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Abstract

The present paper shows that, for a submodular function f on a crossing family $\mathcal{F}$ of subsets of E with $E \in \mathcal{F}$, the polytope defined by

$$B(f) = \{x \mid x \in R^E, x(X) \leq f(X)\, (X \in \mathcal{F}), x(E)=f(E)\}$$

coincides with the base polytope of a submodular function on a distributive lattice. Based on this fact, we also show the relationship between the independent-flow problem considered by the author and the minimum cost flow problem considered by J. Edmonds and R. Giles.

1. Introduction

Let $\mathcal{F}$ be a crossing family of subsets of a finite set E with $E \in \mathcal{F}$ and f a submodular function on $\mathcal{F}$. Define a polytope $B(f)$ by

$$B(f) = \{x \mid x \in R^E, x(X) \leq f(X)\,(X \in \mathcal{F}), x(E)=f(E)\}, \qquad (1.1)$$

where R^E is the set of all functions (or vectors) from E to the set R of reals and for any $X \subseteq E$

$$x(X) = \sum_{e \in X} x(e). \qquad (1.2)$$

Suppose $B(f)$ is nonempty.

The main purpose of the present paper is to show that there exist a distributive lattice $\mathcal{D}$ formed by subsets of E with $E \in \mathcal{D}$ and a submodular function f^* on $\mathcal{D}$ such that the polytope $B(f^*)$ defined by

$$B(f^*) = \{x \mid x \in R^E, x(X) \leq f^*(X) \;\; (X \in \mathcal{D}), x(E)=f^*(E)\} \qquad (1.3)$$

coincides with $B(f)$ defined by (1.1).

Based on this fact, we also show the relationship between the independent-flow problem considered by the author [6] and the minimum cost flow problem considered by J. Edmonds and R. Giles [2].

The relationship between the independent-flow problem and the polymatroidal flow problem of R. Hassin [7] and E. L. Lawler and C. V. Martel [10] has been examined by U. Zimmermann [11]. Recently, A. Frank [3] has considered the minimum cost flow problem of Edmonds and Giles and proposed a solution algorithm for it.

2. Definitions and Preliminaries

Let E be a finite set. We denote the cardinality of E by $|E|$. For a collection of subsets X_i $(i \in I)$ of E, we adopt the notations $\{X_i \mid i \in I\}$ for a set and $(X_i \mid i \in I)$ for a family of subsets X_i $(i \in I)$ of E. We use set-theoretical notations for families as well. For example, "$Y \in (X_i \mid i \in I)$" means "for some $i \in I$, $Y = X_i$". Given two families $\mathcal{G}_1 = (X_i \mid i \in I)$ and $\mathcal{G}_2 = (Y_j \mid j \in J)$, the direct sum of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the family

$$\mathcal{G}_3 = (Z_k \mid k \in I+J), \qquad (2.1)$$

where $I+J = \{(i,1) \mid i \in I\} \cup \{(j,2) \mid j \in J\}$ (the direct sum of I and J) and $Z_k = X_i$ (if $k = (i,1)$ and $i \in I$) and $Z_k = Y_j$ (if $k = (j,2)$ and $j \in J$).

For any $X, Y \subseteq E$, we say that X and Y cross if $X \cap Y$, $X \cap (E-Y)$, $(E-X) \cap Y$ and $(E-X) \cap (E-Y)$ are nonempty. A family $\mathcal{F}$ of subsets of E is called a crossing family if, for any $X, Y \in \mathcal{F}$ which cross, we have $X \cup Y \in \mathcal{F}$ and $X \cap Y \in \mathcal{F}$. A family $\mathcal{F}$ of subsets of E is a crossing family if and only if for any $X, Y \in \mathcal{F}$ with $X \cup Y \neq E$ and $X \cap Y \neq \emptyset$ we have $X \cup Y$, $X \cap Y \in \mathcal{F}$. If, for all $X, Y \in \mathcal{F}$, X and Y do not cross, then $\mathcal{F}$ is called a cross-free family.

We say X and Y intersect if $X \cap Y \neq \emptyset$. A family $\mathcal{F}$ of subsets of E is called an intersecting family if for any $X, Y \in \mathcal{F}$ which intersect we have $X \cup Y$, $X \cap Y \in \mathcal{F}$. By definition an intersecting family is a crossing family.

Let $\mathcal{F}$ be a crossing (or intersecting) family of subsets of E. A function f from $\mathcal{F}$ to the set R of reals is called a submodular function on $\mathcal{F}$ if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{2.2}$$

for any crossing (or intersecting) pair of $X, Y \in \mathcal{F}$.

A set $\{X_i \mid i \in I\}$ of subsets of E is a partition of E if $X_i \subseteq E$ $(i \in I)$ satisfy

$$X_i \neq \emptyset \quad (i \in I), \tag{2.3}$$

$$X_i \cap X_j = \emptyset \quad (i,j \in I,\ i \neq j), \tag{2.4}$$

$$\cup\{X_i \mid i \in I\} = E. \tag{2.5}$$

We call $\{E-X_i \mid i \in I\}$ a co-partition of E if $\{X_i \mid i \in I\}$ is a partition of E.

For a subset $X \subseteq E$ and a partition $\{Y_i \mid i \in I\}$ of $E - X$, we call $\{E-Y_i \mid i \in I\}$ a co-partition of $E - X$ augmented by X.

Let $\mathcal{D} \subseteq 2^E$ be a distributive lattice with respect to set inclusion and f a submodular function on $\mathcal{D}$. We call the pair $(\mathcal{D},f)$ a submodular system. The polytope $P(f)$ defined by

$$P(f) = \{x \mid x \in R^E, x(X) \leq f(X)\ (X \in \mathcal{D})\} \tag{2.6}$$

is called a submodular polytope associated with the submodular system $(\mathcal{D},f)$. Here, R^E is the set of functions (or vectors) from E to R and for any $X \subseteq E$ $x(X)$ is defined by (1.2). Moreover, when $E \in \mathcal{D}$, the polytope $B(f)$ defined by

$$B(f) = \{x \mid x \in R^E, x(X) \leq f(X)\ (X \in \mathcal{D}), x(E)=f(E)\} \tag{2.7}$$

is called a base polytope associated with $(\mathcal{D},f)$. The base polytope $B(f)$ is nonempty for any submodular system $(\mathcal{D},f)$ with $E \in \mathcal{D}$.

Now, we briefly survey the graph-theoretical notations and terminology which will be employed in the paper. Let $T = (V,A)$ be a graph with a vertex set V and an arc set A. Each arc $a \in A$ has an initial vertex (or a tail) denoted by $\partial^+ a$ and a terminal vertex (or a head) denoted by $\partial^- a$. When $\partial^+ a \neq \partial^- a$, we say the vertex $\partial^+ a$ and the vertex $\partial^- a$ are adjacent. For each vertex $v \in V$, we define

$$\delta^+ v = \{a \mid a \in A, \partial^+ a = v\}, \quad (2.8)$$

$$\delta^- v = \{a \mid a \in A, \partial^- a = v\}. \quad (2.9)$$

A path is a sequence $Q = (v_0, a_1, v_1, a_2, \ldots, a_k, v_k)$ of vertices v_i $(0 \leqq i \leqq k)$ and arcs a_j $(1 \leqq j \leqq k)$ for some $k \geqq 0$ such that for each $j = 1, 2, \ldots, k$

$$\{\partial^+ a_j, \partial^- a_j\} = \{v_{j-1}, v_j\}. \quad (2.10)$$

The vertices v_0 and v_k are, respectively, called the initial vertex and a terminal vertex of the path Q. Also we say the path Q connects the vertex v_0 with the vertex v_k. For $j = 1, 2, \ldots, k$, if

$$(\partial^+ a_j, \partial^- a_j) = (v_{j-1}, v_j), \quad (2.11)$$

then we say the arc a_j is positively oriented in the path Q and, otherwise, we say the arc a_j is negatively oriented in Q. If all the arcs in Q are positively oriented, Q is called a directed path.

If, for any vertices $u, v \in V$, there exists one and only one path which connects u with v in $T = (V,A)$, then we call T a tree. A vertex v in a tree T is called an end-vertex of T

if $|(\delta^+ v) \cup (\delta^- v)| = 1$. A tree T is a directed tree if, for each $v \in V$, $|\delta^- v| \leq 1$. By the definition of a directed tree, there exists a unique vertex v^* in a directed tree T such that $|\delta^- v| = 0$, which is called the root of T. For each vertex $v \in V - \{v^*\}$ there exists a unique directed path in T which connects the root v^* of T with v.

We shall use the following theorem due to Edmonds and Giles [1].

Theorem 2.1: Let $\mathcal{F} = (X_i \mid i \in I)$ be a cross-free family of subsets of E. Then $\mathcal{F}$ can be represented by a tree $T = (V,A)$ with a vertex set V and an arc set

$$A = \{a_i \mid i \in I\} \tag{2.12}$$

together with a family

$$\mathcal{P} = (P_v \mid v \in V) \tag{2.13}$$

of subsets of E, where the set of all the nonempty P_v's forms a partition of E and each $X_i \in \mathcal{F}$ $(i \in I)$ is expressed as

$$X_i = \cup\{P_v \mid v \in V, \text{ there exists a path } Q, \text{ in } T, \text{ connecting } v \text{ with } \partial^+ a_i \text{ such that } \partial^- a_i \text{ does not lie on } Q\}. \tag{2.14}$$

3. Polytopes Determined by Submodular Functions on Crossing Families

Let $\mathcal{F}$ be a crossing family of subsets of E and f a submodular function on $\mathcal{F}$. We suppose that $f(\emptyset) = 0$ if $\emptyset \in \mathcal{F}$.

Let us define a polytope $P(f)$ by

$$P(f) = \{x \mid x \in R^E, x(X) \leqq f(X) \ (X \in \mathcal{F})\}. \qquad (3.1)$$

Note that such polytope $P(f)$ is nonempty for every set function f with $f(\emptyset) \geqq 0$.

Furthermore, define

$$\hat{f}(Y) = \max\{x(Y) \mid x \in P(f)\} \qquad (3.2)$$

for any $Y \subseteqq E$. Then, by the LP duality theorem, we have

$$\hat{f}(Y) = \min\left\{\sum_{X \in \mathcal{F}} f(X)c(X) \;\middle|\; (3.4),(3.5)\right\}, \qquad (3.3)$$

where

$$\sum_{e \in X \in \mathcal{F}} c(X) = \delta(e|Y) \equiv \begin{cases} 1 & (e \in Y) \\ 0 & (e \notin Y) \end{cases} \quad (e \in E), \qquad (3.4)$$

$$c(X) \geqq 0 \qquad (X \in \mathcal{F}). \qquad (3.5)$$

Here, if there is no such $c(X)$ $(X \in \mathcal{F})$ that (3.4) and (3.5) are satisfied, we put $\hat{f}(Y) = +\infty$. We thus have a set function $\hat{f}$: $2^E \to R \cup \{+\infty\}$.

Since the minimum value of (3.3) can be attained by rational $c(X)$ $(X \in \mathcal{F})$, (3.3) - (3.5) can be rewritten as follows.

$$\hat{f}(Y) = \min\left\{\frac{1}{\mu(\mathcal{G},Y)} \sum_{i \in I} f(X_i) \;\middle|\; (3.7)\text{-}(3.9)\right\}, \qquad (3.6)$$

where

$$\mathcal{G} = (X_i \mid i \in I) \qquad (3.7)$$

with

$$X_i \in \mathcal{F}, \qquad X_i \subseteqq Y \quad (i \in I) \qquad (3.8)$$

and

$$|\{i \mid i \in I, e \in X_i\}| = \text{const.} \equiv \mu(\mathcal{G},Y) > 0 \ (e \in Y). \qquad (3.9)$$

Informally, conditions (3.7) - (3.9) mean that the family $\mathcal{G}$ is composed of (possibly repeated) elements of $\mathcal{F}$ which are subsets of Y and that elements (subsets of E) of $\mathcal{G}$ uniformly cover each $e \in Y$.

By the definition of the set function $\hat{f}: 2^E \to R \cup \{+\infty\}$, we have

$$\hat{f}(X) \leqq f(X) \quad (X \in \mathcal{F}), \tag{3.10}$$

$$P(f) = \{x \mid x \in R^E, x(X) \leqq \hat{f}(X)\,(X \subseteq E)\}. \tag{3.11}$$

It should be noted that if we decrease any $\hat{f}(X)$ $(X \subseteq E)$, (3.11) does not hold any more and that (3.1) - (3.11) are valid for any set function defined on any family of subsets of E. In the following, we simplify (3.6) - (3.9) by use of the property of the submodular function f on the crossing family $\mathcal{F}$.

From the submodularity of f and (3.6) - (3.9), we can restrict admissible families $\mathcal{G}$ in (3.6) - (3.9) to those which satisfy

(i) $\mathcal{G}$ is a cross-free family, (3.12)

(ii) $\emptyset \notin \mathcal{G}$. (3.13)

<u>Theorem 3.1</u>: Let $\hat{f}_1(E)$ and $\hat{f}_2(E)$ be defined by

$$\hat{f}_1(E) = \min\left\{\sum_{i \in I} f(X_i) \;\middle|\; \{X_i \mid i \in I\}\text{:a partition of } E,\; X_i \in \mathcal{F}(i \in I)\right\}, \tag{3.14}$$

$$\hat{f}_2(E) = \min\left\{\frac{1}{|I|-1}\sum_{i \in I} f(X_i) \;\middle|\; \{E-X_i \mid i \in I\}\text{:a partition of } E,\; X_i \in \mathcal{F}(i \in I), |I| \geqq 3\right\}. \tag{3.15}$$

Then we have

$$\hat{f}(E) = \min\{\hat{f}_1(E), \hat{f}_2(E)\}. \tag{3.16}$$

(Proof) Let us choose an arbitrary family $G = (X_i \mid i \in I)$ which satisfies (3.7) - (3.9), (3.12) and (3.13). We first suppose

$$E \notin G. \tag{3.17}$$

Since $G = (X_i \mid i \in I)$ is a cross-free family because of (3.12), G can be represented by a tree $T = (V, A)$ with a vertex set V and an arc set $A = \{a_i \mid i \in I\}$ together with a family

$$P = (P_v \mid v \in V) \tag{3.18}$$

of subsets of E, by Theorem 2.1. Here, by (3.13) and (3.17), for any end-vertex v of T,

$$P_v \neq \emptyset. \tag{3.19}$$

Now, from the assumption and (3.19) there exists at least one X_i such that $X_i \neq \emptyset, E$. Therefore, there exist distinct vertices v_1 and v_2 of T satisfying the conditions that

$$P_{v_1} \neq \emptyset, \qquad P_{v_2} \neq \emptyset \tag{3.20}$$

and that, for any vertex u ($\neq v_1, v_2$) lying on the unique path $Q(v_1, v_2)$ in T connecting v_1 with v_2,

$$P_u = \emptyset. \tag{3.21}$$

It follows from (3.9) with $Y = E$ that the number of positively oriented arcs in $Q(v_1, v_2)$ is equal to the number of negatively oriented arcs in $Q(v_1, v_2)$. (If this is not the case, the value of (3.9) for any $e_1 \in P_{v_1}$ can not be equal to that for any $e_2 \in P_{v_2}$.) Consequently, there is a vertex u ($\neq v_1, v_2$) on the path $Q(v_1, v_2)$

satisfying (3.21). Let $\hat{u}$ be the vertex on $Q(v_1,v_2)$ adjacent to v_1 and let $\{v_2,v_3,\ldots,v_k\}$ $(k \geqq 2)$ be the maximal set of vertices of T such that, for each $\ell = 2, 3, \ldots, k$, $P_{v_\ell} \neq \emptyset$, the vertex $\hat{u}$ lies on the unique path $Q(v_1,v_\ell)$ connecting v_1 with v_ℓ and any vertex u $(\neq v_1, v_\ell)$ lying on $Q(v_1,v_\ell)$ satisfies (3.21). Moreover, let $a_{j(1)} \in A$ be the arc connecting v_1 with $\hat{u}$ and for each $\ell = 2, 3, \ldots, k$ let $a_{j(\ell)} \in A$ be the arc on $Q(v_1,v_\ell)$ such that the orientation of the arc $a_{j(\ell)}$ is opposite to the orientation of the arc $a_{j(1)}$ and any arc a_i $(\neq a_{j(\ell)})$ between $a_{j(1)}$ and $a_{j(\ell)}$ on $Q(v_1,v_\ell)$ has the same orientation as $a_{j(1)}$. (Note that arcs $a_{j(\ell)}$ $(\ell = 2, 3, \ldots, k)$ are not necessarily distinct.)

Let us put

$$\Pi = \{X_{j(\ell)} \mid X_{j(\ell)} \in G, 1 \leqq \ell \leqq k\}. \tag{3.22}$$

Then

(i) if $\partial^+ a_{j(1)} = v_1$, Π is a partition of E

and

(ii) if $\partial^- a_{j(1)} = v_1$, Π is a co-partition of E.

Therefore, the family G contains a subfamily which forms a partition or a co-partition of E. Define

$$G' = (X_i \mid i \in I - \{j(\ell) \mid \ell = 1,2,\ldots,k\}). \tag{3.23}$$

If G' is not empty, then G' also satisfies (3.7) - (3.9), (3.12) and (3.13). Repeating the above mentioned argument, we see that G, satisfying (3.7) - (3.9) with $Y = E$, (3.12) and (3.13), can be expressed as a direct sum of families which form partitions and co-partitions of E. This is also true for the case where $E \in G$.

Since every subfamily of $\mathcal{G}$ forming a partition or a co-partition of E satisfies (3.7) - (3.9) with Y = E, (3.12) and (3.13), it follows that we can restrict admissible families $\mathcal{G}$ to those which form partitions and co-partitions and satisfy (3.7) - (3.9) with Y = E. Note that the co-partition $\{\emptyset\}$ of cardinality 1 is excluded by (3.13) and that co-partitions of cardinality 2 are also partitions. This completes the proof of Theorem 3.1. Q.E.D.

Theorem 3.2: For each $Y \subsetneqq E$,

$$\hat{f}(Y) = \min\left\{ \sum_{i \in I} f(X_i) \;\middle|\; \{X_i \mid i \in I\}\text{:a partition of } Y, \; X_i \in \mathcal{F}(i \in I) \right\}. \tag{3.24}$$

(Proof) By the same argument as in the case of Y = E in the proof of Theorem 3.1, we see that we can restrict admissible families $\mathcal{G}$ in (3.7) - (3.9) to those which form partitions and co-partitions of Y.

Let $\mathcal{P}^* = \{X_i \mid i \in I\}$ be a co-partition of Y with $X_i \in \mathcal{F}$ $(i \in I)$. Here, $|I| \geqq 3$. (If $|I| = 2$, $\mathcal{P}^*$ is also a partition of Y.) Then, for any distinct $X_i, X_j \in \mathcal{P}^*$, X_i and X_j cross. It follows from (3.12) that we need not consider co-partitions of Y.

Q.E.D.

It should be noted that, if f is integer-valued, $\hat{f}$ is also integer-valued except for $\hat{f}(E)$.

The following lemma will be used in the next section.

Lemma 3.3: For any $X, Y \subseteqq E$ with $X \cup Y \neq E$, if $\hat{f}(X), \hat{f}(Y) < +\infty$, then we have

$$\hat{f}(X) + \hat{f}(Y) \geqq \hat{f}(X \cup Y) + \hat{f}(X \cap Y). \tag{3.25}$$

(Proof) For some partition $\{X_i \mid i \in I\}$ of X and some partition $\{Y_j \mid j \in J\}$ of Y such that $X_i \in F$ $(i \in I)$ and $Y_j \in F$ $(j \in J)$,

$$\hat{f}(X) + \hat{f}(Y) = \sum_{i \in I} f(X_i) + \sum_{j \in J} f(Y_j) \tag{3.26}$$

due to Theorem 3.2. Let $G = (Z_k \mid k \in I+J)$ be the direct sum of families $(X_i \mid i \in I)$ and $(Y_j \mid j \in J)$. Since $X \cup Y \neq E$, for any Z_i, $Z_j \in G$ one of the following (i) - (iii) holds.

(i) Z_i and Z_j are disjoint, (3.27)

(ii) $Z_i \subseteqq Z_j$ or $Z_j \subseteqq Z_i$, (3.28)

(iii) Z_i and Z_j cross. (3.29)

If Z_i and Z_j cross, we replace Z_i and Z_j as

$$Z_i \leftarrow Z_i \cup Z_j, \qquad Z_j \leftarrow Z_i \cap Z_j. \tag{3.30}$$

Repeat this replacement until the obtained family $G = (Z_k \mid k \in I+J)$ becomes a cross-free family and satisfies (3.27) and (3.28) for any Z_i, $Z_j \in G$. We can easily see that this process terminates in finite steps and that the finally obtained G is the direct sum of families which form a partition $\{\hat{X}_i \mid i \in \hat{I}\}$ $(\hat{X}_i \in F$ $(i \in \hat{I}))$ of $X \cup Y$ and a partition $\{\hat{Y}_j \mid j \in \hat{J}\}$ $(\hat{Y}_j \in F$ $(j \in \hat{J}))$ of $X \cap Y$. Since the replacement (3.30) reduces the value of (3.26), we get

$$\hat{f}(X) + \hat{f}(Y) \geqq \sum_{i \in \hat{I}} f(\hat{X}_i) + \sum_{j \in \hat{J}} f(\hat{Y}_j)$$

$$\geqq \hat{f}(X \cup Y) + \hat{f}(X \cap Y). \qquad \text{Q.E.D.}$$

It should be noted that $A = \{X \mid X \subseteq E, f(X) < +\infty\}$ does not form a distributive lattice with respect to set inclusion.

4. From Submodular Functions on Crossing Families to Submodular Functions on Distributive lattices

In this section, we suppose that f is a submodular function on a crossing family F with $E \in F$ and that the polytope $B(f)$ defined by

$$B(f) = \{x \mid x \in R^E, x(X) \leq f(X) \ (X \in F), x(E) = f(E)\} \tag{4.1}$$

is nonempty, i.e., $\hat{f}(E)$ defined by (3.6) (or (3.16)) is given by

$$\hat{f}(E) = f(E). \tag{4.2}$$

For each $Y \subseteq E$, define

$$\hat{f}^*(Y) = \max\{x(Y) \mid x \in B(f)\}. \tag{4.3}$$

Then, by the LP duality theorem, we have

$$\hat{f}^*(Y) = \min\left\{\sum_{X \in F} f(X)c(X) \;\middle|\; (4.5),(4.6)\right\}, \tag{4.4}$$

where

$$\sum_{e \in X \in F} c(X) = \delta(e|Y) \quad (e \in E), \tag{4.5}$$

$$c(X) \geq 0 \quad (X \in F,\ X \neq E). \tag{4.6}$$

Similarly as (3.6), (4.4) can be rewritten as

$$\hat{f}^*(Y) = \min\left\{\frac{1}{\mu(G,Y) - \mu(G,E-Y)} \left[\sum_{i \in I} f(X_i) - \mu(G,E-Y)f(E)\right] \;\middle|\; (4.8) - (4.12)\right\}, \tag{4.7}$$

where

$$G = (X_i \mid i \in I), \tag{4.8}$$

$$X_i \in F, \qquad X_i \neq E \qquad (i \in I), \tag{4.9}$$

$$|\{i \mid e \in X_i, i \in I\}| = \text{const.} \equiv \mu(G,Y) \quad (e \in Y), \tag{4.10}$$

$$|\{i \mid e \in X_i, i \in I\}| = \text{const.} \equiv \mu(G,E-Y) \quad (e \in E-Y), \tag{4.11}$$

$$\mu(G,Y) > \mu(G,E-Y). \tag{4.12}$$

By use of the set function $\hat{f}: 2^E \rightarrow R \cup \{+\infty\}$ defined by (3.6) (or (3.16) and (3.24)), the polytope $B(f)$ of (4.1) can also be expressed as

$$B(f) = \{x \mid x \in R^E, x(X) \leqq \hat{f}(X)\,(X \in F), x(E) = \hat{f}(E)\,(=f(E))\}. \tag{4.13}$$

Therefore, f in (4.7) can be replaced by $\hat{f}$.

<u>Theorem 4.1</u>: For each $Y \subsetneqq E$,

$$\hat{f}^*(Y) = \min\Big\{\sum_{i \in I} \hat{f}(X_i) - (|I|-1)\hat{f}(E) \;\Big|\; \{E-X_i \mid i \in I\}\text{:a partition of } E-Y,\ X_i \in F(i \in I)\Big\}. \tag{4.14}$$

(Proof) If f is replaced by $\hat{f}$ in (4.7), we can restrict admissible families G in (4.8) - (4.12) to those which satisfy (4.8) - (4.12) and the following (i) - (iv):

(i) G is a cross-free family, (4.15)

(ii) for any X_i, $X_j \in G$, $X_i \cap X_j \neq \emptyset$, (4.16)

(iii) G does not contain a subfamily which forms a co-partition of E, (4.17)

(iv) $E \notin G$. (4.18)

(Here, (ii) and (iii) follow from Theorems 3.1 and 3.2.) From (i), the family $G = (X_i \mid i \in I)$ can be represented by a tree $T = (V,A)$ together with a family

$$P = (P_v \mid v \in V), \tag{4.19}$$

where $A = \{a_i \mid i \in I\}$ and nonempty P_v's form a partition of E as in Theorem 2.1. It follows from (4.16) that T is a directed tree. (For if there are distinct arcs a_i and a_j in T with $\partial^- a_i = \partial^- a_j$, then $X_i \cap X_j = \emptyset$.)

Let v_0 be the root of T. If $P_{v_0} = \emptyset$, then G contains a subfamily which forms a co-partition of E. Therefore, $P_{v_0} \neq \emptyset$ from (4.17). Since for each $e \in E$ the number of i's for which $e \in X_i$ should be taken from the fixed set of two distinct values of (4.10) and (4.11), for any end-vertex u of T every vertex w ($\neq u, v_0$) lying on the unique path $Q(u,v_0)$ connecting u with v_0 in T gives

$$P_w = \emptyset. \tag{4.20}$$

This implies that

$$P_{v_0} = Y \tag{4.21}$$

and that

$$\{X_i \mid i \in I, \partial^+ a_i = v_0\} \tag{4.22}$$

is a co-partition of $E - Y$ augmented by Y.

Since any co-partition $\{Z_j \mid j \in J\}$ of $E - Y$ augmented by Y with $Z_j \in F$ $(j \in J)$ satisfies (4.9) - (4.12) with X_i and I replaced by Z_j and J, $G = (X_i \mid i \in I)$ is a direct sum of families which form co-partitions of $E - Y$ augmented by Y. It follows from (4.7) that $G = (X_i \mid i \in I)$ in (4.7) can be restricted to those for which $\{X_i \mid i \in I\}$ is a co-partition of $E - Y$ augmented by Y. This completes the proof of Theorem 4.1. Q.E.D.

We define

$$\hat{f}^*(E) = \hat{f}(E) \ (= f(E)). \tag{4.23}$$

From (4.7) (or (4.14)) and (4.23), $\hat{f}^*$ is a function from 2^E to $R \cup \{+\infty\}$.

Theorem 4.2: For any $X, Y \subseteq E$, if $\hat{f}^*(X), \hat{f}^*(Y) < +\infty$, then

$$\hat{f}^*(X) + \hat{f}^*(Y) \geqq \hat{f}^*(X \cup Y) + \hat{f}^*(X \cap Y). \tag{4.24}$$

(Proof) If $X = E$ or $Y = E$, then (4.24) is trivial. So, we suppose $X \neq E$ and $Y \neq E$. Then, from Theorem 4.1, for some partition $\{E-X_i \mid i \in I\}$ of $E - X$ and some partition $\{E-Y_j \mid j \in J\}$ of $E - Y$ with $X_i \in F$ $(i \in I)$ and $Y_j \in F$ $(j \in J)$, we get

$$\hat{f}^*(X) + \hat{f}^*(Y)$$

$$= \sum_{i \in I} \hat{f}(X_i) - (|I| - 1)\hat{f}(E) + \sum_{j \in J} \hat{f}(Y_j) - (|J| - 1)\hat{f}(E). \tag{4.25}$$

Let $G = (Z_k \mid k \in I+J)$ be the direct sum of families $(X_i \mid i \in I)$ and $(Y_j \mid j \in J)$. If, for $Z_i, Z_j \in G$, $(E-Z_i) \cap (E-Z_j) \neq \emptyset$, $Z_i \cap (E-Z_j) \neq \emptyset$ and $(E-Z_i) \cap Z_j \neq \emptyset$, then replace

$$Z_i \leftarrow Z_i \cup Z_j, \qquad Z_j \leftarrow Z_i \cap Z_j. \tag{4.26}$$

Repeat such replacement until there is no such pair of Z_i and Z_j in G. We can easily see that the finally obtained G is the direct sum of families $\mathcal{G}_1 = (X_i^* \mid i \in I^*)$ and $\mathcal{G}_2 = (Y_j^* \mid j \in J^*)$ such that $\{E - X_i^* \mid i \in I^*\}$ is a partition of $E - (X \cup Y)$ and $\{E - Y_j^* \mid j \in J^*\}$ is a partition of $E - (X \cap Y)$, where, if $X \cap Y = \emptyset$, the family $\mathcal{G}_2$ may be composed of the empty set alone.

For any pair of Z_i and Z_j to be replaced by (4.26), we have

$$Z_i \cup Z_j \neq E. \tag{4.27}$$

Therefore, from the replacement of (4.26), Lemma 3.3 and (4.25),

$$\begin{aligned} &\hat{f}^*(X) + \hat{f}^*(Y) \\ &\geqq \sum_{i \in I^*} \hat{f}(X_i^*) - (|I^*| - 1)\hat{f}(E) + \sum_{j \in J^*} \hat{f}(Y_j^*) - (|J^*| - 1)\hat{f}(E) \\ &\geqq \hat{f}^*(X \cup Y) + \hat{f}^*(X \cap Y). \end{aligned}$$

Q.E.D.

From Theorem 4.2,

$$\mathcal{D}_0 = \{X \mid X \subseteq E, \hat{f}^*(X) < +\infty\} \tag{4.28}$$

is a distributive lattice with respect to set inclusion. Denote by f^* the function obtained by restricting the domain 2^E of $\hat{f}^*$ to $\mathcal{D}_0$. Then f^* is a submodular function on the distributive lattice $\mathcal{D}_0$ and the polytope $B(f)$ defined by (4.1) is also expressed as

$$B(f) = \{x \mid x \in R^E, x(X) \leq f^*(X)\,(X \in \mathcal{D}_0), x(E) = f^*(E)\,(=f(E))\} \tag{4.29}$$

which is the base polytope $B(f^*)$ associated with the submodular system $(\mathcal{D}_0, f^*)$.

We have thus shown the following theorem.

<u>Theorem 4.3</u>: Suppose f is a submodular function on a crossing family

F of subsets of E with $E \in F$. Let f^* be the submodular function on the distributive lattice $\mathcal{D}_0$ defined as above. Then the polytope $B(f)$ defined by (4.1) coincides with the base polytope $B(f^*)$ associated with the submodular system $(\mathcal{D}_0, f^*)$. Furthermore, f^* is integer-valued if f is.

5. Submodular Functions on Intersecting Families

Let F be an intersecting family of subsets of E and $f: F \to R$ a submodular function on F. Then, from Theorems 3.1 and 3.2, the polytope

$$P(f) = \{x \mid x \in R^E, x(X) \leqq f(X) \, (X \in F)\} \tag{5.1}$$

is also expressed as

$$P(f) = \{x \mid x \in R^E, x(X) \leqq \hat{f}(X) \, (X \subseteq E)\}, \tag{5.2}$$

where $\hat{f}$ is defined by (3.14) - (3.16) and (3.24).

Since F is an intersecting family, we can restrict G in (3.6) - (3.9) to those which satisfy (3.8), (3.9), (3.12), (3.13) and the following (i):

(i) for any $X_i, X_j \in G$, if $X_i \cap X_j \neq \emptyset$, then

$$X_i \subseteq X_j \text{ or } X_j \subseteq X_i. \tag{5.3}$$

In particular, $\hat{f}_1(E)$ and $\hat{f}_2(E)$ defined by (3.14) and (3.15) satisfy

$$\hat{f}_1(E) \leqq \hat{f}_2(E). \tag{5.4}$$

It follows from (5.4) and Theorems 3.1 and 3.2 that for any $Y \subseteq E$

$$\hat{f}(Y) = \min\left\{ \sum_{i \in I} f(X_i) \;\middle|\; \{X_i \mid i \in I\}\text{:a partition of } Y, \; X_i \in \mathcal{F}(i \in I) \right\}. \tag{5.5}$$

Lemma 5.1: Suppose f is a submodular function on an intersecting family $\mathcal{F}$. For $X, Y \subseteq E$ and $\hat{f}: 2^E \to R \cup \{+\infty\}$ defined by (5.5), if $\hat{f}(X), \hat{f}(Y) < +\infty$ then

$$\hat{f}(X) + \hat{f}(Y) \geq \hat{f}(X \cup Y) + \hat{f}(X \cap Y). \tag{5.6}$$

(Proof) Since f is a submodular function on an intersecting family $\mathcal{F}$, Lemma 3.3 holds for $X, Y \subseteq E$ with $X \cup Y = E$ as well, which is Lemma 5.1. Q.E.D.

From Lemma 5.1,

$$\mathcal{D}_1 = \{X \mid X \subseteq E, \hat{f}(X) < +\infty\} \tag{5.7}$$

is a distributive lattice. Let us denote by f' the function obtained by restricting the domain of $\hat{f}$ to $\mathcal{D}_1$. Then f' is a submodular function on the distributive lattice $\mathcal{D}_1$ and the polytope $P(f)$ of (5.1) is expressed in terms of f' as

$$P(f) = \{x \mid x \in R^E, x(X) \leq f'(X)\,(X \in \mathcal{D}_1)\} \tag{5.8}$$

which is the submodular polytope $P(f')$ associated with the submodular system $(\mathcal{D}_1, f')$.

Theorem 5.2: Suppose f is a submodular function on an intersecting family $\mathcal{F}$ of subsets of E. Let f' be the submodular function on the distributive lattice $\mathcal{D}_1$ defined as above. Then the polytope

$P(f)$ defined by (5.1) coincides with the submodular polytope $P(f')$ associated with the submodular system $(\mathcal{D}_1, f')$. Furthermore, f' is integer-valued if f is.

6. Relationship Between the Independent-Flow Problem and the Minimum Cost Flow problem of Edmonds and Giles

The author considered in [6] the minimum cost flow problem called the independent-flow problem as follows.

Let $G = (V,A;S^+,S^-)$ be a graph with a vertex set V, an arc set A, an entrance vertex set $S^+ \subseteq V$ and an exit vertex set $S^- \subseteq V$, where we assume $S^+ \cap S^- = \emptyset$ for simplicity. Each arc $a \in A$ is given a capacity $c(a) \geq 0$. Also let $\mathbb{P}^+ = (S^+,\rho^+)$ and $\mathbb{P}^- = (S^-,\rho^-)$ be polymatroids defined on the entrance vertex set S^+ and the exit vertex set S^-, respectively. (For polymatroids, see [1].) We denote the network with these characteristics by $N = (G,c;\mathbb{P}^+,\mathbb{P}^-)$.

An independent flow ϕ in N is a function from A to R such that

$$0 \leq \phi(a) \leq c(a) \qquad (a \in A), \tag{6.1}$$

$$\partial\phi(v) = 0 \qquad (v \in V - (S^+ \cup S^-)), \tag{6.2}$$

$$(\partial\phi(v) \mid v \in S^+) \in P^+(\rho^+), \tag{6.3}$$

$$(-\partial\phi(v) \mid v \in S^-) \in P^+(\rho^-), \tag{6.4}$$

where for each $v \in V$

$$\partial\phi(v) = \sum_{a \in \delta^+ v} \phi(a) - \sum_{a \in \delta^- v} \phi(a), \tag{6.5}$$

and

$$P^+(\rho^+) = \{x \mid x \in R^{S^+}, 0 \leqq x(U) \leqq \rho^+(U)\,(U \subseteqq S^+)\}, \qquad (6.6)$$

$$P^+(\rho^-) = \{x \mid x \in R^{S^-}, 0 \leqq x(U) \leqq \rho^-(U)\,(U \subseteqq S^-)\}. \qquad (6.7)$$

The <u>flow value</u> of ϕ is given by

$$\partial\phi(S^+) = \sum_{v \in S^+} \partial\phi(v). \qquad (6.8)$$

For a given function $\gamma\colon A \to R$, the cost $C(\phi)$ of an independent flow ϕ in N is defined by

$$C(\phi) = \sum_{a \in A} \gamma(a)\phi(a). \qquad (6.9)$$

Given a nonnegative value v_0, the independent-flow problem is to find an independent flow ϕ in N with the flow value v_0 which has the minimum cost among all independent flows in N with the flow value v_0.

On the other hand, Edmonds and Giles considered the minimum cost flow problem as follows [2]. Let $G = (V,A)$ be a graph with a vertex set V and an arc set A, F^* a crossing family of subsets of V, $\overline{f}^*$ a submodular function on F^* and b, c and γ functions from A to $R \cup \{+\infty,-\infty\}$. Denote by $\hat{N}$ the network with these characteristics. A feasible flow ϕ in $\hat{N}$ is a function from A to $R \cup \{+\infty,-\infty\}$ such that

$$b(a) \leqq \phi(a) \leqq c(a) \qquad (a \in A), \qquad (6.10)$$

$$\partial\phi(V-U) \leqq \overline{f}^*(U) \qquad (U \in F^*). \qquad (6.11)$$

The cost $C(\phi)$ of the flow ϕ is given by

$$C(\phi) = \sum_{a \in A} \gamma(a)\phi(a). \qquad (6.12)$$

The problem is to find a feasible flow in $\hat{N}$ of the minimum cost.

The theory and algorithms for the independent-flow problem in [6] can easily be generalized to the case where

(i) the lower capacity in (6.1) of each arc is not necessarily zero (possibly $-\infty$),

(ii) the functions c and γ take values from $R \cup \{+\infty\}$ and from $R \cup \{+\infty, -\infty\}$, respectively,

(iii) $P^+(\rho^+)$ and $P^+(\rho^-)$ in (6.3) and (6.4) are, respectively, replaced by submodular polytopes $P(f^+)$ and $P(f^-)$ associated with submodular systems $(\mathcal{D}^+, f^+)$ and $(\mathcal{D}^-, f^-)$.

Here, $\mathcal{D}^+$ and $\mathcal{D}^-$ are, respectively, distributive lattices formed by subsets of S^+ and S^-. We shall show that the minimum cost flow problem of Edmonds and Giles can be reduced to an independent-flow problem generalized as above.

Let us define

$$\mathcal{F} = \{V-U \mid U \in \mathcal{F}^*\}, \tag{6.13}$$

$$f(V-U) = \overline{f}^*(U) \qquad (U \in \mathcal{F}^*). \tag{6.14}$$

Then, $\mathcal{F}$ is a crossing family of subsets of V, f is a submodular function on $\mathcal{F}$, and (6.11) is rewritten as

$$\partial\phi(U) \leqq f(U) \qquad (U \in \mathcal{F}). \tag{6.15}$$

Since

$$\partial\phi(\emptyset) = \partial\phi(V) = 0, \tag{6.16}$$

we can suppose, without loss of generality (as far as a feasible flow exists in $\hat{N}$), that

$$\emptyset, \quad V \in \mathcal{F}, \tag{6.17}$$

$$f(\emptyset) = f(V) = 0. \tag{6.18}$$

Because of (6.16) and (6.18), the system of inequalities (6.15) is equivalent to

$$\partial\phi(U) \leqq f^*(U) \qquad (U \in \mathcal{D}), \tag{6.19}$$

where f^* is a submodular function on a distributive lattice $\mathcal{D}$ which is defined in terms of f similarly as described in Sections 3 and 4 (see Theorem 4.3). The (6.19) means

$$\partial\phi \in P(f^*), \tag{6.20}$$

where $P(f^*)$ is the submodular polytope associated with the submodular system $(\mathcal{D}, f^*)$. Therefore, the minimum cost flow problem of Edmonds and Giles can be reduced to a generalized version of the independent-flow problem in a network $N = (G=(V,A;S^+,S^-),c;\mathbb{P}^+=(\mathcal{D}^+,f^+),\mathbb{P}^-=(\mathcal{D}^-,f^-))$, where

$$S^+ = V, \qquad S^- = \emptyset \tag{6.21}$$

and the flow value v_0 is taken as zero.

Since f^* in (6.20) is integer-valued if f is integer-valued, the integrality property of the optimal (primal and dual) solutions of the minimum cost flow problem of Edmonds and Giles easily follows from the results of [4] - [6], [8] and [9] (also see [10] and [11]).

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