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DYNAMIC ANALYSIS OF A REWARD PROCESS DEFINED ON A CYCLIC RENEWAL PROCESS WITH APPLICATIONS TO PREVENTIVE MAINTENANCE PROBLEMS

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Abstract   A cyclic renewal process is considered as an extension of an alternating renewal process, where each of the underlying independently and identically distributed (i.i.d.) nonnegative random increments is composed of multiple stages. Such a process may be appropriate for analyzing optimal preventive maintenance policies for production management, where a pair of two stages representing an uptime until a minor failure and the subsequent minimal repair time would be repeated until it is decided to conduct a complete overhaul. In order to address economic problems in such applications, we also introduce a reward process with jumps defined on the cyclic renewal process. When the system is running in stage j, the profit grows linearly at the rate of $\rho(j)$. Upon a minor failure, the subsequent minimal repair in stage $(j+1)$ incurs the linear cost at the rate of $\rho(j+1)$. In addition, the fixed cost may be imposed whenever either a minimal repair or a complete overhaul takes place, resulting in jumps of the reward process. The problem is then to determine when to conduct a complete overhaul so as to maximize the total reward in the time interval $(0, T]$. A multivariate Markov process generated from both the cyclic renewal process and the reward process is studied extensively, yielding various new transform results explicitly and deriving their asymptotic expansions. These results are used to numerically explore optimal preventive maintenance policies for production management.

Keywords: Applied probability, cyclic renewal process, preventive maintenance

1.   Introduction

Renewal theory is the branch of probability theory concerning a variety of problems related to the partial sums of a sequence of i.i.d. nonnegative random variables. More specifically, let $(Y_n)_{n=1}^\infty$ be a sequence of i.i.d. nonnegative random variables and define $S_n = \sum_{j=1}^n Y_j$. The renewal process $\{N(t) : t \geqslant 0\}$ associated with $(Y_n)_{n=1}^\infty$ is a counting process defined by $N(t) = n$ if and only if $S_n \leqslant t < S_{n+1}$. Of interest are the renewal function $H(t) = \mathrm{E}[N(t)]$, the renewal density $h(t) = \frac{d}{dt}H(t)$ if it exists, and other related probabilistic entities. As the name "renewal theory" indicates, the study stemmed from a class of applications involving successive replacements of items subject to failure. Here, Y_n denotes the lifetime of the n-th item and $N(t)$ is the number of replacements that took place by time t.

The renewal theory has been extended in many ways. A delayed renewal process, for example, has the distribution of Y_1 different from that of $Y_n(n > 1)$, and an alternating renewal process deals with a situation where Y_n consists of two stages : the system uptime and the system repair time, see e.g. Cox [4]. A Markov renewal process considers a case where distributions of interfailure times are governed by a Markov chain $\{J(n) : n = 0, 1, 2, \cdots\}$ in discrete time, i.e. if $J(n-1) = i$ and $J(n) = j$, then the distribution of Y_n is given by

$A_{ij}(x)$. The reader is referred to Keilson [12], Keilson and Rao [13, 14], and an excellent survey paper by Çinlar [2] for the study of Markov renewal processes. Keener [11] develops a general renewal theory where i.i.d. increments have support on full continuum. In Kijima and Sumita [15], the renewal theory is extended in that the distribution of the Y_n depends on the partial sum S_n up to the n-th increment.

The purpose of this paper is to introduce a cyclic renewal process as an extension of an alternating renewal process, where each of the underlying i.i.d. nonnegative random increments is composed of multiple stages, i.e. $Y_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$, where Y_n denotes the lifetime of the n-th cycle and Y_n's are i.i.d. with respect to n. Such a process may be appropriate for analyzing optimal preventive maintenance policies for production management, where a pair of two stages representing an uptime until a minor failure and the subsequent minimal repair time would be repeated until it is decided to conduct a complete overhaul. In order to address economic problems in such applications, we also introduce a reward process with jumps defined on the cyclic renewal process. When the system is running in stage j, the profit grows linearly at the rate of $\rho(j)$. Upon a minor failure, the subsequent minimal repair in stage $(j+1)$ incurs the linear cost at the rate of $\rho(j+1)$. In addition, the fixed cost may be imposed whenever either a minimal repair or a complete overhaul takes place, resulting in jumps of the reward process. The problem is then to determine when to conduct a complete overhaul so as to maximize the total reward in the time interval $(0, T]$. A multivariate Markov process generated from both the cyclic renewal process and the reward process is studied extensively, yielding various new transform results explicitly and deriving their asymptotic expansions. These results are used to numerically explore optimal preventive maintenance policies for production management.

When the renewal aspect is suppressed, the above model is reduced to a semi-Markov process. The study of semi-Markov processes dates back to the middle of 1950s, originated by works of Lévy [16], Smith [25] and Takács [29]. Subsequently the scope of the study has been expanded through a series of papers by Pyke [22, 23], Pyke and Schaufele [24], and Moore and Pyke [21]. Since the early 1960s, the field attracted many researchers resulting in a collection of quite extensive results. The reader is referred to two excellent survey papers by Çinlar [1, 2] and references therein for extensive analysis of semi-Markov and related processes. Reward processes defined on semi-Markov processes also have been studied extensively, including the original works by Jewell [8, 9, 10] followed by Howard [5], Mclean and Neuts [20], Çinlar [3], Hunter [6], Sumita and Masuda [27], Masuda and Sumita [19] and Igaki, Sumita and Kowada [7] to name a few. However, to the best knowledge of the authors, the joint distribution of the cyclic renewal process, the underlying semi-Markov process and the reward process has never been studied in the literature.

The structure of this paper is as follows. A cyclic renewal process $\{N(t) : t \geqslant 0\}$ is formally introduced in Section 2 based on a cyclic semi-Markov process $\{J(t) : t \geqslant 0\}$ describing multiple stages to constitute system lifetimes. The associated age process $\{X(t) : t \geqslant 0\}$ and the reward process $\{Z(t) : t \geqslant 0\}$ are also introduced so that the multivariate process $[N(t), J(t), X(t), Z(t)]$ becomes Markov. Section 3 is devoted to analysis of this multivariate process by examining its probabilistic flow in its state space, yielding various new transform results. In Section 4, the asymptotic expansions of $\mathrm{E}[Z(t)|J(0) = i]$ and $\mathrm{Cor}[N(t), Z(t)|J(0) = i]$ as $t \to \infty$ are derived. In Section 5, these results are used to

2

numerically explore optimal preventive maintenance policies for production management. Finally, brief concluding remarks are given in Section 6. Some mathematical details are deferred to Appendix for enhancing the readability of the paper.

2. Model Description

We consider a cyclic renewal process $\{N(t) \; ; \; t \geqslant 0\}$ defined on $\mathcal{N} = \{0, 1, 2, \cdots\}$ where the underlying lifetime consists of J stages and $N(t)$ denotes the number of failures by time t. More specifically, let $\mathcal{J} = \{1, 2, \cdots, J\}$ be the set of the stages and let the dwell time in stage $j \in \mathcal{J}$ be a nonnegative random variable denoted by X_j. Throughout the paper, we assume that X_j $(j \in \mathcal{J})$ are independent of the failure count and also mutually independent. For each $j \in \mathcal{J}$, it is assumed that X_j is absolutely continuous characterized by

$$(2.1) \bar{A}_j(x) = \mathrm{P}[X_j > x]; a_j(x) = -\frac{d}{dx}\bar{A}_j(x); \eta_j(x) = \frac{a_j(x)}{\bar{A}_j(x)}; \alpha_j(v) = \int_0^\infty e^{-vx} a_j(x) dx$$

where $\bar{A}_j(x)$, $a_j(x)$, $\eta_j(x)$ and $\alpha_j(v)$ are the survival function, the probability density function, the hazard function and the Laplace transform of $a_j(x)$ respectively. Here v takes values from the complex plane satisfying $Re(v) > 0$ so that $\alpha_j(v)$ is well defined. A lifetime associated with the cyclic renewal process is given by

$$(2.2) \qquad\qquad Y = \sum_{j=1}^J X_j \; .$$

Let Y_k be the lifetime of the k-th renewal cycle where Y_k's are i.i.d. with common structure of (2.2). For $k = 0$, one then sees that

$$(2.3) \qquad\qquad \mathrm{P}[N(t) = 0] = \mathrm{P}[0 \leqslant t < Y_1]$$

and for $k \geqslant 1$,

$$(2.4) \qquad\qquad \mathrm{P}[N(t) = k] = \mathrm{P}[\sum_{m=1}^k Y_m \leqslant t < \sum_{m=1}^{k+1} Y_m] \; .$$

Let $\{J(t) \; ; \; t \geqslant 0\}$ be a stochastic process describing the stage at time t. We note that $J(t)$ is a cyclic semi-Markov process on $\mathcal{J} = \{1, \cdots, J\}$ governed by the matrix distribution function $\underline{\underline{A}}(x)$ where

$$(2.5) \quad \underline{\underline{A}}(x) \stackrel{\text{def}}{=} \begin{bmatrix} 0 & A_1(x) & 0 & \cdots & 0 \\ 0 & 0 & A_2(x) & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_{J-1}(x) \\ A_J(x) & 0 & \cdots & 0 & 0 \end{bmatrix} \; ; \; A_j(x) \stackrel{\text{def}}{=} 1 - \bar{A}_j(x) \; .$$

Since the bivariate process $[N(t), J(t)]$ is not Markov, we introduce an additional process $\{X(t) \; ; \; t \geqslant 0\}$ on $\mathcal{R}^+$ denoting the elapsed time since the last entry into the current stage at time t, where $\mathcal{R}^+$ is the set of nonnegative real numbers. This process is called the age
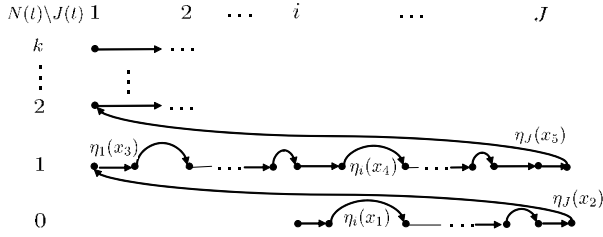
3

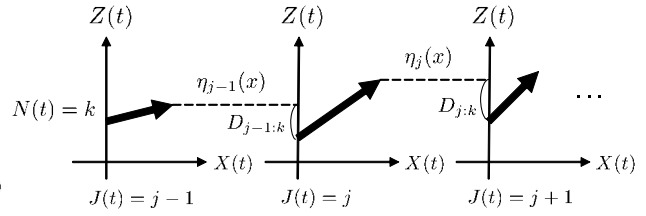Figure 2.1: Typical Sample Path of $[N(t), J(t), X(t)]$

Figure 2.2: Typical Sample Path of $[X(t), Z(t), J(t)]$ with Jumps

process. The trivariate process $[N(t), J(t), X(t)]$ then becomes Markov. A typical sample path of $[N(t), J(t), X(t)]$ is depicted in Figure 2.1 where $N(0) = 0$, $J(0) = i$ and $X(0) = 0$.

From an application point of view, of particular interest is a reward process $\{Z(t)\, ;\, t \geqslant 0\}$ with jumps defined on $[N(t), J(t), X(t)]$. We assume that the reward increases or decreases linearly at the rate of $\rho(j)$ when $J(t)$ is in state $j \in \mathcal{J}$. Furthermore, the reward process jumps in the random amount of D_j when $J(t)$ moves from j to $j + 1$ for $j \in \mathcal{J} \setminus \{J\}$, and D_J for a transition from J to 1. Accordingly, $Z(t)$ takes a value from $\mathcal{R}$ where $\mathcal{R}$ is the set of real numbers. As for X_j $(j \in \mathcal{J})$, it is assumed that D_j $(j \in \mathcal{J})$ are independent of the failure count, mutually independent, and absolutely continuous having

$$(2.6) \qquad \bar{B}_j(z) = \mathrm{P}[D_j > z]\, ;\, b_j(z) = -\frac{d}{dz}\bar{B}_j(z)\, ;\, \beta_j(w) = \int_{-\infty}^{\infty} e^{-wz} b_j(z)dz\, ,$$

where w takes values on the unit circle on the complex plane so that $\beta_j(w)$ is well defined.

In order to describe the reward process $\{Z(t)\, ;\, t \geqslant 0\}$ more formally, let $\{M_j(t)\, ;\, t \geqslant 0\}$ be the stochastic process counting the number of transitions of $J(t)$ from j to $j + 1$ by time t for $j \in \mathcal{J} \setminus \{J\}$. The stochastic process $\{M_J(t)\, ;\, t \geqslant 0\}$ is defined similarly for transitions of $J(t)$ from J to 1. One then has

$$(2.7) \qquad Z(t) = \int_0^t \rho(J(\tau))d\tau + \sum_{j=1}^{J} \sum_{m=1}^{M_j(t)} D_{j:m}\, ,$$

where $D_{j:m}$ denotes the jump amount associated with the m-th transition from j to $j + 1$ for $j \in \mathcal{J} \setminus \{J\}$, and from J to 1 for $j = J$. Following the mathematical convention, we define $\sum_{m=a}^{b} c_m = 0$ whenever $a > b$. It should be noted that, by the assumptions discussed above, $D_{j:m}(m = 1, \cdots, M_j(t))$ are i.i.d. with respect to m. When $J(t)$ is a general semi-Markov process, the expectation of the semi-Markov reward process with jumps is given in Howard [5]. The transform results of $[J(t), Z(t)]$ are derived in McLean and Neuts [20]. The trivariate Markov process $[J(t), X(t), Z(t)]$ is also studied in detail in Sumita and Masuda [27, 26] and Masuda [18]. The thrust of this paper is to analyze the multivariate process $[N(t), J(t), X(t), Z(t)]$ where the cyclic renewal process $N(t)$ is incorporated together with $[J(t), X(t), Z(t)]$, which is new. The results are then used to numerically explore optimal preventive maintenance policies for production management.

4

3. Dynamic Analysis of Multivariate Process $[N(t), J(t), X(t), Z(t)]$

In this section, we analyze the multivariate process $[N(t), J(t), X(t), Z(t)]$ by describing its probabilistic flow in the state space $\mathcal{N} \times \mathcal{J} \times \mathcal{R}^+ \times \mathcal{R}$. For this purpose, let $F_{k:ij}(x, z, t)$ be the joint distribution function of $[N(t), J(t), X(t), Z(t)]$ given $J(0) = i$, $X(0) = Z(0) = 0$. More formally, we define

$$
(3.1) \quad F_{k:ij}(x, z, t)
$$
$$
= \quad P[N(t) = k, J(t) = j, X(t) \leqslant x, Z(t) \leqslant z | J(0) = i, \ X(0) = Z(0) = 0] \ .
$$

The corresponding joint probability density function is given by

$$
(3.2) \qquad\qquad f_{k:ij}(x, z, t) = \frac{\partial^2}{\partial x \partial z} F_{k:ij}(x, z, t) \ .
$$

For the process $[N(t), J(t), X(t), Z(t)]$ to be at $(0, j, x, z)$ at time $t > 0$ given $J(0) = i$, either no transition of $J(t)$ has occurred in the time interval $[0, t]$ with $j = i$, or at least one transition of $J(t)$ from $J(0) = i$ occurred in $[0, t)$, the process entered the state $(0, j, 0+, z - \rho(j)x)$ at time $t - x$, and no transition of $J(t)$ has occurred since then. Accordingly, one has

$$
(3.3) \quad f_{0:ij}(x, z, t) \quad = \quad \delta_{\{j=i\}} \delta(z - \rho(j)t) \delta(t - x) \bar{A}_j(x)
$$
$$
+ \quad \delta_{\{j>i\}} f_{0:ij}(0+, z - \rho(j)x, t - x) \bar{A}_j(x) \ , \quad x > 0 \ , \ j = i, \cdots, J \ .
$$

Here, $\delta_{\{P\}} = 1$ if the statement P holds true, $\delta_{\{P\}} = 0$ otherwise, and $\delta(t)$ is the delta function defined as the unit function associated with the convolution operation, i.e., $f(x) = \int f(y)\delta(x - y)dy$ for any integrable function f. Similarly, for $k > 0$, to be at (k, j, x, z) at time $t > 0$, the process should have entered the state $(k, j, 0+, z - \rho(j)x)$ at time $t - x$ and no transition of $J(t)$ has occurred since then. This then yields

$$
(3.4) \qquad f_{k:ij}(x, z, t) \quad = \quad f_{k:ij}(0+, z - \rho(j)x, t - x) \bar{A}_j(x) \ , \quad x > 0 \ , \ k \geqslant 1 \ .
$$

In order to determine the boundary conditions $f_{k:ij}(0+, z, t)$ associated with the age process $X(t)$, we first consider the case that $k = 0$, $z = 0+$ and $t = 0+$. One then sees that $f_{0:ij}(0+, 0+, 0+) = \delta_{\{j=i\}}\delta(z)\delta(t)$. For $t > 0$ and $j \geqslant i$, the process $[N(t), J(t), X(t), Z(t)]$ just enters the state $(0, j, 0+, z)$ at time t only if the dwell time of $J(t)$ in state $j - 1$ expires at time t with the reward at $z - D_{j-1}$ followed by the instantaneous jump of size D_{j-1} so that $Z(t) = z$. Combining the two cases, one observes that

$$
(3.5) \qquad f_{0:ij}(0+, z, t)
$$
$$
= \quad \delta_{\{j=i\}}\delta(z)\delta(t)
$$
$$
+ \quad \delta_{\{j>i\}} \int_0^\infty \int_{-\infty}^\infty f_{0:i,j-1}(x, z - z', t)\eta_{j-1}(x)b_{j-1}(z')dz' dx \ , \quad j = i, \cdots, J \ .
$$

For $k \geqslant 1$, similar arguments lead to

$$
(3.6) \qquad f_{k:ij}(0+, z, t)
$$
$$
= \begin{cases} \int_0^\infty \int_{-\infty}^\infty f_{k-1:iJ}(x, z - z', t)\eta_J(x)b_J(z')dz' dx \ , & j = 1 \\ \int_0^\infty \int_{-\infty}^\infty f_{k:i,j-1}(x, z - z', t)\eta_{j-1}(x)b_{j-1}(z')dz' dx \ , & 2 \leqslant j \leqslant J \end{cases} .
$$

We are now in a position to prove the key theorem of this paper. For notational convenience, the following matrix Laplace-Fourier transforms are introduced.

$$(3.7) \qquad \underline{\hat{\varphi}}_k(x,z,s) \overset{\text{def}}{=} [\hat{\varphi}_{k:ij}(x,z,s)] \; ; \; \hat{\varphi}_{k:ij}(x,z,s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f_{k:ij}(x,z,t)dt \; ,$$

$$(3.8) \qquad \underline{\hat{\hat{\varphi}}}_k(x,w,s) \overset{\text{def}}{=} [\hat{\hat{\varphi}}_{k:ij}(x,w,s)] \; ; \; \hat{\hat{\varphi}}_{k:ij}(x,w,s) \overset{\text{def}}{=} \int_{-\infty}^\infty e^{-wz} \hat{\varphi}_{k:ij}(x,z,s)dz \; ,$$

$$(3.9) \qquad \underline{\hat{\hat{\hat{\varphi}}}}_k(v,w,s) \overset{\text{def}}{=} [\hat{\hat{\hat{\varphi}}}_{k:ij}(v,w,s)] \; ; \; \hat{\hat{\hat{\varphi}}}_{k:ij}(v,w,s) \overset{\text{def}}{=} \int_0^\infty e^{-vx} \hat{\hat{\varphi}}_{k:ij}(x,w,s)dx \; ,$$

$$(3.10) \qquad \underline{\hat{\xi}}_k(0+,z,s) \overset{\text{def}}{=} [\hat{\xi}_{k:ij}(0+,z,s)] \; ; \; \hat{\xi}_{k:ij}(0+,z,s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f_{k:ij}(0+,z,t)dt \; ,$$

$$(3.11) \qquad \underline{\hat{\hat{\xi}}}_k(0+,w,s) \overset{\text{def}}{=} [\hat{\hat{\xi}}_{k:ij}(0+,w,s)] \; ; \; \hat{\hat{\xi}}_{k:ij}(0+,w,s) \overset{\text{def}}{=} \int_{-\infty}^\infty e^{-wz} \hat{\xi}_{k:ij}(0+,z,s)dz \; ,$$

$$(3.12) \qquad \underline{\beta}_D(w,s) \overset{\text{def}}{=} \left[\delta_{\{i=j\}} \frac{1-\alpha_j(s+\rho(j)w)}{s+\rho(j)w} \right],$$

$$(3.13) \qquad \zeta_{ij}(w,s) \overset{\text{def}}{=} \prod_{n=i}^j \alpha_n(s+\rho(n)w)\beta_n(w) \; ; \; \zeta_{ij}(w,s) = 1 \;\; \text{for} \;\; i > j \; ,$$

$$(3.14) \qquad \underline{\alpha}^*(w,s) \overset{\text{def}}{=} \begin{bmatrix} 0 & \zeta_{11}(w,s) & 0 & \cdots & 0 \\ 0 & 0 & \zeta_{22}(w,s) & \cdots & 0 \\ \vdots & \cdots & & \cdots & \zeta_{J-1,J-1}(w,s) \\ 0 & \cdots & & \cdots & 0 \end{bmatrix} ,$$

$$(3.15) \qquad \underline{\alpha}_D^*(w,s) \overset{\text{def}}{=} \begin{bmatrix} \zeta_{1J}(w,s) & & & \underline{0} \\ & \zeta_{2J}(w,s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{JJ}(w,s) \end{bmatrix}.$$

We also define the following matrices.

$$(3.16) \qquad \underline{1} \overset{\text{def}}{=} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \; , \; \underline{I} \overset{\text{def}}{=} \begin{bmatrix} 1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & 1 \end{bmatrix}.$$

A few preliminary lemmas are needed.

Lemma 3.1 *For $\underline{\hat{\hat{\xi}}}_k(0+,w,s)$ defined in (3.11), one has*

$$(3.17) \qquad \underline{\hat{\hat{\xi}}}_k(0+,w,s) = \begin{cases} [\underline{I} - \underline{\alpha}^*(s,w)]^{-1} \; , & k=0 \\ \{\zeta_{1J}(w,s)\}^k \underline{\alpha}_D^*(w,s)\underline{1} \, \underline{\alpha}_D^{*-1}(w,s) \; , & k \geqslant 1 \end{cases}.$$

Proof

Substituting (3.3) into (3.5), it can be seen that

$$(3.18) \qquad f_{0:ij}(0+,z,t)$$
$$= \delta_{\{j=i\}}\delta(z)\delta(t) + \delta_{\{j>i\}} \int_0^\infty \int_{-\infty}^\infty \{\delta_{\{j-1=i\}}\delta(z-z'-\rho(j-1)t)\delta(t-x)\bar{A}_{j-1}(x)$$
$$+ \delta_{\{j-1>i\}} f_{0:i,j-1}(0+,z-z'-\rho(j-1)x,t-x)\bar{A}_{j-1}(x)\}\eta_{j-1}(x)b_{j-1}(z')dz'dx \; .$$

6

Similarly, substitution of (3.4) into (3.6) yields

(3.19) $f_{k:ij}(0+, z, t)$

$$= \begin{cases} \int_0^\infty \int_{-\infty}^\infty f_{k-1:iJ}(0+, z - z' - \rho(J)x, t - x)a_J(x)b_J(z')dz'dx , & j = 1 \\ \int_0^\infty \int_{-\infty}^\infty f_{k:i,j-1}(0+, z - z' - \rho(j-1)x, t - x)a_{j-1}(x)b_{j-1}(z')dz'dx , & 2 \leqslant j \leqslant J \end{cases} .$$

By taking Laplace transforms with respect to t in (3.18) and (3.19), it then follows that

(3.20) $\hat{\xi}_{0:ij}(0+, z, s)$

$$= \delta_{\{j=i\}}\delta(z) + e^{-sx}\delta_{\{j>i\}} \int_0^\infty \int_{-\infty}^\infty \{\delta_{\{j-1=i\}}\delta(z - z' - \rho(j-1)x)\bar{A}_{j-1}(x)$$

$$+ \delta_{\{j-1>i\}}e^{-sx}\hat{\xi}_{0:i,j-1}(0+, z - z' - \rho(j-1)x, s)\bar{A}_{j-1}(x)\}\eta_{j-1}(x)b_{j-1}(z')dz'dx ,$$

and

(3.21) $\hat{\xi}_{k:ij}(0+, z, s)$

$$= \begin{cases} \int_0^\infty \int_{-\infty}^\infty e^{-sx}\hat{\xi}_{k-1:iJ}(0+, z - z' - \rho(J)x, s)a_J(x)b_J(z')dz'dx , & j = 1 \\ \int_0^\infty \int_{-\infty}^\infty e^{-sx}\hat{\xi}_{k:i,j-1}(0+, z - z' - \rho(j-1)x, s)a_{j-1}(x)b_{j-1}(z')dz'dx , & 2 \leqslant j \leqslant J \end{cases} .$$

If we again take Laplace-Fourier transforms with respect to z in (3.20) and (3.21), one has

(3.22) $$\hat{\hat{\xi}}_{0:ij}(0+, w, s) = \delta_{\{j=i\}} + \delta_{\{j-1=i\}}\zeta_{j-1,j-1}(w, s)$$

$$+ \delta_{\{j-1>i\}}\hat{\hat{\xi}}_{0:i,j-1}(0+, w, s)\zeta_{j-1,j-1}(w, s)$$

and

(3.23) $$\hat{\hat{\xi}}_{k:ij}(0+, w, s) = \begin{cases} \hat{\hat{\xi}}_{k-1:iJ}(0+, w, s)\zeta_{JJ}(w, s) , & j = 1 \\ \hat{\hat{\xi}}_{k:i,j-1}(0+, w, s)\zeta_{j-1,j-1}(w, s) , & 2 \leqslant j \leqslant J \end{cases}$$

since $\zeta_{jj}(w, s) = \alpha_j(s + \rho(j)w)\beta_j(w)$.

Equations in (3.22) and (3.23) can be rewritten in matrix form using $\hat{\underline{\underline{\hat{\xi}}}}_k(0+, w, s)$ defined in (3.11) in the following manner. From (3.22), we first note that

$$\hat{\underline{\underline{\hat{\xi}}}}_0(0+, w, s) = \begin{bmatrix} 1 & \hat{\hat{\xi}}_{0:12}(0+, w, s) & \cdots & \hat{\hat{\xi}}_{0:1J}(0+, w, s) \\ & 1 & \cdots & \hat{\hat{\xi}}_{0:2J}(0+, w, s) \\ & & \ddots & \vdots \\ & \underline{0} & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \zeta_{11}(w, s) & \hat{\hat{\xi}}_{0:12}(0+, w, s)\zeta_{22}(w, s) & \cdots & \hat{\hat{\xi}}_{0:1,J-1}(0+, w, s)\zeta_{J-1,J-1}(w, s) \\ & 1 & \zeta_{22}(w, s) & \cdots & \hat{\hat{\xi}}_{0:2,J-1}(0+, w, s)\zeta_{J-1,J-1}(w, s) \\ & & 1 & \ddots & \vdots \\ & \underline{0} & & & 1 \end{bmatrix} .$$

The last matrix in the above expression can be written from (3.14) as $\underline{\underline{I}} + \hat{\underline{\underline{\hat{\xi}}}}_0(0+, w, s)\underline{\underline{\alpha}}^*(w, s)$, so that

(3.24) $$\hat{\underline{\underline{\hat{\xi}}}}_0(0+, w, s) = [\underline{\underline{I}} - \underline{\underline{\alpha}}^*(w, s)]^{-1} ,$$

7

proving the case for $k = 0$.

For $k \geqslant 1$, we prove by induction. When $k = 1$, one sees that

$$
\hat{\underline{\xi}}_1(0+, w, s)
$$

$$
= \begin{bmatrix} \hat{\xi}_{1:11}(0+, w, s) & \cdots & \hat{\xi}_{1:1J}(0+, w, s) \\ \vdots & \cdots & \vdots \\ \hat{\xi}_{1:J1}(0+, w, s) & \cdots & \hat{\xi}_{1:JJ}(0+, w, s) \end{bmatrix}
$$

$$
= \begin{bmatrix} \hat{\xi}_{0:1J}(0+, w, s)\zeta_{JJ}(w, s) & \hat{\xi}_{1:11}(0+, w, s)\zeta_{11}(w, s) & \cdots & \hat{\xi}_{1:1,J-1}(0+, w, s)\zeta_{J-1,J-1}(w, s) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\xi}_{0:JJ}(0+, w, s)\zeta_{JJ}(w, s) & \hat{\xi}_{1:J1}(0+, w, s)\zeta_{11}(w, s) & \cdots & \hat{\xi}_{1:J,J-1}(0+, w, s)\zeta_{J-1,J-1}(w, s) \end{bmatrix}.
$$

By employing (3.22) in the above expression, it follows that

$$
\hat{\underline{\xi}}_1(0+, w, s) = \begin{bmatrix} \zeta_{1J}(w, s) & \zeta_{1J}(w, s)\zeta_{11}(w, s) & \cdots & \zeta_{1J}(w, s)\zeta_{1,J-1}(w, s) \\ \zeta_{2J}(w, s) & \zeta_{2J}(w, s)\zeta_{11}(w, s) & \cdots & \zeta_{2J}(w, s)\zeta_{1,J-1}(w, s) \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_{JJ}(w, s) & \zeta_{JJ}(w, s)\zeta_{11}(w, s) & \cdots & \zeta_{JJ}(w, s)\zeta_{1,J-1}(w, s) \end{bmatrix}.
$$

From (3.15) and (3.16), this then leads to

$$
(3.25) \qquad \hat{\underline{\xi}}_1(0+, w, s) = \underline{\alpha}_D^*(w, s)\underline{1} \begin{bmatrix} 1 & & & \underline{0} \\ & \zeta_{11}(w, s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{1,J-1}(w, s) \end{bmatrix}.
$$

It should be noted from (3.13) and (3.15) that

$$
\underline{\alpha}_D^*(s, w) \begin{bmatrix} 1 & & & \underline{0} \\ & \zeta_{11}(w, s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{1,J-1}(w, s) \end{bmatrix} = \zeta_{1J}(w, s)\underline{I},
$$

so that one has

$$
(3.26) \qquad \begin{bmatrix} 1 & & & \underline{0} \\ & \zeta_{11}(w, s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{1,J-1}(w, s) \end{bmatrix} = \zeta_{1J}(w, s)\underline{\alpha}_D^{*-1}(w, s) .
$$

Substituting (3.26) into (3.25), one concludes that

$$
(3.27) \qquad \hat{\underline{\xi}}_1(0+, w, s) = \zeta_{1J}(w, s)\underline{\alpha}_D^*(w, s)\underline{1}\,\underline{\alpha}_D^{*-1}(w, s) ,
$$

proving for $k = 1$.

Suppose the statement holds true for $k-1$ and consider the case for k. It can be seen from (3.23) that

$$\hat{\underline{\hat{\xi}}}_k(0+,w,s)$$

$$= \begin{bmatrix} \hat{\hat{\xi}}_{k:11}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:1J}(0+,w,s) \\ \hat{\hat{\xi}}_{k:21}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:2J}(0+,w,s) \\ \vdots & \vdots & \vdots \\ \hat{\hat{\xi}}_{k:J1}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:JJ}(0+,w,s) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\hat{\xi}}_{k-1:1J}(0+,w,s)\zeta_{JJ}(w,s) & \cdots & \hat{\hat{\xi}}_{k:1,J-1}(0+,w,s)\zeta_{J-1,J-1}(w,s) \\ \hat{\hat{\xi}}_{k-1:2J}(0+,w,s)\zeta_{JJ}(w,s) & \cdots & \hat{\hat{\xi}}_{k:2,J-1}(0+,w,s)\zeta_{J-1,J-1}(w,s) \\ \vdots & \cdots & \vdots \\ \hat{\hat{\xi}}_{k-1:JJ}(0+,w,s)\zeta_{JJ}(w,s) & \cdots & \hat{\hat{\xi}}_{k:J,J-1}(0+,w,s)\zeta_{J-1,J-1}(w,s) \end{bmatrix}.$$

The last matrix in the above expression can be written in matrix product form as

$$\hat{\underline{\hat{\xi}}}_k(0+,w,s) = \begin{bmatrix} \hat{\hat{\xi}}_{k-1:1J}(0+,w,s) & \hat{\hat{\xi}}_{k:11}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:1,J-1}(0+,w,s) \\ \hat{\hat{\xi}}_{k-1:2J}(0+,w,s) & \hat{\hat{\xi}}_{k:21}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:2,J-1}(0+,w,s) \\ \vdots & \cdots & \vdots & \vdots \\ \hat{\hat{\xi}}_{k-1:JJ}(0+,w,s) & \hat{\hat{\xi}}_{k:J1}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:J,J-1}(0+,w,s) \end{bmatrix}$$

$$\cdot \begin{bmatrix} \zeta_{JJ}(w,s) & & & \underline{0} \\ & \zeta_{11}(w,s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{J-1,J-1}(w,s) \end{bmatrix}.$$

By applying (3.23) to the first matrix in the above expression, one sees that

$$\hat{\underline{\hat{\xi}}}_k(0+,w,s)$$

$$= \begin{bmatrix} \hat{\hat{\xi}}_{k-1:1J}(0+,w,s) & \hat{\hat{\xi}}_{k:11}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:1,J-1}(0+,w,s) \\ \hat{\hat{\xi}}_{k-1:2J}(0+,w,s) & \hat{\hat{\xi}}_{k:21}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:2,J-1}(0+,w,s) \\ \vdots & \cdots & \vdots & \vdots \\ \hat{\hat{\xi}}_{k-1:JJ}(0+,w,s) & \hat{\hat{\xi}}_{k:J1}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:J,J-1}(0+,w,s) \end{bmatrix} \cdot \begin{bmatrix} \zeta_{JJ}(w,s) & & & \underline{0} \\ & \zeta_{11}(w,s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{J-1,J-1}(w,s) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\hat{\xi}}_{k-1:1,J-1}(0+,w,s) & \hat{\hat{\xi}}_{k-1:1J}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:1,J-2}(0+,w,s) \\ \hat{\hat{\xi}}_{k-1:2,J-1}(0+,w,s) & \hat{\hat{\xi}}_{k-1:2J}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:2,J-2}(0+,w,s) \\ \vdots & \cdots & \vdots & \vdots \\ \hat{\hat{\xi}}_{k-1:J,J-1}(0+,w,s) & \hat{\hat{\xi}}_{k-1:JJ}(0+,w,s) & \cdots & \hat{\hat{\xi}}_{k:J,J-2}(0+,w,s) \end{bmatrix}$$

$$\cdot \begin{bmatrix} \zeta_{J-1,J-1}(w,s) & & & \underline{0} \\ & \zeta_{JJ}(w,s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{J-2,J-2}(w,s) \end{bmatrix} \cdot \begin{bmatrix} \zeta_{JJ}(w,s) & & & \underline{0} \\ & \zeta_{11}(w,s) & & \\ & & \ddots & \\ \underline{0} & & & \zeta_{J-1,J-1}(w,s) \end{bmatrix}.$$

By repeating this procedure, it follows that

(3.28) $$\hat{\underline{\hat{\xi}}}_k(0+,w,s) = \zeta_{1J}(w,s)\hat{\underline{\hat{\xi}}}_{k-1}(0+,w,s),$$

9

where $\zeta_{1J}(w,s) = \prod_{j=1}^{J} \zeta_{jj}(w,s)$ is employed from (3.13). From the induction hypothesis, the lemma now follows. $\qquad\square$

Lemma 3.2 *For the multivariate process $[N(t), J(t), X(t), Z(t)]$ with $N(0) = X(0) = Z(0) = 0$ and $J(0) = i$, let $\underline{\underline{\hat{\hat{\varphi}}}}_k(v,w,s)$ be defined as in (3.9). Then*

$$(3.29) \quad \underline{\underline{\hat{\hat{\varphi}}}}_k(v,w,s) = \begin{cases} [\underline{\underline{I}} - \underline{\underline{\alpha}}^*(w,s)]^{-1}\underline{\underline{\beta}}_D(w,v+s) , & k=0 \\ \{\zeta_{1J}(w,s)\}^k \underline{\underline{\alpha}}_D^*(w,s)\underline{\underline{1}}\,\underline{\underline{\alpha}}_D^{*-1}(w,s)\underline{\underline{\beta}}_D(w,v+s) , & k \geqslant 1 \end{cases}.$$

Proof

By taking Laplace transforms of (3.3) and (3.4) with respect to t, one sees that

$$(3.30) \quad \hat{\varphi}_{k:ij}(x,z,s) = \begin{cases} \delta_{\{j=i\}}\delta(z - \rho(j)x)e^{-sx}\bar{A}_j(x) \\ \quad +\delta_{\{j>i\}}e^{-sx}\hat{\xi}_{0:ij}(0+, z-\rho(j)x, s)\bar{A}_j(x) , & k=0 \\ e^{-sx}\hat{\xi}_{k:ij}(0+, z-\rho(j)x, s)\bar{A}_j(x) , & k \geqslant 1 \end{cases}.$$

If Laplace-Fourier transforms are taken again with respect to z in (3.30), one has

$$(3.31)\,\hat{\hat{\varphi}}_{k:ij}(x,w,s) = \begin{cases} \left[\delta_{\{j=i\}} + \delta_{\{j>i\}}\hat{\hat{\xi}}_{0:ij}(0+,w,s)\right]e^{-(s+\rho(j)w)x}\bar{A}_j(x) , & k=0 \\ \hat{\hat{\xi}}_{k:ij}(0+,w,s)e^{-(s+\rho(j)w)x}\bar{A}_j(x) , & k \geqslant 1 \end{cases}.$$

By taking Laplace transforms one more time with respect to x in (3.31), it follows that

$$(3.32) \quad \hat{\hat{\hat{\varphi}}}_{k:ij}(v,w,s) = \begin{cases} \left[\delta_{\{j=i\}} + \delta_{\{j>i\}}\hat{\hat{\xi}}_{0:ij}(0+,w,s)\right]\frac{1-\alpha_j(v+\rho(j)w+s)}{v+\rho(j)w+s} , & k=0 \\ \hat{\hat{\xi}}_{k:ij}(0+,w,s)\cdot\frac{1-\alpha_j(v+\rho(j)w+s)}{v+\rho(j)w+s} , & k \geqslant 1 \end{cases},$$

which can be rewritten in matrix form as

$$(3.33) \qquad \underline{\underline{\hat{\hat{\hat{\varphi}}}}}_k(v,w,s) = \underline{\underline{\hat{\hat{\xi}}}}_k(0+,w,s)\underline{\underline{\beta}}_D(w,v+s) , \quad k \geqslant 0 .$$

Substituting (3.17) of Lemma 3.1 into (3.33), the theorem follows. $\qquad\square$

By taking the generating function of $\underline{\underline{\hat{\hat{\hat{\varphi}}}}}_k(v,w,s)$ in (3.29) with respect to k ($k = 0, 1, 2 \cdots$), the joint transform of $[N(t), J(t), X(t), Z(t)]$ can be obtained.

Theorem 3.3 *Let $\underline{\underline{\hat{\hat{\hat{\varphi}}}}}(v,w,s,u)$ be the matrix generating function of $\underline{\underline{\hat{\hat{\hat{\varphi}}}}}_k(v,w,s)$ in (3.29) defined by*

$$(3.34) \quad \underline{\underline{\hat{\hat{\hat{\varphi}}}}}(v,w,s,u) \stackrel{def}{=} [\hat{\hat{\hat{\varphi}}}_{ij}(v,w,s,u)] ; \quad \hat{\hat{\hat{\varphi}}}_{ij}(v,w,s,u) \stackrel{def}{=} \sum_{k=0}^{\infty} \hat{\hat{\hat{\varphi}}}_{k:ij}(v,w,s)u^k .$$

Then one has

$$(3.35) \qquad \underline{\underline{\hat{\hat{\hat{\varphi}}}}}(v,w,s,u) = \underline{\underline{\chi}}(w,s,u)\underline{\underline{\beta}}_D(w,v+s) ,$$

10

where $\underset{=D}{\beta}(w, v+s)$ is as given in (3.12),

(3.36)
$$\underset{=}{\chi}(w, s, u) \overset{def}{=} [\underset{=}{I} - \underset{=}{\zeta}(w, s, u)]^{-1},$$

and

(3.37) $\quad \underset{=}{\zeta}(w, s, u) \overset{def}{=} \begin{bmatrix} 0 & \zeta_{11}(w, s) & 0 & \cdots & 0 \\ 0 & 0 & \zeta_{22}(w, s) & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \zeta_{J-1, J-1}(w, s) \\ u\zeta_{JJ}(w, s) & 0 & \cdots & 0 & 0 \end{bmatrix}.$

Proof

Multiplying u^k to both sides of (3.29) and then summing from $k = 0$ to ∞, one finds that

$$
\begin{aligned}
(3.38)\, \underset{=}{\hat{\hat{\varphi}}}(v, w, s, u) \;=\; & [\underset{=}{I} - \underset{=}{\alpha}^*(w, s)]^{-1} \underset{=D}{\beta}(w, v+s) \\
& + \; u\zeta_{1J}(w, s) \sum_{k=1}^{\infty} \left\{ u\zeta_{1J}(w, s) \right\}^{k-1} \underset{=D}{\alpha}^*(w, s) \underset{=}{1}\, \underset{=D}{\alpha}^{*-1}(w, s) \underset{=D}{\beta}(w, v+s) \\
\;=\; & \left[[\underset{=}{I} - \underset{=}{\alpha}^*(w, s)]^{-1} + \frac{u\zeta_{1J}(w, s)}{1 - u\zeta_{1J}(w, s)} \underset{=D}{\alpha}^*(w, s) \underset{=}{1}\, \underset{=D}{\alpha}^{*-1}(w, s) \right] \\
& \cdot \; \underset{=D}{\beta}(w, v+s) \; .
\end{aligned}
$$

From (3.13), (3.14), (3.15), (3.36) and (3.37), it should be noted that

$$(3.39) \quad \left[[\underset{=}{I} - \underset{=}{\alpha}^*(w, s)]^{-1} + \frac{u\zeta_{1J}(w, s)}{1 - u\zeta_{1J}(w, s)} \underset{=D}{\alpha}^*(w, s) \underset{=}{1}\, \underset{=D}{\alpha}^{*-1}(w, s) \right] = \underset{=}{\chi}(w, s, u) \; .$$

Substituting (3.39) into (3.38) then yields (3.35), completing the proof. $\quad\square$

Remark 3.4 *By setting $u = 1$ in (3.35), Theorem 3.3 is reduced to a special case of Theorem 2.8.1 of Masuda [17]. Indeed, $\underset{=}{\zeta}(w, s, 1)$ is the bivariate transform of $J(t)$ and $Z(t)$, where $\underset{=}{\alpha}^{**}(w, s)$ of Masuda [17] is equal to $\underset{=}{\zeta}(w, s, 1)$.*

4. Asymptotic Expansion of $E[Z(t)|J(0) = i]$ and $Cor[N(t), Z(t)|J(0) = i]$

The purpose of this section is to establish the asymptotic expansions of $E[Z(t)|J(0) = i]$ and $Cor[N(t), Z(t)|J(0) = i]$ as $t \to \infty$. To accomplish this, we introduce Theorem 1 of Keilson [12]. Let $A_{j:k}$ be the k-th moment of X_j. More formally, we define

(4.1)
$$A_{j:k} \overset{def}{=} \int_0^\infty x^k a_j(x) dx \; .$$

Also the following matrix is employed.

(4.2) $\quad \underset{=}{A}_k \overset{def}{=} \begin{bmatrix} 0 & A_{1:k} & 0 & \cdots & 0 \\ 0 & 0 & A_{2:k} & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_{J-1:k} \\ A_{J:k} & 0 & \cdots & 0 & 0 \end{bmatrix} .$

11

If $\int_0^\infty x^2 dA_j(x) < 0$ for all j and $A_j(x)$ are not lattice distribution with a common span, one has

(4.3)
$$\underline{\underline{\chi}}(0, s, 1) = \frac{1}{s}\underline{\underline{H}}_1 + \underline{\underline{H}}_0 + \underline{\underline{o}}(1)$$

as $s \to 0+$ where, for $\underline{e}_d^\top = \underline{e}_d^\top \underline{\underline{A}}_0$, one has

(4.4)
$$\underline{\underline{H}}_1 \overset{def}{=} \frac{1}{m_1}\underline{\underline{J}} \;;\; \underline{\underline{J}} \overset{def}{=} \underline{1}\,\underline{e}_d^\top \;;\; m_1 \overset{def}{=} \underline{e}_d^\top \underline{\underline{A}}_1 \underline{1}$$

and

(4.5)
$$\underline{\underline{H}}_0 \overset{def}{=} \underline{\underline{H}}_1\left(-\underline{\underline{A}}_1 + \frac{1}{2}\underline{\underline{A}}_2\underline{\underline{H}}_1\right) + \left(\underline{\underline{Z}} - \underline{\underline{H}}_1\,\underline{\underline{A}}_1\underline{\underline{Z}}\right)\left(\underline{\underline{A}}_0 - \underline{\underline{A}}_1\underline{\underline{H}}_1\right) + \underline{\underline{I}} \;.$$

Here, $\underline{\underline{Z}}$ is the fundamental matrix associated with the Markov chain governed by $\underline{\underline{A}}_0$, i.e.

(4.6)
$$\underline{\underline{Z}} = \left[\underline{\underline{I}} - \underline{\underline{A}}_0 + \underline{\underline{J}}\right]^{-1} \;.$$

Using Lemmas A.1, A.2 and A.3 in Appendix, the following theorem holds.

Theorem 4.1 *For the matrices in Lemmas A.2 and A.3, we define*

$$\underline{\underline{X}}_1 \overset{def}{=} \underline{\underline{H}}_1(\underline{\underline{A}}_{D:1}\underline{\rho}^\# + \underline{\underline{D}}^\#)\underline{\underline{H}}_1\underline{\underline{A}}_{D:1} \;;$$

$$\underline{\underline{X}}_0 \overset{def}{=} \frac{1}{2}\underline{\underline{V}}_2\underline{\underline{A}}_{D:2} - \underline{\underline{V}}_1\underline{\underline{A}}_{D:1} + \frac{1}{2}\underline{\underline{H}}_1\underline{\rho}_D\underline{\underline{A}}_{D:2} \;;$$

$$n_2 \overset{def}{=} \underline{p}^\top(0)\underline{\underline{S}}_2\underline{1} - (\underline{p}^\top(0)\underline{\underline{L}}_1\underline{1})^2 \;;$$

$$n_1 \overset{def}{=} \underline{p}^\top(0)\underline{\underline{S}}_1\underline{1} - 2\underline{p}^\top(0)\underline{\underline{L}}_1\underline{1}\underline{p}^\top(0)\underline{\underline{L}}_0\underline{1} \;;$$

$$z_2 \overset{def}{=} \underline{p}^\top(0)\underline{\underline{T}}_2\underline{1} - (\underline{p}^\top(0)\underline{\underline{X}}_1\underline{1})^2 \;;$$

$$z_1 \overset{def}{=} \underline{p}^\top(0)\underline{\underline{T}}_1\underline{1} - 2\underline{p}^\top(0)\underline{\underline{X}}_1\underline{1}\underline{p}^\top(0)\underline{\underline{X}}_0\underline{1} \;;$$

$$Co_2 \overset{def}{=} \underline{p}^\top(0)\underline{\underline{U}}_2\underline{1} - \underline{p}^\top(0)\underline{\underline{L}}_1\underline{1}\underline{p}^\top(0)\underline{\underline{X}}_1\underline{1} \;;\; and$$

$$Co_1 \overset{def}{=} \underline{p}^\top(0)\underline{\underline{U}}_1\underline{1} - \underline{p}^\top(0)\underline{\underline{L}}_1\underline{1}\underline{p}^\top(0)\underline{\underline{X}}_0\underline{1} - \underline{p}^\top(0)\underline{\underline{L}}_0\underline{1}\underline{p}^\top(0)\underline{\underline{X}}_1\underline{1} \;,$$

where $\underline{p}^\top(0)$ is the initial probability vector of $J(t)$ and $\underline{1}^\top \overset{def}{=} [1 \cdots 1]$. As $t \to \infty$, one has

a) $E[Z(t)|J(0) = i] = \underline{p}^\top(0)(\underline{\underline{X}}_1 t + \underline{\underline{X}}_0)\underline{1} + o(1)$

b) $Cor[N(t), Z(t)|J(0) = i] = \dfrac{Co_2 t^2 + Co_1 t + o(t)}{\sqrt{n_2 z_2 t^4 + (n_2 z_1 + n_1 z_2)t^3 + n_1 z_1 t^2 + o(t^2)}}$

Proof

a) Setting $v = w = 0$ in (3.35) leads to

$$\underline{\hat{\hat{\underline{\varphi}}}}(0, 0, s, u) = \underline{\underline{\chi}}(0, s, u)\underline{\underline{\beta}}_D(0, s) \;.$$

12

By differentiating the above expression with respect to u and setting $u = 1$, it can be seen that

$$(4.7) \qquad \frac{\partial}{\partial u}\Big\{\hat{\underline{\hat{\varphi}}}(0,0,s,u)\Big\}\Big|_{u=1} = \frac{\partial}{\partial u}\Big\{\underline{\chi}(0,s,u)\Big\}\Big|_{u=1}\underline{\beta}_D(0,s) \ .$$

Applying Lemmas A.1 c) and A.2 a) to (4.7), it then follows that

$$(4.8) \qquad \frac{\partial}{\partial u}\Big\{\hat{\underline{\hat{\varphi}}}(0,0,s,u)\Big\}\Big|_{u=1} = \Big\{\frac{1}{s^2}\underline{Q}_2 + \frac{1}{s}\underline{Q}_1 + \underline{o}\Big(\frac{1}{s}\Big)\Big\}\Big\{\underline{A}_{D:1} - \frac{1}{2}\underline{A}_{D:2} + \underline{o}(s)\Big\}$$

$$= \frac{1}{s^2}\underline{Q}_2\underline{A}_{D:1} + \frac{1}{s}(\underline{Q}_1\underline{A}_{D:1} - \frac{1}{2}\underline{Q}_2\underline{A}_{D:2}) + \underline{o}\Big(\frac{1}{s}\Big) \ .$$

Hence one has

$$\mathrm{E}[Z(t)|J(0) = i] = \underline{p}^\top(0)\mathcal{L}^{-1}\Big\{\frac{\partial}{\partial u}\hat{\underline{\hat{\varphi}}}(0,0,s,u)\Big|_{u=1}\Big\}\underline{1}$$

$$= \underline{p}^\top(0)(\underline{Q}_2\underline{A}_{D:1}t + \underline{Q}_1\underline{A}_{D:1} - \frac{1}{2}\underline{Q}_2\underline{A}_{D:2})\underline{1} + o(1)$$

$$= \underline{p}^\top(0)(\underline{X}_1 t + \underline{X}_0)\underline{1} + o(1) \quad \text{as } t \to \infty \ ,$$

where $\mathcal{L}^{-1}$ means the inversion of the Laplace transform, i.e., $\mathcal{L}^{-1}\{\alpha(s)\} = a(t)$ with $\alpha(s) = \mathcal{L}\{a(t)\} = \int_0^\infty e^{-st}a(t)dt$, proving part a).

For part b), we first note that

$$\mathcal{L}\{\mathrm{E}[N(t)Z(t)|J(0) = i]\} = \underline{p}^\top(0)\mathcal{L}^{-1}\Big\{\frac{\partial^2}{\partial u \partial w}\hat{\underline{\hat{\varphi}}}(0,w,s,u)\Big|_{u=1,w=0}\Big\}\underline{1} \ .$$

The asymptotic expansion of the above expression is given in Lemma A.3 d), which in turn yields that of

$$\mathrm{Cov}[N(t), Z(t)|J(0) = i] = \mathrm{E}[N(t)Z(t)|J(0) = i] - \mathrm{E}[N(t)|J(0) = i]\mathrm{E}[Z(t)|J(0) = i] \ .$$

More specifically, using Theorem 4.1 a) and Lemma A.3 a) and d), one finds that

$$(4.9) \qquad \mathrm{Cov}[N(t), Z(t)|J(0) = i] = Co_2 t^2 + Co_1 t + o(t) \ .$$

One also sees from Lemma A.3 that

$$(4.10) \qquad \mathrm{V}[N(t)|J(0) = i] = n_2 t^2 + n_1 t + o(t) \ ,$$
$$(4.11) \qquad \mathrm{V}[Z(t)|J(0) = i] = z_2 t^2 + z_1 t + o(t) \ .$$

Part b) then follows from (4.10), (4.11) and (4.9) since $\mathrm{Cor}[N(t), Z(t)|J(0) = i] = \mathrm{Cov}[N(t), Z(t)|J(0) = i]/\sqrt{\mathrm{V}[N(t)|J(0) = i]\mathrm{V}[Z(t)|J(0) = i]}$. $\qquad \square$

 In the next section, as an example of applications of these asymptotic results, we investigate optimal preventive maintenance policies for production systems where the opportunity cost for system down is huge.

5. Numerical Exploration of Optimal Preventive Maintenance Policies for Production Management

We consider a production system where the system down cost is huge. A typical example may be the production of semi-conductor chips because the production machines are extremely expensive and the repair takes a long time since vendor engineers often have to be called in once the system fails. In such a situation, preventive maintenance is widely practiced where minimal repairs take place as minor problems occur, which can be addressed by on-site engineers. A complete overhaul demanding the presence of vendor engineers is conducted only after minimal repairs are repeated certain many times, as depicted in Figure 5.1. The question then is to determine when to conduct a complete overhaul. The reward process defined on the cyclic renewal process proposed in this paper provides a useful computational vehicle for numerically exploring optimal preventive maintenance policies of this sort in a dynamic environment. In this section, we demonstrate this claim using Theorem 4.1 a).
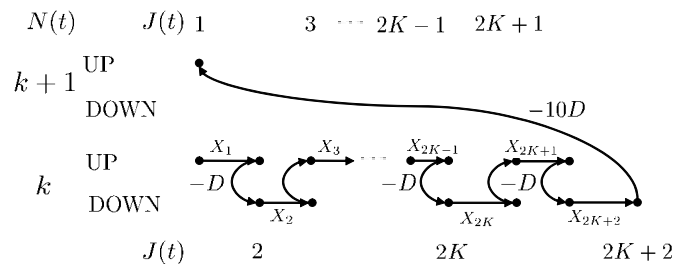
Figure 5.1: Typical Sample Path of $[N(t), J(t), X(t)]$ for Preventive Maintenance Model

The idea behind minimal repairs is to prolong the availability of the system in the time interval $(0, T]$ by accommodating a partial system adjustment from time to time. This approach can be effective since minimal repairs can be done at much lower cost and in much shorter time in comparison with a complete overhaul. Starting with a fresh system lifetime, it is natural to assume that the time until the next minimal repair becomes shorter while the subsequent minimal repair time becomes longer as this cycle is repeated. When it is decided to conduct a complete overhaul, the system is brought back to its original fresh state upon completion of the overhaul.

In order to incorporate this probabilistic structure, we employ Gamma variates. More specifically, let $\{\widehat{X}_i\}_{i=1}^{\infty}$ and $\{\widetilde{X}_i\}_{i=1}^{\infty}$ be sequences of i.i.d. exponential random variables with parameters λ and μ respectively, where the former is used to construct system lifetimes while the latter is employed to structure repair times. The system lifetime X_1 when it is in the fresh state is assumed to be a Gamma variate of integral order $K(0)$ with scaling parameter λ, i.e.,

$$(5.1) \qquad X_1 = \sum_{i=1}^{K(0)} \widehat{X}_i \ .$$

We also assume that the time required for conducting a complete overhaul is a Gamma variate of integral order $K(1)$ with scaling parameter μ. Assuming that K minimal repairs

would take place, one has

$$(5.2) \qquad X_{2(K+1)} = \sum_{i=1}^{K(1)} \widetilde{X}_i .$$

So as to reflect the fact that the time until the next minimal repair becomes shorter while the subsequent minimal repair time becomes longer as this cycle is repeated, we define

$$(5.3) \qquad X_j = \begin{cases} \widehat{X}_1 + \widehat{X}_2 + \cdots + \widehat{X}_{K(2)+2-(j+1)/2} & \text{if } j = 3, 5, \cdots, 2K+1 \\ \widetilde{X}_1 + \widetilde{X}_2 + \cdots + \widetilde{X}_{j/2} & \text{if } j = 2, 4, \cdots, 2K \end{cases} ,$$

where $K(2)$ is a parameter satisfying $K \leqslant K(2) \leqslant K(0)$. For j odd, X_j is the time until the next minimal repair which decreases stochastically with respect to j. For j even, X_j is the subsequent minor repair time which increases stochastically in j.

Let $\alpha_j(s)$ be the Laplace transform of the p.d.f of X_j. From (5.1), (5.2) and (5.3), it can be seen that

$$(5.4) \qquad \alpha_j(s) = \begin{cases} \left(\dfrac{\lambda}{s+\lambda}\right)^{K(0)} & \text{if } j = 1 \\[2mm] \left(\dfrac{\lambda}{s+\lambda}\right)^{K(1)} & \text{if } j = 2(K+1) \\[2mm] \left(\dfrac{\lambda}{s+\lambda}\right)^{K(2)+1-\frac{j+1}{2}} & \text{if } j = 3, 5, \cdots, 2K+1 \\[2mm] \left(\dfrac{\mu}{s+\mu}\right)^{\frac{j}{2}} & \text{if } j = 2, 4, \cdots, 2K \end{cases} .$$

By differentiating (5.4) with respect to s once or twice and setting $s = 0$, one finds that

$$(5.5) \qquad \mathrm{E}[X_j] = \begin{cases} \dfrac{K(0)}{\lambda} & \text{if } j = 1 \\[2mm] \dfrac{K(1)}{\mu} & \text{if } j = 2(K+1) \\[2mm] \dfrac{1}{\lambda}\left(K(2) + 2 - \dfrac{j+1}{2}\right) & \text{if } j = 3, 5, \cdots, 2K+1 \\[2mm] \dfrac{j}{2\mu} & \text{if } j = 2, 4, \cdots, 2K \end{cases} ,$$

and

$$(5.6)\mathrm{E}[X_j^2] = \begin{cases} \dfrac{1}{\lambda}K(0)(K(0)+1) & \text{if } j = 1 \\[2mm] \dfrac{1}{\mu}K(1)(K(1)+1) & \text{if } j = 2(K+1) \\[2mm] \dfrac{1}{\lambda}\left(K(2)+2-\dfrac{j+1}{2}\right)\left(K(2)+3-\dfrac{j+1}{2}\right) & \text{if } j = 3, 5, \cdots, 2K+1 \\[2mm] \dfrac{j}{2\mu}\left(\dfrac{j}{2}+1\right) & \text{if } j = 2, 4, \cdots, 2K \end{cases} .$$

15

We next turn our attention to the reward structure. The reward rate function $\rho(j)$ is defined as

$$(5.7) \qquad \rho(j) = \begin{cases} \rho_{\mathrm{UP}} & \text{if } j = 1, 3, \cdots, 2K+1 \\ -\rho_{\mathrm{DOWN}} & \text{if } j = 2, 4, \cdots, 2K+2 \end{cases},$$

where ρ_{UP} and ρ_{DOWN} are parameters satisfying $\rho_{\mathrm{UP}} > 0$ and $\rho_{\mathrm{DOWN}} > 0$. The fixed cost for calling in on-site engineers for a minimal repair and that for calling in vendor engineers for a complete overhaul can be expressed in terms of random reward jumps D_j. The associated means are defined as

$$(5.8) \qquad \mathrm{E}[D_j] = \begin{cases} -D & \text{if } j = 1, 3, \cdots, 2K+1 \\ 0 & \text{if } j = 2, 4, \cdots, 2K \\ -10D & \text{if } j = 2K+2 \end{cases}.$$

In what follows, a set of parameter values for λ, ρ_{DOWN}, D, i, $K(0)$, $K(1)$ and $K(2)$ would be fixed as specified in Table 5.1 below. Numerical experiments are then conducted to explore the optimal value of K, which maximizes the expected reward per unit time in the time interval $(0, T]$ as a function of K and T for given values of μ and ρ_{UP}.

Table 5.1: Parameter Values for λ, ρ_{DOWN}, D, i, $K(0)$, $K(1)$ and $K(2)$

λ	ρ_{DOWN}	D	i	$K(0)$	$K(1)$	$K(2)$
3	10	100	1	100	100	50

More specifically, let $C_0(K)$ and $C_1(K)$ be defined as

$$(5.9) \qquad C_0(K) \overset{\mathrm{def}}{=} \underline{p}^\top(0)\underline{\underline{X}}_0\underline{1}$$

and

$$(5.10) \qquad C_1(K) \overset{\mathrm{def}}{=} \underline{p}^\top(0)\underline{\underline{X}}_1\underline{1} ,$$

so that one has from Theorem 4.1 $a)$

$$(5.11) \qquad \frac{\mathrm{E}[Z(T)|J(0) = 1]}{T} = C_1(K) + \frac{1}{T}C_0(K) + o(1) .$$

The optimal number of minimal repairs, denoted by K_T^*, is now given as

$$(5.12) \qquad K_T^* \overset{\mathrm{def}}{=} \underset{K}{\arg\max}\, \mathrm{E}[Z(T)|J(0) = 1]/T$$

$$= \underset{K}{\arg\max} \left\{ C_1(K) + \frac{1}{T}C_0(K) + o(1) \right\} .$$

Of interest is to understand the behavior of K_T^* as K and T are varied for given values of μ and ρ_{UP}.

Figures 5.2 through 5.7 exhibit $\mathrm{E}[Z(T)|J(0) = 1]/T$ as a function of K and T for each pair of $\mu = 3, 5, 15$ and $\rho_{\mathrm{UP}} = 15, 20$ arranged in lexicographic order.
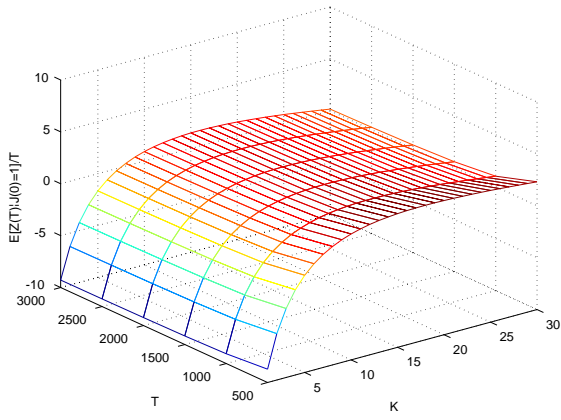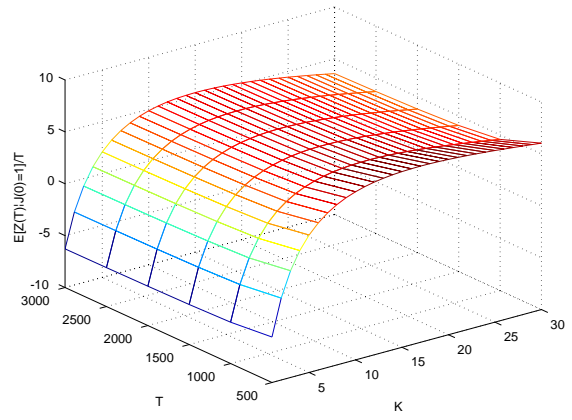
Figure 5.2: $E[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\mathrm{UP}}) = (3, 15)$

Figure 5.3: $E[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\mathrm{DOWN}}) = (3, 20)$

Figure 5.4: $E[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\mathrm{UP}}) = (5, 15)$

Figure 5.5: $E[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\mathrm{DOWN}}) = (5, 20)$

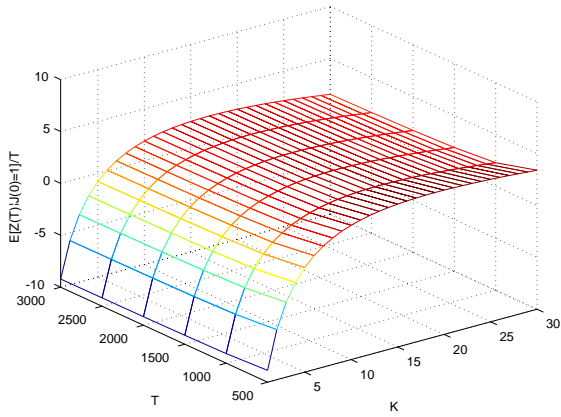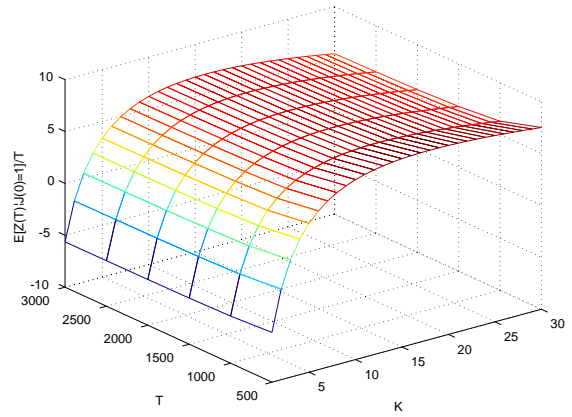Figure 5.6: $E[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\mathrm{UP}}) = (15, 15)$

Figure 5.7: $E[Z(T)|J(0) = 1]/T$ for $(\mu, \rho_{\mathrm{DOWN}}) = (15, 20)$
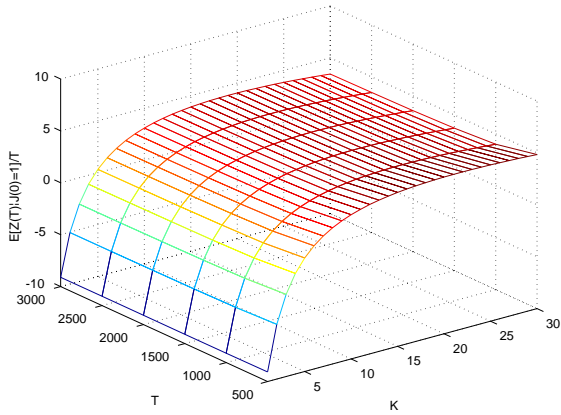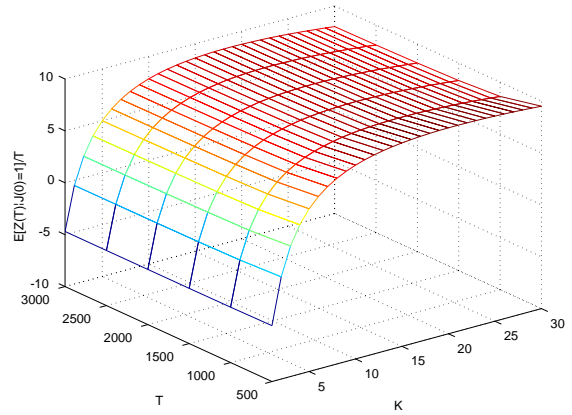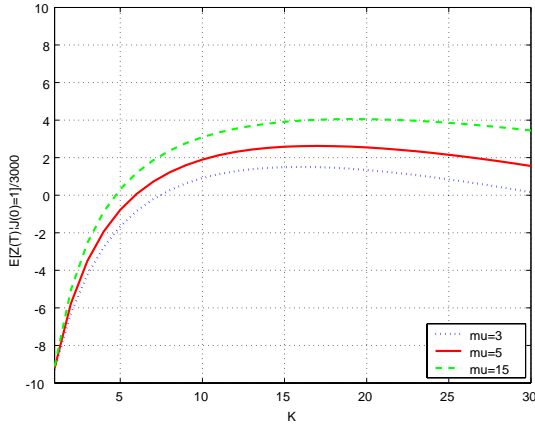
Figure 5.8: $\mathrm{E}[Z(T)|J(0) = 1]/T$ for $T = 3000$, $\rho_{\mathrm{UP}} = 15$ and $\mu = 3, 5, 15$
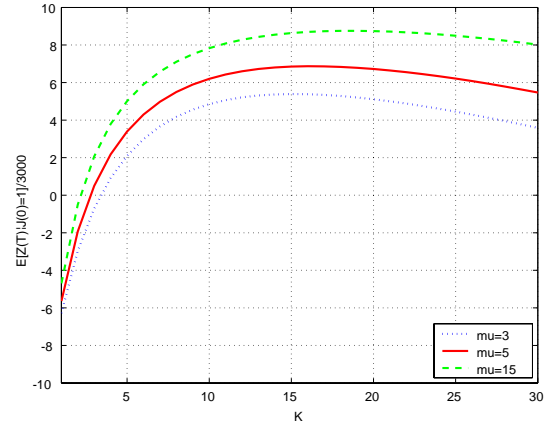
Figure 5.9: $\mathrm{E}[Z(T)|J(0) = 1]/T$ for $T = 3000$, $\rho_{\mathrm{UP}} = 20$ and $\mu = 3, 5, 15$

Table 5.2: $K_T^*(T = 500, 1000, 1500, 2000, 2500, 3000)$ for each pair of $\mu = 3, 5, 15$ and $\rho_{\mathrm{UP}} = 15, 20$

$(\mu, \rho_{\mathrm{UP}}) \setminus T$	500	1000	1500	2000	2500	3000
$(3, 15)$	18	17	16	16	16	16
$(3, 20)$	17	16	15	15	15	15
$(5, 15)$	19	18	17	17	17	17
$(5, 20)$	18	17	17	16	16	16
$(15, 15)$	22	20	20	19	19	19
$(15, 20)$	21	19	19	19	19	19

Table 5.3: K_∞^* for $\mu = 3, 5, 15$ and $\rho_{\mathrm{UP}} = 15, 20$

$\rho_{\mathrm{UP}} \setminus \mu$	3	5	15
15	16	17	19
20	15	16	18

In order to facilitate the understanding of the functional behavior, $\mathrm{E}[Z(T)|J(0) = 1]/T$ are plotted in Figures 5.8 and 5.9 as the marginal functions of K for $T = 3000$ and $\rho_{\mathrm{UP}} = 15, 20$. The values of the optimal number of minimal repairs K_T^* are given in Table 5.2 for $T = 500, 1000, \cdots, 3000$. From Table 5.2 and Figures 5.2 through 5.7, one observes that K_T^* decreases as T increases and $\mathrm{E}[Z(T)|J(0) = 1]/T$ appears to be a concave function of $K(1 \leqslant K \leqslant 30)$ for all $\mu = 3, 5, 15$ and $\rho_{\mathrm{UP}} = 15, 20$. The optimal K_∞^* in the long run average is summarized in Table 5.3, showing that K_∞^* increases as μ increases or ρ_{UP} decreases. When T is relatively small, the optimal K_∞^* may not be optimal as can be seen from Tables 5.2 and Table 5.3, demonstrating the importance of dynamic analysis.

We next turn our attention to $\mathrm{Cor}[N(T), Z(T)|J(0) = i]$ for capturing the time-dependent correlation structure numerically based on Theorem 4.1 b). Parameter values for λ, ρ_{DOWN}, D, i, $K(0)$, $K(1)$ and $K(2)$ are again as in Table 5.1. Figures 5.10 through 5.15 illustrate $\mathrm{Cor}[N(T), Z(T)|J(0) = i]$ as a function of K and T for each pair of $\mu = 3, 5, 15$ and

$\rho_{\mathrm{UP}} = 15, 20$. They are also exhibited as marginal functions of K in Figures 5.16 through 5.21.

Figure 5.10: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (3, 15)$

Figure 5.11: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (3, 20)$

Figure 5.12: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (5, 15)$

Figure 5.13: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (5, 20)$

Figure 5.14: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (15, 15)$

Figure 5.15: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (15, 20)$
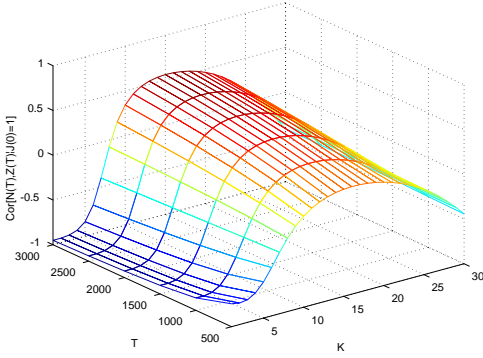
Figure 5.16: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (3, 15)$
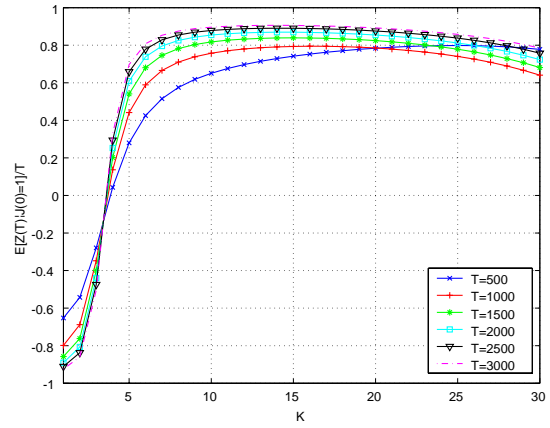
Figure 5.17: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (3, 20)$
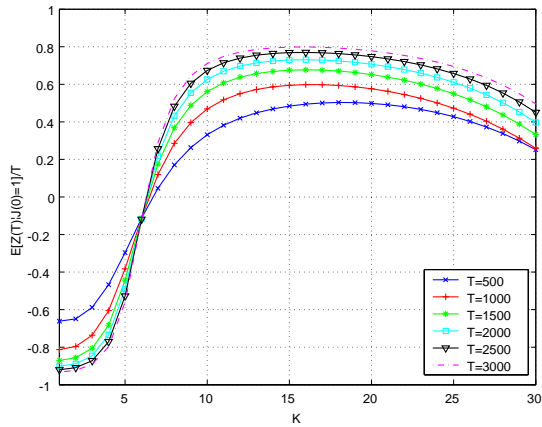
Figure 5.18: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (5, 15)$
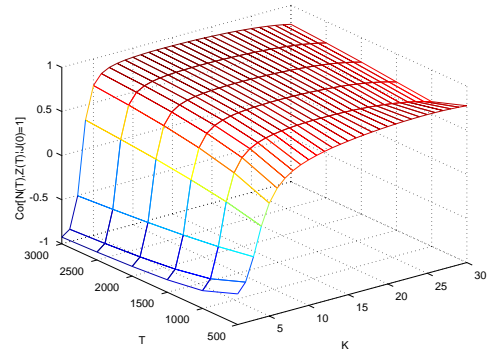
Figure 5.19: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (5, 20)$
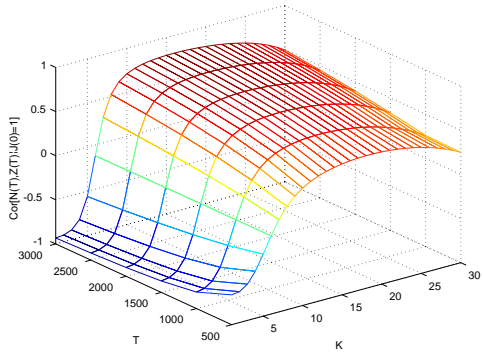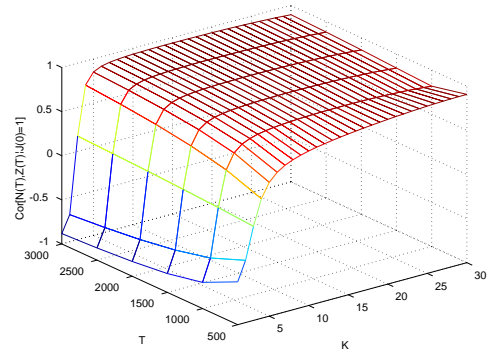
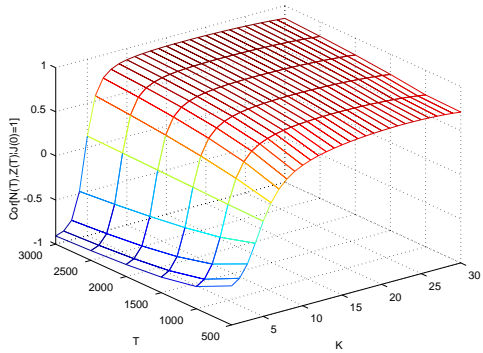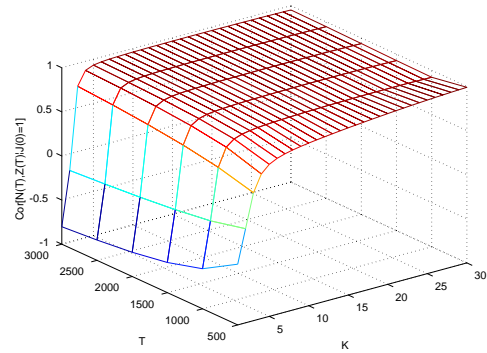Figure 5.20: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (15, 15)$

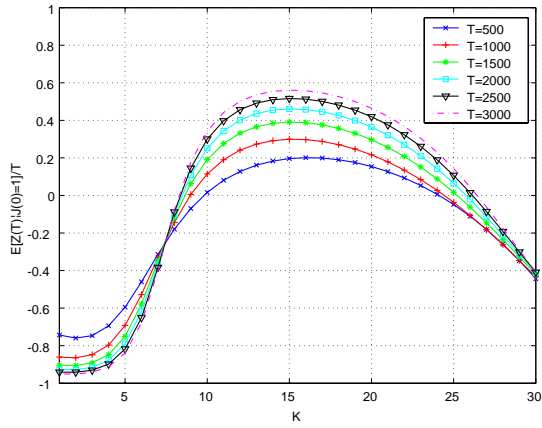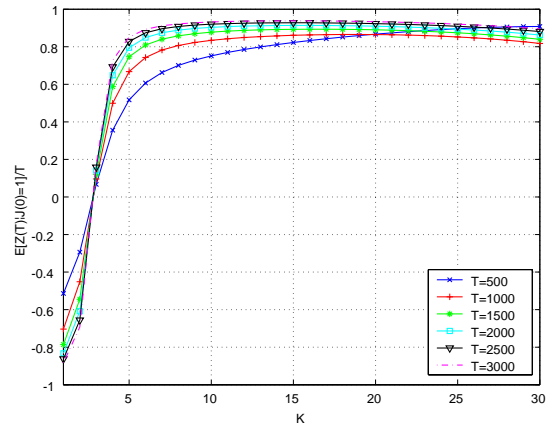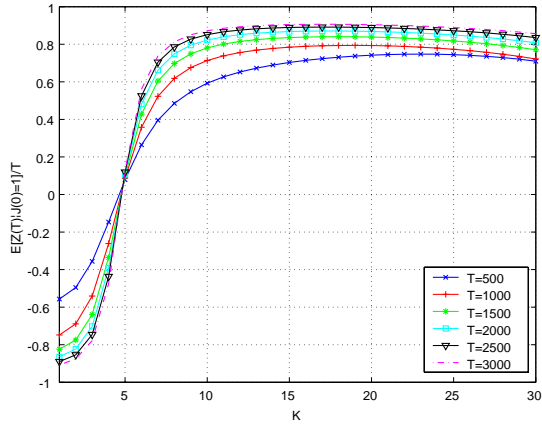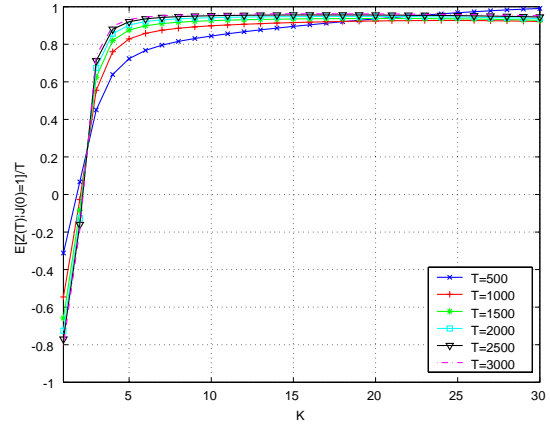Figure 5.21: $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ for $(\mu, \rho_{\mathrm{UP}}) = (15, 20)$

From Figures 5.16, 5.18 and 5.20, one finds that $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ is unimodal with respect to K. Figures 5.17, 5.19 and 5.21 for $\mu = 3, 5, 15$ and $\rho_{\mathrm{UP}} = 20$ show that $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ increases as μ increases. In the cases of $T = 500$, $\mathrm{Cor}[N(T), Z(T)|J(0) = 1]$ is monotonically increasing as a function of K for all values of $\mu = 3, 5, 15$.

6.   Concluding Remarks

In this paper, a cyclic renewal process is considered as an extension of an alternating renewal process where each of the underlying i.i.d. nonnegative random increments is composed of multiple stages. Such a process may be appropriate for analyzing optimal preventive maintenance policies for production management, where a pair of two stages representing an uptime until a minor failure and the subsequent minimal repair time would be repeated until it is decided to conduct a complete overhaul. In order to address economic problems in such applications, also introduced is a reward process with jumps defined on the cyclic renewal process. When the system is running in stage j, the profit grows linearly at the rate of $\rho(j)$. Upon a minor failure, the subsequent minimal repair in stage $(j + 1)$ incurs the linear cost at the rate of $\rho(j + 1)$. In addition, the fixed cost may be imposed whenever either a minimal repair or a complete overhaul takes place, resulting in jumps of the reward process. The problem is then to determine when to conduct a complete overhaul so as to maximize the total reward in the time interval $(0, T]$.

The multivariate Markov process generated from both the cyclic renewal process and the reward process is studied extensively, yielding various transform results explicitly and deriving their asymptotic expansions. These results are used to numerically explore optimal preventive maintenance policies for production management, demonstrating the usefulness of the cyclic renewal model.

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Appendix

In this appendix, we establish various lemmas concerning the asymptotic expansions of the transform results obtained in Section 3. These lemmas can be proven from (4.3) combined with appropriate differentiation in a straightforward manner, and the proofs are omitted. The asymptotic theorems needed for numerically exploring the underlying reward and correlation structure are derived in Section 4 using these lemmas.

Let $\underline{\underline{\alpha}}^{\#}(s)$ and $\underline{\underline{1}}^{\#}$ be the matrices defined by

$$
\underline{\underline{\alpha}}^{\#}(s) \overset{\text{def}}{=}
\begin{bmatrix}
0 & \alpha_1(s) & 0 & \cdots & 0 \\
0 & 0 & \alpha_2(s) & \cdots & 0 \\
0 & 0 & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \alpha_{J-1}(s) \\
\alpha_J(s) & 0 & \cdots & 0 & 0
\end{bmatrix}
, \quad
\underline{\underline{1}}^{\#} \overset{\text{def}}{=}
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix} .
$$

Lemma A.1 *As $s \to 0+$, the following expressions hold true.*

a) $\quad \underline{\underline{\alpha}}^{\#}(s) = \underline{\underline{1}}^{\#} - s\underline{\underline{A}}_1 + \dfrac{1}{2}s^2\underline{\underline{A}}_2 + \underline{\underline{o}}(s^2)$

b) $\quad \dfrac{d}{ds}\underline{\underline{\alpha}}^{\#}(s) = -\underline{\underline{A}}_1 + s\underline{\underline{A}}_2 + \underline{\underline{o}}(s) \ ; \ \left(\dfrac{d}{ds}\right)^2 \underline{\underline{\alpha}}^{\#}(s) = \underline{\underline{A}}_2 + \underline{\underline{o}}(1)$

c) $\quad \underline{\underline{\beta}}_D(0, s) = \underline{\underline{A}}_{D:1} - \dfrac{1}{2}s\underline{\underline{A}}_{D:2} + \underline{\underline{o}}(s)$

d) $\quad \dfrac{\partial}{\partial w}\underline{\underline{\beta}}_D(w, s)\bigg|_{w=0} = -\dfrac{1}{2}\underline{\underline{\rho}}_D\underline{\underline{A}}_{D:2} + \underline{\underline{o}}(1) \ ; \ \dfrac{\partial^2}{\partial w^2}\underline{\underline{\beta}}_D(w, s)\bigg|_{w=0} = \underline{\underline{o}}\left(\dfrac{1}{s}\right)$

Lemma A.2 *As $s \to 0+$, the following statements hold.*

a) $\left\{ \dfrac{\partial}{\partial u} \underline{\underline{\chi}}(0, s, u) \right\}\Big|_{u=1} = \dfrac{1}{s^2} \underline{\underline{Q}}_2 + \dfrac{1}{s} \underline{\underline{Q}}_1 + \underline{\underline{o}}\left(\dfrac{1}{s}\right)$

b) $\left\{ \dfrac{\partial^2}{\partial u^2} \underline{\underline{\chi}}(0, s, u) \right\}\Big|_{u=1} = \dfrac{1}{s^3} \underline{\underline{K}}_3 + \dfrac{1}{s^2} \underline{\underline{K}}_2 + \underline{\underline{o}}\left(\dfrac{1}{s^2}\right)$

c) $\left\{ \dfrac{\partial}{\partial w} \underline{\underline{\zeta}}(w, s, 1) \right\}\Big|_{w=0} = -(\underline{\underline{A}}_{D:1} \underline{\underline{\rho}}^{\#} + \underline{\underline{D}}^{\#}_1) + s(\underline{\underline{A}}_{D:2} \underline{\underline{\rho}}^{\#} + \underline{\underline{A}}_{D:1} \underline{\underline{D}}^{\#}_1) + \underline{\underline{o}}(s)$

d) $\left\{ \dfrac{\partial^2}{\partial w^2} \underline{\underline{\zeta}}(w, s, 1) \right\}\Big|_{w=0} = \underline{\underline{A}}_{D:2} \underline{\underline{\rho}}_D \underline{\underline{\rho}}^{\#} + 2\underline{\underline{A}}_{D:1} \underline{\underline{\rho}}_D \underline{\underline{D}}^{\#}_1 + \underline{\underline{D}}^{\#}_2 + \underline{\underline{o}}(1)$

e) $\left\{ \dfrac{\partial}{\partial w} \underline{\underline{\chi}}(w, s, 1) \right\}\Big|_{w=0} = \dfrac{1}{s^2} \underline{\underline{V}}_2 + \dfrac{1}{s} \underline{\underline{V}}_1 + \underline{\underline{o}}\left(\dfrac{1}{s}\right)$

f) $\left\{ \dfrac{\partial^2}{\partial w^2} \underline{\underline{\chi}}(w, s, 1) \right\}\Big|_{w=0} = \dfrac{1}{s^3} \underline{\underline{W}}_3 + \dfrac{1}{s^2} \underline{\underline{W}}_2 + \underline{\underline{o}}\left(\dfrac{1}{s^2}\right)$

g) $\left\{ \dfrac{\partial^2}{\partial u \partial w} \underline{\underline{\chi}}(w, s, u) \right\}\Big|_{u=1, w=0} = \dfrac{1}{s^3} \underline{\underline{R}}_3 + \dfrac{1}{s^2} \underline{\underline{R}}_2 + \underline{\underline{o}}\left(\dfrac{1}{s^2}\right)$

where

$$
\underline{\underline{\rho}}_D \overset{def}{=} \begin{bmatrix} \rho(1) & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & \rho(J) \end{bmatrix}, \quad
\underline{\underline{\rho}}^{\#} \overset{def}{=} \begin{bmatrix} 0 & \rho(1) & 0 & \cdots & 0 \\ 0 & 0 & \rho(2) & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \rho(J-1) \\ \rho(J) & 0 & \cdots & 0 & 0 \end{bmatrix},
$$

$$
\underline{\underline{D}}^{\#}_1 \overset{def}{=} \begin{bmatrix} 0 & E[D_1] & 0 & \cdots & 0 \\ 0 & 0 & E[D_2] & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & E[D_{J-1}] \\ E[D_J] & 0 & \cdots & 0 & 0 \end{bmatrix},
$$

$$
\underline{\underline{D}}^{\#}_2 \overset{def}{=} \begin{bmatrix} 0 & E[D_1^2] & 0 & \cdots & 0 \\ 0 & 0 & E[D_2^2] & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & E[D_{J-1}^2] \\ E[D_J^2] & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad
\underline{\hat{\underline{1}}} \overset{def}{=} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \underline{\underline{0}},
$$

24

$$\underline{\underline{Q}}_2 \overset{def}{=} \underline{\underline{H}}_1 \hat{\underline{\underline{1}}} \underline{\underline{H}}_1 \ , \ \ \underline{\underline{Q}}_1 \overset{def}{=} \underline{\underline{H}}_1 \hat{\underline{\underline{1}}} \underline{\underline{H}}_0 + (\underline{\underline{H}}_0 - A_{J:1}\underline{\underline{H}}_1)\hat{\underline{\underline{1}}} \underline{\underline{H}}_1 \ ,$$

$$\underline{\underline{K}}_3 \overset{def}{=} 2\underline{\underline{Q}}_2 \hat{\underline{\underline{1}}} \underline{\underline{H}}_1 \ , \ \ \underline{\underline{K}}_2 \overset{def}{=} 2\underline{\underline{Q}}_2 \hat{\underline{\underline{1}}} \underline{\underline{H}}_0 + (\underline{\underline{Q}}_1 - A_{J:1}\underline{\underline{Q}}_2)\hat{\underline{\underline{1}}} \underline{\underline{H}}_1 \ ,$$

$$\underline{\underline{V}}_2 \overset{def}{=} -\underline{\underline{H}}_1(\underline{\underline{A}}_{D:1}\underline{\rho}^{\#} + \underline{\underline{D}}_1^{\#})\underline{\underline{H}}_1 \ ,$$

$$\underline{\underline{V}}_1 \overset{def}{=} \{\underline{\underline{H}}_1(\underline{\underline{A}}_{D:2}\underline{\rho}^{\#} + \underline{\underline{A}}_{D:1}\underline{\underline{D}}_1^{\#}) - \underline{\underline{H}}_0(\underline{\underline{A}}_{D:1}\underline{\rho}^{\#} + \underline{\underline{D}}_1^{\#})\}\underline{\underline{H}}_1 - \underline{\underline{H}}_1(\underline{\underline{A}}_{D:1}\underline{\rho}^{\#} + \underline{\underline{D}}_1^{\#})\underline{\underline{H}}_0 \ ,$$

$$\underline{\underline{W}}_3 \overset{def}{=} -2\underline{\underline{H}}_1(\underline{\underline{A}}_{D:1}\underline{\rho}^{\#} + \underline{\underline{D}}_1^{\#})\underline{\underline{V}}_2 \ ,$$

$$\underline{\underline{W}}_2 \overset{def}{=} -2\underline{\underline{H}}_0(\underline{\underline{A}}_{D:1}\underline{\rho}^{\#} + \underline{\underline{D}}_1^{\#})\underline{\underline{V}}_2 + \underline{\underline{H}}_1\{-2(\underline{\underline{A}}_{D:1}\underline{\rho}^{\#} + \underline{\underline{D}}_1^{\#})\underline{\underline{V}}_1 + 2(\underline{\underline{A}}_{D:2}\underline{\rho}^{\#} + \underline{\underline{A}}_{D:1}\underline{\underline{D}}_1^{\#})\underline{\underline{V}}_2$$
$$+(\underline{\underline{A}}_{D:2}\underline{\rho}_D\underline{\rho}^{\#} + 2\underline{\underline{A}}_{D:1}\underline{\rho}_D\underline{\underline{D}}_1^{\#} + \underline{\underline{D}}_2^{\#})\underline{\underline{H}}_1\} \ ,$$

$$\underline{\underline{R}}_3 \overset{def}{=} \underline{\underline{V}}_2 \hat{\underline{\underline{1}}} \underline{\underline{H}}_1 + \underline{\underline{H}}_1 \hat{\underline{\underline{1}}} \underline{\underline{V}}_2 \ ,$$

$$\underline{\underline{R}}_2 \overset{def}{=} (\underline{\underline{V}}_2 \hat{\underline{\underline{1}}} \underline{\underline{H}}_0 + \underline{\underline{H}}_0 \hat{\underline{\underline{1}}} \underline{\underline{V}}_2) + (\underline{\underline{V}}_1 \hat{\underline{\underline{1}}} \underline{\underline{H}}_1 + \underline{\underline{H}}_1 \hat{\underline{\underline{1}}} \underline{\underline{V}}_1) - A_{J:1}(\underline{\underline{V}}_2 \hat{\underline{\underline{1}}} \underline{\underline{H}}_1 + \underline{\underline{H}}_1 \hat{\underline{\underline{1}}} \underline{\underline{V}}_2)$$
$$-\{\rho(J)A_{J:1} + E[D_J]\}\underline{\underline{H}}_1 \hat{\underline{\underline{1}}} \underline{\underline{H}}_1 \ .$$

Lemma A.3 *As* $t \to \infty$,

$$
\begin{aligned}
a) \quad & E[N(t)|J(0) = i] = \underline{p}^{\top}(0)(\underline{\underline{L}}_1 t + \underline{\underline{L}}_0)\underline{1} + o(1) \\
b) \quad & E[N^2(t)|J(0) = i] = \underline{p}^{\top}(0)(\underline{\underline{S}}_2 t^2 + \underline{\underline{S}}_1 t)\underline{1} + o(t) \\
c) \quad & E[Z^2(t)|J(0) = i] = \underline{p}^{\top}(0)(\underline{\underline{T}}_2 t^2 + \underline{\underline{T}}_1 t)\underline{1} + o(t) \\
d) \quad & E[N(t)Z(t)|J(0) = i] = \underline{p}^{\top}(0)(\underline{\underline{U}}_2 t^2 + \underline{\underline{U}}_1 t)\underline{1} + o(t)
\end{aligned}
$$

where

$$\underline{\underline{L}}_1 \overset{def}{=} \underline{\underline{Q}}_2\underline{\underline{A}}_{D:1} \ , \ \ \underline{\underline{L}}_0 \overset{def}{=} \underline{\underline{Q}}_1\underline{\underline{A}}_{D:1} - \frac{1}{2}\underline{\underline{Q}}_2\underline{\underline{A}}_{D:2} \ ,$$

$$\underline{\underline{S}}_2 \overset{def}{=} \underline{\underline{K}}_3\underline{\underline{A}}_{D:1} \ , \ \ \underline{\underline{S}}_1 \overset{def}{=} (\underline{\underline{K}}_2 + \underline{\underline{Q}}_2)\underline{A}_{D:1} - \frac{1}{2}\underline{\underline{K}}_3\underline{\underline{A}}_{D:2} \ ,$$

$$\underline{\underline{T}}_2 \overset{def}{=} \underline{\underline{W}}_3\underline{\underline{A}}_{D:1} \ , \ \ \underline{\underline{T}}_1 \overset{def}{=} \underline{\underline{W}}_2\underline{\underline{A}}_{D:1} - \frac{1}{2}\underline{\underline{W}}_3\underline{\underline{A}}_{D:2} - \underline{\underline{V}}_2\underline{\rho}_D\underline{\underline{A}}_{D:2} \ ,$$

$$\underline{\underline{U}}_2 \overset{def}{=} -\underline{\underline{R}}_3\underline{\underline{A}}_{D:1} \ , \ \ \underline{\underline{U}}_1 \overset{def}{=} -\underline{\underline{R}}_2\underline{\underline{A}}_{D:1} + \frac{1}{2}\underline{\underline{R}}_3\underline{\underline{A}}_{D:2} + \frac{1}{2}\underline{\underline{Q}}_2\underline{\rho}_D\underline{\underline{A}}_{D:2} \ .$$