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by

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STRUCTURAL ANALYSIS OF FIRST PASSAGE TIMES OF SKIP-FREE SEMI-MARKOV PROCESSES AND RELATED LIMIT THEOREMS

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ABSTRACT. A skip-free semi-Markov process is considered, which moves through its state space $\mathcal{N} = \{0, 1, 2, \cdots\}$ only in a lattice continuous manner. Such processes can be considered to generalize a class of birth-death processes. Of interest, then, is to see how the first passage time structure of birth-death processes may or may not be inherited in that of skip-free semi-Markov processes. It is shown that a recursion formula satisfied by the first passage times T_n^+ from n to $n+1$ for skip-free semi-Markov processes is quite similar to that for birth-death processes. While the distributional properties of the first passage times of birth-death processes are not necessarily present for skip-free semi-Markov processes, the limit theorems hold true almost in an identical form.

Keywords skip-free semi-Markov processes, first passage times, distributional properties, limit theorems.

0. INTRODUCTION

The study of semi-Markov processes dates back to the middle of 1950's represented by the original papers by Léby[11], Smith[18] and Takács[26]. Subsequently various aspects of these processes were studied in a series of papers by Pyke[13, 14], Pyke and Schaufele[15, 16] and Moore and Pyke[12]. Since the early 1960's, the field attracted many researchers, resulting in a collection of quite extensive results. A bibliography on semi-Markov processes in 1976 by Teugels[27] contains about 600 papers by some 300 authors. The reader is referred to two excellent survey papers by Çinlar[1, 2] for the summary of these results. To the best knowledge of the authors, however, there exists no literature specifically focusing on the first passage time structure of skip-free semi-Markov processes, which can be considered as a generalization of a class of birth-death processes.

A Markov chain in continuous time defined on $\{0, 1, 2, \cdots\}$ is called a birth-death process when transitions occur only in a lattice continuous manner. Because of this skip-free property, the first passage times of birth-death processes possess a variety of peculiar results as shown in Keilson[6, 8, 9, 10], Sumita[21, 25] and Sumita and Masuda[22] to name only a few. The purpose of this paper is to analyze the first passage time structure of skip-free semi-Markov processes, and to see how the first passage time structure of birth-death processes can be carried over to that of skip-free semi-Markov processes. It is also examined whether certain limit theorems involving first passage times of birth-death processes still hold true for skip-free semi-Markov processes.

The structure of this paper is as follows. In Section 1, we formally define a semi-Markov process and summarize some well known results. Using these results, the Laplace transforms of the first passage times of skip-free semi-Markov processes are explicitly derived in Section 2. Moments, a recursion formula and its probabilistic interpretation are also discussed. Section 3 establishes a sufficient condition under which the first passage time T_n^+ from n to $n+1$ of a skip-free semi-Markov process is a mixture of independent exponential random variables. Finally in Section 4, it is shown under certain conditions that $T_n^+/\mathrm{E}[T_n^+]$ for skip-free semi-Markov processes converges in law to the exponential variate of mean one as $n \to \infty$, as for birth-death processes. Other related limit theorems are also discussed.

Throughout the paper, the following notation is employed. Vectors and matrices are distinguished by a single underline and double underlines respectively. For an $(N+1)$ dimensional vector $\underline{a}^T$ and an $(N+1)\times(N+1)$ matrix $\underline{\underline{a}}$, subvectors and submatrices are denoted by $\underline{a}_G^T = [a_i]_{i\in G}$, $\underline{\underline{a}}_{GB} = [a_{ij}]_{i\in G, j\in B}$, etc. where $G,\ B \subset \{0,1,\cdots,N\}$. We also define $\underline{1}$ as a column vector having all components equal to 1 , and $\underline{\underline{0}}$ as a matrix having all components equal to zero.

1. Finite Semi-Markov Process $J(t)$ and First Passage Times

Let $\{(J_n,T_n) : n=0,1,\cdots\}$ be a Markov renewal process where J_n is a Markov chain in discrete time on $\mathcal{N}=\{0,1,\cdots,N\}$ and T_n is the n-th transition epoch with $T_0=0$. The behavior of the Markov renewal process is governed by a semi-Markov matrix $\underline{\underline{A}}(x) = [A_{ij}(x)]$, where

$$A_{ij}(x) = \mathrm{P}[J_{n+1}=j,\ T_{n+1}-T_n \le x \big| J_n = i] \ . \tag{1.1}$$

If $\sup_n T_n = +\infty$, then the process $\{J(t) : t \ge 0\}$ where $J(t) = J_n$ for $T_n \le t \le T_{n+1}$ is called the minimal semi-Markov process associated with the Markov renewal process $\{(J_n,T_n) : n=0,1,\cdots\}$, see Çinlar[1].

We assume that the stochastic matrix $\underline{\underline{A}}_0 = [A_{0:ij}] = \underline{\underline{A}}(\infty)$ governing the embedded Markov chain $\{J_n : n=0,1,\cdots\}$ is ergodic, having the ergodic vector $\underline{\check{e}}^T$, i.e.

$$\underline{\check{e}}^T \underline{\underline{A}}_0 = \underline{\check{e}}^T \ ; \ \underline{\check{e}}^T > \underline{0}^T \ ; \ \underline{\check{e}}^T \cdot \underline{1} = 1 \ . \tag{1.2}$$

For notational convenience, let $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$, and define

$$\underline{\underline{A}}_D(x)=[\delta_{ij}A_i(x)]\ ;\ A_i(x)=\sum_{j\in\mathcal{N}}A_{ij}(x)\ , \tag{1.3}$$

$$\underline{\underline{\bar{A}}}_D(x)=\left[\delta_{ij}\bar{A}_i(x)\right]\ ;\ \bar{A}_i(x)=1-A_i(x)\ , \tag{1.4}$$

$$\underline{\underline{A}}_k=[A_{k:ij}]\ ;\ A_{k:ij}=\int_0^\infty x^k dA_{ij}(x)\ , \tag{1.5}$$

$$\underline{\underline{A}}_{D:k}=[\delta_{ij}A_{k:i}]\ ;\ A_{k:i}=\int_0^\infty x^k dA_i(x)\ ,\ k\geq 0\ . \tag{1.6}$$

We note that $A_i(x)=\mathrm{P}[T_{n+1}-T_n\leq x\big|J_n=i]$ is the c.d.f of the dwell time of the semi-Markov process at state i and $\bar{A}_i(x)$ is the corresponding survival function. Throughout the paper, we assume that $A_{k:i}<\infty$ for all $i\in\mathcal{N}$ for $0\leq k\leq 2$. It should be noted that $\underline{\underline{A}}_{D:0}=\underline{\underline{I}}$.

The transform of $\underline{\underline{A}}(x)$ is denoted by

$$\underline{\underline{\alpha}}(s)=[\alpha_{ij}(s)]\ ;\quad \alpha_{ij}(s)=\int_0^\infty e^{-sx}dA_{ij}(x)\ . \tag{1.7}$$

Laplace-Stieltjes transforms $\alpha_i(s)$, $\underline{\underline{\alpha}}_D(s)$, etc. are defined similarly. The transition probability matrix of $J(t)$ is written by $\underline{\underline{P}}(t)$, i.e.,

$$\underline{\underline{P}}(t)=[P_{ij}(t)]\ ;\quad P_{ij}(t)=\mathrm{P}\left[J(t)=j\big|J(0)=i\right]\ . \tag{1.8}$$

Correspondingly, the state probability vector $\underline{p}^T(t)=[p_0(t),\cdots,p_N(t)]$ at time t is given by

$$\underline{p}^T(t)=\underline{p}^T(0)\underline{\underline{P}}(t)\ . \tag{1.9}$$

The Laplace transforms of $\underline{\underline{P}}(t)$ and $\underline{p}^T(t)$ are denoted by

$$\underline{\underline{\Pi}}(s)=\int_0^\infty e^{-st}\underline{\underline{P}}(t)dt\ ,\quad \underline{\pi}^T(s)=\int_0^\infty e^{-st}\underline{p}^T(t)dt\ . \tag{1.10}$$

From the classical renewal argument, one has (see e.g. Çinlar[1, 2])

$$P_{ij}(t)=\delta_{ij}\bar{A}_i(t)+\sum_{k\in\mathcal{N}}\int_0^t P_{kj}(t-\tau)dA_{ik}(\tau)\ ,\quad i,\ j\in\mathcal{N}\ . \tag{1.11}$$

By taking the Laplace transform on both sides of (1.11), it follows that

$$\underline{\underline{\Pi}}(s)=\frac{1}{s}[\underline{\underline{I}}-\underline{\underline{\alpha}}(s)]^{-1}[\underline{\underline{I}}-\underline{\underline{\alpha}}_D(s)]\ . \tag{1.12}$$

It has been shown in Keilson[7] that

$$\underline{\underline{\alpha}}(s)\left[\underline{\underline{I}}-\underline{\underline{\alpha}}(s)\right]^{-1}=\frac{1}{s}\underline{\underline{H}}_1+\underline{\underline{H}}_0+\underline{\underline{o}}(1)\text{ as }s\to 0+\ , \tag{1.13}$$

where the two matrices $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_0$ are given by

$$\underline{\underline{H}}_1=\frac{1}{m}\underline{\underline{J}}\ ;\ \underline{\underline{J}}=\underline{1}\cdot\underline{\check{e}}^T\ ;\ m=\underline{\check{e}}^T\underline{\underline{A}}_1\underline{1}\ , \tag{1.14}$$

with $\underline{\check{e}}^T$ as given in (1.2) and

$$(1.15)\qquad \underline{\underline{H}}_0 = \underline{\underline{H}}_1 \left(-\underline{\underline{A}}_1 + \frac{1}{2}\underline{\underline{A}}_2\underline{\underline{H}}_1 \right) + \left(\underline{\underline{Z}} - \underline{\underline{H}}_1\underline{\underline{A}}_1\underline{\underline{Z}} \right) \left(\underline{\underline{A}}_0 - \underline{\underline{A}}_1\underline{\underline{H}}_1 \right) \ .$$

Here, $\underline{\underline{Z}}$ is the fundamental matrix associated with $\underline{\underline{A}}_0$ defined by

$$(1.16)\qquad \underline{\underline{Z}} = \left[\underline{\underline{I}} - \underline{\underline{A}}_0 + \underline{\underline{J}} \right]^{-1} \ .$$

Since $\left[\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right]^{-1} = \underline{\underline{I}} + \underline{\underline{\alpha}}(s) \left[\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right]^{-1}$ and $\underline{\underline{I}} - \underline{\underline{\alpha}}_D(s) = s\underline{\underline{A}}_{D:1} + \underline{\underline{o}}(s)$, it can be readily seen from (1.12) and (1.13) that

$$(1.17)\qquad \underline{\underline{\Pi}}(s) = \frac{1}{s}\underline{\underline{H}}_1 \cdot \underline{\underline{A}}_{D:1} + \underline{\underline{A}}_{D:1} + \underline{\underline{o}}(1) \text{ as } s \to 0+ \ .$$

It then follows from (1.14) that

$$(1.18)\qquad \lim_{t\to\infty} \underline{\underline{P}}(t) = \lim_{s\to 0} s\underline{\underline{\Pi}}(s) = \underline{\underline{H}}_1 \cdot \underline{\underline{A}}_{D:1} = \frac{1}{m}\underline{1} \cdot \underline{\check{e}}^T \underline{\underline{A}}_{D:1} \ .$$

Accordingly, $J(t)$ is ergodic and the ergodic probability vector $\underline{e}^T$ of $J(t)$ can be written as

$$(1.19)\qquad \underline{e}^T = \frac{1}{m}\underline{\check{e}}^T \underline{\underline{A}}_{D:1} = \left[\frac{\check{e}_0 A_{1:0}}{\sum_{i\in\mathcal{N}} \check{e}_i A_{1:i}}, \cdots, \frac{\check{e}_N A_{1:N}}{\sum_{i\in\mathcal{N}} \check{e}_i A_{1:i}} \right] \ .$$

From an application point of view, it may be useful to decompose the state space $\mathcal{N}$ into two subsets G and B where the good set G is the set of desirable states and the bad set B is the set of undesirable states. Then of interest is the first passage time of $J(t)$ from $i \in G$ to the bad set B. More formally, let this first passage time be denoted by T_{iB} so that

$$(1.20)\qquad T_{iB} = \inf\{t : J(t) \in B \big| J(0) = i \in G\} \ .$$

When the bad set B is a singleton set, the Laplace transform $\sigma_{iB}(s) = \mathrm{E}[e^{-sT_{iB}}]$ is given in (5.17) of Çinlar[2]. A more general result has been obtained by Sumita and Masuda[21] and we follow their proof here so as to facilitate further discussions.

For the study of the first passage time T_{iB} of $J(t)$ from $i \in G$ to any state in B, we consider the absorbing process $\tilde{J}(t)$ obtained from the original process by making all states in B absorbing. Let $\underline{\tilde{p}}^T(t) = [\tilde{p}_0(t), \cdots, \tilde{p}_N(t)]$ where $\tilde{p}_j(t) = \mathrm{P}[\tilde{J}(t) = j \big| \tilde{J}(0) = i]$ with the Laplace transform $\underline{\tilde{\pi}}^T(s)$. It should be noted that $\tilde{p}_j(t)$ for $j \in B$ is the probability that the process $J(t)$ hits the bad set B at $j \in B$ for the first time in $[0, t)$. Without loss of generality, we assume that $G = \{0, 1, \cdots, K\}$ and $B = \{K + 1, \cdots, N\}$. It is easy to see that

$$(1.21)\qquad \mathrm{P}[T_{iB} \le t] = \underline{\tilde{p}}_B^T(t)\underline{1}_B \ .$$

Let

$$(1.22)\qquad \underline{\underline{\tilde{\alpha}}}(s) = [\tilde{\alpha}_{ij}(s)] = \begin{bmatrix} \underline{\underline{\alpha}}_{GG}(s) & \underline{\underline{\alpha}}_{GB}(s) \\ \underline{\underline{0}}_{BG} & \underline{\underline{0}}_{BB} \end{bmatrix} \ ; \ \ \underline{\underline{\tilde{\alpha}}}_D(s) = \left[\delta_{ij} \sum_{k\in\mathcal{N}} \tilde{\alpha}_{ik}(s) \right] .$$

If we denote the i-th unit vector by $\underline{u}_i$, it can be seen from (1.10), with $\underline{\underline{\Pi}}(s)$, $\underline{\underline{\alpha}}(s)$ and $\underline{\underline{\alpha}}_D(s)$ replaced by $\tilde{\underline{\underline{\Pi}}}(s)$, $\tilde{\underline{\underline{\alpha}}}(s)$ and $\tilde{\underline{\underline{\alpha}}}_D(s)$ respectively, that

$$\tilde{\underline{\pi}}_G^T(s) = \frac{1}{s}\underline{u}_{i:G}^T[\underline{\underline{I}}_{GG} - \underline{\underline{\alpha}}_{GG}(s)]^{-1}[\underline{\underline{I}}_{GG} - \underline{\underline{\alpha}}_{D:GG}(s)] \ , \tag{1.23}$$

and

$$\tilde{\underline{\pi}}_B^T(s) = \tilde{\underline{\pi}}_G^T(s)[\underline{\underline{I}}_{GG} - \underline{\underline{\alpha}}_{D:GG}(s)]^{-1}\underline{\underline{\alpha}}_{GB}(s) \ . \tag{1.24}$$

Substituting (1.23) into (1.24), one finally has

$$\tilde{\underline{\pi}}_B^T(s) = \frac{1}{s}\underline{u}_{i:G}^T\left[\underline{\underline{I}}_{GG} - \underline{\underline{\alpha}}_{GG}(s)\right]^{-1}\underline{\underline{\alpha}}_{GB}(s) \ . \tag{1.25}$$

From (1.21), one easily sees that $\sigma_{iB}(s) = s\tilde{\underline{\pi}}_B^T(s)\underline{1}_B$. The following theorem then holds true from (1.25).

Theorem 1.1 (Theorem 4.1 of Sumita and Masuda[23])

Let $\sigma_{iB}(s) = \mathrm{E}[e^{-sT_{iB}}]$ and define $\underline{\sigma}_G(s) = [\sigma_{iB}(s)]_{i\in G}$. Then

$$\underline{\sigma}_G(s) = [\underline{\underline{I}}_{GG} - \underline{\underline{\alpha}}_{GG}(s)]^{-1}\underline{\underline{\alpha}}_{GB}(s)\underline{1}_B \ . \tag{1.26}$$

By differentiating $\underline{\sigma}_G(s)$ in Theorem 1.1 with respect to s at $s = 0$, the next corollary is immediate.

Corollary 1.2 (Sumita and Masuda[23])

Let $\mathrm{E}[\underline{T}_G^k] = \left[\mathrm{E}[T_{iB}^k]\right]_{i\in G}$. Then for $k = 1, 2, \cdots$

$$\mathrm{E}[\underline{T}_G^k] = [\underline{\underline{I}}_{GG} - \underline{\underline{A}}_{0:GG}]^{-1}\left[\underline{\underline{A}}_{k:GB}\underline{1}_B + \sum_{l=1}^{k}\binom{k}{l}\underline{\underline{A}}_{l:GG}\mathrm{E}[\underline{T}_G^{k-l}]\right] \ .$$

2. First Passage Time Structure of Skip-Free Semi-Markov Processes

In this section, we focus on the first passage time structure of skip-free semi-Markov processes, and examine to what extent the basic properties of the first passage time structure of birth-death processes can be carried over. A semi-Markov process $J(t)$ on $\mathcal{N} = \{0, 1, \cdots\}$ governed by a semi-Markov matrix $\underline{\underline{A}}(x) = [A_{ij}(x)]$ is said to be skip-free if

$$A_{ij}(x) = \begin{cases} q_i A_i^-(x) & j = i-1 \ , \quad i \geq 1 \ , \\ r_i A_i^0(x) & j = i \ , \quad i \geq 0 \ , \\ p_i A_i^+(x) & j = i+1 \ , \quad i \geq 0 \ , \\ 0 & \text{else} \ , \end{cases} \tag{2.1}$$

where $q_i, r_i, p_i \geq 0$, $r_0 + p_0 = 1$ and $q_i + r_i + p_i = 1$, $i \geq 1$. Here, $A_i^-(x), A_i^0(x)$ and $A_i^+(x)$ are probability distribution functions having Laplace-Stieltjes transforms $\alpha_i^-(s) = \int_0^\infty e^{-sx}dA_i^-(x)$, $\alpha_i^0(s) = \int_0^\infty e^{-sx}dA_i^0(x)$ and $\alpha_i^+(s) = \int_0^\infty e^{-sx}dA_i^+(x)$

respectively. For notational convenience, we define $q_0 = 0$. Then, the dwell time of $J(t)$ in state i is given by

$$A_i(x) = q_i A_i^-(x) + r_i A_i^0(x) + p_i A_i^+(x) \ , \ i \geq 0 \ , \tag{2.2}$$

and the corresponding Laplace transform is denoted by $\alpha_i(s) = \int_0^\infty e^{-sx} dA_i(x)$. The semi-Markov process $J(t)$ is skip-free in that it moves only in a lattice continuous manner in either direction.

Let T_n^+ be the first passage time of $J(t)$ from n to $n+1$. In what follows, we establish some structural properties of $\sigma_n^+(s) = \mathrm{E}[e^{-sT_n^+}]$ based on Theorem 1.1. The following matrix is frequently employed in this study.

$$\underline{\underline{\alpha}}_n(s) = \begin{bmatrix} r_0\alpha_0^0(s) & p_0\alpha_0^+(s) & & \underline{\underline{0}} \\ q_1\alpha_1^-(s) & r_1\alpha_1^0(s) & p_1\alpha_1^+(s) & \\ \ddots & \ddots & \ddots & \\ & q_{n-1}\alpha_{n-1}^-(s) & r_{n-1}\alpha_{n-1}^0(s) & p_{n-1}\alpha_1^+(s) \\ \underline{\underline{0}} & & q_n\alpha_n^-(s) & r_n\alpha_n^0(s) \end{bmatrix} . \tag{2.3}$$

For convenience, we define $\underline{\underline{\alpha}}_n(s) = \underline{\underline{0}}$ for $n < 0$. Our first theorem describes $\sigma_n^+(s)$ in terms of determinants of matrices of the form specified in (2.3).

Theorem 2.1

$$\sigma_n^+(s) = p_n\alpha_n^+(s)\frac{\det\left(\underline{\underline{I}} - \underline{\underline{\alpha}}_{n-1}(s)\right)}{\det\left(\underline{\underline{I}} - \underline{\underline{\alpha}}_n(s)\right)}, \ n \geq 0 \ .$$

Proof. Let $G = \{0, 1, 2, \cdots, n\}$ and $B = \{n+1, n+2, \cdots\}$. Since $J(t)$ is skip-free, an entry into B is possible only through $n+1$. Hence we can redefine B as a singleton set $B = \{n+1\}$. Let $T_{j,n+1}$ be the first passage time of $J(t)$ from j to $n+1$, and define $\underline{\sigma}_G^T(s) = [\sigma_{0,n+1}(s), \cdots, \sigma_{n,n+1}(s)]$ where $\sigma_{j,n+1}(s) = \mathrm{E}[e^{-sT_{j,n+1}}]$. We note that $\sigma_n^+(s) = \sigma_{n,n+1}(s)$. From Theorem 1.1, one sees that

$$\left[\underline{\underline{I}} - \underline{\underline{\alpha}}_n(s)\right] \underline{\sigma}_G(s) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_n\alpha_n^+(s) \end{pmatrix} . \tag{2.4}$$

Since $\sigma_n^+(s)$ is the last element in $\underline{\sigma}_G(s)$, the theorem follows from (2.3) by applying Cramer's rule to (2.4). □

Theorem 2.1 then leads to the following recursion formula for $\sigma_n^+(s)$.

Theorem 2.2

$$\sigma_n^+(s) = \frac{p_n\alpha_n^+(s)}{1 - r_n\alpha_n^0(s) - q_n\alpha_n^-(s)\sigma_{n-1}^+(s)} \ , \ n \geq 0 \ ,$$

where $\alpha_n^-(s) = \sigma_n^+(s) = 0$ for $n < 0$.

Proof. From (2.3), it can be readily seen that

$$\det\left(\underline{\underline{I}} - \underline{\underline{\alpha}}_n(s)\right) = \left(1 - r_n\alpha_n^0(s)\right)\det\left(\underline{\underline{I}} - \underline{\underline{\alpha}}_{n-1}(s)\right) - q_n\alpha_n^-(s)p_{n-1}\alpha_{n-1}^+(s)\det\left(\underline{\underline{I}} - \underline{\underline{\alpha}}_{n-2}(s)\right) . \tag{2.5}$$

Substituting (2.5) into Theorem 2.1, and then dividing both numerator and denominator by $\det\left(\underline{\underline{I}} - \underline{\underline{\alpha}}_{n-1}(s)\right)$, the theorem follows. □

The recursion formula given in Theorem 2.2 can be rewritten as

$$\sigma_n^+(s) = p_n\alpha_n^+(s) + r_n\alpha_n^0(s)\sigma_n^+(s) + q_n\alpha_n^-(s)\sigma_{n-1}^+(s)\sigma_n^+(s) . \tag{2.6}$$

The probabilistic interpretation of (2.6) is now clear. Namely, suppose that the process $J(t)$ has entered state n at time $t = 0$. It goes to state $n+1$ directly with probability p_n. In this case the first passage time from n to $n+1$ is given by the dwell time of $J(t)$ at state n given that the upward transition occurs before any self- or downward transition, characterized by $\alpha_n^+(s)$. The first transition of $J(t)$ is a self-transition with probability r_n. The time required to reach $n+1$ is then the sum of the dwell time of $J(t)$ given that the self-transition occurs before any downward or upward transition, having the Laplace transform $\alpha_n^0(s)$, and the first passage time from n to $n+1$. If $J(t)$ goes down to $n-1$ first, which occurs with probability q_n, the dwell time of $J(t)$ given that the downward transition occurs before any self- or upward transition, characterized by the Laplace transform $\alpha_n^-(s)$, has to be added to the first passage time from $n-1$ to n and that from n to $n+1$.

We next derive the mean and variance of T_n^+. For $k \geq 0$, let $\mu_{n:k}^-, \mu_{n:k}^0, \mu_{n:k}^+$ and $\mu_{n:k}$ be defined by

$$\begin{cases} \mu_{n:k}^- = \int_0^\infty x^k dA_n^-(x) \; ; \\ \mu_{n:k}^0 = \int_0^\infty x^k dA_n^0(x) \; ; \\ \mu_{n:k}^+ = \int_0^\infty x^k dA_n^+(x) \; ; \\ \mu_{n:k} = \int_0^\infty x^k dA_n(x) = q_n\mu_{n:k}^- + r_n\mu_{n:k}^0 + p_n\mu_{nk}^+ \; . \end{cases} \tag{2.7}$$

Theorem 2.3

(a) $\mathrm{E}[T_n^+] = \dfrac{1}{p_n}\left\{\mu_{n:1} + q_n\mathrm{E}[T_{n-1}^+]\right\}, \; n \geq 0$,

where $\mathrm{E}[T_n^+] = 0$ for $n < 0$.

(b) $$\mathrm{Var}[T_n^+] = \frac{q_n}{p_n}\mathrm{Var}[T_{n-1}^+] + \frac{q_n}{p_n}\mathrm{E}^2[T_{n-1}^+] + \mathrm{E}^2[T_n^+] + \frac{1}{p_n}\left\{\mu_{n:2} - 2\mu_{n:1}\mu_{n:1}^+ + 2q_n(\mu_{n:1}^- - \mu_{n:1}^+)\mathrm{E}[T_{n-1}^+]\right\} \; , \; n \geq 0 \; ,$$

where $\mathrm{Var}[T_n^+] = 0$ for $n < 0$.

Proof. Let

$$D_n(s) = 1 - r_n\alpha_n^0(s) - q_n\alpha_n^-(s)\sigma_{n-1}^+(s) \ , \tag{2.8}$$

so that $\sigma_n^+(s) = p_n\alpha_n^+(s)/D_n(s)$. We easily see that

$$D_n(0) \ = \ p_n \ , \tag{2.9}$$

$$\frac{d}{ds}D_n(s)\bigg|_{s=0} \ = \ \mu_{n:1} - p_n\mu_{n:1}^+ + q_n\mathrm{E}[T_{n-1}^+] \ , \tag{2.10}$$

and

$$\left(\frac{d}{ds}\right)^2 D_n(s)\bigg|_{s=0} \ = \ p_n\mu_{n:2}^+ - \left(\mu_{n:2} + 2q_n\mu_{n:1}^-\mathrm{E}[T_{n-1}^+] + q_n\mathrm{E}[{T_{n-1}^+}^2]\right) . \tag{2.11}$$

The theorem follows by differentiating $\log\sigma_n^+(s) = \log p_n + \log\alpha_n^+(s) - \log D_n(s)$ twice at $s = 0$. □

Remark 2.4

We note that, with $\nu_n = \lambda_n + \mu_n$, $r_0 = 0$, $p_n = \lambda_n/\nu_n$, $q_n = \mu_n/\nu_n$ and $\alpha_n(s) = \alpha_n^+(s) = \alpha_n^-(s) = \nu_n/(s+\nu_n)$, the semi-Markov process $J(t)$ is reduced to a birth-death process governed by upward transition rates λ_n $(n \geq 0)$ and downward transition rates μ_n $(n \geq 1)$. Theorems 2.2 and 2.3 are then rewritten as

$$\sigma_n^+(s) = \frac{\lambda_n}{s + \nu_n - \mu_n\sigma_{n-1}^+(s)} \ , \tag{2.12}$$

$$\mathrm{E}[T_n^+] = \frac{1}{\lambda_n}\left\{1 + \mu_n\mathrm{E}[T_{n-1}^+]\right\} \ , \ n \geq 0 \ , \tag{2.13}$$

where $\mathrm{E}[T_n^+] = 0$ for $n < 0$, and

$$\mathrm{Var}[T_n^+] = \frac{\mu_n}{\lambda_n}\mathrm{Var}[T_{n-1}^+] + \frac{\mu_n}{\lambda_n}\mathrm{E}^2[T_{n-1}^+] + \mathrm{E}^2[T_n^+] \ , \ n \geq 0 \ , \tag{2.14}$$

where $\mathrm{Var}[T_n^+] = 0$ for $n < 0$. While (2.12) and (2.13) are more or less similar to the counterparts of the semi-Markov process, the variance formula of (2.14) is simpler than Theorem 2.3 (b) where the last term in Theorem 2.3 (b) vanishes.

One sees that the first passage time of $J(t)$ from 0 to n , denoted by $T_{0,n}$, can be expressed as the sum of T_j^+ for $j = 0, \cdots, n-1$, i.e.

$$T_{0,n} \stackrel{\mathrm{def}}{=} \sum_{j=0}^{n-1} T_j^+ \ . \tag{2.15}$$

Let $s_{0,n}(\tau)$ be the p.d.f. of $T_{0,n}$ with $\sigma_{0,n}(s) = \int_0^\infty \mathrm{e}^{-s\tau}s_{0,n}(\tau)d\tau$. From (2.15), it follows that

$$\sigma_{0,n}(s) = \prod_{j=0}^{n-1}\sigma_j^+(s) \ , \ n \geq 1 \ . \tag{2.16}$$

It is obvious that the expectation and the variance of $T_{0,n}$ would be the sums of those of T_j^+ for $j = 0, \cdots, n-1$. Namely, one has

$$\mathrm{E}[T_{0,n}] = \sum_{j=0}^{n-1} \mathrm{E}[T_j^+] \ , \tag{2.17}$$

and

$$\mathrm{Var}[T_{0,n}] = \sum_{j=0}^{n-1} \mathrm{Var}[T_j^+] \ . \tag{2.18}$$

3. Distributional Properties of First Passage Times of Skip-Free Semi-Markov Processes

For any birth-death process on $\{0, 1, \cdots\}$, Keilson[8, 9] shows that the first passage time T_n^+ from n to $n+1$ is always a mixture of $n+1$ independent exponential random variables and the first passage time $T_{0,n+1}$ from 0 to $n+1$ is always a sum of $n+1$ independent exponential random variables. It has been also shown in Keilson[10], Rösler[17], and Sumita and Masuda[24] that the p.d.f. of the first passage time $T_{m,n}$ from state m to state n is always unimodal. For skip-free semi-Markov processes, however, since $\sigma_n^-(s)$, $\sigma_n^0(s)$ and $\sigma_n^+(s)$ can be Laplace transforms of arbitrary p.d.f.'s, it is virtually impossible to establish general structural properties concerning zeros of $\det\left(\underline{\underline{I}} - \underline{\underline{\alpha}}_n(s)\right)$ in (2.3). Consequently, the study of distributional properties of first passage times of skip-free semi-Markov processes is quite difficult. In order to demonstrate this difficulty, we report here some results under rather restrictive conditions.

Let CM be the class of completely monotone p.d.f.'s on $[0, \infty)$ defined by

Definition 3.1

$$CM = \left\{ f \middle| f(x) \geq 0,\ \int_0^\infty f(x)dx = 1,\ (-1)^k \left(\frac{d}{dx}\right)^k f(x) \geq 0,\ k = 0, 1, \cdots \right\}.$$

Of related interest is a class of p.d.f.'s each of which is a finite mixture of exponential densities. Formally, we define

$$CM_n = \left\{ f \middle| f(x) = \sum_{i=1}^n p_i \theta_i \mathrm{e}^{-\theta_i x},\ 0 < n < \infty,\ \theta_i, p_i > 0,\ \sum_{i=1}^n p_i = 1,\ \theta_i \neq \theta_j,\ \text{for } i \neq j \right\}.$$

The union of CM_n is denoted by

$$CM^* = \bigcup_{n=0}^\infty CM_n. \tag{3.1}$$

This class is contained in the class CM as a subset. Following Sumita and Masuda[24], we also introduce SCM defined by

$$SCM = \{ f | f = f_1 * f_2 * \cdots * f_n,\ f_i \in CM_{m_i},\ m_i > 0,\ 1 \leq i \leq n \} \ , \tag{3.2}$$

where the asterisk denotes convolution, i.e. $f * g(x) = \int_0^x f(x-y)g(y)dy$. The next proposition is well known, see e.g. Lemma 2.12.1 of Steutel [19].

Proposition 3.2

$$[f \in CM_n] \Leftrightarrow \left[\phi_f(s) = \frac{\prod_{i=1}^{n-1}(1+s/\eta_i)}{\prod_{j=1}^{n}(1+s/\theta_i)},\ 0 < \theta_1 < \eta_1 < \theta_2 < \eta_2 < \cdots < \eta_{n-1} < \theta_n\right].$$

One sees from Theorem 2.2 that

$$\sigma_0^+(s) = \frac{p_0\alpha_0^+(s)}{1 - r_0\alpha_0^0(s)} \ . \tag{3.3}$$

The next theorem then directly follows from Theorem 3.2 of Sumita and Masuda[24]. The p.d.f. of T_n^+ is denoted by $s_n^+(\tau)$ so that $\sigma_n^+(s) = \int_0^\infty \mathrm{e}^{-s\tau} s_n^+(\tau)d\tau$.

Theorem 3.3
Let $a_0^+ \in CM_n$ and $a_0^0 \in CM_m$. Let $\rho(s) = \big(1 - r_0\alpha_0^0(s)\big)^{-1}$. Then $s_0^+ \in CM_r$ for some r with $0 < r < m+n$ if and only if $\big[\rho(s) \leq 0 \Rightarrow \alpha_0^+(s) \geq 0\big]$.

It may be worth noting that even when both a_0^+ and a_0^0 are exponential, the p.d.f. s_0^+ still may not be a mixture of exponential p.d.f.'s. As an example, let

$$\alpha_0^+(s) = \frac{\lambda}{s+\lambda};\ \alpha_0^0(s) = \frac{\mu}{s+\mu},\ \lambda \neq \mu \ . \tag{3.4}$$

One then easily sees that

$$\sigma_0^+(s) = \frac{p_0(\lambda-\mu)}{\lambda - p_0\mu} \cdot \frac{\lambda}{s+\lambda} + \frac{\lambda(1-p_0)}{\lambda - p_0\mu} \cdot \frac{p_0\mu}{s+p_0\mu} \ . \tag{3.5}$$

Hence $s_0^+(\tau)$ is a mixture of exponential p.d.f.'s if and only if $\lambda > \mu$.

In general, sufficient conditions under which $s_n^+(\tau)$ is a mixture of exponential p.d.f.'s are hard to come by. We therefore consider skip-free semi-Markov processes with a special structure where the dwell times of the process at any state j ($0 \leq j \leq n$) do not depend on the next destination. This is equivalent to saying that

$$\begin{cases} a_0(\tau) = a_0^0(\tau) = a_0^+(\tau); \\ a_j(\tau) = a_j^-(\tau) = a_j^0(\tau) = a_j^+(\tau),\ 1 \leq j \leq n. \end{cases} \tag{3.6}$$

We next show that, under the conditions of (3.6), if $a_j(\tau)$ are finite mixtures of exponential p.d.f.'s for $0 \leq j \leq n$, so are the first passage time p.d.f.'s $s_j^+(\tau)$ for $0 \leq j \leq n$.

Theorem 3.4
Under the conditions of (3.6), let $a_j(\tau) \in CM^*$ for $0 \leq j \leq n$. Then $s_j^+(\tau) \in CM^*$ for $0 \leq j \leq n$.

Proof. We prove the theorem by induction. For $j = 0$, one sees from (3.3) that $\sigma_0^+(s) = p_0\alpha_0(s)/\{1 - r_0\alpha_0(s)\}$. Hence from Theorem 3.3, $s_0^+(\tau) \in CM^*$. Suppose $s_{j-1}^+(\tau) \in CM^*$ for $j \le n$ and consider s_j^+. We note from Theorem 2.2 that

$$\sigma_j^+(s) = p_j\alpha_j^+(s)/D_j(s) \tag{3.7}$$

where $D_j(s)$ is defined in (2.8). One easily sees that

$$\frac{d}{ds}D_j(s) = -\left[r_j\frac{d}{ds}\alpha_j(s) + q_j\left\{\sigma_{j-1}^+(s)\frac{d}{ds}\alpha_j(s) + \alpha_j(s)\frac{d}{ds}\sigma_{j-1}^+(s)\right\}\right]. \tag{3.8}$$

It then follows from (3.7) and (3.8) that

$$\frac{d}{ds}\sigma_j^+(s) = \frac{p_j}{D_j(s)^2}\left[\frac{d}{ds}\alpha_j(s) + q_j\alpha_j^2(s)\frac{d}{ds}\sigma_{j-1}^+(s)\right]. \tag{3.9}$$

By the assumption, for $0 \le j \le n$, one has $a_j \in CM^*$ so that $\frac{d}{ds}\alpha_j(s) < 0$ for real s apart from singularities. For $j \le n$, the induction hypothesis assures that $s_{j-1}^+ \in CM^*$ and hence $\frac{d}{ds}\sigma_{j-1}^+(s) < 0$ for real s apart from singularities. Consequently $\frac{d}{ds}\sigma_j^+(s) < 0$ for real s apart from singularities. This means that $\sigma_j^+(s)$ is strictly decreasing in real s apart from singularities.

Since both $a_j(\tau)$ and $s_{j-1}^+(\tau)$ belong to CM^*, from Proposition 3.2 there exist polynomials $f_{j:K-1},\ f_{j:K}$ and $g_{j-1:L-1},\ g_{j-1:L}$ such that

$$\alpha_j(s) = \frac{f_{j:K-1}(s)}{f_{j:K}(s)};\ \sigma_{j-1}^+(s) = \frac{g_{j-1:L-1}(s)}{g_{j-1:L}(s)}. \tag{3.10}$$

Here K and L denote the order of polynomials. All of the four polynomials have only zeros of multiplicity one on the negative real axis. Furthermore zeros of $f_{j:K}(s)$ ($g_{j-1:L}(s)$) interleave those of $f_{j:K-1}(s)$ ($g_{j-1:L-1}(s)$). From (3.7), it can be readily seen that

$$\sigma_j^+(s) = \frac{p_j\frac{f_{j:K-1}(s)}{f_{j:K}(s)}}{1 - r_j\frac{f_{j:K-1}(s)}{f_{j:K}(s)} - q_j\frac{f_{j:K-1}(s)g_{j-1:L-1}(s)}{f_{j:K}(s)g_{j-1:L}(s)}}. \tag{3.11}$$

Any common factors between $f_{j:K-1}(s)$ and $g_{j-1:L}(s)$ should be cancelled in the last term of the denominator. Let l be the number of common factors which the two polynomials share where $0 \le l \le \min\{K-1, L\}$. Then the last term of the denominator can be rewritten as

$$\frac{f_{j:K-1}(s)g_{j-1:L-1}(s)}{f_{j:K}(s)g_{j-1:L}(s)} = \frac{\tilde{f}_{j:K-1-l}(s)g_{j-1:L-1}(s)}{f_{j:K}(s)\tilde{g}_{j-1:L-l}(s)}.$$

Multiplying both the numerator and the denominator of (3.11) by $f_{j:K}(s)\tilde{g}_{j-1:L-l}(s)$, one finds that

$$\sigma_j^+(s) = \frac{g_{j:M-1}(s)}{g_{j:M}(s)} \tag{3.12}$$

where

$$g_{j:M-1}(s) = p_j f_{j:K-1}(s)\tilde{g}_{j-1:L-l}(s) \tag{3.13}$$

and

$$g_{j:M}(s) = f_{j:K}(s)\tilde{g}_{j-1:L-l}(s) - r_j f_{j:K-1}(s)\tilde{g}_{j-1:L-l}(s) - q_j \tilde{f}_{j:K-1-l}(s) g_{j-1:L-1}(s) \ . \tag{3.14}$$

We note that $M = K + L - l$ and $\sigma_j^+(s)$ has exactly $M-1$ zeros of multiplicity one, all of which are located on the negative real axis. We have seen that $\frac{d}{ds}\sigma_j^+(s) < 0$ for real s apart from singularities. Hence as a real function of s, $\sigma_j^+(s)$ is strictly decreasing between singularities. Furthermore from Theorem 1.1 of Sumita and Masuda[24], $\sigma_j^+(s)$ cannot have a singular point in the positive right half plane. Hence $g_{j:M}(s)$ has M distinct zeros on the negative real axis which interleave zeros of $g_{j:M-1}(s)$. The theorem then follows from Proposition 3.2. □

The next corollary follows immediately from Theorem 3.4 and the definition of SCM in (3.2).

Corollary 3.5
Under the conditions of (3.6), let the p.d.f. $s_j^+(\tau)$ of T_j^+ for $0 \le j \le n$ be as in Theorem 3.4. Then the p.d.f. $s_{0,n+1}(\tau)$ of $T_{0,n+1}$ satisfies $s_{0,n+1} \in SCM$.

Proof. Since $T_{0,n+1}$ is defined by $T_{0,n+1} = T_0^+ + T_1^+ + \cdots T_n^+$, the p.d.f. $s_{0,n+1}(\tau)$ of $T_{0,n+1}$ is given by $s_{0,n+1}(\tau) = s_0^+ * s_1^+ * \cdots * s_n^+(\tau)$, where the asterisk denotes convolution. From Theorem 3.4 , for each j , $0 \le j \le n$, there exists $m(j)$ such that $s_j^+ \in CM_{m(j)}$. Hence $s_{0,n+1}(\tau) \in SCM$ from (3.2), completing the proof. □

The discussions in this section reveal that the distributional properties of first passage times for birth-death processes are not necessarily inherited in those for skip-free semi-Markov processes. Their limiting behaviors, however, seem to be quite similar as we show in the next section.

4. Limiting Behavior of First Passage Times for Skip-Free Semi-Markov Processes

Let $N(t)$ be a birth-death process governed by upward transition rates $\lambda_n > 0,\ n \ge 0$ and downward transition rates $\mu_n > 0,\ n \ge 1$ satisfying $\lambda_n \to \lambda > 0$ and $\mu_n \to \mu > 0$. When $\rho = \lambda/\mu < 1$, $N(t)$ is ergodic. In this case, Keilson[9] has shown the following limit theorems concerning the first passage times T_n^+ and $T_{0,n}$.

Theorem 4.1 (Theorem 8.2B of Keilson[9])
If $0 < \rho < 1$, then

(a) $\dfrac{T_{0,n}}{\mathrm{E}[T_{0,n}]} \xrightarrow{\mathrm{d}} E$ as $n \to \infty,$ where $\mathrm{P}[E > x] = \mathrm{e}^{-x}$;

(b) $\dfrac{T_n^+}{\mathrm{E}[T_n^+]} \xrightarrow{\mathrm{d}} X$ as $n \to \infty,$ where $\mathrm{P}[X > x] = (1-\rho)\mathrm{e}^{-(1-\rho)x}$.

It should be noted that the random variable X has a mass ρ at $X = 0$, describing the jitter effect. This means that, when n is large, the sum of the ergodic probabilities from state 0 up to state n is very close to 1 and $\mathrm{E}[T_{0,n+1}]$ becomes very large. If the process enters the state n, one observes clustering of the epochs at which the process crosses from n to $n+1$ within a time interval much smaller compared to $\mathrm{E}[T_{0,n+1}]$. As this scaling factor $\mathrm{E}[T_{0,n+1}]$ goes to infinity as $n \to \infty$, these multiple crossings amount to the mass ρ of the limiting distribution at $X = 0$.

When $\rho > 1$, $N(t)$ is not ergodic. For this non-ergodic case, Sumita[25] has proven that the following limit theorems hold true.

Theorem 4.2 (Theorem 0.1 (2) of Sumita[25])
If $\rho > 1$, then

(a) $\quad T_n^+ \xrightarrow{\mathrm{d}} T_{BP(\mu,\lambda)}$;

(b) $\quad \dfrac{T_{0,n}}{\mathrm{E}[T_{0,n}]} \to 1$ with probability 1 as $n \to \infty$.

Here, $T_{BP(\mu,\lambda)}$ is the server busy period of $M/M/1$ queueing system with Poisson arrivals of intensity μ and the exponential service rate λ.

The purpose of this section is to show that the limit theorems in Theorem 4.1 and Theorem 4.2 for birth-death processes can be more or less carried over to skip-free semi-Markov processes. Throughout this section, we assume that,
for $n \in \mathcal{N} = \{0, 1, 2, \cdots\}$,

$$(4.1)\qquad \begin{gathered} p_n > 0\ ,\ \lim_{n\to\infty} p_n = p > 0\ ,\ q_n > 0\ ,\ \lim_{n\to\infty} q_n = q > 0\ ;\ \text{and} \\ \lim_{n\to\infty} \alpha_n^-(s) = \alpha^-(s)\ ,\ \lim_{n\to\infty} \alpha_n^0(s) = \alpha^0(s)\ ,\ \lim_{n\to\infty} \alpha_n^+(s) = \alpha^+(s)\ . \end{gathered}$$

For notational convenience, we also define

$$(4.2)\qquad \hat{\rho} \stackrel{\text{def}}{=} \frac{p}{q}\ .$$

We first show that the counterpart of Theorem 4.1 for skip-free semi-Markov processes is present under the ergodic condition $0 < \hat{\rho} < 1$.

Theorem 4.3
Let $T_{0,n}$ be the first passage time of a skip-free semi-Markov process from 0 to n satisfying (4.1). If $0 < \hat{\rho} < 1$, then

$$\frac{T_{0,n}}{\mathrm{E}[T_{0,n}]} \xrightarrow{\mathrm{d}} E \text{ as } n \to \infty,\ \text{where } \mathrm{P}[E > x] = \mathrm{e}^{-x}.$$

Proof. We consider the regenerative process representing the return of the underlying skip-free semi-Markov process from state 0 to state 0. Let $\underline{V}_k = [Y_k, Z_k]$ where Z_k is the length of the k-th regenerative cycle and Y_k is the largest state visited within this regenerative cycle. It should be noted that Y_k and Z_k are correlated but $\underline{V}_k$ ($k =$

$1, 2, 3, \cdots$) constitute a sequence of independently and identically distributed (i.i.d.) random vectors.

Let the domain of $\underline{V}_k$ be decomposed into $\mathcal{N} \times R^+ = G(n) \bigcup B(n)$ where $G(n) = \{(m,x) \;:\; 0 \leq m < n \;, x \geq 0\}$ and $B(n) = \{(m,x) \;:\; n \leq m \;, x \geq 0\}$. We now consider the following experiment. If $\underline{V}_k \in G(n)$, then the experiment continues and $\underline{V}_{k+1}$ is chosen. The experiment stops when a random vector falls in $B(n)$. In other words, the experiment stops with probability $\eta_n = \mathrm{P}[\underline{V}_k \in B(n)]$. It should be noted that the first passage time $T_{0,n}$ from state 0 to state n is the sum of Z_k's until the experiment stops, and η_n is the probability that the state n is visited before returning to state 0 within a regenerative cycle. We next show that $\eta_n \to 0$ as $n \to \infty$ under the conditions of (4.1). The limit theorem then follows by Theorem 1.A4 of Shanthikumar and Sumita[20].

Let $\xi_{m,n}$ be the probability that the process $J(t)$ reaches state n before state 0 , given that $J(0) = m$, $1 \leq m < n$. One then easily sees that

$$(4.3)\quad \begin{cases} \xi_{1,n} &= r_1\xi_{1,n} + p_1\xi_{2,n} \;; \\ \xi_{m,n} &= q_m\xi_{m-1,n} + r_m\xi_{m,n} + p_m\xi_{m+1,n} \;, \qquad 2 \leq m \leq n-2 \;; \\ \xi_{n-1,n} &= q_{n-1}\xi_{n-2,n} + r_{n-1}\xi_{n-1,n} + p_{n-1} \;. \end{cases}$$

Let $\Delta\xi_{m,n+1} \stackrel{\text{def}}{=} \xi_{m,n+1} - \xi_{m-1,n+1}$ and define $\hat{\rho}_j \stackrel{\text{def}}{=} p_j/q_j$. Substituting $r_m = 1 - q_m - p_m$ for $1 \leq m < n$ into (4.3), it follows that

$$(4.4)\quad \begin{cases} \xi_{1,n} &= \hat{\rho}_1\Delta\xi_{2,n} \;; \\ \Delta\xi_{m,n} &= \hat{\rho}_m\Delta\xi_{m+1,n}, \qquad 2 \leq m \leq n-2 \;; \\ \Delta\xi_{n-1,n} &= \hat{\rho}_{n-1}(1 - \xi_{n-1,n}) \;, \end{cases}$$

which in turn leads to

$$(4.5)\qquad \xi_{1,n} = (1 - \xi_{n-1,n}) \prod_{j=1}^{n-1} \hat{\rho}_j \;.$$

From (4.1), for arbitrarily small $\varepsilon > 0$, there exists $M(\varepsilon)$ such that $|q_n - q| < \varepsilon$ and $|p_n - p| < \varepsilon$ for $n > M(\varepsilon)$. One then sees from (4.5) that, for $n > M(\varepsilon) + 1$,

$$(4.6)\quad \left(\prod_{j=1}^{M(\varepsilon)} \hat{\rho}_j\right) \left(\frac{p-\varepsilon}{q+\varepsilon}\right)^{n-M(\varepsilon)-2} < \prod_{j=1}^{n-1} \hat{\rho}_j < \left(\prod_{j=1}^{M(\varepsilon)} \hat{\rho}_j\right) \left(\frac{p+\varepsilon}{q-\varepsilon}\right)^{n-M(\varepsilon)-2} .$$

Under the assumption that $0 < \hat{\rho} < 1$, one sees for sufficiently small $\varepsilon > 0$ that $(p-\varepsilon)/(q+\varepsilon) < 1$ and $(p+\varepsilon)/(q-\varepsilon) < 1$, so that $\xi_{1,n} \to 0$ as $n \to \infty$ from (4.5) and (4.6). Since $\eta_n = r_0\eta_n + p_0\xi_{1,n}$ and therefore $\eta_n = p_0\xi_{1,n}/(1-r_0)$, one concludes that $\eta_n \to 0$ as $n \to \infty$, completing the proof. □

We next turn our attention to the exponential limit theorem of Theorem 4.1 (b) with the jitter effect. As for birth-death processes, this limit theorem still holds true for skip-free semi-Markov processes. A preliminary lemma is needed.

Lemma 4.4

Let T_n^+ be the first passage time of a skip-free semi-Markov process from n to $n+1$ satisfying (4.1). If $0 < \hat{\rho} < 1$, then

$$\text{(a)} \quad \frac{\mathrm{E}[T_{n-1}^+]}{\mathrm{E}[T_n^+]} \;\rightarrow\; \hat{\rho} \text{ as } n \rightarrow \infty \;;$$

$$\text{(b)} \quad \frac{\mathrm{E}[T_n^+]}{\mathrm{E}[T_{0,n+1}]} \;\rightarrow\; 1-\hat{\rho} \text{ as } n \rightarrow \infty \;.$$

Proof. From the recursion formula in Theorem 2.3 (a), one sees for $n \geq 1$ that

$$\frac{\mathrm{E}[T_{n-1}^+]}{\mathrm{E}[T_n^+]} = \frac{p_n}{q_n} - \frac{\mu_{n:1}}{q_n \mathrm{E}[T_n^+]} \leq \hat{\rho}_n \;. \tag{4.7}$$

We next show that $\frac{\mu_{n:1}}{q_n \mathrm{E}[T_n^+]}$ vanishes as $n \rightarrow \infty$. To do so, we write

$$\frac{\mu_{n:1}}{q_n \mathrm{E}[T_n^+]} = \frac{\mu_{n:1}}{q_n \mathrm{E}[T_0^+]} \cdot \frac{\mathrm{E}[T_0^+]}{\mathrm{E}[T_1^+]} \cdot \frac{\mathrm{E}[T_1^+]}{\mathrm{E}[T_2^+]} \cdots \frac{\mathrm{E}[T_{n-2}^+]}{\mathrm{E}[T_{n-1}^+]} \cdot \frac{\mathrm{E}[T_{n-1}^+]}{\mathrm{E}[T_n^+]} \;.$$

From (4.7), this leads to

$$\frac{\mu_{n:1}}{q_n \mathrm{E}[T_n^+]} \leq \frac{p_n \mu_{n:1}}{q_n \mathrm{E}[T_0^+]} \cdot \prod_{j=1}^{n} \hat{\rho}_j \;. \tag{4.8}$$

For arbitrarily small $\varepsilon > 0$, let $M(\varepsilon)$ be as in the proof of Theorem 4.3. From (4.6) and (4.7), it then follows that $\mu_{n:1}/q_n \mathrm{E}[T_n^+] \rightarrow 0$ as $n \rightarrow \infty$. This in turn implies from (4.7) that $\mathrm{E}[T_{n-1}^+]/\mathrm{E}[T_n^+] \rightarrow \hat{\rho}$ as $n \rightarrow \infty$, proving (a). For (b), it can be seen from (2.17) that

$$\frac{\mathrm{E}[T_{0,n+1}]}{\mathrm{E}[T_n^+]} = \frac{\sum_{j=0}^{n} \mathrm{E}[T_j^+]}{\mathrm{E}[T_n^+]} = 1 + \sum_{j=0}^{n-1} \frac{\mathrm{E}[T_j^+]}{\mathrm{E}[T_n^+]} = 1 + \sum_{k=1}^{n} \prod_{j=k}^{n} \frac{\mathrm{E}[T_{j-1}^+]}{\mathrm{E}[T_j^+]} \;.$$

From (a), one has $\mathrm{E}[T_{j-1}^+]/\mathrm{E}[T_j^+] \rightarrow \hat{\rho}$ as $j \rightarrow \infty$. From Lemma 8.3B of Keilson[9], it then follows that

$$\lim_{n\rightarrow\infty} \frac{\mathrm{E}[T_{0,n+1}]}{\mathrm{E}[T_n^+]} = 1 + \frac{\hat{\rho}}{1-\hat{\rho}} = \frac{1}{1-\hat{\rho}} \;, \tag{4.9}$$

completing the proof □

We are now in a position to prove the exponential limit theorem with the jitter effect for skip-free semi-Markov processes.

Theorem 4.5

If $0 < \hat{\rho} < 1$, then

$$\frac{T_n^+}{\mathrm{E}[T_n^+]} \xrightarrow{\mathrm{d}} X \text{ as } n \rightarrow \infty \;, \text{ where } \mathrm{P}[X > x] = (1-\hat{\rho})\mathrm{e}^{-(1-\hat{\rho})x} \;.$$

Proof. From (2.16), one has

$$\sigma_n^+(s) = \frac{\sigma_{0,n+1}(s)}{\sigma_{0,n}(s)} ,$$

so that

$$\sigma_n^+\left(\frac{s}{\mathrm{E}[T_n^+]}\right) = \frac{\sigma_{0,n+1}\left(\frac{s}{\mathrm{E}[T_{0,n+1}]}\frac{\mathrm{E}[T_{0,n+1}]}{\mathrm{E}[T_n^+]}\right)}{\sigma_{0,n}\left(\frac{s}{\mathrm{E}[T_{0,n}]}\frac{\mathrm{E}[T_{0,n}]}{\mathrm{E}[T_n^+]}\right)} . \tag{4.10}$$

Let E be the exponential variate of mean 1 as in Theorem 4.3.It should be noted from Lemma 4.4 that

$$\lim_{n\to\infty}\frac{\mathrm{E}[T_{0,n}]}{\mathrm{E}[T_n^+]} = \lim_{n\to\infty}\frac{\mathrm{E}[T_{0,n+1}] - \mathrm{E}[T_n^+]}{\mathrm{E}[T_n^+]} = \frac{1}{1-\hat{\rho}} - 1 = \frac{\hat{\rho}}{1-\hat{\rho}} . \tag{4.11}$$

It then follows from (4.10), Theorem 4.3, Lemma 4.4 and (4.11) that

$$\sigma_n^+\left(\frac{s}{\mathrm{E}[T_n^+]}\right) \to \frac{\mathrm{E}[\mathrm{e}^{-s\left(\frac{1}{1-\hat{\rho}}\right)E}]}{\mathrm{E}[\mathrm{e}^{-s\left(\frac{\hat{\rho}}{1-\hat{\rho}}\right)E}]} = \frac{\left[\frac{1}{1-\hat{\rho}}s+1\right]^{-1}}{\left[\frac{\hat{\rho}}{1-\hat{\rho}}s+1\right]^{-1}} = \hat{\rho} + (1-\hat{\rho})\frac{(1-\hat{\rho})}{s+(1-\hat{\rho})} ,$$

as $n \to \infty$, completing the proof. □

Unlike a birth-death process satisfying $\lambda_n \to \lambda > 0$ and $\mu_n \to \mu > 0$ as $n \to \infty$, a skip-free semi-Markov process satisfying (4.1), in general, cannot be related to a queueing system easily. When $\hat{\rho} > 1$, the counterpart of the limit theorem of Sumita[21] in Theorem 4.2 (a) for skip-free semi-Markov processes, therefore, has to take a different form.

Theorem 4.6
If $\hat{\rho} > 1$, then T_n^+ converges in distribution to T^+ where $\sigma^+(s) = \mathrm{E}[\mathrm{e}^{-sT^+}]$ is given by

$$\sigma^+(s) = \frac{1 - r\alpha^0(s) - \sqrt{\{1 - r\alpha^0(s)\}^2 - 4pq\alpha^+(s)\alpha^-(s)}}{2q\alpha^-(s)} . \tag{4.12}$$

Proof. Let $g_n(x,s)$ be defined as

$$g_n(x,s) \stackrel{\mathrm{def}}{=} \frac{p_n\alpha_n^+(s)}{1 - r_n\alpha_n^0(s) - q_n\alpha_n^-(s)x},\ 0 \le x \le 1 . \tag{4.13}$$

From Theorem 2.2, one sees that $\sigma_n^+(s) = g(\sigma_{n-1}^+(s), s)$ so that

$$\sigma_n^+(s) - \sigma_{n-1}^+(s) = \int_{\sigma_{n-2}^+(s)}^{\sigma_{n-1}^+(s)} \frac{d}{dx}g(x,s)dx . \tag{4.14}$$

By differentiating (4.13) with respect to x , one has

$$\frac{d}{dx}g_n(x,s) = \frac{p_n\alpha_n^+(s)q_n\alpha_n^-(s)}{\left\{1 - r_n\alpha_n^0(s) - q_n\alpha_n^-(s)x\right\}^2} . \tag{4.15}$$

Since the domain of $\frac{d}{dx}g_n(x,s)$ is $0 \le x \le 1$, its range is given by

$$(4.16)\quad \frac{p_n\alpha_n^+(s)q_n\alpha_n^-(s)}{\{1-r_n\alpha_n^0(s)\}^2} \le \frac{d}{dx}g_n(x,s) \le \frac{p_n\alpha_n^+(s)q_n\alpha_n^-(s)}{\{1-r_n\alpha_n^0(s)-q_n\alpha_n^-(s)\}^2} \stackrel{\text{def}}{=} \theta_n(s).$$

It can be readily seen that $\theta_n(s)$ is monotone decreasing for real $s > 0$, and therefore $\theta_n(s) < \theta_n(0) \stackrel{\text{def}}{=} \theta_n$. From (4.14) it then follows that

$$\begin{aligned} \left|\sigma_n^+(s) - \sigma_{n-1}^+(s)\right| &= \left|\int_{\sigma_{n-2}^+(s)}^{\sigma_{n-1}^+(s)} \frac{d}{dx}g(x,s)dx\right| \\ &\le \theta_n(s)\left|\sigma_{n-1}^+(s) - \sigma_{n-2}^+(s)\right| \\ &< \theta_n\left|\sigma_{n-1}^+(s) - \sigma_{n-2}^+(s)\right| . \end{aligned}$$

Under the conditions of (4.1), one has $\theta_n = \theta_n(0) = p_nq_n/(1-r_n-q_n)^2 = q_n/p_n \to 1/\hat{\rho} \stackrel{\text{def}}{=} \theta$ as $n \to \infty$. Since $\hat{\rho} > 1$, for sufficiently small $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $\theta - \varepsilon < \theta_n < \theta + \varepsilon < 1$ for $n > N(\varepsilon)$. This implies, for $n > N(\varepsilon)$, that

$$\left|\sigma_n^+(s) - \sigma_{n-1}^+(s)\right| < (\theta+\varepsilon)\left|\sigma_{n-1}^+(s) - \sigma_{n-2}^+(s)\right| .$$

From the contraction mapping theorem, it then follows that $\sigma_n^+(s)$ converges uniquely to a function of s denoted by $\sigma^+(s)$ as $n \to \infty$. It remains to show that $\sigma^+(s)$ satisfies (4.12).

From Theorem 2.2, the unique convergence of $\sigma_n^+(s)$ to $\sigma^+(s)$ implies that $\sigma^+(s)$ has to satisfy

$$\sigma^+(s) = \frac{p\alpha^+(s)}{1 - r\alpha^0(s) - q\alpha^-(s)\sigma^+(s)} ,$$

which in turn leads to the following functional equation.

$$(4.17)\qquad q\alpha^-(s)\sigma^+(s)^2 - \{1 - r\alpha^0(s)\}\sigma^+(s) + p\alpha^+(s) = 0 .$$

This quadratic equation has two solutions given by

$$\frac{1 - r\alpha^0(s) \pm \sqrt{\{1-r\alpha^0(s)\}^2 - 4pq\alpha^+(s)\alpha^-(s)}}{2q\alpha^-(s)} .$$

Since $\sigma^+(0) = 1$, one concludes that

$$\sigma^+(s) = \frac{1 - r\alpha^0(s) - \sqrt{\{1-r\alpha^0(s)\}^2 - 4pq\alpha^+(s)\alpha^-(s)}}{2q\alpha^-(s)} ,$$

completing the proof. □

We next derive the mean and variance of T^+ based on Theorem 4.6. For notational convenience, we denote the limits of the first and second moments given in (2.7) as

$$(4.18)\qquad \begin{aligned} &\hat{\mu}_k^- \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu_{n:k}^- \ ; \ \hat{\mu}_k^0 \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu_{n:k}^0 \ ; \ \hat{\mu}_k^+ \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu_{n:k}^+ \ ; \\ &\hat{\mu}_k = \hat{\mu}_k^- + \hat{\mu}_k^0 + \hat{\mu}_k^+ \ , \ \text{for } k = 1, 2 \ . \end{aligned}$$

It is assumed that

$$\hat{\mu}_k < \infty \ , \ \text{for } k = 1, 2 \ . \tag{4.19}$$

Theorem 4.7

(4.20) (a) $\mathrm{E}[T^+] = \dfrac{\hat{\mu}_1}{p-q}$

(4.21) (b) $\mathrm{Var}[T^+] = \dfrac{\hat{\mu}_2}{p-q} + \left\{ \dfrac{2p(q\hat{\mu}_1^+ + r\hat{\mu}_1^0 + p\hat{\mu}_1^-)}{(p-q)^3} - \dfrac{\hat{\mu}_1}{(p-q)^2} + \dfrac{2\hat{\mu}_1^-}{p-q} \right\} \hat{\mu}_1$

Proof. Let $C(s)$ be defined as

$$C(s) = 1 - r\alpha^0(s) - \sqrt{\{1 - r\alpha^0(s)\}^2 - 4pq\alpha^+(s)\alpha^-(s)} \ , \tag{4.22}$$

so that $\sigma^+(s) = C(s)/2q\alpha^-(s)$. One easily sees that

$$C(0) = -2q \ , \tag{4.23}$$

$$\frac{d}{ds}C(s)\bigg|_{s=0} = \frac{-2q(p\hat{\mu}_1^+ + r\hat{\mu}_1^0 + p\hat{\mu}_1^-)}{p-q} \ , \tag{4.24}$$

and

$$\left(\frac{d}{ds}\right)^2 C(s)\bigg|_{s=0} = \frac{-2q(p\hat{\mu}_2^+ + r\hat{\mu}_2^0 + p\hat{\mu}_2^-)}{p-q} + \frac{4pq\hat{\mu}_1(q\hat{\mu}_1^+ + r\hat{\mu}_1^0 + p\hat{\mu}_1^-)}{(p-q)^3} \ . \tag{4.25}$$

The theorem now follows by differentiating $\log \sigma^+(s) = \log C(s) - \log 2q\alpha^-(s)$ twice at $s = 0$. □

In order to prove the counterpart of Theorem 4.2 (b) for skip-free semi-Markov processes, a preliminary lemma is needed.

Lemma 4.8
Under the condition of (4.19), the following statement holds true.

$$\frac{1}{n}\mathrm{E}[T_{0,n}] \ \to \ \frac{\hat{\mu}_1}{p-q} \ \text{as } n \to \infty \ .$$

Proof. From Theorem 4.7 (a), for arbitrarily small $\varepsilon > 0$, there exists $M(\varepsilon)$ such that $|\mathrm{E}[T_n^+] - \hat{\mu}_1/(p-q)| < \varepsilon$ for $n > M(\varepsilon)$. From (2.17), it can be seen that

$$\frac{1}{n}\mathrm{E}[T_{0,n}] = \frac{1}{n}\sum_{j=0}^{n-1}\mathrm{E}[T_j^+] = \frac{1}{n}\sum_{j=0}^{M(\varepsilon)}\mathrm{E}[T_j^+] + \frac{1}{n}\sum_{j=M(\varepsilon)+1}^{n-1}\mathrm{E}[T_j^+] \ ,$$

so that one has, for $n > M(\varepsilon) + 1$,

$$(4.26)\quad \frac{1}{n}\sum_{j=0}^{M(\varepsilon)} \mathrm{E}[T_j^+] + \frac{n - M(\varepsilon) - 2}{n}\left(\frac{\hat{\mu}_1}{p-q} - \varepsilon\right) < \frac{1}{n}\sum_{j=0}^{n-1}\mathrm{E}[T_j^+]$$

$$< \frac{1}{n}\sum_{j=0}^{M(\varepsilon)} \mathrm{E}[T_j^+] + \frac{n - M(\varepsilon) - 2}{n}\left(\frac{\hat{\mu}_1}{p-q} + \varepsilon\right) ,$$

and hence $(1/n)\mathrm{E}[T_{0,n}] \to \hat{\mu}_1/(p-q)$ as $n \to \infty$, completing the proof. □

The following theorem can now be proven.

Theorem 4.9

$$\frac{T_{0,n}}{\mathrm{E}[T_{0,n}]} \to 1 \text{ with probability 1 as } n \to \infty.$$

Proof. Let $X_n \stackrel{\text{def}}{=} T_n^+ - \mathrm{E}[T_n^+]$ for $n \geq 0$. Clearly X_n are mutually independent and $\mathrm{E}[X_n] = 0$ so that $\mathrm{E}[X_n^2] = \mathrm{Var}[T_n^+]$ for $n \geq 0$. Under the condition of (4.19), it can be seen from Theorem 4.7 (b) that $\lim_{n\to\infty} \mathrm{Var}[T_n^+] < \infty$. This then implies that

$$\sum_{m=0}^{\infty} \frac{1}{m^2}\mathrm{E}[X_{m-1}^2] < \infty .$$

Hence from VII.8 Theorem 3 of Feller[3], one has

$$\frac{1}{n}\sum_{m=1}^{n} X_m = \frac{\mathrm{E}[T_{0,n}]}{n}\left\{\frac{T_{0,n}}{\mathrm{E}[T_{0,n}]} - 1\right\} \to 0$$

with probability 1 as $n \to \infty$. The theorem now follows from Lemma 4.8. □

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