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ON NON-EXISTENCE OF NASH EQUILIBRIUM OF AN M PERSON GAME WITH PURE STRATEGY FOR ENERGY SUPPLY

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Abstract A competitive market model is considered with M suppliers and N customers, where each supplier provides a homogeneous service such as energy supply and has to offer a uniform price upon delivery to all customers. Given a price upper bound U, the model is formulated as an M person game with pure strategy. It is shown that the M person game has the unique Nash equilibrium if and only if each customer can be serviced by at most one supplier. Furthermore, this unique Nash equilibrium is peculiar in that all suppliers adopt the same upper bound price U. In general, the M person game does not have any Nash equilibrium. For such a case it is demonstrated that the suppliers continue to exercise their price strategies in a cyclic manner indefinitely.

Keywords: Energy, N person game, Non-existence of Nash equilibrium

1. Introduction

Competitive market models for homogeneous products and services such as the energy supply can be traced back to 1920's. The pioneering paper by Hotelling [5] develops a duopoly model where customers are distributed uniformly over a finite line and serviced by two suplliers who choose their locations and prices so as to maximize their profit. Non-existence of Nash equilibrium, unless the two suppliers are located relatively for apart, is shown by D'Aspremont et al [2]. Subsequently, the Hotteling model has been extended in several directions. Economides [3] deals with the case where customers are distributed uniformly on a bounded plane. Anderson [1] incorporate stackelberg leadership within the context of the Hotelling model. Other variations include Thisse and Vives [9], Zhang and Teraoka [10] and Rath [7]. Gabszewicz and Thisse [4] provide an excellent review of the literature. More recently, for a spatially duopoly model with customers located at different nodes having separate demand functions, Matsubayashi et al.[6] establish a necessary and sufficient condition for the existence of Nash equilibrium and develop computational algorithms for finding the equilibrium point. When mixed strategies are allowed, Takahashi and Sumita [8] derive two type of Nash equilibriums explicitly for a two person model.

The purpose of this paper is to develope an M person game with pure strategy, describing a competitive market for homogeneous products and services such as the energy supply. The market consists of M suppliers and N customers, where each supplier offers a uniform price upon delivery to all customers. Such a uniform price practice in the energy supply industry is still in effect to some extent even after deregulation of the industry in Japan. Locations of suppliers and customers are fixed and the competitive structures are characterized in terms of costs and prices. Analysis of the price strategy in this realistic setting has been increasing its importance in the energy supply industry because of deregulation. The deregulation in principle is intended to derive a variety of ways to lower barriers for new entry. Large-scale industrial customers are now quite sensitive to prices of the energy they need, and the industry has been exposed to growing severe price competition. The thrust of this paper is to show that, except under a rather peculiar necessary and safficient condition, Nash equilibrium does not exist, demonstrating that the suppliers exercise their price strategies in a cyclic manner indefinitely.

The structure of this paper is as follows. In seciton 2, a competitive market model is formally introduced and the game-theoretic framework is established. A necessary and sufficient condition is derived in Section 3 for existence of Nash equilibrium. It is shown that the Nash equilibrium is unique, if any, and rather peculiar in that all suppliers adopt the price upper bound U. Finally in Section 4, a duoply model is discussed explicitly demonstrating the cyclic phenomenon of the suppliers in exercising their price strategies so as to maximize their profits.

2. Model Description

We consider a market consisting of M suppliers and N customers as depicted in Figure 2.1, where each supplier provides a homogeneous service such as propane gas or LNG transportation by tank lorry. Each customer may represent one large industry or a group of residents in the same district. Let $\mathcal{M} = \{1, 2, \cdots, M\}$ and $\mathcal{N} = \{1, 2, \cdots, N\}$ be a set of suppliers and a set of customers repectively. The cost for supplier $i \in \mathcal{M}$ to provide a unit of service to customer $j \in \mathcal{N}$ is denoted by c_{ij} .

Since the service under consideration is typically an energy supply service, it is natural

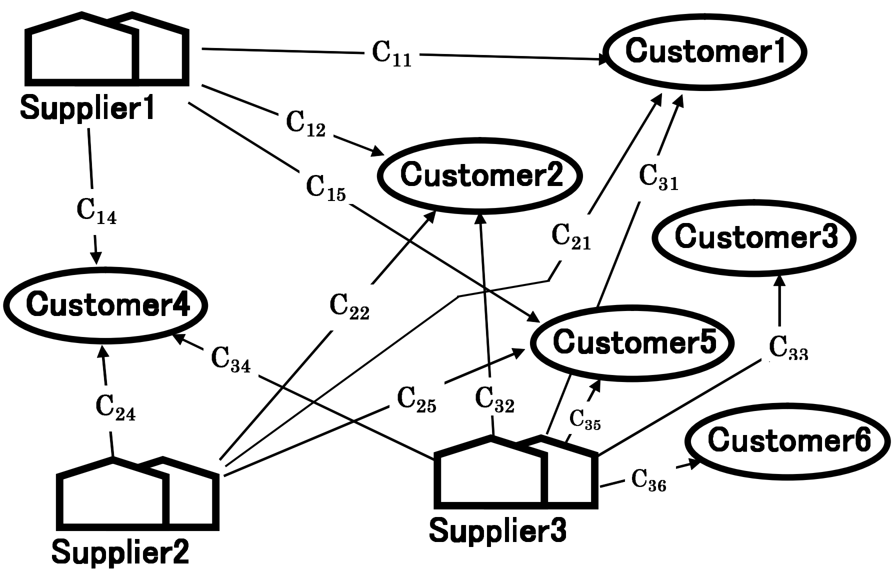

Figure 2.1: M Supplier N Customer Model with M=3 and N=6

to assume that there exists a price upper bound U. In our model each supplier has to offer a uniform price upon delivery to all customers, denoted by $\pi_i, i \in \mathcal{M}$. Supplier i may offer the service to customer j only when it results in a positive return to do so. In other words, a supplier i may offer the service to customer j only if $c_{ij} < \pi_i$. In order to avoid trivial cases, we assume that each supplier can offer the service to at least one of the customers so that

$$\min_{j \in \mathcal{N}} c_{ij} \stackrel{def}{=} c_i < \pi_i \leq U \qquad for\ all \qquad i \in \mathcal{M} \quad . \tag{2.1}$$

Let D_j be the total demand of customer j. We assume that the production capacity of each supplier is large enough to cover the entire demand $\sum_{j \in \mathcal{N}} D_j$. If there exists only one supplier who offers the lowest price to customer j, the supplier monopolizes the demand of customer j. Should k different suppliers offer the same lowest price to customer j, then each of such suppliers would sell D_j/k to customer j .

In what follows, we describe an M person game defined on the strategy set S where

$$S = \Pi_{i=1}^{M} S_i \quad ; \quad S_i = [c_i, U] \quad i \in \mathcal{M} \quad .$$

Given $\underline{\pi}^T \stackrel{def}{=} [\pi_1, \pi_2, \cdots, \pi_M] \in S$, let $P_i(\underline{\pi})$ be the payoff function of supplier i . In order to define the function specifically, the following index sets are introduced. Given $\underline{\pi}^T \in S$, we define for $j \in \mathcal{N}$ the set of suppliers not available to provide service to customer j by

$$NA_j(\underline{\pi}) = \{m \in \mathcal{M} \mid \pi_m \leq c_{mj}\} \quad . \tag{2.2}$$

We also define for $i \in \mathcal{M}$,

$$\begin{aligned} LE_i(\underline{\pi}) &= \{m \in \mathcal{M} \mid \pi_i > \pi_m\} \quad ; & (2.3) \\ LA_i(\underline{\pi}) &= \{m \in \mathcal{M} \mid \pi_i < \pi_m\} \quad ; \\ EQ_i(\underline{\pi}) &= \{m \in \mathcal{M} \mid \pi_i = \pi_m\} \quad . & (2.4) \end{aligned}$$

It should be noted that $NA_j(\underline{\pi})$ consists of those suppliers who cannot offer the service to customer j because a positive return does not result from doing so, and $LE_i(\underline{\pi})$ is the set of those suppliers who would eliminate supplier i if they happen to offer the service to the same customer. Similarly $LA_i(\underline{\pi})$ consists of those suppliers who would be eliminated by supplier i. With those suppliers in $EQ_i(\underline{\pi})$, supplier i would split the demand equally, should they offer the lowest price to the same customer simultaneously.

Let $W_{ij}(\underline{\pi})$ be the set of suppliers who would offer the service to customer j together with supplier i . Using the above notation, $W_{ij}(\underline{\pi})$ can be written as

$$W_{ij}(\underline{\pi}) = \begin{cases} \{m \in \mathcal{M} \mid m \in EQ_i(\underline{\pi}) \cap \overline{NA_j}(\underline{\pi})\} & \text{if} \quad \overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset \\ & \qquad \text{and} \quad i \in \overline{NA_j}(\underline{\pi}) \\ \emptyset & \text{if} \quad \overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi}) \neq \emptyset \\ & \qquad \text{or} \quad i \in NA_j(\underline{\pi}) \end{cases} \tag{2.5}$$

where $\overline{NA_j}(\underline{\pi}) \stackrel{def}{=} \mathcal{M} \setminus NA_j(\underline{\pi})$. It should be noted that $W_{ij}(\underline{\pi}) = \emptyset$ if either supplier i cannot gain positive profit by offering service to customer j so that $i \in NA_j(\underline{\pi})$, or supplier i does not offer the lowest price to customer j. In the latter case, there exists $m' \in \mathcal{M}$ satisfying

$m' \in LE_i(\underline{\pi})$ and $m' \in \overline{NA_j}(\underline{\pi})$, and hence $\overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi}) \neq \emptyset$. When supplier i offer the lowest price to customer j, one sees that $\overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset$ and $i \in \overline{NA_j}(\underline{\pi})$ so that $i \in W_{ij}(\underline{\pi})$.

Based on these index sets, the following index functions are now introduced.

$$I_{ij}(\underline{\pi}) = \begin{cases} 1 & if \quad |W_{ij}(\underline{\pi})| = 1 \\ 0 & else \end{cases} \tag{2.6}$$

$$J_{ij}(\underline{\pi}) = \begin{cases} 1 & if \quad |W_{ij}(\underline{\pi})| > 1 \\ 0 & else \end{cases} \tag{2.7}$$

where $|A|$ denotes the cardinality of a set A. It should be noted from (2.5) that if $W_{ij}(\underline{\pi}) \neq \emptyset$ then $i \in \overline{NA_j}(\underline{\pi})$ so that $i \in W_{ij}(\underline{\pi})$. Hence if $I_{ij}(\underline{\pi}) = 1$, then $W_{ij}(\underline{\pi}) = \{i\}$, i.e. $I_{ij}(\underline{\pi}) = 1$ if and only if supplier i exclusively provides the service to customer j. Similarly, one has $J_{ij}(\underline{\pi}) = 1$ if and only if supplier i jointly provides the service to customer j with another suppliers. When a price vector $\underline{\pi} = [\pi_1, \cdots, \pi_M]^T$ is given, the payoff function $P_i(\underline{\pi})$ of supplier i is then given by

$$P_i(\underline{\pi}) = \sum_{j \in \mathcal{N}} D_j(\pi_i - c_{ij}) \left\{ I_{ij}(\underline{\pi}) + \frac{J_{ij}(\underline{\pi})}{|W_{ij}(\underline{\pi})|} \right\}, \ i \in \mathcal{M} \tag{2.8}$$

where $J_{ij}(\underline{\pi})/|W_{ij}(\underline{\pi})| \stackrel{def}{=} 0$ if $J_{ij}(\underline{\pi}) = 0$ and $W_{ij}(\underline{\pi}) = \emptyset$.

The following conventional notion in game theory is employed. Given $\underline{\pi} = [\pi_1, \cdots, \pi_M]^T$, we write $\underline{\pi}_{\setminus i} = [\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_M]^T$ and $(a_i, \underline{\pi}_{\setminus i}) = [\pi_1, \cdots, \pi_{i-1}, a_i, \pi_{i+1}, \cdots, \pi_M]^T$.

Definition 2.1

a) For $i \in \mathcal{M}, \pi_i^*$ is a best reply against $\underline{\pi}_{\setminus i}$ if $P_i(\pi_i^*, \underline{\pi}_{\setminus i}) = \max_{\pi_i \in S_i}[P_i(\pi_i, \underline{\pi}_{\setminus i})]$.

b) For $i \in \mathcal{M}, B_i(\underline{\pi}_{\setminus i}) = \{\pi_i^* \mid \pi_i^*$ is a best reply against $\underline{\pi}_{\setminus i}\}$ is called the set of best replies of supplier i against $\underline{\pi}_{\setminus i}$.

c)The best reply correspondence $B : S \to S$ is defined as $B(\underline{\pi}) = \prod_{i=1}^M B_i(\underline{\pi}_{\setminus i})$.

d)$\underline{\pi}^*$ is a Nash equilibrium, denoted by $\underline{\pi}^* \in \mathcal{NE}$, if and only if $\underline{\pi}^* \in B(\underline{\pi}^*)$.

Of interest is to see whether one or more than one Nash equilibrium points exist, i.e. $\mathcal{NE} \neq \emptyset$. In the next section, a necessary and sufficient condition is given under which $\mathcal{NE} \neq \emptyset$. An example is provided for illustrating this case. This condition is rather restrictive however and normally one has $\mathcal{NE} = \emptyset$. Section 4 is devoted to exhibit typical strategies of suppliers, when $\mathcal{NE} = \emptyset$, through a numerical example.

3. A Necessary and Sufficient Condition for Existence of Nash Equilibrium

In this section we prove a necessary and sufficient condition under which Nash Equilibrium points exist for the model defined in the previous section. A few preliminary lemmas are needed. The first lemma states that, if supplier i is the only supplier to service customer j when all suppliers offer the maximum price U, then supplier i remains to be the unique supplier to customer j for any price vector $\underline{\pi}$ as long as supplier i could generate a positive return from π_i.

Lemma 3.1 *Let $\underline{U} = [U, \cdots, U]$. If $\overline{NA_j}(\underline{U}) = \{i\}$ for some $j \in \mathcal{N}$, then, for any price vector $\underline{\pi}$ satisfying $i \in \overline{NA_j}(\underline{\pi})$, one has*

$$W_{ij}(\underline{\pi}) = \{i\} \quad .$$

Proof: From (2.2), it can be readily seen that $\overline{NA_j}(\underline{\pi}) \subset \overline{NA_j}(\underline{U})$ for any $\underline{\pi} \in S$. Since $\overline{NA_j}(\underline{U}) = \{i\}$ and $i \in \overline{NA_j}(\underline{\pi})$, this then implies that $\overline{NA_j}(\underline{\pi}) = \{i\}$. Hence, one has $\overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset$ so that $W_{ij}(\underline{\pi}) = \{i\}$ from (2.5). □

The next lemma states that if $\underline{\pi} \neq \underline{U}$, then at least one supplier could serve at least one customer with price less than the upper limit U.

Lemma 3.2 *If $\underline{\pi}$ satisfies (2.1) and $\underline{\pi} \neq \underline{U}$, then there exists at least one pair of supplier i and customer j such that $|W_{ij}(\underline{\pi})| \geq 1$ and $\pi_i < U$.*

Proof: Since $\underline{\pi} \neq \underline{U}$, there exists at least one i satisfying $\pi_i < U$. Let j be such that $c_{ij} = \min_{n \in \mathcal{N}} c_{in}$. Then one has $c_{ij} < \pi_i$ from (2.1) so that $i \in \overline{NA_j}(\underline{\pi})$. We consider the following two cases.

Case1: $\overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi}) = \emptyset$

Since $i \in \overline{NA_j}(\underline{\pi})$, one has $i \in W_{ij}(\underline{\pi})$ from (2.5) and hence $|W_{ij}(\underline{\pi})| \geq 1$.

Case2: $\overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi}) \neq \emptyset$

Let i' be such that $\pi_{i'} = \min_{m \in \overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi})} \pi_m$. Then $\overline{NA_j}(\underline{\pi}) \cap LE_{i'}(\underline{\pi}) = \emptyset$. One also sees that $i' \in \overline{NA_j}(\underline{\pi}) \cap LE_i(\underline{\pi})$ implies $i' \in \overline{NA_j}(\underline{\pi})$. These observations together with (2.5) imply that $i' \in W_{i'j}(\underline{\pi})$ and $|W_{i'j}(\underline{\pi})| \geq 1$. □

The third and last lemma implies that if supplier i is the unique supplier for customer j, then supplier i could increase its price, while remaining to be the single service provider to customer j, as long as the increased price is less than the nearest price of the competitors.

Lemma 3.3 *For $\underline{\pi}^* = [\pi_1^*, \pi_2^*, \cdots, \pi_M^*]$ with $\pi_i^* < U$ for some $i \in \mathcal{M}$, let $\Delta > 0$ be sufficiently small so that*

$$\pi_i^\sharp \overset{def}{=} \pi_i^* + \Delta < \min_{m \in LA_i(\underline{\pi}^*)} \{\pi_m^*\} \quad . \tag{3.1}$$

Then the following statements hold true for all $j \in \mathcal{N}$.

1) $|W_{ij}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)| \leq 1$

2) *If* $|W_{ij}(\underline{\pi}^*)| = 1$, *then* $|W_{ij}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)| = 1$

Proof: We first prove part 1) by contraposition. Suppose $|W_{ij}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)| \geq 2$ for some j. Then from the definition of $W_{ij}(\underline{\pi})$ in (2.5), one has $|EQ_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)| \geq 2$. From (3.1) it is clear that

$$LA_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) = LA_i(\underline{\pi}^*) \quad . \tag{3.2}$$

Since $\Delta > 0$, it follows from (2.3) and (2.4) that $LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) = LE_i(\underline{\pi}^*) \cup (EQ_i(\underline{\pi}^*) \setminus \{i\})$. From this and (3.2), it is readily seen that

$$\begin{aligned} EQ_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) &= \mathcal{M} \setminus [LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \cup LA_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)] \\ &= \mathcal{M} \setminus [LE_i(\underline{\pi}^*) \cup (EQ_i(\underline{\pi}^*) \setminus \{i\}) \cup LA_i(\underline{\pi}^*)] \\ &= \mathcal{M} \setminus [(LE_i(\underline{\pi}^*) \cup EQ_i(\underline{\pi}^*) \cup LA_i(\underline{\pi}^*)) \setminus (\overline{LE_i}(\underline{\pi}^*) \cap \{i\} \cap \overline{LA_i}(\underline{\pi}^*))] \\ &= \mathcal{M} \setminus (\mathcal{M} \setminus \{i\}) = \{i\} \quad , \end{aligned}$$

which contradicts to $|EQ_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)| \geq 2$.

For part 2), suppose $|W_{ij}(\underline{\pi}^*)| = 1$ and $|W_{ij}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)| \neq 1$ for some $j \in \mathcal{N}$. Then from part 1), one has $|W_{ij}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)| = 0$, and hence $W_{ij}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) = \emptyset$. Accordingly from (2.5), one has either

$$\overline{NA_j}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \cap LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \neq \emptyset \quad \text{or} \quad i \in NA_j(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \quad . \tag{3.3}$$

Similarly from (2.5), since $W_{ij}(\underline{\pi}^*) \neq \emptyset$ from the assumption, one has

$$\overline{NA_j}(\underline{\pi}^*) \cap LE_i(\underline{\pi}^*) = \emptyset \ ; \quad \text{and} \tag{3.4}$$

$$i \in \overline{NA}_j(\pi^*) \qquad . \tag{3.5}$$

From (2.2) and (3.5), it is clear that

$$\overline{NA_j}(\underline{\pi}^*) \subset \overline{NA_j}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \quad . \tag{3.6}$$

Hence one has $i \in \overline{NA_j}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)$ from (3.5) and (3.6). This, in turn, implies from (3.3) that

$$\overline{NA_j}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \cap LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \neq \emptyset. \tag{3.7}$$

It then follows from (3.4), (3.6) and (3.7) that

$$\begin{aligned} SE &\stackrel{def}{=} \overline{NA_j}(\underline{\pi}^*) \cap [LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \setminus LE_i(\underline{\pi}^*)] \\ &= \{\overline{NA_j}(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) \cap LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)\} \setminus \{\overline{NA_j}(\underline{\pi}^*) \cap LE_i(\underline{\pi}^*)\} \neq \emptyset \quad . \end{aligned} \tag{3.8}$$

Suppose $i' \in SE$. It is clear from (3.8), that $i' \in LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)$ and hence $i' \notin LA_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*) = LA(\underline{\pi}^*)$ from (3.2). Since $i' \in SE$, one sees that $i' \notin LE_i(\underline{\pi}^*)$. Consequently, one has $i' \in EQ_i(\underline{\pi}^*)$. Thus $i' \in (EQ_i(\underline{\pi}^*) \cap \overline{NA_j}(\underline{\pi}^*))$ so that $i' \in W_{ij}(\underline{\pi}^*)$. Since $i' \in LE_i(\pi_i^\sharp, \underline{\pi}_{\setminus i}^*)$, one has $i' \neq i$, so that $W_{ij}(\underline{\pi}^*) \supset \{i, i'\}$ and hence $|W_{ij}(\underline{\pi}^*)| \geq 2$, which contradicts to $|W_{ij}(\underline{\pi}^*)| = 1$, completing the proof. □

We are now in a position to prove the main theorem of this section.

Theorem 3.4 *For the game defined in Section 2, the following two statements hold true.*

1) $\mathcal{NE} \neq \emptyset$ *if and only if* $|\overline{NA_j}(\underline{U})| \leq 1$ *for all* $j \in \mathcal{N}$
2) *If* $\mathcal{NE} \neq \emptyset$, *then* $\mathcal{NE} = \{\underline{U}\}$

Proof: We first prove part 2) by contraposition. Suppose $\underline{\pi}^* \in \mathcal{NE}$ and $\underline{U} \neq \underline{\pi}^*$. From Lemma 3.2, there exists $\hat{i} \in \mathcal{M}$ and $\hat{j} \in \mathcal{N}$ such that $|W_{\hat{i}\hat{j}}(\underline{\pi}^*)| \geq 1$ and $\pi_{\hat{i}}^* < U$. We consider the following two cases.

Case1: $J_{\hat{i}j}(\underline{\pi}^*) = 0$ for all $j \in \mathcal{M}$

From the definition of $P_i(\underline{\pi})$ in (2.8), one sees that

$$P_{\hat{i}}(\underline{\pi}^*) = \sum_{j \in \mathcal{N}} D_j(\pi_{\hat{i}}^* - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) \qquad . \tag{3.9}$$

Let $\pi_{\hat{j}}^\sharp$ be as in (3.1). Then from 1) of Lemma 3.3, one has $J_{\hat{i}j}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) = 0$ for all $j \in \mathcal{M}$. It then follows from this and (2.8) that

$$P_{\hat{i}}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) = \sum_{j \in \mathcal{N}} D_j(\pi_{\hat{i}}^* + \Delta - c_{\hat{i}j}) I_{\hat{i}j}(\pi_{\hat{i}}^\sharp, \underline{\pi}_{\setminus\hat{i}}) \quad . \tag{3.10}$$

From 2) of lemma 3.3, $I_{ij}(\underline{\pi}^*) = 1$ implies $I_{\hat{i}j}(\pi^\sharp_{\hat{i}}, \underline{\pi}_{\backslash\hat{i}}) = 1$ so that $I_{\hat{i}j}(\pi^\sharp_{\hat{i}}, \underline{\pi}_{\backslash\hat{i}}) - I_{\hat{i}j}(\underline{\pi}^*) \geq 0$ for all $j \in \mathcal{N}$. Since $|W_{\hat{i}\hat{j}}(\underline{\pi}^*)| \geq 1$ and $J_{\hat{i}\hat{j}}(\underline{\pi}^*) = 0$, it is clear that $I_{\hat{i}\hat{j}}(\underline{\pi}^*) = 1$. These observations together with (3.9) and (3.10) then yield that

$$\begin{aligned}
&P_{\hat{i}}(\pi^\sharp_{\hat{i}}, \underline{\pi}_{\backslash\hat{i}}) - P_{\hat{i}}(\underline{\pi}^*) \\
&\quad = \sum_{j\in\mathcal{N}} \Big[D_j(\pi^*_{\hat{i}} + \Delta - c_{\hat{i}j}) I_{\hat{i}j}(\pi^\sharp_{\hat{i}}, \underline{\pi}_{\backslash\hat{i}}) - D_j(\pi^*_{\hat{i}} - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) \Big] \\
&\qquad = \sum_{j\in\mathcal{N}} \Big[D_j(\pi^*_{\hat{i}} + \Delta - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) - D_j(\pi^*_{\hat{i}} - c_{\hat{i}j}) I_{\hat{i}j}(\underline{\pi}^*) \\
&\qquad\qquad + D_j(\pi^*_{\hat{i}} + \Delta - c_{\hat{i}j}) \{ I_{\hat{i}j}(\pi^\sharp_{\hat{i}}, \underline{\pi}_{\backslash\hat{i}}) - I_{\hat{i}j}(\underline{\pi}^*) \} \Big] \\
&\qquad\qquad \geq \sum_{j\in\mathcal{N}} D_j \Delta I_{\hat{i}j}(\underline{\pi}^*) \ \geq \ D_{\hat{j}} \Delta I_{\hat{i}\hat{j}}(\underline{\pi}^*) \ > 0 \quad ,
\end{aligned}$$

which contradicts to $\underline{\pi}^* \in \mathcal{NE}$.

Case2: $J_{\hat{i}j}(\underline{\pi}^*) = 1$ for some $j \in \mathcal{N}$

Since $\pi^*_{\hat{i}} > c_{\hat{i}n}$ for any customer n supplied by supplier $\hat{i}$, and $\pi^*_{\hat{i}} > \pi^*_m$ for any $m \in LE_{\hat{i}}(\underline{\pi}^*)$, one can choose $\Delta > 0$ sufficiently small so that $\pi^\dagger_{\hat{i}} = \pi^*_{\hat{i}} - \Delta$ satisfies

$$\max \left[\max_{n\in\{n:(I_{\hat{i}n}(\underline{\pi}^*)=1)\vee(J_{\hat{i}n}(\underline{\pi}^*)=1)\}} \{c_{\hat{i}n}\} \quad , \qquad \max_{m\in LE_{\hat{i}}(\underline{\pi}^*)} \{\pi^*_m\} \right] < \pi^\dagger_{\hat{i}} \quad , \tag{3.11}$$

where the second maximum in (3.11) is ignored if $LE_{\hat{i}}(\underline{\pi}^*) = \emptyset$. One then sees that $LE_{\hat{i}}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = LE_{\hat{i}}(\underline{\pi}^*)$, $EQ_{\hat{i}}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = \{\hat{i}\}$ and $LA_{\hat{i}}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = LA_{\hat{i}}(\underline{\pi}^*) \cup \big(EQ_{\hat{i}}(\underline{\pi}^*) \setminus \{\hat{i}\}\big)$. From (2.6) and (2.7), these observations imply that the following statements hold true for all $j \in \mathcal{N}$.

$$a) \quad \text{If} \ \ I_{\hat{i}j}(\underline{\pi}^*) = 1 \quad \text{then} \quad I_{\hat{i}j}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = 1 \tag{3.12}$$

$$b) \quad \text{If} \ \ J_{\hat{i}j}(\underline{\pi}^*) = 1 \quad \text{then} \quad I_{\hat{i}j}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = 1 \ \text{and} \quad J_{\hat{i}j}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = 0 \tag{3.13}$$

$$c) \quad \text{If} \ \ \big[I_{\hat{i}j}(\underline{\pi}^*) = 0 \wedge J_{\hat{i}j}(\underline{\pi}^*) = 0\big], \text{then} \ \big[I_{\hat{i}j}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = 0 \wedge J_{\hat{i}j}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) = 0\big] \tag{3.14}$$

From the definition of $P_{\hat{i}}(\underline{\pi})$ in (2.8) together with (3.12), (3.13) and (3.14), one then sees that

$$\begin{aligned}
P_{\hat{i}}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) - P_{\hat{i}}(\underline{\pi}^*) &= \sum_{j\in\mathcal{N}} D_j(\pi^\dagger_{\hat{i}} - c_{\hat{i}j}) I_{\hat{i}j}(\pi^\dagger_{\hat{i}}, \underline{\pi}^*_{\backslash\hat{i}}) - \sum_{j\in\mathcal{N}} D_j(\pi^\dagger_{\hat{i}} - c_{\hat{i}j}) \Big[I_{\hat{i}j}(\underline{\pi}^*) + \frac{J_{\hat{i}j}(\underline{\pi}^*)}{|W_{\hat{i}j}(\underline{\pi}^*)|} \Big] \\
&\quad + \sum_{j\in\mathcal{N}} D_j(\pi^\dagger_{\hat{i}} - c_{\hat{i}j}) \Big[I_{\hat{i}j}(\underline{\pi}^*) + \frac{J_{\hat{i}j}(\underline{\pi}^*)}{|W_{\hat{i}j}(\underline{\pi}^*)|} \Big] - \sum_{j\in\mathcal{N}} D_j(\pi^*_{\hat{i}} - c_{\hat{i}j}) \Big[I_{\hat{i}j}(\underline{\pi}^*) + \frac{J_{\hat{i}j}(\underline{\pi}^*)}{|W_{\hat{i}j}(\underline{\pi}^*)|} \Big] \\
&= \sum_{j\in\mathcal{N}} D_j(\pi^\dagger_{\hat{i}} - c_{\hat{i}j}) \Big[1 - \frac{1}{|W_{\hat{i}j}(\underline{\pi}^*)|} \Big] J_{\hat{i}j}(\underline{\pi}^*) - \Delta \sum_{j\in\mathcal{N}} D_j \Big[I_{\hat{i}j}(\underline{\pi}^*) + \frac{J_{\hat{i}j}(\underline{\pi}^*)}{|W_{\hat{i}j}(\underline{\pi}^*)|} \Big] \\
&= \sum_{j\in\mathcal{N}} D_j(\pi^*_{\hat{i}} - c_{\hat{i}j}) \Big[1 - \frac{1}{|W_{\hat{i}j}(\underline{\pi}^*)|} \Big] J_{\hat{i}j}(\underline{\pi}^*) - \Delta \sum_{j\in\mathcal{N}} D_j \{ I_{\hat{i}j}(\underline{\pi}^*) + J_{\hat{i}j}(\underline{\pi}^*) \} \\
&\geq D_{\hat{j}}(\pi^*_{\hat{i}} - c_{\hat{i}\hat{j}}) \Big[1 - \frac{1}{|W_{\hat{i}\hat{j}}(\underline{\pi}^*)|} \Big] - \Delta \sum_{j\in\mathcal{N}} D_j \{ I_{\hat{i}j}(\underline{\pi}^*) + J_{\hat{i}j}(\underline{\pi}^*) \} \quad .
\end{aligned}$$

Since the first component in the last term is positive, one can choose Δ sufficiently small so that $P_{\hat{i}}(\pi_{\hat{i}}^{\dagger}, \underline{\pi}_{\setminus \hat{i}}^{*}) > P_{\hat{i}}(\underline{\pi}^{*})$, which contradicts to $\underline{\pi}^{*} \in \mathcal{NE}$, completing the proof for part 2).

We next prove "if part" of part 1). If $|\overline{NA_j}(\underline{U})| \leq 1$ for all $j \in \mathcal{N}$, then from Lemma 3.1, one has $|W_{ij}(\underline{\pi})| \leq 1$ for all $\underline{\pi} \in S$ and $i \in \mathcal{M}$. Hence, for all $\underline{\pi} \in S$ and $i \in \mathcal{M}$, one has $P_i(\underline{\pi}) = \sum_{j \in \mathcal{N}} D_j(\underline{\pi} - c_{ij}) I_{ij}(\underline{\pi})$. It then follows that

$$P_i(\underline{U}) - P_i(\underline{\pi}) = \sum_{j \in \mathcal{N}} D_j(U - c_{ij})\{I_{ij}(\underline{U}) - I_{ij}(\underline{\pi})\} + \sum_{j \in \mathcal{N}} D_j(U - \pi_i) I_{ij}(\underline{\pi}) \ . \tag{3.15}$$

If $U \leq c_{ij}$ for some $i \in \mathcal{M}$, then $W_{ij}(\underline{\pi}) = \emptyset$, and hence $I_{ij}(\underline{\pi}) = 0$ for all $\underline{\pi} \in S$. This then implies that the payoff difference in (3.15) is non-negative for all $\underline{\pi} \in S$. If $U > c_{ij}$ (and hence $NA_j(\underline{U}) = \{i\}$) for some $i \in \mathcal{M}$, then for any price vector $\underline{\pi}$ with $\pi_i > c_{ij}$ so that $i \in NA_j(\underline{\pi})$, one has $NA_j(\underline{\pi}) = \{i\}$ from Lemma 3.1. In this case, $I_{ij}(\underline{U}) = 1$ and $I_{ij}(\underline{\pi}) = 1$ and again the payoff difference in (3.15) is positive for all $\underline{\pi} \in S$. It then follows that $U \in B_i(\underline{U}_{\setminus i})$ for all $i \in \mathcal{M}$, hence one has $\underline{U} \in \mathcal{NE}$, proving "if part".

For "only if part", suppose $\mathcal{NE} \neq \emptyset$ and $|\overline{NA_{\hat{j}}}(\underline{U})| \geq 2$ for some $\hat{j} \in \mathcal{N}$. From part 2) of this theorem one has $\mathcal{NE} = \{\underline{U}\}$. To emphasize this, we write $\underline{\pi}^{*} = \underline{U}$. Let $\hat{i}, \hat{i}' \in \overline{NA_{\hat{j}}}(\underline{\pi}^{*})$. Since $LE_i(\underline{\pi}^{*}) = \emptyset$ from (2.3), the definition of $W_{ij}(\underline{\pi})$ in (2.5) implies $\hat{i} \in W_{\hat{i}\hat{j}}(\underline{\pi}^{*})$. Since $\pi_{\hat{i}} = \pi_{\hat{i}'} = U$, it is clear that $\hat{i}' \in EQ_{\hat{i}}(\underline{\pi}^{*})$ thus $\hat{i}' \in W_{\hat{i}\hat{j}}(\underline{\pi}^{*})$, so that $J_{\hat{i}\hat{j}}(\underline{\pi}^{*}) = 1$. Let $\pi_i^{\dagger} = \pi_i^{*} - \Delta$ for sufficiently small Δ as in (3.11). Similarly as in the proof of Case2 of part 2), statements (3.12), (3.13) and (3.14) hold true. These together with the definition of $P_i(\underline{\pi})$ in (2.8) imply that

$$\begin{aligned} & P_{\hat{i}}(\pi_{\hat{i}}^{\dagger}, \underline{\pi}_{\setminus i}^{*}) - P_{\hat{i}}(\underline{\pi}^{*}) \\ & \quad \geq D_{\hat{j}}(\pi_{\hat{i}}^{*} - c_{\hat{i}\hat{j}})\Big[1 - \frac{1}{|W_{\hat{i}\hat{j}}(\underline{\pi}^{*})|}\Big] J_{\hat{i}\hat{j}}(\underline{\pi}^{*}) - \Delta \sum_{j \in \mathcal{N}} D_j \Big[I_{\hat{i}j}(\underline{\pi}^{*}) + J_{\hat{i}j}(\underline{\pi}^{*})\Big] \ . \end{aligned} \tag{3.16}$$

Since the first component in the last term in (3.16) is positive, one can choose Δ sufficiently small so that $P_{\hat{i}}(\pi_{\hat{i}}^{\dagger}, \underline{\pi}_{\setminus i}^{*}) > P_{\hat{i}}(\underline{\pi}^{*})$, which contradicts $\underline{\pi}^{*} \in \mathcal{NE}$, proving "only if part" of part 2) . □

From Theorem 3.4, one sees that $\underline{U}$ is the only candidate to be the Nash equilibrium point. If $\underline{U}$ is not Nash equilibrium, then this game has no equilibriums. The algorithm to determine whether this game has Nash equilibrium or not is quite simple, as presented below.

Algorithm

step1: Let $j = 1$

step2: Determine whether $|NA_j(\underline{\pi})| \geq 1$. If $|NA_j(\underline{\pi})| > 1$, then one concludes $\mathcal{NE} = \emptyset$. Else go to step 3

step3: $j = j + 1$. If $j > N$, Then $\mathcal{NE} \neq \emptyset$. Else go to step 2

In the remainder of this section, an example of Nash equilibrium is provided. We consider a case that there are three suppliers providing LNG by lorry tankers to six customers, who are middle-sized industrial users as depicted in Figure 2.1. It should be noted that, unlike usual city gas through pipeline networks, the transportation costs are considered to be marginal costs. Although the price and cost vary depending on the condition and demand pattern, for the sake of convenience, we suppose here $U = 60(\text{Yen/m}^3)$, and the values of c_{ij} are supposed to be as shown in Table 3.1. When the equilibrium is realized, the

Table 3.1: The value of c_{ij} when Nash equilibrium is realized

i \ j	1	2	3	4	5	6
1	55	45	100	61	70	100
2	70	61	100	48	66	100
3	65	63	58	100	20	20

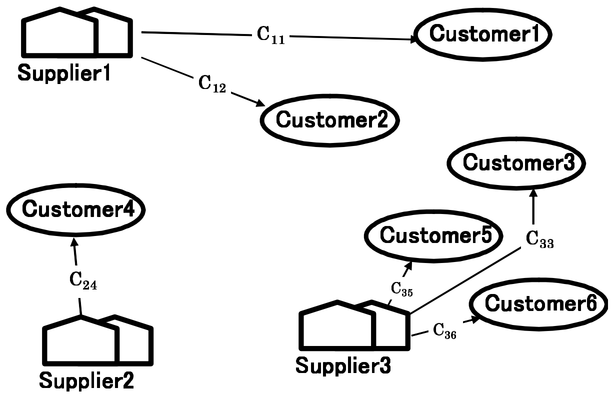

Figure 3.1: Nash equilibrium with 3 Supplier 6 Customer Model

resulting supplier-customer combinations are shown in Figure 3.1. In this case, the market is completely separated by the suppliers, where there is only one supplier for each customer. The rest of the suppliers cannot offer the customer since the cost is above the upperbound price. However, these situation is rather unnatural. In the next section we show the case of $\mathcal{NE} = \emptyset$ and illustrate how players may continue to behave forever in a cyclic manner in pursuit of maximizing their profit.

4. Cyclic Phenomenon for Case of Two Person Game

In this section, we illustrate typical strategies of suppliers, when $\mathcal{NE} = \emptyset$. We assume that there are two suppliers and three customers who are middle-sized industrial users recieving LNG by lorry tankers, where $U = 50\,(\text{Yen/m}^3)$, $D_1 = 100\,(\text{Mcm/y})$, $D_2 = 200\,(\text{Mcm/y})$ and $D_3 = 150\,(\text{Mcm/y})$ (Mcm/y=thousand cubic meter per year). The cost variables are given in Table 4.1. Theorem 3.4 shows that, if $\underline{U} \notin \mathcal{NE}$, this game has no Nash equilibriums. In

Table 4.1: The values of c_{ij} when $\mathcal{NE} = \emptyset$

i \ j	1	2	3
1	37	40	44
2	35	47	40

this example, each supplier tries to obtain the furthest customer demand by setting lower price than its competitor. This supplier acquires the new distant customer at the expense of losing profits of the existing near customers since each supplier must set the same delivery price to all customers. We show this situation through a numerical example as depicted in Figure 4.1. Since $|\overline{NA}_j(\underline{U})| = 2 > 1$ for all $j = 1, 2, 3$, one has $\mathcal{NE} = \emptyset$ from Theorem 3.4.

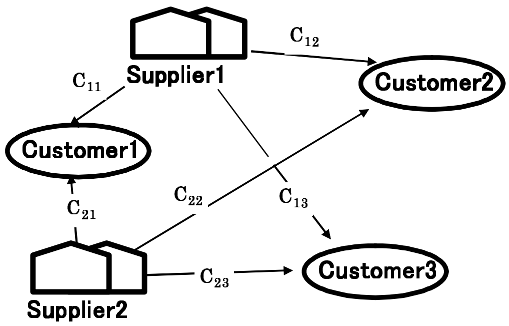

Figure 4.1: Non-Existence of Nash Equilibrium with 2 Supplier 3 Customer Model

Let $\underline{\pi} = \underline{U}$ be an initial price vector. For the sake of convenience, we discretize the strategy set so that each supplier can only take integer prices, and suppose each supplier changes its strategy in turn so as to maximize its profit. Table 4.2 and Figure 4.2 show the results of this simulation. In Figure 4.2, the cyclic behaviour of each supplier under the conditions of Table 4.1 is depicted. Here the initial price vector is $\underline{U} = [50\,,\,50]$, and the first action is taken by Supplier 1. At the first step, Supplier 1 tries to maximize its profit by setting lower price of 49 (Yen/m^3) than its competitor and eliminate Supplier 2. In return, Supplier 2 also takes a similar action by setting the price of 48 (Yen/m^3). This process continues several times. At the 4th step, Supplier 2 has no choice but to set the lower price of 46 (Yen/m^3) to secure Customers 1 and 3 at the expense of giving up Customer 2. Since it does not result in a positive return to provide service to Customer 2 at the price of 46 (Yen/m^3), Supplier 2 cannot offer the service to Customer 2. However it is better off to acquire the other customers even with low average earning per unit instead of losing all customers or splitting demands of all customers. At this point, Supplier 1 already monopolizes Customer 2, and it is in a position to enjoy the highest per-unit earning without losing the customer by setting the upper-bound price of 50 (Yen/m^3). And this cyclic process is repeated indefinitely.

Table 4.2: Each Supplier's behaviour when $\mathcal{NE} = \emptyset$

	Offering Price		Acquired Customers		Payoff Value (Thousand YEN)	
Step \ Supplier	1	2	1	2	1	2
Initial	50	50	1,2,3	1,2,3	2,100	1,800
1st	49	50	1,2,3	None	3,750	0
2nd	49	48	None	1,2,3	0	2,700
3rd	47	48	1,2,3	None	2,850	0
4th	47	46	2	1,3	1,400	2,000
5th	50	46	2	1,3	2,000	2,000
6th	50	49	None	1,2,3	0	3,150
7th	48	49	1,2,3	None	3,300	0
8th	48	47	2	1,3	1,600	2,250
9th	46	47	1,2,3	None	2,400	0
10th	46	45	2	1,3	1,200	1,750
11th	50	45	2	1,3	2,000	1,750
12th	50	49	None	1,2,3	0	3,150

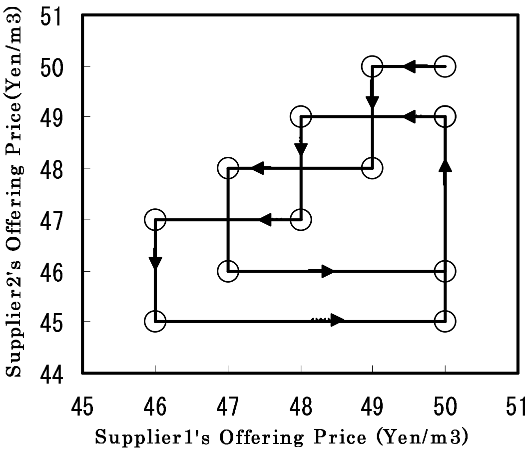

Figure 4.2: Cyclic Phenomenon with 2 Supplier and 3 Customer Model when $\mathcal{NE} = \emptyset$

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