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Danger of Exclusive Reliance on Ergodic Analysis for a Class of Preventive Maintenance Policies : Dynamic Semi-Markov Process Approach

by

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DANGER OF EXCLUSIVE RELIANCE ON ERGODIC ANALYSIS FOR A CLASS OF PREVENTIVE MAINTENANCE POLICIES : DYNAMIC SEMI-MARKOV PROCESS APPROACH

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ABSTRACT. In the literature, the study of preventive maintenance policies has been largely focused on ergodic analysis where the expected economic performance measure per unit time would be optimized in a long run. When the planning horizon τ is not large, however, the optimal preventive maintenance policy in $[0, \tau]$ could be significantly different from that under ergodicity. In this paper, the classical semi-Markov model of Makabe[4] is first examined thoroughly at ergodicity, yielding many new results. Then, through the dynamic analysis of the semi-Markov model, the asymptotic expansion of the expected reward in $[0, \tau]$ is obtained explicitly in an affine form. The optimal preventive maintenance policies in $[0, \tau]$ are then compared with the ergodic counterparts, thereby demonstrating danger of exclusive reliance on ergodic analysis when τ is not sufficiently large.

Key words: semi-Markov process, dynamic analysis, preventive maintenance (PM), corrective maintenance (CM), ergodic optimal PM policy, non-ergodic optimal PM policy

1. INTRODUCTION

The study of preventive maintenance policies in manufacturing dates back to late 1950's stemmed from the original work by Morse[7] and Barlow and Hunter[2]. In these papers, the preventive maintenance (PM) takes place for overhauling the system as soon as the system lifetime exceeds the prespecified time T and the system is renewed upon completion of PM. If the system fails before T, the corrective maintenance (CM) takes place, bringing the system back to the fresh state upon completion. The former focused on analysis of the optimal PM policy maximizing the expected profit per unit time at ergodicity where two different variable costs per unit time for PM and CM are involved, while the latter was concerned with the optimal PM policy which maximizes the availability of the system at ergodicity.

Since then, the study has been expanded in several different directions. Makabe[4] generalized the original model of Morse[7] by additionally incorporating the fixed costs for both PM and CM. It was shown that if the optimal PM policy T^* exists and the system lifetime has an increasing hazard function $\eta_L(x)$ with $\lim_{x \to \infty} \eta_L(x) = \infty$, then T^* is unique and can be computed numerically. Nakagawa[8] proposed a periodic checking model where the system is inspected periodically, making the system

anew with probability q and resulting in no effect with probability p upon completion. The inspection model was combined with the original PM model subsequently by Nakagawa and Yasui[9] where a PM takes place after every K inspections. This line of research has been further developed by Vaurio[12, 13] and Badía, Berrade and Campos[1].

All of the above papers are restricted to ergodic analysis. A rare exception dealing with finite horizon optimization is an age reduction approach proposed by Dedopoulos and Smeers[3] which was subsequently generalized by Samrout, Châtelet, Kouta and Chebbo[10]. The thrust of the age reduction model can be found in that PM activities do not necessarily result in bringing the system back to the fresh state but make the system younger to the extent determined by the level of PM quality. The optimal PM policy problem is then to determine when to implement PM activities at what quality. While these papers addressed themselves to find the optimal PM policy within a finite planning horizon, all system failures were assumed to be minimal, enabling one to restart the system instantaneously with the same system state at the time of failure. Because of this limitation, the ergodic analysis is totally irrelevant to the age reduction model.

To the best knowledge of the authors, there has been no research available in the literature concerning how to asses the danger of exclusive reliance on ergodic analysis for a class of PM policies, where the results of ergodic analysis for one model should be compared with the results of dynamic analysis for the same model. The purpose of this paper is to fill this gap by analyzing the classical semi-Markov model of Makabe[4] dynamically as well as at ergodicity. The underlying semi-Markov model is first examined thoroughly at ergodicity, yielding many new results. Then, through the dynamic analysis of the semi-Markov model, the asymptotic expansion of the expected reward in $[0, \tau]$ is obtained explicitly in an affine form. The optimal preventive maintenance policies in $[0, \tau]$ are then compared with the ergodic counterparts, thereby demonstrating the danger of exclusive reliance on ergodic analysis when τ is not sufficiently large.

The structure of this paper is as follows. The classical semi-Markov model of Makabe[4] is introduced in Section 2. Dynamic analysis of the model is discussed in Section 3 by examining the trivariate process $[N(t), X(t), Z(t)]$ where, at time t, $N(t)$ is the state of the semi-Markov process, $X(t)$ describes the time spent in the current state since the last transition to it, and $Z(t)$ gives the cumulative reward. The asymptotic expansion of $\mathrm{E}[Z(t)]$ as $t \to \infty$ is obtained explicitly in an affine form. Section 4 is dedicated to ergodic analysis, yielding many new results. In particular, sufficient conditions are given for the existence of the ergodic optimal PM policy in terms of hazard rate properties of system lifetimes. In Section 5, non-ergodic optimal PM policy is introduced based on the asymptotic expansion of $\mathrm{E}[Z(t)]$ obtained in Section 3. Numerical results are presented for demonstrating

the danger of exclusive reliance on ergodic analysis when the planning horizon is not sufficiently large. Finally, some concluding remarks are given in Section 6. In order to enhance the readability of the paper, proofs are given in Appendix.

2. Model Description

We consider a production system generating the profit of p per unit time while it is working. The system lifetime X_L, which is the time until failure since its fresh start, is assumed to be an absolutely continuous positive random variable with p.d.f. (probability density function) $a_L(x)$. The associated distribution function, the survival function, and the hazard rate function are denoted by

$$(2.1) \quad \begin{cases} A_L(x) & = & P[X_L \le x] & = & \int_0^x a_L(y)dy \; ; \\ \bar{A}_L(x) & = & P[X_L > x] & = & \int_x^\infty a_L(y)dy \\ & = & 1 - A_L(x) \; ; \\ \eta_L(x) & = & \dfrac{a_L(x)}{\bar{A}_L(x)} & = & -\dfrac{d}{dx}\log \bar{A}_L(x) \; . \end{cases}$$

If the system fails before T, CM takes place where the CM time X_R is assumed to be an absolutely continuous positive random variable with p.d.f. $a_R(x)$. We define $A_R(x)$, $\bar{A}_R(x)$ and $\eta_R(x)$ similarly to (2.1). CM requires the fixed cost c_R for each CM and the variable cost v_R per unit time under CM.

For PM, throughout this paper, we define PM to mean that the system is over-hauled when the system lifetime exceeds a prespecified level T. The PM time X_M, which is the time under overhaul, is also assumed to be an absolutely continuous positive random variable with $a_M(x)$, $A_M(x)$, $\bar{A}_M(x)$ and $\eta_M(x)$ defined similarly to (2.1). The cost structure of PM is in parallel with that of CM, having the fixed cost c_M for each PM and the variable cost v_M per unit time under PM. Throughout the paper, we assume that all random variables involved are mutually independent.

It is natural to assume that CM upon failure is "more costly" than PM, where the term "more costly" is interpreted in the following manner. Let the moments of the random variables X_L, X_R and X_M for k=1,2 be denoted by

$$(2.2) \qquad\qquad \mu_{L:k} \stackrel{\text{def}}{=} \int_0^\infty x^k a_L(x)dx \; ;$$

$$(2.3) \qquad\qquad \mu_{R:k} \stackrel{\text{def}}{=} \int_0^\infty x^k a_R(x)dx \; ;$$

$$(2.4) \qquad\qquad \mu_{M:k} \stackrel{\text{def}}{=} \int_0^\infty x^k a_M(x)dx \; .$$

The expected total cost $\hat{C}_R$ for each CM and the expected total cost $\hat{C}_M$ for each PM are then given by

$$(2.5) \qquad \hat{C}_R \stackrel{\text{def}}{=} c_R + v_R\mu_{R:1} \; ; \; \hat{C}_M \stackrel{\text{def}}{=} c_M + v_M\mu_{M:1} \; .$$

We assume that the expected CM time is larger than the expected PM time and CM upon failure is "more costly" than PM in that the expected total cost per unit time under CM is larger than that under PM. More specifically, throughout the paper, it is assumed that

$$(2.6) \qquad \mu_{R:1} > \mu_{M:1} \text{ and } u_{diff} = \frac{\hat{C}_R}{\mu_{R:1}} - \frac{\hat{C}_M}{\mu_{M:1}} > 0 \ .$$

As we will see, it is also useful to introduce l_R and l_M representing the expected actual total cost plus the expected opportunity cost for each CM and each PM respectively, and their difference. Formally, we define

$$l_{diff} \stackrel{\text{def}}{=} l_R - l_M \ ;$$
$$(2.7) \qquad l_R \stackrel{\text{def}}{=} \hat{C}_R + p\mu_{R:1} \ ; \ l_M \stackrel{\text{def}}{=} \hat{C}_M + p\mu_{M:1} \ .$$

Concerning l_{diff}, the following proposition holds, where proof is provided in Appendix.

Proposition 2.1.

Under the assumption (2.6), one has $l_{diff} > 0$.

Let $\{N(t) : t \geq 0\}$ be a stochastic process describing the state of the system at time t, where

$$(2.8) \qquad N(t) = \begin{cases} 0 & \text{if the system is under CM at time } t \ ; \\ 1 & \text{if the system is working at time } t \ ; \\ 2 & \text{if the system is under PM at time } t \ . \end{cases}$$

The dwell time of the system in state 1, denoted by X_W, is then given by

$$(2.9) \qquad X_W = \min\{X_L, T\}.$$

For the time being, we assume that T is also an absolutely continuous positive random variable, having $a_T(x)$, $A_T(x)$, $\bar{A}_T(x)$ and $\eta_T(x)$ defined similarly to (2.1). The case of T being constant can be treated by choosing an appropriate sequence of absolutely continuous distributions which would converge in distribution to the desired constant as we will see. For X_W, we define $a_W(x)$, $A_W(x)$, $\bar{A}_W(x)$ and $\eta_W(x)$ similarly to (2.1). From (2.9), it can be readily seen that,

$$(2.10) \qquad \begin{cases} \bar{A}_W(x) = \bar{A}_L(x)\bar{A}_T(x) \ ; \\ \eta_W(x) = \eta_L(x) + \eta_T(x) \ ; \\ a_W(x) = \bar{A}_W(x)\{\eta_L(x) + \eta_T(x)\} \ . \end{cases}$$

It then follows that $\{N(t) : t \geq 0\}$ is a semi-Markov process on $\{0, 1, 2\}$ governed by the matrix p.d.f. $\underline{a}(x)$ given by

$$
\begin{aligned}
\underline{a}(x) &= \begin{bmatrix} 0 & a_R(x) & 0 \\ \bar{A}_W(x)\eta_L(x) & 0 & \bar{A}_W(x)\eta_T(x) \\ 0 & a_M(x) & 0 \end{bmatrix} \\
(2.11) \qquad\qquad &= \begin{bmatrix} 0 & a_R(x) & 0 \\ \bar{A}_T(x)a_L(x) & 0 & \bar{A}_L(x)a_T(x) \\ 0 & a_M(x) & 0 \end{bmatrix}.
\end{aligned}
$$

The state transition diagram of $\{N(t) : t \geq 0\}$ is depicted in Figure 1, where $X(t)$ is the age process of the semi-Markov process describing the elapsed time at time t since the last transition into the current state.
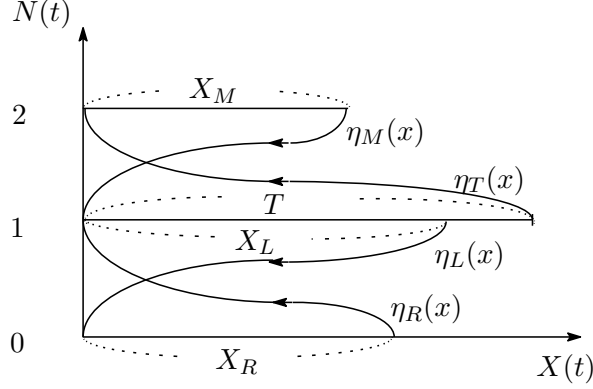

FIGURE 1. The state transition diagram of $N(t)$

It should be noted that $N(t)$ itself is not Markov, but the bivariate process $[N(t), X(t)]$ is Markov. Throughout the paper, we assume that the system is fresh and working at time $t = 0$ with $X(0) = 0$.

Let $M_{ij}(t)$ be the number of transitions from state i to state j in the time interval $[0, t)$. Then the reward process $Z(t)$ is defined by

$$
(2.12) \qquad Z(t) \overset{\text{def}}{=} \int_0^t \rho(N(\tau))d\tau + \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \sum_{m=1}^{M_{ij}(t)} D_{ij:m}
$$

where $\rho : \{0, 1, 2\} \to R$ is the reward rate function and, for $m = 1, 2, \cdots$, $D_{ij:m}$ is the cost of the m-th transition from state i to state j. In our model, the reward process $Z(t)$ grows continuously with rate $\rho(i)$ when $N(t) = i$, where

$$
(2.13) \qquad \rho(i) = \begin{cases} -v_R & \text{if } i = 0 ; \\ p & \text{if } i = 1 ; \\ -v_M & \text{if } i = 2 . \end{cases}
$$

The transition cost $D_{ij:m}$ is specified by

$$(2.14) \qquad D_{ij:m} = \begin{cases} -c_R & \text{if } i = 1, \ j = 2 \ ; \\ -c_M & \text{if } i = 1, \ j = 0 \ ; \\ 0 & \text{else} \ . \end{cases}$$

A sample path of $Z(t)$ is depicted in Figure 2.

FIGURE 2. A sample path of $Z(t)$

3. Dynamic Analysis of $[N(t), X(t), Z(t)]$

The purpose of this section is to analyze the trivariate process $[N(t), X(t), Z(t)]$ defined on $\mathcal{N} \times R_+ \times R$, yielding the matrix trivariate Laplace transform with respect to t, x and z explicitly. Here, $\mathcal{N} = \{0, 1, 2\}$, R_+ is the set of nonnegative real numbers and R is the set of real numbers. For $i, j \in \mathcal{N}$, we first define

$$(3.1) \quad F_{ij}(x, z, t) \stackrel{\text{def}}{=} \mathrm{P}[N(t) = j, X(t) \le x, Z(t) \le z | N(0) = i, X(0) = 0] \ .$$

Let $\delta(t)$ be the delta function defined as the unit operator for convolution, i.e. $g(t) = \int_0^\infty \delta(t - x)g(x)dx$ for an arbitrary function $g(t)$ integrable on $[0, \infty)$. Exploiting the delta function whenever necessary, the generalized joint p.d.f. can be written as

$$(3.2) \qquad f_{ij}(x, z, t) \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x \partial z} F_{ij}(x, z, t) \ .$$

In matrix form, the corresponding trivariate Laplace transform with respect to t, x and z is denoted by

$$(3.3) \qquad \hat{\hat{\underline{\underline{\varphi}}}}(v, w, s) \stackrel{\text{def}}{=} \int_0^\infty dt \ e^{-st} \int_{-\infty}^\infty \int_0^\infty e^{-vx - wz} \underline{\underline{F}}(dx, dz, t) \ ,$$

where $Re(s) > 0$, $Re(v) > 0$ and $w = e^{i\theta}$ ($\theta \in R$).

For $\underline{a}(x)$ given in (2.11), we define

$$(3.4) \qquad \underline{\underline{\alpha}}(s) = \int_0^\infty e^{-sx} \underline{\underline{a}}(x) dx = \begin{bmatrix} 0 & \alpha_R(s) & 0 \\ \alpha_{10}(s) & 0 & \alpha_{12}(s) \\ 0 & \alpha_M(s) & 0 \end{bmatrix},$$

where

$$(3.5) \qquad \alpha_R(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} a_R(t) dt \; ;$$

$$(3.6) \qquad \alpha_M(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} a_M(t) dt \; ;$$

$$(3.7) \qquad \alpha_{10}(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} a_L(t) \bar{A}_T(t) dt \; ;$$

$$(3.8) \qquad \alpha_{12}(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} a_T(t) \bar{A}_L(t) dt \; .$$

Let $\underline{\underline{g}}(z)$ be the generalized matrix function describing the fixed costs c_R and c_M associated with transitions of $N(t)$ from 1 to 0 and those from 1 to 2 respectively. Namely, $\underline{\underline{g}}(z)$ can be written as

$$(3.9) \qquad \underline{\underline{g}}(z) = \begin{bmatrix} \delta(z) & \delta(z) & \delta(z) \\ \delta(z + c_R) & \delta(z) & \delta(z + c_M) \\ \delta(z) & \delta(z) & \delta(z) \end{bmatrix}$$

with the matrix Laplace transform $\underline{\underline{\gamma}}(w) = \int_{-\infty}^\infty e^{-wz} \underline{\underline{g}}(z) dz$ given by

$$(3.10) \qquad \underline{\underline{\gamma}}(w) = \begin{bmatrix} 1 & 1 & 1 \\ e^{c_R w} & 1 & e^{c_M w} \\ 1 & 1 & 1 \end{bmatrix}.$$

A continuous reward process defined on a semi-Markov process has been studied extensively in the literature, yielding the trivariate matrix Laplace transform $\hat{\hat{\underline{\underline{\phi}}}}(v, w, s)$ of (3.3), see e.g. Theorem 2.1 of Sumita and Masuda[11]. This result is extended to incorporate a reward process with jumps by Masuda[5, 6]. More specifically, the next theorem holds true where the following notation is employed.

$$(3.11) \qquad \underline{\underline{\alpha}}^{**}(w, s) = [\alpha_{ij}(\rho_j w + s)\gamma_{ij}(w)] \; ;$$

$$(3.12) \qquad \hat{\hat{\underline{\underline{\beta}}}}_D(w, s) = \left[\delta_{\{i=j\}} \frac{1 - \alpha_j(\rho_j w + s)}{\rho_j w + s} \right] \; ;$$

$$(3.13) \qquad \hat{\hat{\underline{\underline{\chi}}}}_{\alpha^{**}}(w, s) = \left[\underline{\underline{I}} - \underline{\underline{\alpha}}^{**}(w, s) \right]^{-1} \; .$$

Theorem 3.1. *(Theorem 2.8.1 of Masuda[5])*
For the trivariate stochastic process $[N(t), X(t), Z(t)]$ *with* $X(0) = 0$ *and* $Z(0) = 0$,

let $\hat{\underline{\hat{\varphi}}}(v, w, s)$ be defined as in (3.3). One then has

$$\hat{\underline{\hat{\varphi}}}(v, w, s) = \hat{\underline{\hat{\chi}}}_{\underline{\alpha}^{**}}(w, s)\hat{\underline{\hat{\beta}}}_{\underline{D}}(w, v + s) \ .$$

Of particular interest is the bivariate stochastic process $[N(t), Z(t)]$ characterized by

(3.14) $$\hat{\underline{\hat{\varphi}}}_{Z}(w, s) \overset{\text{def}}{=} \hat{\underline{\hat{\varphi}}}(0, w, s) = \hat{\underline{\hat{\chi}}}_{\underline{\alpha}^{**}}(w, s)\hat{\underline{\hat{\beta}}}_{\underline{D}}(w, s) \ .$$

In what follows, we exploit the specific structure of $\underline{\alpha}(s)$ given in (3.4) so as to evaluate $\hat{\underline{\hat{\varphi}}}_{Z}(w, s)$ explicitly. This in turn enables one to obtain

(3.15) $$\zeta_Z(s) \overset{\text{def}}{=} \int_0^\infty e^{-st}\text{E}[Z(t)]dt = -\underline{p}^T(0)\frac{\partial}{\partial w}\hat{\underline{\hat{\varphi}}}_{Z}(w, s)\Big|_{w=0}\underline{1} \ .$$

A preliminary lemma is needed. It is easy to confirm this lemma and proof is omitted.

Lemma 3.2. *Let $\underline{\underline{A}}$ be defined as*

$$\underline{\underline{A}} \overset{\text{def}}{=} \begin{bmatrix} 0 & b & 0 \\ a & 0 & d \\ 0 & c & 0 \end{bmatrix} \ .$$

If $|ab + cd| < 1$, then $\left[\underline{\underline{I}} - \underline{\underline{A}}\right]^{-1}$ exists and is given by

$$\left[\underline{\underline{I}} - \underline{\underline{A}}\right]^{-1} = \frac{1}{1 - (ab + cd)}\begin{bmatrix} 1 - cd & b & bd \\ a & 1 & d \\ ac & c & 1 - ab \end{bmatrix} \ .$$

From (3.4), (3.10) and (3.11), one sees that

(3.16) $$\underline{\alpha}^{**}(w, s) = \begin{bmatrix} 0 & \alpha_R(s - v_R w) & 0 \\ e^{c_R w}\alpha_{10}(s + pw) & 0 & e^{c_M w}\alpha_{12}(s + pw) \\ 0 & \alpha_M(s - v_M w) & 0 \end{bmatrix} \ .$$

For $Re(s) > 0$ and $w = e^{i\theta}$ with $\theta \in R$, it can be readily seen that

$$|e^{c_R w}\alpha_{10}(s + pw)\alpha_R(s - v_R w) + \alpha_M(s - v_M w)e^{c_M w}\alpha_{12}(s + pw)| < 1 \ .$$

Hence from (3.14) and Lemma 3.2, the next theorem holds true.

Theorem 3.3.

$$\hat{\underline{\hat{\varphi}}}_{Z}(w, s) = \frac{1}{1 - q(w, s)}\hat{\underline{\hat{\varepsilon}}}(w, s)\hat{\underline{\hat{\beta}}}_{D}(w, s) \ ,$$

where $\hat{\underline{\hat{\beta}}}_{\underline{D}}(w, s)$ is as in (3.12) and

$$q(w, s) = b_0(w, s)b_{10}(w, s) + b_{12}(w, s)b_2(w, s)$$

with

$$
\begin{aligned}
b_0(w,s) &= \alpha_R(s - v_R w) \ ; \\
b_{10}(w,s) &= e^{c_R w}\alpha_{10}(s + pw) \ ; \\
b_{12}(w,s) &= e^{c_M w}\alpha_{12}(s + pw) \ ; \\
b_2(w,s) &= \alpha_M(s - v_M w) \ ;
\end{aligned}
$$

and

$$
\hat{\underline{\underline{\hat{\varepsilon}}}}(w,s) = \begin{bmatrix}
1 - b_{12}(w,s)b_2(w,s) & b_0(w,s) & b_0(w,s)b_2(w,s) \\
b_{10}(w,s) & q(w,s) & b_{12}(w,s) \\
b_{10}(w,s)b_{12}(w,s) & b_2(w,s) & 1 - b_{10}(w,s)b_0(w,s)
\end{bmatrix}.
$$

In order to determine the optimal PM triggering time as a constant, we consider a sequence of distribution functions $(A_{T:j}(t))_{j=1}^{\infty}$ satisfying $A_{T:j}(t) \to U(t - T)$ as $j \to \infty$ where $U(x) = 1$ if $x \geq 0$ and $U(x) = 0$ else. It is clear that Theorem 3.3 still holds true at the limit. In this case, one has $a_T(t) = \delta(t-T)$ and $\bar{A}_T(t) = 1-U(t-T)$ so that, from (3.7) and (3.8),

(3.17) $$\alpha_{10}(s) = \int_0^T e^{-st}a_L(t)dt \ ; \ \alpha_{12}(s) = e^{-sT}\bar{A}_L(T) \ .$$

Employing (3.17) in Theorem 3.3, differentiating $\hat{\underline{\hat{\varphi}}}_Z(w,s)$ with respect to w at $w = 0$, and then exploiting the Taylor expansion of the resulting equation at $s = 0$, the following asymptotic expansion of $E[Z(t)]$ can be obtained, where proof is given in Appendix.

Theorem 3.4.

$$
E[Z(t)] = C_1(T)t + C_2(T) + o(t) \ as \ t \to \infty \ ,
$$

where

(3.18) $\qquad G(T) = \int_0^T \bar{A}_L(x)dx + \mu_{R:1}A_L(T) + \mu_{M:1}\bar{A}_L(T)$;

(3.19) $\qquad C_1(T) = \dfrac{p\int_0^T \bar{A}_L(x)dx - \left\{\hat{C}_R A_L(T) + \hat{C}_M \bar{A}_L(T)\right\}}{G(T)}$;

(3.20) $\qquad C_2(T) = \dfrac{C_1(T)C_{21}(T) + C_{22}(T)}{G(T)}$;

$$C_{21}(T) = -\frac{1}{2}\left\{\mu_{L:2} + \mu_{R:2}A_L(T) + \mu_{M:2}\bar{A}_L(T)\right\}$$

$$+ \int_T^\infty xa_L(x)dx$$

(3.21) $\qquad -\mu_{R:1}\int_0^T xa_L(x)dx - \mu_{M:1}T\bar{A}_L(T)$;

$$C_{22}(T) = 3p\left\{\frac{1}{2}\mu_{L:2} + \int_0^T xa_L(x)dx - \mu_{L:1}\right\}$$

$$+\hat{C}_R\int_0^T xa_L(x)dx$$

(3.22) $\qquad +\frac{1}{2}\left\{v_R\mu_{R:2} - v_M\mu_{M:2} - (\hat{C}_R - \hat{C}_M)T\right\}\bar{A}_L(T)$.

It should be noted that $C_1(T)$ characterizes the ergodic behavior of the reward rate per unit time, while $C_2(T)$ dictates the speed of its convergence to ergodicity. In the subsequent two sections, we study $C_1(T)$ theoretically and $C_1(T)+\{C_2(T)/\tau\}$ numerically so as to explore the ergodic optimal PM policy T^* which maximizes the former and the non-ergodic optimal PM policy $T^{**}(\tau)$ which achieves the maximum of the latter, where τ denotes the planning horizon. It will be shown that T^* could be significantly different from $T^{**}(\tau)$, thereby demonstrating danger of exclusive reliance on ergodic analysis and providing some useful managerial implications.

4. Ergodic Optimal PM Policy T^*

From Theorem 3.4, it can be readily seen that

(4.1) $$C_1(T) = \lim_{t\to\infty} \frac{\mathrm{E}[Z(t)]}{t} \ .$$

In other words, given a PM policy $T > 0$, $C_1(T)$ is the reward rate per unit time at ergodicity. In this section, we first establish the probabilistic interpretation of $C_1(T)$. The conditions for the existence of the ergodic optimal PM policy T^* are then investigated in terms of distributional properties of the system lifetime X_L, where

(4.2) $$T^* = \operatorname*{argmax}_{T \geq 0}\{C_1(T)\} \ .$$

As can be seen from Figure 1, the point (1,0) is a regenerative point of the bivariate process $[N(t), X(t)]$. The regenerative cycle time $X_{cycle}(T)$ is given by

$$
(4.3) \qquad X_{cycle}(T) \stackrel{\text{def}}{=} \begin{cases} T + X_M & \text{if} \quad X_L \geq T \\ X_L + X_R & \text{if} \quad X_L < T \end{cases} .
$$

The following proposition then holds, where proof is given in Appendix.

Proposition 4.1.

(a) Let $\bar{H}_{cycle}(x, T) = P[X_{cycle}(T) > x]$. Then

$$
\bar{H}_{cycle}(x, T) = \int_0^T \bar{A}_R(x - y) a_L(y) dy + \int_T^\infty \bar{A}_M(x - T) a_L(y) dy .
$$

(b) Let $G(T)$ be as in (3.18). Then

$$
E[X_{cycle}(T)] = G(T) .
$$

Let $X_{cycle:up}(T)$ be the system running time within a regenerative cycle. Then, as in (4.3), one sees that

$$
(4.4) \qquad X_{cycle:up}(T) \stackrel{\text{def}}{=} \begin{cases} T & \text{if} \quad X_L \geq T \\ X_L & \text{if} \quad X_L < T \end{cases} .
$$

It then follows that

$$
E[X_{cycle:up}(T)] = \int_0^T x a_L(x) dx + T \bar{A}_L(T) .
$$

Using integration by parts, this then leads to

$$
(4.5) \qquad E[X_{cycle:up}(T)] = \int_0^T \bar{A}_L(x) dx .
$$

The probabilistic interpretation of $C_1(T)$ in (3.19) is now clear. The numerator consists of the expected profit $p \times E[X_{cycle:up}]$, the expected cost $\hat{C}_R$ per CM with probability $A_L(T)$, and the expected cost $\hat{C}_M$ per PM with probability $\bar{A}_L(T)$, representing the expected reward within a regenerative cycle. Since the denominator is $E[X_{cycle}]$ from Proposition 4.1 (b), $C_1(T)$ is the expected reward rate per unit time within a regenerative cycle, which coincides with the ergodic reward rate from (4.1) as it should be.

We next turn our attention to investigate the conditions under which the ergodic optimal PM policy T^* as defined in (4.2) exists. For this purpose, we first note from (3.19) that

$$
(4.6) \qquad \frac{d}{dT} C_1(T) = \frac{\bar{A}_L(T)}{\{G(T)\}^2} \xi(T) ,
$$

where

$$
(4.7) \quad \xi(T) \;=\; l_R - l_{diff} \cdot \eta_L(T) E[X_{cycle:up}(T)] - l_{diff} \bar{A}_L(T) - S \cdot \eta_L(T)
$$

and

$$(4.8) \qquad S \stackrel{\text{def}}{=} \mu_{R:1}\mu_{M:1}u_{diff} ,$$

with u_{diff} as defined in (2.6). It can be seen from (2.5), (2.7) and (4.7) that $\xi(T) = 0$ at $T = T^*$ if and only if

$$(4.9) \qquad \frac{l_R}{l_M} - 1 = K(T^*) ;$$

$$(4.10) \qquad K(T) = \frac{\eta_L(T)(\mu_{R:1} - \mu_{M:1}) + 1}{\eta_L(T)\left\{\mathrm{E}[X_{cycle:up}(T)] + \mu_{M:1}\right\} - A_L(T)} .$$

In this regard, Makabe[4] has shown the following theorem. The class of random variables with increasing failure rate function (decreasing failure rate function) is denoted by IFR (DFR). By increasing (decreasing), we actually mean non-decreasing (non-increasing). In addition, the class of exponential random variables with constant failure rate function is denoted by EXP. The set difference between two sets $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \setminus \mathcal{B} = \{x : x \in \mathcal{A} \text{ and } x \notin \mathcal{B}\}$.

Theorem 4.2. *(Makabe[4])*

    1) If $X_L \in IFR\backslash EXP$ (DFR\EXP), then $K(T)$ in (4.10) is decreasing (increasing).

    2) If the optimal PM policy T^* exists, and $X_L \in IFR\backslash EXP$ and $\eta_L(T) \to \infty$ as $T \to \infty$, then T^* is unique and is decreasing in $\frac{l_R}{l_M}$.

In this paper, we elaborate further concerning the ergodic optimal PM policy T^*. In particular, sufficient conditions are given explicitly for the existence of T^*. Proofs of the following theorems are given in Appendix. Our first theorem characterizes the monotonicity properties of $\xi(T)$ in (4.7) in terms of the IFR\EXP, DFR\EXP and EXP properties of the system lifetime X_L.

Theorem 4.3. *Let $\xi(T)$ be as in (4.7).*

    a) $\xi(T)$ is decreasing in T if and only if $X_L \in IFR\backslash EXP$.

    b) $\xi(T)$ is increasing in T if and only if $X_L \in DFR\backslash EXP$.

    c) $\xi(T)$ is constant if and only if $X_L \in EXP$.

Theorem 4.3 enables one to capture the functional behavior of $\frac{d}{dT}C_1(T)$ through (4.6), which in turn leads to sufficient conditions for the existence of T^*, as we see next. For this purpose, the following identities play a key role.

$$(4.11) \qquad \begin{cases} \xi(0+) = l_M - \eta_L(0+)S ; \\ \xi(+\infty) = l_R - \eta_L(+\infty)(l_{diff}\mu_{L:1} + S) . \end{cases}$$

In what follows, $T^* = +\infty$ means that the preventive maintenance should not be implemented. We also note from (3.18) and (3.19) that $C_1(0) = -\frac{\hat{C}_M}{\mu_{M:1}} < 0$. Accordingly, $T^* = 0$ if and only if $C_1(T) < 0$ for all $T \geq 0$ and therefore the system should not be run at all for this case.

Theorem 4.4. *Let T^* be defined as in (4.2) and suppose $X_L \in$ IFR\EXP.*

    A) If $\eta_L(+\infty) < \frac{l_R}{l_{diff}\mu_{L:1}+S}$, then $C_1(T)$ is increasing in T and $T^* = +\infty$.

    B) If $\eta_L(0) < \frac{l_M}{S}$ and $\eta_L(+\infty) > \frac{l_R}{l_{diff}\mu_{L:1}+S}$, then T^* exists uniquely satisfying $\frac{d}{dT}C_1(T)\big|_{T=T^*} = 0$.

    C) If $\eta_L(0) > \frac{l_M}{S}$, then $C_1(T)$ is decreasing in T and $T^* = 0$.

When $X_L \in$ DFR\EXP, as summarized in the next theorem, either the preventive maintenance should not be implemented or the system should not be run at all. This theorem can be proven similarly to Theorem 4.4 and proof is omitted.

Theorem 4.5. *Let T^* be defined as in (4.2) and suppose $X_L \in$ DFR\EXP.*

    A) If $\eta_L(0) < \frac{l_M}{S}$, then $C_1(T)$ is increasing in T and $T^* = +\infty$.

    B) If $\eta_L(0) > \frac{l_M}{S}$ and $\eta_L(+\infty) < \frac{l_R}{l_{diff}\mu_{L:1}+S}$, then $\tilde{T}$ exists uniquely satisfying $\frac{d}{dT}C_1(T)\big|_{T=\tilde{T}} = 0$. In this case, $C_1(\tilde{T})$ is the global minimum of $C_1(T)$. For the global maximum point, $T^* = +\infty$ if $\frac{p\mu_{L:1}-\hat{C}_R}{\mu_{L:1}+\mu_{R:1}} \geq -\frac{\hat{C}_M}{\mu_{M:1}}$, and $T^* = 0$ else.

    C) If $\eta_L(+\infty) > \frac{l_R}{l_{diff}\mu_{L:1}+S}$, then $C_1(T)$ is decreasing in T and $T^* = 0$.

For the case of $X_L \in$ EXP, one also sees that either the preventive maintenance should not be implemented or the system should not be run at all. Since the underlying p.d.f. is exponential, however, the conditions can be simplified. This theorem can be also proven similarly to Theorem 4.4 and proof is omitted.

Theorem 4.6. *Let T^* be defined as in (4.2) and suppose $X_L \in$ EXP with p.d.f. $a_L(x) = \theta e^{-\theta x}$.*

    A) If $\theta \leq \frac{l_M}{S}$, then $C_1(T)$ is increasing in T and $T^* = +\infty$.

    B) If $\theta > \frac{l_M}{S}$, then $C_1(T)$ is decreasing in T and $T^* = 0$.

We now illustrate Theorems 4.3 and 4.4 of primary concern numerically. Let the exponential variate of mean $1/\lambda$ be denoted by $E(\lambda)$. Throughout the rest of the paper, we assume that $X_R = E(\lambda_{R1}) + E(\lambda_{R2})$ and $X_M = E(\lambda_{M1}) + E(\lambda_{M2})$ where the underlying random variables are independent. The corresponding p.d.f's are given by

$$(4.12) \qquad a_R(x) \;=\; \frac{\lambda_{R1}\lambda_{R2}}{\lambda_{R2} - \lambda_{R1}}\left(e^{-\lambda_{R1}x} - e^{-\lambda_{R2}x}\right)\;;$$

$$(4.13) \qquad a_M(x) \;=\; \frac{\lambda_{M1}\lambda_{M2}}{\lambda_{M2} - \lambda_{M1}}\left(e^{-\lambda_{M1}x} - e^{-\lambda_{M2}x}\right)\;.$$

We also adopt the following parameter values.

$$(4.14) \qquad \begin{cases} p = 25.0 \;; \\ v_R = 20.0 \;;\; v_M = 0.5 \;; \\ c_R = 10.0 \;;\; c_M = 5.0 \;. \end{cases}$$

Case 1 : IFR-1 for Theorem 4.4 A)

We suppose that the lifetime X_L is the sum of two independent exponential variates, i.e, $X_L = E(\lambda_{L1}) + E(\lambda_{L2})$. The corresponding p.d.f. is then given by

$$(4.15) \qquad a_L(x) = \frac{\lambda_{L1}\lambda_{L2}}{\lambda_{L2} - \lambda_{L1}} \left(e^{-\lambda_{L1}x} - e^{-\lambda_{L2}x} \right) .$$

For the parameters to specify X_L, X_R and X_M, we set

$$\begin{cases} \lambda_{L1} = 0.15 \; ; \; \lambda_{L2} = 0.1 \; ; \\ \lambda_{R1} = 0.8 \; ; \; \lambda_{R2} = 0.5 \; ; \\ \lambda_{M1} = 0.95 \; ; \; \lambda_{M2} = 0.55 \; . \end{cases}$$

Accordingly, one has

$$\eta_L(\infty) = 0.1 \; < \; 0.104 = \frac{l_R}{l_{diff}\mu_{L:1} + S}$$

so that the conditions of Theorem 4.4 A) are satisfied. In this case, $\xi(T)$ is decreasing while $C_1(T)$ is increasing with $T^* = +\infty$ as depicted in Figures 3 and 4.
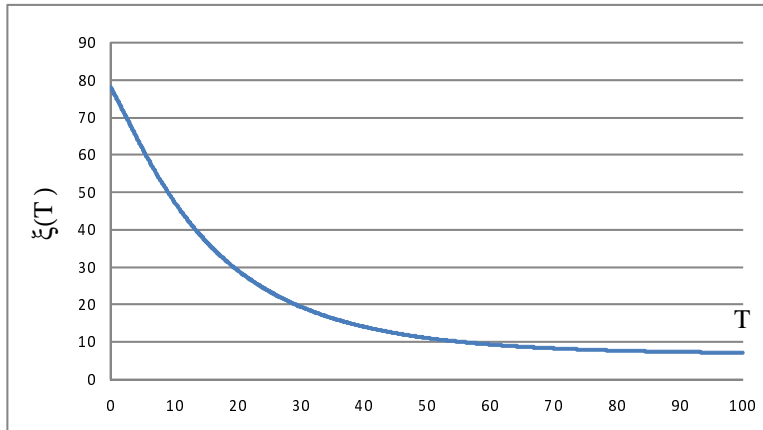
FIGURE 3. $\xi(T)$ IFR-1
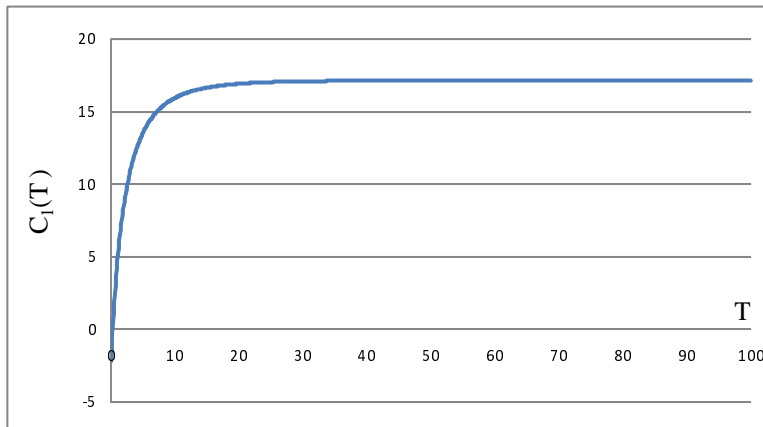
FIGURE 4. $C_1(T)$ IFR-1

Case 2 : IFR-2 for Theorem 4.4 B)

Let X_L be as in Case 1. However, the parameters for X_L, X_R and X_M are set differently as specified below.

$$\begin{cases} \lambda_{L1} = 0.15 \; ; \; \lambda_{L2} = 0.1 \; ; \\ \lambda_{R1} = 0.6 \; ; \; \lambda_{R2} = 0.1 \; ; \\ \lambda_{M1} = 0.9 \; ; \; \lambda_{M2} = 0.5 \; . \end{cases}$$

It then follows that

$$\eta_L(0) = 0 < 0.124 = \frac{l_M}{S},$$

and

$$\eta_L(\infty) = 0.1 > 0.065 = \frac{l_R}{l_{diff}\mu_{L:1} + S},$$

and the conditions for Theorem 4.4 B) are met. In this case, $\xi(T)$ is decreasing while $C_1(T)$ is unimodal with $T^* = 5.7$ and $C_1(T^*) = 9.3$ as depicted in Figures 5 and 6.
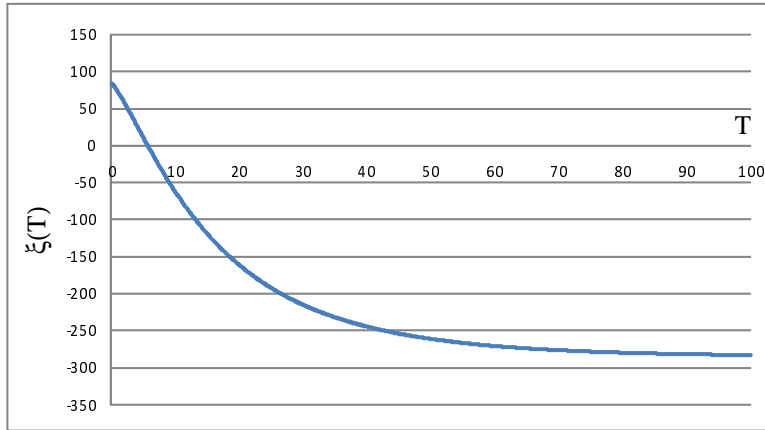
FIGURE 5. $\xi(T)$ IFR-2
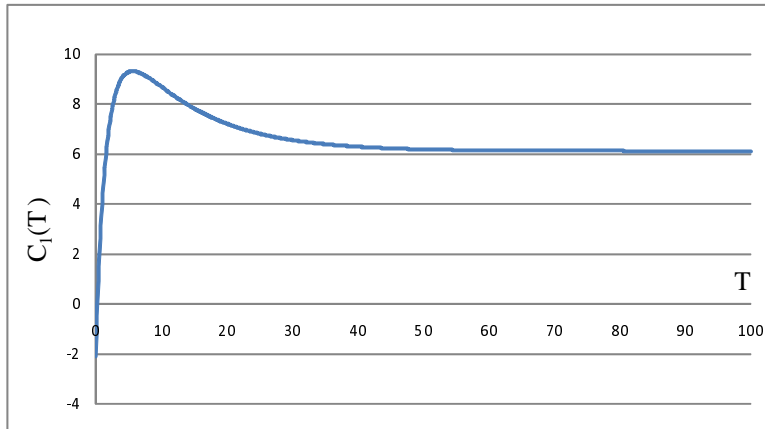
FIGURE 6. $C_1(T)$ IFR-2

Case 3 : IFR-3 for Theorem 4.4 C)

In order to illustrate Theorem 4.4 C), we define the p.d.f. of X_L as

$$a_L(x) = \frac{1}{\lambda_{L1} - \lambda_{L2}} \left\{ \lambda_{L1}(\lambda_{L2} + c)e^{-(\lambda_{L2}+c)x} - \lambda_{L2}(\lambda_{L1} + c)e^{-(\lambda_{L1}+c)x} \right\} .$$

It should be noted that, under the conditions $\lambda_{L1} > \lambda_{L2} > 0$ and $c > 0$, one has

$$
\begin{aligned}
a_L(x) &= \frac{\lambda_{L1}\lambda_{L2}e^{-cx}}{\lambda_{L1} - \lambda_{L2}} \left\{ \left(1 + \frac{c}{\lambda_{L2}}\right) e^{-\lambda_{L2}x} - \left(1 + \frac{c}{\lambda_{L1}}\right) e^{-\lambda_{L1}x} \right\} \\
&> \frac{\lambda_{L1}\lambda_{L2}e^{-cx}}{\lambda_{L1} - \lambda_{L2}} \left\{ \left(1 + \frac{c}{\lambda_{L1}}\right) e^{-\lambda_{L1}x} - \left(1 + \frac{c}{\lambda_{L1}}\right) e^{-\lambda_{L1}x} \right\} = 0 ,
\end{aligned}
$$

so that $a_L(x)$ is well defined. Let the parameters for X_L, X_R and X_M be given by

$$
\begin{cases}
\lambda_{L1} = 0.02 \; ; \; \lambda_{L2} = 0.01 \; ; \; c = 0.275 \; ; \\
\lambda_{R1} = 0.4 \; ; \; \lambda_{R2} = 0.3 \; ; \\
\lambda_{M1} = 0.9 \; ; \; \lambda_{M2} = 0.5 .
\end{cases}
$$

It can be seen that

$$\eta_L(0) = 0.275 \; > \; 0.274 = \frac{l_M}{S} ,$$

and the conditions for Theorem 4.4 C) are satisfied. In this case, both $C_1(T)$ and $\xi(T)$ are decreasing with $T^* = 0$ so that the system should not be run at all. Figures 7 and 8 illustrate these observations.
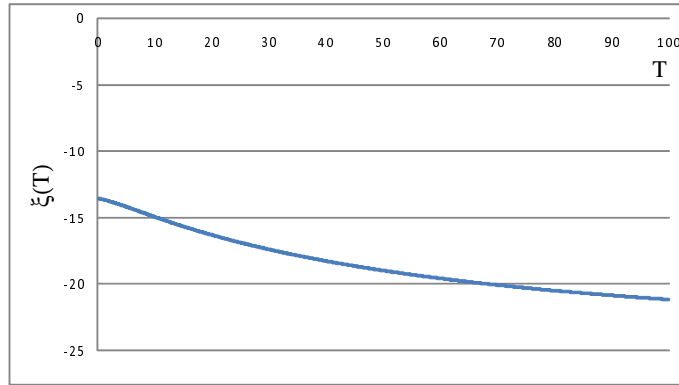
FIGURE 7. $\xi(T)$ IFR-3
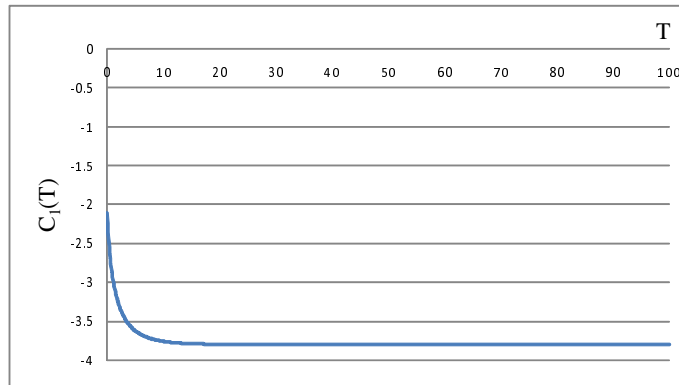
FIGURE 8. $C_1(T)$ IFR-3

5. Non-ergodic Optimal PM Policy $T^{**}(\tau)$ and Danger of Exclusive Reliance on Ergodic Analysis

In this section, we discuss the non-ergodic optimal PM policy $T^{**}(\tau)$ defined by

$$(5.1) \qquad T^{**}(\tau) = \operatorname*{argmax}_{T \geq 0} \left\{ C_1(T) + \frac{C_2(T)}{\tau} \right\} ,$$

where τ denotes the planning horizon. Given $\tau > 0$, we are interested in exploring differences between the optimal ergodic PM policy T^* defined in (4.2) and the optimal non-ergodic PM policy $T^{**}(\tau)$ in (5.1). For this purpose, the expected cumulative reward over the planning horizon τ for each optimal policy is introduced as

$$(5.2) \qquad \mathrm{E_{erg}}[\tau, T^*] \stackrel{\text{def}}{=} C_1(T^*)\tau ;$$

$$(5.3) \qquad \mathrm{E_{non\text{-}erg}}[\tau, T^{**}(\tau)] \stackrel{\text{def}}{=} C_1(T^{**}(\tau))\tau + C_2(T^{**}(\tau)) .$$

We now demonstrate the danger of exclusive reliance on ergodic analysis using Case 2 of Section 4. For this case, we recall that $T^* = 5.7$ and $C_1(T^*) = 9.3$ so that $\mathrm{E_{erg}}[\tau, T^*]$ is linear in τ with slope of 9.3. Figure 9 depicts $T^{**}(\tau)$ as a function of τ, and both $\mathrm{E_{erg}}[\tau, T^*]$ and $\mathrm{E_{non\text{-}erg}}[\tau, T^{**}(\tau)]$ are exhibited in Figure 10.

As far as the non-ergodic analysis is concerned, we observe that $T^{**}(\tau) = \tau$ for $0 \leq \tau \leq 59.4$. This means that no preventive maintenance is required when the planning horizon is within 59.4. For $\tau > 59.4$, the non-ergodic optimal PM policy $T^{**}(\tau)$ emerges with $T^{**}(\tau) < \tau$, which monotonically decreases to the limit $T^* = 5.7$ as $\tau \to \infty$. One has, for example, $T^{**}(59.5) = 9.0$ while $T^{**}(100) = 7.2$. Because of this difference, the expected cumulative reward over the planning horizon τ under the ergodic optimal policy T^* differs from that under the non-ergodic optimal policy $T^{**}(\tau)$ when τ is not sufficiently large, as shown in Figure 10, demonstrating the danger of exclusive reliance on ergodic analysis.
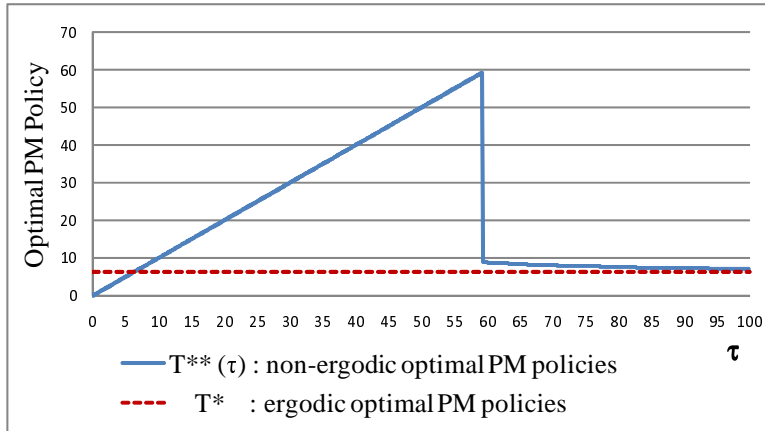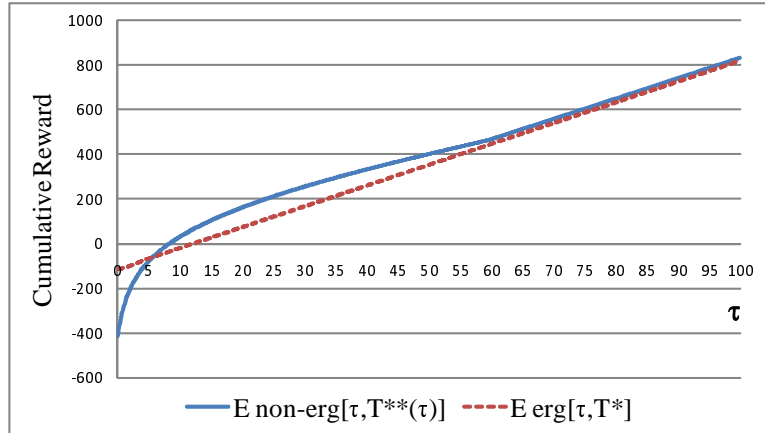

Figure 9. Optimal PM policies

FIGURE 10. $E_{\text{non-erg}}[\tau, T^{**}]$ and $E_{\text{erg}}[\tau, T^*]$

6. CONCLUSION

In this paper, the danger of exclusive reliance on ergodic analysis in devising optimal PM policies is addressed. The classical semi-Markov model of Makabe[4] is first examined thoroughly at ergodicity, yielding many new results. In particular, sufficient conditions are given for the existence of the ergodic optimal PM policy in terms of hazard rate properties of system lifetimes. Then, dynamic analysis of the model is discussed by examining the trivariate process $[N(t), X(t), Z(t)]$, where $N(t)$ is the underlying semi-Markov process describing the state of the production system, $X(t)$ is the associated age process, i.e. the elapsed time at time t since the last transition into the current state, and $Z(t)$ is the reward process with jumps defined on $[N(t), X(t)]$. The asymptotic expansion of $E[Z(t)]$ as $t \to \infty$ is obtained explicitly in an affine form, and non-ergodic optimal PM policy is introduced based on the asymptotic expansion. Numerical results are presented for demonstrating the danger of exclusive reliance on ergodic analysis when the planning horizon is not sufficiently large.

Acknowledgements

Appendix

Proof of Proposition 2.1

We note that

$$\frac{l_{diff}}{\mu_{R:1}\mu_{M:1}} = \frac{\hat{C}_R}{\mu_{R:1}\mu_{M:1}} + \frac{p}{\mu_{M:1}} - \frac{\hat{C}_M}{\mu_{R:1}\mu_{M:1}} - \frac{p}{\mu_{R:1}} \ .$$

Since $\mu_{R:1} > \mu_{M:1} > 0$, by replacing $\mu_{R:1}$ in the denominators of $\hat{C}_M$ and p by $\mu_{M:1}$, one then sees that

$$\frac{l_{diff}}{\mu_{R:1}\mu_{M:1}} > \frac{\hat{C}_R}{\mu_{R:1}\mu_{M:1}} - \frac{\hat{C}_M}{\mu_{M:1}^2} = \frac{1}{\mu_{M:1}} u_{diff} \ ,$$

and the proposition follows since $u_{diff} > 0$.   □

Proof of Theorem 3.4

Let

$$\frac{\partial}{\partial w} \underline{\hat{\underline{\varphi}}}_Z(w,s)\bigg|_{w=0} \overset{\text{def}}{=} \begin{bmatrix} \hat{\mu}_{1:00}(s) & \hat{\mu}_{1:01}(s) & \hat{\mu}_{1:02}(s) \\ \hat{\mu}_{1:10}(s) & \hat{\mu}_{1:11}(s) & \hat{\mu}_{1:12}(s) \\ \hat{\mu}_{1:20}(s) & \hat{\mu}_{1:21}(s) & \hat{\mu}_{1:22}(s) \end{bmatrix} .$$

Since we assume that the system starts from state 1, the initial vector is $\underline{p}^T(0) = [0, 1, 0]$. From (3.15), one then has

$$\int_0^\infty e^{-st} \mathrm{E}[Z(t)]dt = \hat{\mu}_{1:10}(s) + \hat{\mu}_{1:11}(s) + \hat{\mu}_{1:12}(s) \ .$$

In order to find $\hat{\mu}_{1:1j}(s)$ for $j = 0, 1, 2$, we differentiate $\underline{\hat{\underline{\varphi}}}_Z(w,s)$ in Theorem 3.3 with respect to w and setting $w = 0$, yielding

$$\hat{\mu}_{1:10}(s) = -\frac{1}{\{G(s)\}^2} \{H(s)G(s) + H_0(s)B(s)\} \; ;$$

$$\hat{\mu}_{1:11}(s) = -\frac{1}{\{G(s)\}^2} \left\{\tilde{k}^{'}(s)G(s) + \tilde{k}(s)B(s)\right\} \; ;$$

$$\hat{\mu}_{1:12}(s) = -\frac{1}{\{G(s)\}^2} \{L(s)G(s) + L_0(s)B(s)\} \; ,$$

where, with $f^{'}(x) = \frac{d}{dx}f(x)$,

$$G(s) = 1 - \left\{\tilde{a}(s)\tilde{b}(s) + \tilde{c}(s)\tilde{d}(s)\right\} \; ;$$

$$B(s) = \tilde{a}^{'}(s)\tilde{b}(s) + \tilde{a}(s)\tilde{b}^{'}(s) + \tilde{c}^{'}(s)\tilde{d}(s) + \tilde{c}(s)\tilde{d}^{'}(s) \; ;$$

$$H(s) = \tilde{a}^{'}(s)\tilde{h}(s) + \tilde{a}(s)\tilde{h}^{'}(s) \; ;$$

$$H_0(s) = \tilde{a}(s)\tilde{h}(s);$$

$$L(s) = \tilde{d}^{'}(s)\tilde{l}(s) + \tilde{d}(s)\tilde{l}^{'}(s) \; ;$$

$$L_0(s) = \tilde{d}(s)\tilde{l}(s) \; ;$$

$$\tilde{a}(s) = \alpha_{10}(s) \; ; \; \tilde{b}(s) = \alpha_R(s) \; ;$$

$$\tilde{c}(s) = \alpha_M(s) \; ; \; \tilde{d}(s) = \alpha_{12}(s) \; ;$$

$$\tilde{a}^{'}(s) = c_R\alpha_{10}(s) + p\frac{d}{ds}\alpha_{10}(s) \; ;$$

$$\tilde{b}^{'}(s) = -v_R\frac{d}{ds}\alpha_R(s) \; ;$$

$$\tilde{c}^{'}(s) = -v_M\frac{d}{ds}\alpha_M(s) \; ;$$

$$\tilde{d}^{'}(s) = c_M\alpha_{12}(s) + p\frac{d}{ds}\alpha_{12}(s);$$

$$\tilde{h}(s) = \bar{\alpha}_R(s) \; ; \; \tilde{k}(s) = \bar{\alpha}_W(s) \; ; \; \tilde{l}(s) = \bar{\alpha}_M(s) \; ;$$

$$\tilde{h}^{'}(s) = \frac{v_R}{s}\left\{\left(\frac{d}{ds}\alpha_R(s) + \bar{\alpha}_R(s)\right)\right\} \; ;$$

$$\tilde{k}^{'}(s) = -\frac{p}{s}\left\{\left(\frac{d}{ds}\alpha_1(s)\right) + \bar{\alpha}_1(s)\right\} \; ;$$

$$\tilde{l}^{'}(s) = \frac{v_M}{s}\left\{\left(\frac{d}{ds}\alpha_M(s) + \bar{\alpha}_M(s)\right)\right\} \; .$$

By exploiting the Taylor expansions of these functions at $s = 0$, one has

$$(6.1) \quad \begin{cases}
\tilde{a}(s) & = & \mu_{10:0} - \mu_{10:1}s + \frac{1}{2}\mu_{10:2}s^2 + o(s^2) \; ; \\
\tilde{b}(s) & = & 1 - \mu_{R:1}s + \frac{1}{2}\mu_{R:2}s^2 + o(s^2) \; ; \\
\tilde{c}(s) & = & 1 - \mu_{M:1}s + \frac{1}{2}\mu_{M:2}s^2 + o(s^2) \; ; \\
\tilde{d}(s) & = & \mu_{12:0} - \mu_{12:1}s + \frac{1}{2}\mu_{12:2}s^2 + o(s^2) \; ; \\
\tilde{a}'(s) & = & \check{a}(0) - \check{a}(1)s + \frac{1}{2}\check{a}(2)s^2 + o(s^2) \; ; \\
\tilde{b}'(s) & = & v_R \left\{ \mu_{R:1} - \mu_{R:2}s + \frac{1}{2}\mu_{R:3}s^2 \right\} + o(s^2) \; ; \\
\tilde{c}'(s) & = & v_M \left\{ \mu_{M:1} - \mu_{M:2}s + \frac{1}{2}\mu_{M:3}s^2 \right\} + o(s^2) \; ; \\
\tilde{d}'(s) & = & \check{d}(0) - \check{d}(1)s + \frac{1}{2}\check{d}(2)s^2 + o(s^2) \; ; \\
\tilde{h}(s) & = & \mu_{R:1} - \frac{1}{2}\mu_{R:2}s + \frac{1}{6}\mu_{R:3}s^2 + o(s^2) \; ; \\
\tilde{k}(s) & = & \mu_{1:1} - \frac{1}{2}\mu_{1:2}s + \frac{1}{6}\mu_{1:3}s^2 + o(s^2) \; ; \\
\tilde{l}(s) & = & \mu_{M:1} - \frac{1}{2}\mu_{M:2}s + \frac{1}{6}\mu_{M:3}s^2 + o(s^2) \; ; \\
\tilde{h}'(s) & = & -v_R \left\{ \frac{1}{2}\mu_{R:2} - \frac{1}{3}\mu_{R:3}s + \frac{1}{12}\mu_{R:4}s^2 \right\} \\
& & +o(s^2) \; ; \\
\tilde{k}'(s) & = & -p \left\{ \frac{1}{2}\mu_{1:2} - \frac{1}{3}\mu_{1:3}s + \frac{1}{12}\mu_{1:4}s^2 \right\} \\
& & +o(s^2) \; ; \\
\tilde{l}'(s) & = & -v_M \left\{ \frac{1}{2}\mu_{M:2} - \frac{1}{3}\mu_{M:3}s + \frac{1}{12}\mu_{M:4}s^2 \right\} \\
& & +o(s^2) \; ,
\end{cases}$$

where

$$\begin{cases}
\check{a}(0) & = & c_R\mu_{10:0} - p\mu_{10:1} \; ; \\
\check{a}(1) & = & c_R\mu_{10:1} + p\mu_{10:2} \; ; \\
\check{a}(2) & = & c_R\mu_{10:2} - p\mu_{10:3} \; ,
\end{cases}$$

$$\begin{cases}
\check{d}(0) & = & c_M\mu_{12:0} - p\mu_{12:1} \; ; \\
\check{d}(1) & = & c_M\mu_{12:1} + p\mu_{12:2} \; ; \\
\check{d}(2) & = & c_M\mu_{12:2} - p\mu_{12:3} \; .
\end{cases}$$

Since

$$\begin{cases}
a_T(x) & = & \delta(x - T); \\
\bar{A}_T(x) & = & 1 - U(x - T),
\end{cases} \quad U(x - T) = \begin{cases} 1 & \text{if} & x \geq T; \\ 0 & \text{if} & x < T, \end{cases}$$

one can sees that

$$\mu_{10:0} = A_L(T) \; ; \; \mu_{12:0} = \bar{A}_L(T) \; ;$$

$$\mu_{10:1} = \int_0^\infty dx \int_x^\infty \bar{A}_L(y)dy - T\bar{A}_L(T) \; ;$$

$$\mu_{12:1} = T\bar{A}_L(T) \; ; \; \mu_{1:1} = m_{10}(1) + m_{12}(1) \; .$$

From (6.1), it then leads to

$$\int_0^\infty e^{-st}\mathrm{E}[Z(t)]dt = \frac{1}{s^2}C_1(T) + \frac{1}{s}C_2(T) + o\left(\frac{1}{s^2}\right) \; .$$

The theorem now follows by inverting the above equation into the real domain. □

Proof of Proposition 4.1

Using the law of total probability and then employing Bayes' rule, the survival function of the cycle time can be evaluated as

$$
\begin{aligned}
\mathrm{P}[X_{cycle} > x] &= \mathrm{P}[X_{cycle} > x, X_L \le T] + \mathrm{P}[X_{cycle} > x, X_L > T] \\
&= \int_0^\infty \mathrm{P}[X_L + X_R > x, X_L \le T | X_L = y] a_L(y) dy \\
&\quad + \int_0^\infty \mathrm{P}[T + X_M > x, X_L > T | X_L = y] a_L(y) dy \\
&= \int_0^T \bar{A}_R(x - y) a_L(y) dy + \int_T^\infty \bar{A}_M(x - T) a_L(y) dy \ .
\end{aligned}
$$

This then leads to

$$
\begin{aligned}
\mathrm{E}[X_{cycle}] &= \int_0^\infty \mathrm{P}[X_{cycle} > x] dx \\
&= \int_0^T dy \, a_L(y) \int_0^\infty \bar{A}_R(x - y) dx + \int_T^\infty dy \, a_L(y) \int_0^\infty \bar{A}_M(x - T) dx \\
&= \int_0^T dy \, a_L(y) \left\{ \int_{-y}^0 \bar{A}_R(z) dz + \mu_{R:1} \right\} + \bar{A}_L(T) \left\{ \int_{-T}^0 \bar{A}_M(z) dz + \mu_{M:1} \right\} \ .
\end{aligned}
$$

Since $\bar{A}_R(z) = \bar{A}_M(z) = 1$ for $z \le 0$, one has

$$
\begin{aligned}
\mathrm{E}[X_{cycle}] &= \int_0^T y \, a_L(y) dy + \mu_{R:1} A_L(T) + T \bar{A}_L(T) + \mu_{M:1} \bar{A}_L(T) \\
&= \int_0^T \bar{A}_L(y) dy + \mu_{R:1} A_L(T) + \mu_{M:1} \bar{A}_L(T) \ ,
\end{aligned}
$$

completing the proof. □

Proof of Theorem 4.3

By differentiating $\xi(T)$ in (4.7) with respect to T, one finds that

$$
\frac{d}{dT} \xi(T) = -\left\{ \frac{d}{dT} \eta_L(T) \right\} \left(l_{diff} \mathrm{E}[X_{cycle:up}(T)] + S \right) \ .
$$

Since $l_{diff} > 0$ from Propsition 2.1 and $S > 0$ from (4.8), it then follows that

$$
X_L \in IFR \setminus EXP \Leftrightarrow \frac{d}{dT} \eta_L(T) > 0 \Leftrightarrow \frac{d}{dT} \xi(T) < 0 \ ;
$$

$$
X_L \in DFR \setminus EXP \Leftrightarrow \frac{d}{dT} \eta_L(T) < 0 \Leftrightarrow \frac{d}{dT} \xi(T) > 0 \ ;
$$

$$
X_L \in EXP \Leftrightarrow \frac{d}{dT} \eta_L(T) = 0 \Leftrightarrow \frac{d}{dT} \xi(T) = 0 \ ,
$$

completing the proof. □

Proof of Theorem 4.4

Theorem 4.3 states that $X_L \in$ IFR\EXP implies $\xi(T)$ is increasing in T. The following three cases are then considered.

Case A) $\xi(T) > 0$ for $T \geq 0$

Because of the monotonicity of $\xi(T)$, the above condition equivalent to

$$\eta_L(\infty) < \frac{l_R}{l_{diff}\mu_{L:1} + S}$$

Since the sign of $\frac{d}{dT}C_1(T)$ coincides with that of $\xi(T)$ from (4.6), one has $C_1(T)$ is increasing in T with $T^* = +\infty$.

Case B) $\xi(0) > 0$ and $\xi(+\infty) < 0$

The above conditions are satisfied if and only if

$$\eta_L(0) < \frac{l_M}{S} \text{ and } \eta_L(\infty) > \frac{l_R}{l_{diff}\mu_{L:1} + S} \ .$$

In this case, $\xi(T)$ crosses zero exactly once at $T = T^*$. From (4.6), this then implies that $C_1(T)$ achieves the unique global maximum at T^*.

Case C) $\xi(T) < 0$ for $T \geq 0$

Since $\xi(T)$ is decreasing, the above condition can be rewritten as

$$\eta_L(0) > \frac{l_M}{S} \ .$$

Clearly, one has $\xi(T) < 0$ for $T \geq 0$ so that $C_1(T)$ is strictly decreasing in T from (4.6). Accordingly, $C_1(T)$ achieves the unique global maximum at $T^* = 0$, completing the proof. □

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