$

$$(4.11) \qquad x \in |T(k, \alpha)| = X(k, \alpha)$$

$$y \in |B(k, \alpha)| = B \cap Y(k, \alpha) .$$

Dividing both sides of the equations in (4.11) by $1 - t \neq 0$ we can see that x and $y' = (t/(1 - t))y$ satisfy the following system (4.12)

$$(4.12) \quad \begin{array}{l} F(x) + y = 0 \\ x \in X(k, \alpha), \quad y \in Y(k, \alpha) \end{array}$$

or equivalently the basic system (2.12) of the 2-ray method. Conversely if x and y' satisfy (4.12) and $y' \neq 0$, then x and $y = y'/ \|y'\|_1$ satisfy (4.11) togeter with $t = \|y'\|_1 / (1 + \|y'\|_1)$, and hense $z = (1 - t)x + ty$ $= (1 / (1 + \|y'\|_1))(x + y')$ and t satisfy $K(z, t) = 0$. In this way a solution of the PL mapping $K: |L| \longrightarrow R^n$ corresponds to a solution of the basic system (2.12) of the 2-ray method. Thus we can apply the index theory to the 2-ray method. Suppose that a solution path of (4.12) leads to $(x^1, y^1) = (x^1, 0)$ $\in R^{2n}$ from a starting point $(x^0, y^0) = (0, -F(0)) \in R^{2n}$. Then the solution path S_K^0 connects $(x^0, t^0) = (0, 1) \in R^n \times [0, 1]$ with $(x^1, t^1) = (x^1, 0) \in R^n$ $\times [0, 1]$. Let I_n be the $n \times n$ identity matrix, then for $(z, t) \in B(0) \# \{0\}$

$$(4.13) \quad K(z, t) = I_n z + (-f(0))t + f(0) .$$

On the other hand let $(z, t) \in R^n \times [0, 1]$ be a point in $|B(n, \alpha)| \# \tau$ $= \{(0, \ldots, 0, \alpha)\} \# \tau$ for some $\tau \in T(n, \alpha) \subseteq T$. Then

$$(z, t) = (1 - t)(x, 0) + t(0, \ldots, 0, \alpha, 1),$$

or

$$x = (1 / (1 - t))(z - \alpha t e_n),$$

where $e_n \in R^n$ is the n-th unit vector. Hence by letting $A(\tau) \in R^{n \times n}$ and $b(\tau) \in R^n$ be such that

$$F(x) = A(\tau)x + b(\tau) \quad \text{for } \forall x \in \tau ,$$

we have

$$(4.14) \quad K(z, t) = A(\tau)z + (\alpha(I - A(\tau))e_n - b(\tau))t + b(\tau) .$$

Let τ_1 be an n-dimensional simplex of T which contains x^1. Then applying the index theory (Theorem 5.4 in Eaves and Scarf [2]) from (4.13) and (4.14) we obtain that

$$\text{sign det } A(\tau_1) = \text{sign det } I_n = +1.$$

Thus the 2-ray method provides an approximate solution x^1 at which the Jacobian matrix of F has positive determinant.

5. APPLICATIONS TO SOME PROBLEMS

5.1. TWO-POINT BOUNDARY VALUE PROBLEM

We shall consider the two-point boundary value problem:

$$(5.1)\qquad \begin{aligned} &u''(t) = r(t, u(t)) \qquad \text{for } t \in [0, 1] \\ &u(0) = b_0, \quad u(1) = b_1, \end{aligned}$$

where $u: R^1 \to R^1$ is a twice continuously differentiable mapping and $r: R^2 \to R^1$ is a continuous mapping. Let $h = 1/(n+1)$ for an integer $n \geqq 1$ and $t_j = jh$ for $j = 0,1,\ldots,n+1$, i.e., t_j's are equally spaced grid points in the unit interval $[0, 1]$. Then $u''(t_j)$ is approximated by the central difference quotient

$$(1/h^2)\{u(t_{j-1}) - 2u(t_j) + u(t_{j+1})\},$$

hence we obtain the following finite difference approximation (5.2) of the two-point boundary value problem (see Ortega and Rheinboldt [13] for details):

$$(5.2)\qquad f(x) = Ax + \phi(x) = 0, \quad x \in R^n,$$

where $A = (a_{ij})$ is an $n \times n$ matrix such that

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $\phi: R^n \to R^n$ is such that

$$\phi_j(x) = \begin{cases} h^2 r(t_j, x_j) - b_0 & \text{if } j = 1 \\ h^2 r(t_j, x_j) - b_1 & \text{if } j = n \\ h^2 r(t_j, x_j) & \text{otherwise.} \end{cases}$$

Since $r: R^2 \to R^1$ is assumed to be continuous, f is a continuous mapping from R^n into R^n. Hence we can apply the 2-ray method to finding a solution of the system of equations (5.2). To obtain a sufficient condition under which the 2-ray method terminates after a finitely many iterations, we shall introduce some definitions.

Definition 5.1. Let W be a subset of R^n and x^0 be a point of R^n. $f: R^n \to R^n$ is said to be a monotone mapping on W with respect to x^0 if

$$(x - x^0)^t (f(x) - f(x^0)) \geqq 0$$

for any $x \in W$. f is said to be a uniformly monotone mapping on W with respect to x^0 if there is an $\varepsilon > 0$ so that

$$(x - x^0)^t (f(x) - f(x^0)) \geqq \varepsilon \| x - x^0 \|^2$$

for any $x \in W$.

Lemma 5.2. Let x^0 be a point of R^n and V be a bounded subset of R^n such that $x^0 \in V$. Suppose that $f: R^n \to R^n$ is a uniformly monotone on $W = R^n \setminus V$ with respect to x^0 and Lipschitz continuous mapping. Then for any $\delta_0 > 0$ there exists a positive real number η such that

$$\| x - x^0 \| > \eta \quad \text{and} \quad \| x - v \| \leqq \delta_0$$

always imply that

$$(x - x^0)^t f(v) > 0.$$

proof. Let $\eta' = \sup_{x \in V} \| x - x^0 \|$, and suppose that $\| x - x^0 \| > \eta'$ and $\| x - v \| \leqq \delta_0$. Then

$$\begin{aligned} &(x - x^0)^t f(v) \\ &= (x - x^0)^t (f(x) - f(x^0)) + (x - x^0)^t (f(x^0) - f(x) + f(v)) \\ &\geqq \varepsilon \| x - x^0 \|^2 - \| x - x^0 \| (\| f(x^0) \| + \| f(x) - f(v) \|) \\ &\geqq \varepsilon \| x - x^0 \|^2 - (\| f(x^0) \| + \mu \delta_0) \| x - x^0 \|, \end{aligned}$$

where μ is the Lipschitz constant. Therefore

$$\eta = \max \{ \eta', (\| f(x^0) \| + \mu \delta_0) / \varepsilon \}$$

is the desired number. Q.E.D.

Theorem 5.3. Let V be a bounded subset of R^n containing the starting point x^0. Assume that $f: R^n \to R^n$ is uniformly monotone on $W = R^n \setminus V$ with respect to x^0 and Lipschitz continuous. Then for any mesh size δ_0 of the triangulation T the 2-ray method provides an approximate solution of the system of equations

$f(x) = 0$, $x \in R^n$ after a finitely many iterations. Furthermore the approximate solution obtained lies in the closed ball

$$O(x^0, \eta) = \{ x \in R^n : \| x - x^0 \| \leqq \eta \},$$

where $\eta = \max \{ \sup_{x \in V} \| x - x^0 \|, (\| f(x^0) \| + \mu \delta_0) / \varepsilon \}$.

proof. Let S be the solution set of the basic system (2.12), i.e.,

$$S = \{ (x, y) \in |M| : H(x, y) = 0 \}$$

and let

$$S' = \{ x \in R^n : (x, y) \in S \text{ for some } y \in R^n \} .$$

Then it suffices to show that S' is contained in the closed ball $O(x^0, \eta)$. Suppose on the contrary that there exists an $(\bar{x}, \bar{y}) \in S$ such that $\| \bar{x} - x^0 \| > \eta$. Let $\sigma = co\{ v^0, v^1, \ldots, v^n \}$ be an n-simplex of T containing $\bar{x}$. Then by Lemma 5.2

$$(\bar{x} - x^0)^t f(v^i) > 0$$

for $i = 0, 1, \ldots, n$, consequently we have

$$(\bar{x} - x^0)^t F(\bar{x}) > 0.$$

Since $(\bar{x}, \bar{y}) \in |M|$, we can redily see that $(\bar{x} - x^0)^t \bar{y} \geqq 0$. Therefore

$$(\bar{x} - x^0)^t H(\bar{x}, \bar{y}) = (\bar{x} - x^0)^t (F(\bar{x}) + \bar{y}) > 0.$$

This is contrary to the assumption that $(\bar{x}, \bar{y}) \in S$. Q.E.D.

Note that $f(x) = Ax + \phi(x)$ is a separable mapping, so that we can apply the computational procedure developed in Section 3 of this paper for separable mappings. Let δ_j be the mesh size of the 1-dimensional triangulation T_j of the j-th coordinate axis. Then we have the following corollary.

Corollary 5.4. Suppose that $r(t, u)$ is monotone nondecreasing and Lipschitz continuous in u for each $t \in [0, 1]$. Then the 2-ray method locates an approximate solution of (5.2) after a finitely many iterations.

proof. Note that the matrix A in (5.2) is positive definite (see Ortega and Rheinboldt [13] for the proof). Then by the assumption that r is monotone nondecreasing we readily see that there exists an $\varepsilon > 0$ such that

$$(x - x^0)^t (f(x) - f(x^0)) \geqq \varepsilon \| x - x^0 \|^2$$

for any $x \in R^n$. Hence f is uniformly monotone on R^n with respect to x^0. Furthermore

$$\begin{aligned} \| f(x) - f(y) \| &\leqq \| A \| \, \|x - y\| + \| \phi(x) - \phi(y) \| \\ &= \| A \| \, \|x - y\| + h^2 \left\{ \sum_{j=1}^{n} (r(t_j, x_j) - r(t_j, y_j))^2 \right\}^{1/2} \\ &\leqq (\| A \| + h^2 \mu') \| x - y \| , \end{aligned}$$

where μ' is the Lipschitz constant of r. Thus the assertion directly follows from Theorem 5.3.

Q.E.D.

5.2. DIMENSION DOUBLING DISCRETIZATION OF THE TWO-POINT BOUNDARY VALUE PROBLEM

Now consider the case where we obtain an approximate solution of the system (5.2) by the separable version of the 2-ray method developed in Section 3. Then we have an $x \in R^n$ and $[l_j, u_j] \in T_j$ for $j = 1,2,\dots,n$ such that

$$(5.3) \qquad Ax + h^2 \begin{bmatrix} R_1(x_1) \\ \vdots \\ R_n(x_n) \end{bmatrix} - b = 0$$

$$x_j \in [l_j, u_j] \quad \text{for } j = 1,2,\dots,n,$$

where $R_j : R^1 \to R^1$ is a PL approximation of $r(t_j,\cdot)$ on T_j and $b = (b_0, 0,\dots, 0, b_1)^t \in R^n$. For $j = n+1, n+2,\dots, 2n+1$ let

$$t_j = (1/2)h + (j - (n+1))h.$$

Then we obtain a more accurate finite difference approximation (5.4) of (5.1):

$$(5.4) \qquad \bar{A}z + (h^2/4) \begin{bmatrix} r(t_1, z_1) \\ \vdots \\ r(t_n, z_n) \\ r(t_{n+1}, z_{n+1}) \\ \vdots \\ r(t_{2n+1}, z_{2n+1}) \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \\ \bar{b} \\ \\ \end{bmatrix} = 0, \qquad z \in R^{2n+1}$$

where

$$\bar{A} = \left[\begin{array}{c|c} 2I_n & \bar{B} \\ \hline \bar{B}^t & 2I_{n+1} \end{array}\right] \in R^{(2n+1)\times(2n+1)}$$

$$\bar{B} = \begin{bmatrix} -1 & -1 & & & 0 \\ & \ddots & \ddots & & \\ 0 & & & -1 & -1 \end{bmatrix} \in R^{n\times(n+1)}, \quad \bar{b} = (b_0, 0,\dots, 0, b_1)^t \in R^{n+1}$$

and I_n is the $n \times n$ identity matrix. For the sake of notational convenience let

$$w_1 = (z_1, z_2,\dots, z_n)^t \in R^n$$

$w_2 = z_{n+1} \in R^1$

$w_3 = (z_{n+2}, z_{n+3}, \ldots, z_{2n+1})^t \in R^n$

$e^1 = (1, 0, \ldots, 0) \in R^n$

$$B = \begin{bmatrix} -1 & & & 0 \\ -1 & -1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & -1 \end{bmatrix} \in R^{n \times n}.$$

Then the fist n equations of (5.4) approximated by using 1-dimensional triangulations T_j's are written as

$$(5.5) \qquad 2I_n w_1 - ew_2 + Bw_3 + (h^2/4) \begin{bmatrix} R_1(z_1) \\ \vdots \\ R_n(z_n) \end{bmatrix} = 0.$$

Lemma 5.5. Let $x \in R^n$ be a solution of (5.3) and $C = (c_{ij})$ be an $n \times n$ lower triangular matrix such that

$$c_{ij} = \begin{cases} 0 & \text{if } i < j \\ (-1)^{i+j+1} & \text{otherwise,} \end{cases}$$

and let T_j be a 1-dimensional triangulation of the j-th coordinate axis for j = n+1, n+2, ..., 2n+1. Suppose that $z = (w_1, w_2, w_3) \in R^{2n+1}$ satisfies the following relations:

$$(5.6) \qquad \begin{aligned} w_1 &= x \\ w_3 &= -C\left\{ (1/4)(b - Ax) + 2x \right\} + Ce^1 w_2. \end{aligned}$$

Then z satisfies the first n equations of the system (5.4) with $r(t_j,\cdot)$ replaced by its PL approximation $R_j(\cdot)$ on T_j for j = 1,2,..., 2n+1.

proof. Since x satisfies (5.3) and $B^{-1} = C$, the assertion is straightforward from (5.5). Q.E.D.

Now we shall explain how the 2-ray method is applied to solving the system (5.4). First set w_2 ($= z_{n+1}$) to an appropriate value w_2^0, for instance $(1/2)(b_0 + x_1)$. We denote by $z^0 = (w_1^0, w_2^0, w_3^0)$ the vector (w_1, w_2, w_3) satisfying

(5.6). For $j = n+1, n+2, \dots, 2n+1$ let T_j be a 1-dimensional triangulation of the j-th coordinate axis of R^{2n+1} having the j-th component z_j^0 of z^0 as a vertex. Let

$$y^0 = -\left\{ \bar{A}z^0 + (h^2/4) \begin{bmatrix} R_1(z_1^0) \\ \cdot \\ \cdot \\ R_{2n+1}(z_{2n+1}^0) \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{b} \end{bmatrix} \right\} ,$$

then we readily see that y^0 together with z^0 satisfies the following system (5.7):

$$\bar{A}z + (h^2/4) \begin{bmatrix} R_1(z_1) \\ \cdot \\ \cdot \\ R_{2n+1}(z_{2n+1}) \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{b} \end{bmatrix} + y = 0$$

$$z_j \in [l_j, u_j] \in T_j \qquad \text{for } j \leqq n+1$$

$$\text{(5.7)} \qquad z_j \in \{ z_j^0 \} \in \bar{T}_j \qquad \text{for } j > n+1$$

$$\prod_{j=1}^{n+1} [l_j, u_j] \times \prod_{j=n+2}^{2n+1} \{ z_j^0 \} \subseteq \bar{X}(n+1,\alpha)$$

$$y \in \bar{Y}(n+1,\alpha),$$

where $\alpha = \text{sign } y_{n+1}^0$ and

$$\bar{X}(k,\alpha) = \{ z \in R^{2n+1}: \alpha(z_k - z_k^0) \geqq 0, \quad z_j - z_j^0 = 0 \text{ for } j > k \}$$

$$\bar{Y}(k,\alpha) = \{ y \in R^{2n+1}: \alpha y_k \geqq 0, \qquad y_j = 0 \qquad \text{for } j < k \} .$$

Since y_j's are not restricted at all for $j > n+1$, (5.7) is equivalent to (5.8).

$$\text{(5.8)} \quad \left[\begin{array}{c|c} 2I_n & -e^1 \\ \hline -(e^1)^t & 2 \end{array}\right] \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_{n+1} \end{bmatrix} + (h^2/4) \begin{bmatrix} R_1(z_1) \\ \cdot \\ \cdot \\ \cdot \\ R_{n+1}(z_{n+1}) \end{bmatrix} + Bw_3^0 - \left[\begin{array}{c} 0 \\ \hline b_0 \end{array}\right] + \left[\begin{array}{c} 0 \\ \hline \alpha \end{array}\right] s_{n+1} = 0$$

$$l_j \leqq z_j \leqq u_j \qquad \text{for } j \leqq n+1$$

$$\alpha [l_{n+1}, u_{n+1}] \geqq 0$$

$$s_{n+1} \geqq 0.$$

Thus we are at the stage in which the first n equations of (5.4) with r_j replaced by R_j are solved, so that we can restart the 2-ray method by increasing or decreasing z_{n+1} from z_{n+1}^0.

5.3. VARIATIONAL PROBLEM AND RITZ METHOD

Let U be a linear subspace of a certain class of functions u and J(u) be a functional defined on U. We here consider the following optimization problem

$$(5.9)\qquad \begin{array}{ll} \text{minimize} & J(u) \\ \text{subject to} & u \in U. \end{array}$$

Suppose we have a infinite sequence $u_1, u_2, \ldots$ of functions of U. Restricting J(u) on the subspace spanned by the first n functions of the sequence we obtain the n-dimensional optimization problem $(5.10)_n$ which approximates the problem (5.9).

$$(5.10)_n\qquad \begin{array}{ll} \text{minimize} & J\left(\sum_{j=1}^{n} x_j u_j \right) \\ \text{subject to} & x \in R^n. \end{array}$$

Hence if we find an optimum solution of $(5.10)_n$ for a large n, we can locate an optimum solution of (5.9) with desired accuracy (the choice of the functions $u_1, u_2, \ldots,$ of course, affects the convergence and accuracy). This is the basic idea of Ritz method (see, for example Gelfand and Fomin [4]).

Now suppose the differentiability of J and denote $(\partial/\partial x_j) J(\sum_{j=1}^{n} x_j u_j)$ by $f_j^{(n)}(x)$. Then if $x \in R^n$ is a solution of $(5.10)_n$, it satisfies the following system of n equations in n variables:

$$(5.11)_n\qquad f^{(n)}(x) = 0, \quad x \in R^n,$$

where $f^{(n)}(\cdot) = (f_1^{(n)}(\cdot), \ldots, f_n^{(n)}(\cdot))^t$. For k = 0, 1,.... let $T^{(k)}$ be a triangulation of R^k such that

$$\{ \sigma \in \overline{T}^{(k+1)} : \sigma \subseteq R^k \times \{0\},\ \dim \sigma = k \} = T^{(k)},$$

and $F^{(k)}$ be the PL approximation of $f^{(k)}$ on $T^{(k)}$. Consider the case where we have reached a solution $x \in R^n$ of the system of PL equations

$$(5.12)_n\qquad F^{(n)}(x) = 0, \quad x \in R^n.$$

Let $z^0 = (x, 0) \in R^{n+1}$, then it is clear that z^0 satisfies the first n equations of the augmented system

$$F^{(n+1)}(z) = 0, \quad z \in R^{n+1}.$$

Hence z^0 and $y^0 = -F^{(n+1)}(z^0)$ satisfy the basic system

$$F^{(n+1)}(z) + y = 0$$

$$z \in \overline{X}(n+1, \alpha)$$

$$y \in \overline{Y}(n+1, \alpha)$$

for $\alpha = \text{sign } y^0_{n+1}$. Therefore we can start the 2-ray method for a solution of $(5.12)_{n+1}$ with increasing or decreasing z_{n+1} form zero.

As a concrete example let us consider the following variational problem:

$$\text{minimize} \quad J(u) = \int_0^1 \phi(t, u(t), u'(t))dt$$

(5.13)

$$\text{subject to} \quad u \in U = \{ u \in C^1[0,1] : u(0) = u(1) = 0 \},$$

where $C^1[0,1]$ is the linear space of continuously differentiable functions $u : [0, 1] \to R^1$. To apply Ritz method to this problem we first approximate the integration in the objective function by a quadrature formula

$$\int_0^1 \phi(t, u(t), u'(t)) \doteq \sum_{i=1}^{m} \gamma_i \phi(t_i, u(t_i), u'(t_i)).$$

Then for given n functions of U we have an n-dimensional approximation of (5.13) as follows:

$$\text{minimize} \quad \sum_{i=1}^{m} \gamma_i \phi(t_i, \sum_{j=1}^{n} x_j u_j(t_i), \sum_{j=1}^{n} x_j u'_j(t_i))$$

$$\text{subject to} \quad x \in R^n.$$

Hence the system of equations to be solved is

$$\sum_{i=1}^{m} \gamma_i \{ (\partial \phi / \partial u)(t_i, \psi(t_i), \psi'(t_i))$$
$$+ (\partial \phi / \partial u')(t_i, \psi(t_i), \psi'(t_i)) \} = 0, \quad i = 1,2,\ldots,n,$$

where $\psi(t) = \sum_{j=1}^{n} x_j u_j(t)$.

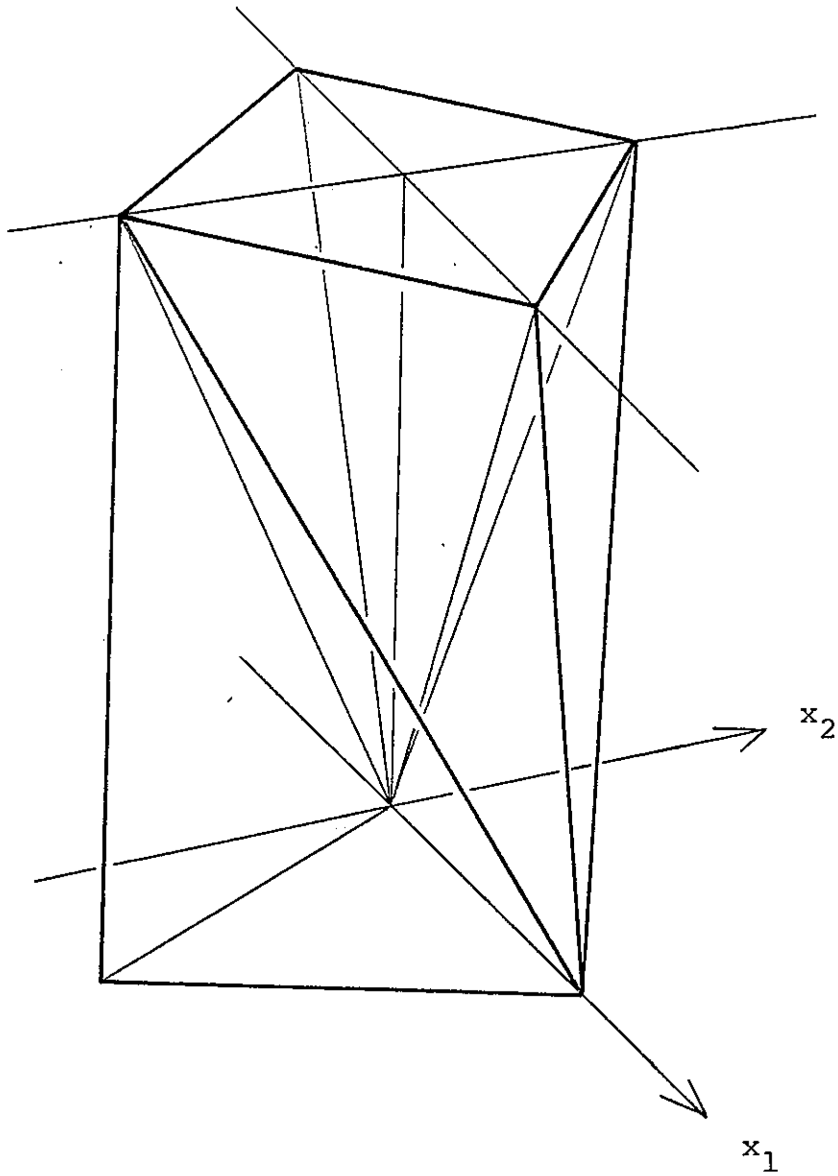

Fig. 1

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