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New Bounds on the Minimum Calls of Failure-Tolerant Gossiping

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Abstract

Gossiping is an extensively investigated information dissemination process. In gossiping, every vertex holds a message which has to be transmitted to all other vertices. This paper deals with k-failure tolerant gossiping, which investigates the minimum number of transmissions (calls) required by the communication process, provided that at most k transmissions may fail. We show new bounds for the number of transmissions, which are better than the previous results if k is sufficiently large. In addition, some examples of k-failure tolerant gossiping with fewer transmissions are shown for small graphs.

*Keywoards:*Communication network; Gossiping; Reliability

1 Introduction

A gossiping problem and its various variations have been extensively studied for several decades (See, for example, [3, 5, 6] for surveys). In the gossiping problem, first proposed by A. Boyd in 1971, there are n ladies, each of whom knows a unique message that is not known by any of the others. They communicate by telephone. Whenever two ladies make a call, they pass on to each other all information they know at that time. The gossiping problem is to find the minimum number of calls required for all ladies to know all massages. It has been proven that the solution to the problem is $2n-4$ for $n \geq 4$.

Gossiping is a fundamental task in network communication. This type of network communication often occurs in distributed computing. As communication networks grow in size, they become increasingly vulnerable to component failures. Berman and Hawrylaycz [1] introduced the additional feature that as many as k of the calls may fail in the sense that no information is exchanged, where k is a second parameter of the problem. We assume that the ladies cannot attempt different calls depending on which ones have failed previously. Berman and Hawrylaycz [1] have sought bounds on $\tau(n,k)$, the number of telephone calls needed to ensure that all n ladies possess all n messages even if some arbitrary k calls fail. They established

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the bounds of $\tau(n,k)$: $\left\lceil \left(\frac{k+4}{2}\right)(n-1)\right\rceil - 2\lceil \sqrt{n}\ \rceil + 1 \leq \tau(n,k) \leq \left\lfloor \left(k+\frac{3}{2}\right)(n-1)\right\rfloor$ for $k \leq n-2$, and $\left\lceil \left(\frac{k+3}{2}\right) n\right\rceil - 2\lceil \sqrt{n}\ \rceil \leq \tau(n,k) \leq \left\lfloor \left(k+\frac{3}{2}\right)(n-1)\right\rfloor$ for $k \geq n-2$. By constructing a communication network on a hypercube system, Haddad, Roy and Schäffer[4] showed the upper bound, $\tau(n,k) \leq \frac{nk}{2} + \mathrm{O}(k\sqrt{n} + n\log_2 n)$ which was an improvement for almost all k. Recently, Berman and Paul[2] proved that $2n-2+\left\lceil \frac{k(n-1)}{2}\right\rceil - \lfloor \log_2 n \rfloor \leq \tau(n,k)$ using stronger reliable scheme, k-verifiable gossiping.

In this paper, we proposed a new upper bound $\tau(n,k) \leq \frac{n(n-1)}{2} + \left\lceil \frac{nk}{2} \right\rceil$, which improves the previous upper bounds for a sufficiently large k. Moreover, we give tighter bounds on $\tau(n,k)$ for small n.

Although it may not so be attractive in application to research in gossipings with small n and large k, it is important theoretically. One of the purposes of this field is to get tighter bound of $\tau(n,k)$. But the gap between the lower and upper bounds is bigger when n and k become larger. The exact bound we know so far is only the case of $k=0$. So, we shall investigate a case of small n, which is one step in order to get a tighter bound of $\tau(n,k)$ for any n,k.

Gossiping is modeled by an ordered multigraph $G=(V,E)$, where $V=\{v_0,v_1,\ldots,v_{n-1}\}$ is a vertex set and E is an edge set on which a linear ordering is imposed. The vertices of G represent ladies, the edges represent telephone calls between pairs of ladies, and the linear order determines the turn of telephone calls. A message from $u \in V$ to $v \in V$ must proceed along a *u-v ascending path*, i.e., a path from u to v such that for any two edges in the path the edge closer to u is smaller in the linear order. We say an ordered multigraph G is *gossiping* if there is a u-v ascending path for every ordered pair of distinct vertices $u,v \in V$. If an ordered multigraph G is still gossiping whenever any k edges are deleted, G is called *k-failure tolerant gossiping*. Note that G is k-failure tolerant gossiping if and only if G has at least $k+1$ edge-disjoint u-v ascending paths for every ordered pair of distinct vertices $u,v \in V$.

Let a *class* be a subset of edges whose calls can be made in any order among themselves. We denote by E_i the ith class. Edges in the same class are ordered arbitrarily, but, for $i<j$, all the edges in E_i are ordered before any edge in E_j. Since we specify no order for the calls with in any particular class, only one member of each class can appear in any ascending path.

2 A construction of k-failure tolerant gossiping

This section shows how to construct k-failure tolerant gossiping that establishes $\tau(n,k) \leq n(n-1)/2 + \lceil nk/2 \rceil$.

At first, we prepare a complete graph K_n in which each vertex is adjacent to every other vertex with one edge. All edges of this complete graph belong to E_0. Obviously, this complete graph is (0-failure tolerant) gossiping of redundancies. We next define $n-1$ graphs $G_1,\ldots,G_{n-1}$, where each edge set, denoted by $E(G_i)$, is parceled to a class E_i for $1 \leq i \leq n-1$. When n is even, $G_1,\ldots,G_{n-1}$ are defined by edge-disjoint 1-factors, i.e., spanning subgraphs induced by perfect matchings such that $\bigcup_{i=1}^{n-1} G_i$ is a complete graph K_n. When n is odd, we can decompose the complete graph K_n into $n-1$ spanning subgraphs $G_1,\ldots,G_{n-1}$, where the degree of each vertex in G_i except for a specified vertex v' is exactly one, and, if i is odd, the degree of v' in G_i is two, otherwise, it is zero. At the end of this section, we shall show an example for decomposing K_n into $G_1, \ldots, G_{n-1}$ according to the above condition.

Lemma 1 *For $0 \le k \le n-1$, the ordered multigraph obtained by the collection of all the edges in $\bigcup_{i=0}^{k} E_i$ is k-failure tolerant gossiping.*

Proof. Since each vertex $v \in V$ is adjacent to at least k vertices by edges belonging to $\bigcup_{i=1}^{k} E_i$, there are at least k edge-disjoint ascending paths form any vertex $u \in V$ to v using edges in $\bigcup_{i=0}^{k} E_i$. These u-v ascending paths have at most two edges: the first edge belongs to E_0 and the second edge in $\bigcup_{i=1}^{k} E_i$; or u and v are adjacent by an edge in $\bigcup_{i=1}^{k} E_i$. In addition, since there is an edge between u and v in E_0, we have $k+1$ edge-disjoint u-v ascending paths in the obtained multigraph. ∎

When $nr \le k \le 2nr-1$ for $r \ge 1$, we make r copies $(G_1^t, G_2^t, \ldots, G_{n-1}^t)$ $(1 \le t \le r)$ of the spanning subgraphs $(G_1, G_2, \ldots, G_{n-1})$ obtained by the above process, and parcel $E(G_i^t)$ to E_{nt+i} for $1 \le i \le n-1$ and $1 \le t \le r$.

Theorem 2 *For any k, the ordered multigraph obtained by the collection of all the edges in $\bigcup_{i=0}^{k} E_i$ is k-failure tolerant gossiping. Therefore, we obtain $\tau(n,k) \le \frac{n(n-1)}{2} + \lceil \frac{nk}{2} \rceil$.*

Proof. Let $\mathcal{P}_{uv}$ be a set of u-v ascending paths in $(V, \bigcup_{i=0}^{nr-1} E_i)$. Suppose that for each pair of vertices $u, v \in V$, $\mathcal{P}_{uv}$ contains nr edge-disjoint ascending paths so that we obtain $(nr-1)$-failure tolerant gossiping. Since, for any k with $nr \le k \le 2nr-1$, each vertex $v \in V$ is adjacent to at least $k' = k - nr + 1$ vertices by edges in $\bigcup_{i=nr}^{k} E_i$, there are at least k' edge-disjoint ascending paths form $u \in V$ to v, where the first edge belongs to $\bigcup_{i=n(r-1)}^{nr-1} E_i$ and the second edge in $\bigcup_{i=nr}^{k} E_i$; or u and v are adjacent in $\bigcup_{i=nr}^{k} E_i$. These ascending paths do not have the same edges in any path belonging to $\mathcal{P}_{uv}$. So, we have $k+1$ edge-disjoint u-v ascending paths in the obtained multigraph.

Obviously, the number of edges in E_0 is $n(n-1)/2$. Since other even class has $\lfloor n/2 \rfloor$ edges and any odd class has $\lceil n/2 \rceil$ edges, we can establish the upper bound of the minimum calls. ∎

This result improves the previous upper bounds for a sufficiently large k.

We next consider a parallel complexity of the problem. Assume that each call takes unit time and that each lady participates in at most one call at a time. When n is even, since $n-1$ units of time are needed to construct E_0 and other class can be fulfilled by only one unit, $(n-1)+k$ units of time are needed to establish k-failure tolerant gossiping. When n is odd, $3(n-1)/2 + \lceil 3k/2 \rceil$ units are needed. Our result competes with Hadda, Roy and Schäffer [4]'s parallel complexity, $2k + 5\log_2 n + 10$.

Finally, the following shows a rule so as to decompose $E(K_n)$ into $E_1, \ldots, E_{n-1}$ holding the condition we described before, when n is odd. Let a function $\rho : \mathbf{Z} \to \{0, 1, \ldots, n-3\}$ be defined by $\rho(x) = x - (n-2)\lfloor \frac{x}{n-2} \rfloor (= x \bmod (n-2))$. We then put $E_1 = \{(v_{2j-1}, v_{2j}) \mid j = 1, \ldots, \frac{n-1}{2}\} \cup \{(v_0, v_{n-1})\}$, $E_{2i} = \{(v_{\rho(2i-2-j)}, v_{\rho(2i-1+j)}) \mid j = 0, \ldots, \frac{n-5}{2}\} \cup \{(v_{\rho(2i+\frac{n-5}{2})}, v_{n-2})\}$ for $i = 1, \ldots, \frac{n-1}{2}$, and $E_{2i+1} = \{(v_{\rho(2i-1-j)}, v_{\rho(2i+j)}) \mid j = 1, \ldots, \frac{n-5}{2}\} \cup \{(v_{\rho(2i+\frac{n-3}{2})}, v_{n-2}), (v_{2i-1}, v_{n-1}), (v_{2i}, v_{n-1})\}$ for $i = 1, \ldots, \frac{n-3}{2}$. It is easy to check that the degree constraint of each E_i is satisfied. That is to say, the degree of v_i $(i = 0, 1, \ldots, n-1)$ is exactly one on each (V, E_i), and the degree of v_{n-1} is two on (V, E_{2j-1}) and is zero on (V, E_{2j}) for integer j. This rule is based on a one factorization of a complete graph with the node set $\{v_0, \ldots, v_{n-2}\}$, as we can see in Figure 1 that shows the case of $n = 9$. So, it is obvious that $\bigcup_{i=1}^{n-1} E_i$ induces a complete graph K_n.

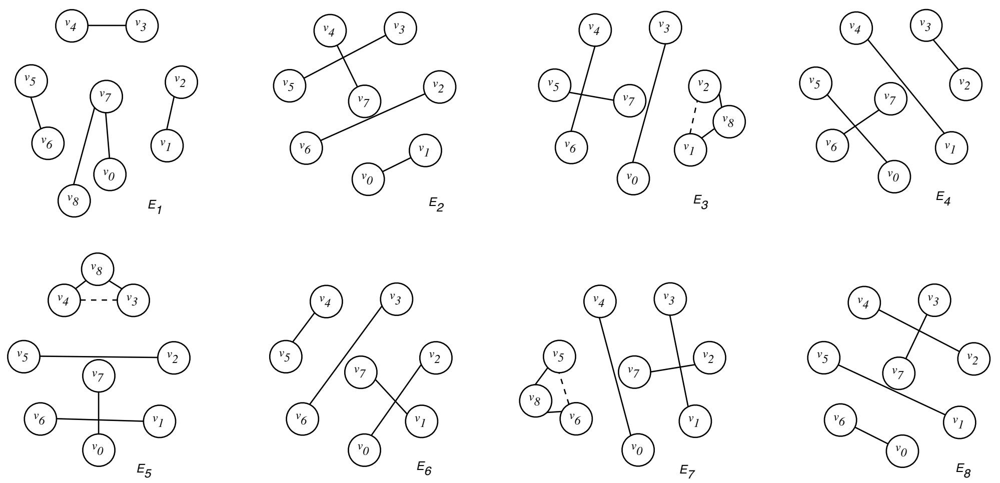

Figure 1: A decomposition of $E(K_9)$ into $E_1, \ldots, E_8$.

3 Tighter bounds for small graphs

Although the previous section gives a new upper bound on $\tau(n, k)$, there are differences between the lower and the upper bounds even if n is small. This section obtains tighter ranges of $\tau(n, k)$ for n fixed to some small values.

We first show a new lower bound on $\tau(n, k)$. If G is k-failure tolerant gossiping, since there are at least $k + 1$ edge-disjoint ascending paths between every pair of vertices, the degree of each vertex is at least $k + 1$. Assume that there are two vertices v, v' whose degrees are exactly $k+1$. Since there are $k+1$ edge-disjoint v-v' (resp. v'-v) ascending paths, the orders of the all edges incident to v must be the same as ones incident to v'. That is to say, all edges incident to v are also incident to v'. Hence, if there are more than two vertices whose degrees are exactly $k + 1$, then the graph becomes disconnected. Thus, we have the following result.

Theorem 3 *It holds that* $\lfloor \frac{n}{2}(k + 2) \rfloor \leq \tau(n, k)$. ■

This lower bound is tighter than the previous ones when $n < 20$.

For several small n, we next show tighter upper bounds on $\tau(n, k)$: $\tau(4, k) \leq 2k + 4$, $\tau(n, k) \leq \frac{n}{2}(k + 3)$ for $n = 6$ and 8, and $\tau(n, k) \leq \frac{n}{2}(k + 4)$ for $n = 10, 12, 14$ and 16. These bounds are established by constructing k-failure tolerant gossipings.

Unfortunately, a common constructing rule for these case of n has not been found. So, in the following, we show a constructing rule for each case. It is left with a future research to construct a general rule which can be applied to any n.

The case of $n = 4$ is very simple. Let $E_{2i-1} = \{(v_0, v_1), (v_2, v_3)\}$, and $E_{2i} = \{(v_1, v_2), (v_3, v_0)\}$ for any $i \geq 1$. Then it is obvious that the ordered multigraph $(V, \bigcup_{i=1}^{k+2} E_i)$ becomes k-failure tolerant gossiping with $2k + 4$ edges. Together with Theorem 3, we obtain $\tau(4, k) = 2k + 4$.

The cases of $n = 6, 8$ and 10 are proved by the similar construction. Let E_{3i-2}, E_{3i-1} and E_{3i} be

$$\{(v_{2j}, v_{2j+1}) \mid j = 0, \ldots, \frac{n}{2} - 1\}, \tag{1}$$

$$\{(v_{2j-1}, v_{2j}) \mid j = 1, \ldots, \frac{n}{2} - 1\} \cup \{(v_{n-1}, v_0)\}, \text{ and,} \tag{2}$$

$$\{(v_j, v_{j+n/2}) \mid j = 0, \ldots, \frac{n}{2} - 1\}, \tag{3}$$

respectively, for $i \geq 1$. Figure 2 illustrates the ordered multigraph $(V, \bigcup_i E_i)$ for $n = 6, 8$ and 10. For convenience, we draw one edge to present multiedges. The number on each edge represents classes to which the multiedges belong, for $i \geq 1$. We can verify that the ordered multigraphs $(V, \bigcup_{i=1}^{k+3} E_i)$ are k-failure tolerant gossipings for $n = 6$ and 8, and $(V, \bigcup_{i=1}^{k+4} E_i)$ is k-failure tolerant gossiping for $n = 10$.

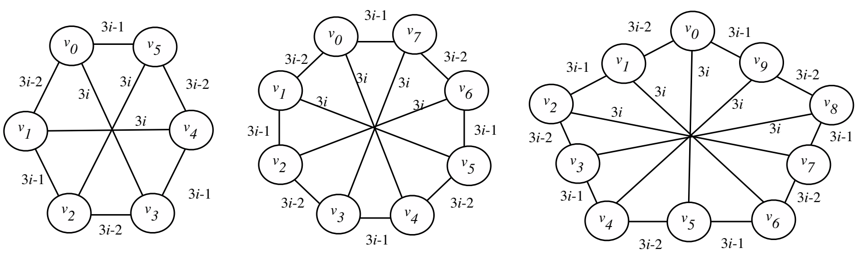

Figure 2: Construction for k-failure tolerant gossipings with $n = 6, 8$ and 10

Because the obtained ordered multigraph $(V, \bigcup_i E_i)$ is vertex transitive, if there are $k + 1$ edge-disjoint v_0-u ascending paths for every $u \in V \setminus \{v_0\}$, the graph is k-failure tolerant gossiping. Let $\mathcal{P}_j(h)$ be a set of v_0-v_j ascending paths using edges in $\bigcup_{i=1}^{h} E_i$. Since every vertex is incident to exactly one edge in any class, we can identify a path by the head and tail vertices and sequence of the numbers of classes. So, a path in $\mathcal{P}_j(h)$ is represented by the sequence of the numbers of classes. For convinience, a set $\mathcal{P}_j(h) \setminus \mathcal{P}_j(h-1)$ is denoted by $\mathcal{P}_j(h^-)$.

Here we justify the case of $n = 8$. Table 1 shows v_0-v_j ascending paths in $\mathcal{P}_j(3)$ $\mathcal{P}_j(4^-), \mathcal{P}_j(5^-)$ and $\mathcal{P}_j(6^-)$. From the second column of Table 1, we can see that the ordered multigraph $(V, \bigcup_{i=1}^{3} E_i)$ is 0-failure tolerant gossiping. Moreover, for each $j = 1, \ldots, 7$, it can be verified that the given four paths in $\mathcal{P}_j(3)$, $\mathcal{P}_j(4^-), \mathcal{P}_j(5^-)$ and $\mathcal{P}_j(6^-)$, respectively, are pairewise edge-disjoint. Hence, the ordered multigraphs $(V, \bigcup_{i=1}^{k+3} E_i)$ are k-failure tolerant gossiping for $k = 1, 2, 3$. We next consider the case of $k \geq 4$. If $(l_1, l_2, \ldots, l_p)$ is a path in $P_j(h)$, then the path obtained by adding

3 to each element, $(l_1+3, l_2+3, \ldots, l_p+3)$ belongs to $P_j(h+3)$. Hence, for $h \geq 7$, we can obtain a path in $P_j(h^-)$ from a path in $P_j((h-3)^-)$. In addition, the obtained path has no common edge with any path in $P_j(h')$ for $h' < h$. Therefore, the ordered multigraphs $(V, \bigcup_{i=1}^{k+3} E_i)$ are k-failure tolerant gossipings for $n = 8$, which establishes $\tau(8,k) \leq 4(k+3)$.

The justifications for $n = 6$ and 10 are given in Appendix.

Table 1: An ascending path in $\mathcal{P}_j(3)$ and $\mathcal{P}_j(h^-)$ when $n = 8$.

j	$\mathcal{P}_j(3)$	$\mathcal{P}_j(4^-)$	$\mathcal{P}_j(5^-)$	$\mathcal{P}_j(6^-)$
1	(1)	(4)	(2,3,4,5)	(3,4,6)
2	(1,2)	(2,3,4)	(4,5)	(3,4,5,6)
3	(2,3)	(1,2,4)	(3,5)	(5,6)
4	(3)	(1,3,4)	(2,3,5)	(6)
5	(1,3)	(3,4)	(2,4,5)	(4,6)
6	(1,2,3)	(2,4)	(3,4,5)	(4,5,6)
7	(2)	(1,2,3,4)	(5)	(3,5,6)

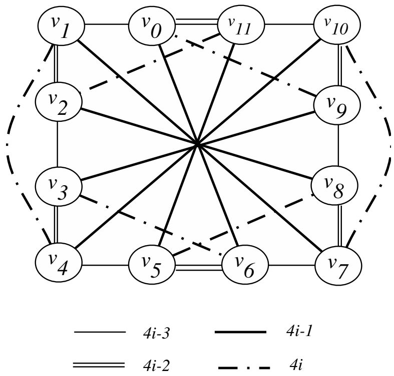

Figure 3: Construction for a k-failure tolerant gossiping with $n = 12$

In the cases of $n = 12, 14$ and 16, we need other types of edge classes. New classes of edges are defined by

$$\{(v_{2j+1}, v_{\rho'(2j+4)}) \mid j = 0, \ldots, \frac{n}{2} - 1\}, \text{ and} \quad (4)$$

$$\{(v_{2j}, v_{\rho'(2j+3)}) \mid j = 0, \ldots, \frac{n}{2} - 1\}, \quad (5)$$

where $\rho'(x) = x \bmod n$.

When $n = 12$, we prepare classes E'_{4i-3}, E'_{4i-2}, E'_{4i-1} and E'_{4i} by Eqs. (1), (2),(3) and (4), respectively, for $i \geq 1$. Then the obtained ordered multigraph $(V, \bigcup_{i=1}^{k+4} E'_i)$, which is illustrated in Figure 3, is k-failure tolerant gossiping. Hence we establish $\tau(12,k) \leq 6(k+4)$. In the case of $n = 14$, E''_{6i-5}, E''_{6i-4}, E''_{6i-2} and E''_{6i-1} are given by Eqs. (4), (2), (5) and (1), respectively, and both of E''_{6i-3} and E''_{6i} by Eq. (3). Then $(V, \bigcup_{i=1}^{k+4} E''_i)$ is k-failure tolerant gossiping, and $\tau(14,k) \leq 7(k+4)$. When $n = 16$, we use new classes

$$E'''_{4i-3} = \{(v_{4j+3}, v_{\rho'(4j+5)}), (v_{4j+2}, v_{\rho'(4j+4)}) \mid j = 0, \ldots, \frac{n}{4} - 1\},$$

$$E'''_{4i-2} = \{(v_{4j}, v_{4j+2}), (v_{4j+1}, v_{4j+3}) \mid j = 0, \ldots, \frac{n}{4} - 1\}.$$

Classes E'''_{4i-1} and E'''_{4i} are given by Eqs. (3) and (4). Then $(V, \bigcup_{i=1}^{k+4} E'''_i)$ is k-failure tolerant gossiping, and $\tau(16,k) \leq 8(k+4)$. The justifications for $n = 12, 14$ and 16 are also given in Appendix.

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Appendix

Given E_{3i-2}, E_{3i-1} and E_{3i} by Eqs. (1), (2) and (3), respectively, for $i \geq 1$, we can justify that $(V, \bigcup_i E_i)$ is k-failure tolerant gossiping when $n = 6$ and 10, by the similar way as $n = 8$. When $n = 6$, Table 2 shows v_0-v_j ascending paths in $\mathcal{P}_j(3)$ and in $\mathcal{P}_j(h^-)$ for $h = 4, 5, 6$. Since, for $h \geq 7$, a path in $\mathcal{P}_j(h^-)$ can be obtaind from a path in $\mathcal{P}_j((h-3)^-)$, the ordered multigraphs $(V, \bigcup_{i=1}^{k+3} E_i)$ are k-failure tolerant gossipings for $n = 6$. When $n = 10$, Table 3 provides ascending paths in $\mathcal{P}_j(4)$ and in $\mathcal{P}_j(h^-)$. Although a set of $\mathcal{P}_1(5^-)$ is empty, $(V, \bigcup_{i=1}^{k+4} E_i)$ is k-failure tolerant gossiping for $k \leq 3$, because there are two edge-disjoint ascending paths in $\mathcal{P}_1(4)$. When we obtain a path in $\mathcal{P}_j(h)$ for $h \geq 8$ from $\mathcal{P}_j((h-3)^-)$, it seems to be the matter that $\mathcal{P}_1(5^-)$ is empty. There are, however, another path $(2, 4, 5, 6, 8)$ in $\mathcal{P}_1(8^-)$. So we conclude $(V, \bigcup_{i=1}^{k+4} E_i)$ is k-failure tolerant gossiping for any k when $n = 10$.

Table 2: An ascending path in $\mathcal{P}_j(3)$ and $\mathcal{P}_j(h^-)$ when $n = 6$.

j	$\mathcal{P}_j(3)$	$\mathcal{P}_j(4^-)$	$\mathcal{P}_j(5^-)$	$\mathcal{P}_j(6^-)$
1	(1)	(4)	$(3,4,5)$	$(2,4,6)$
2	$(1,2)$	$(3,4)$	$(4,5)$	$(5,6)$
3	(3)	$(1,2,4)$	$(2,4,5)$	(6)
4	$(1,3)$	$(2,4)$	$(3,5)$	$(4,6)$
5	(2)	$(1,3,4)$	(5)	$(4,5,6)$

Table 3: An ascending path in $\mathcal{P}_j(4)$ and $\mathcal{P}_j(h^-)$ when $n = 10$.

j	$\mathcal{P}_j(4)$	$\mathcal{P}_j(5^-)$	$\mathcal{P}_j(6^-)$	$\mathcal{P}_j(7^-)$
1	$(1)(4)$	$\emptyset$	$(3,5,6)$	(7)
2	$(1,2)$	$(4,5)$	$(2,4,5,6)$	$(3,4,5,7)$
3	$(1,2,4)$	$(3,4,5)$	$(2,4,6)$	$(4,5,7)$
4	$(2,3)$	$(1,2,4,5)$	$(5,6)$	$(6,7)$
5	(3)	$(1,3,5)$	(6)	$(5,6,7)$
6	$(1,3)$	$(3,5)$	$(4,6)$	$(2,4,5,7)$
7	$(1,2,3)$	$(2,4,5)$	$(4,5,6)$	$(3,5,7)$
8	$(2,4)$	$(1,2,3,5)$	$(3,4,5,6)$	$(5,7)$
9	(2)	(5)	$(3,4,6)$	$(1,3,4,5,7)$

For the cases of $n = 12, 14$ and 16, Tables 5, 4 and 6 show v_0-v_j ascending paths in $\mathcal{P}_j(4)$ and in $\mathcal{P}_j(h^-)$. When $n = 12$ and $n = 16$, a path in $\mathcal{P}_j(h^-)$ for $h \geq 9$ can be obtained from a path in $\mathcal{P}_j((h-4)^-)$. Then we can see that the obtained ordered multigraphs $(V, \bigcup_{i=1}^{k+4} E_i')$ and $(V, \bigcup_{i=1}^{k+4} E_i''')$ are k-failure tolerant gossipings for any k. In the case of $n = 14$, we can get a path in $\mathcal{P}_j(h)$ for $h \geq 11$ from $\mathcal{P}_j((h-6)^-)$. The only one problem is that the path $(7, 8, 9, 11)$ in $\mathcal{P}_4(11^-)$ has a common edge with $(7, 9) \in \mathcal{P}_4(9^-)$. We, however, find another path $(6, 7, 8, 9, 10, 11)$ in $\mathcal{P}_4(11^-)$. So we conclude $(V, \bigcup_{i=1}^{k+4} E_i)$ is k-failure tolerant gossiping for any k.

Table 4: An ascending path in $\mathcal{P}_j(4)$ and $\mathcal{P}_j(h^-)$ when $n = 14$.

j	$\mathcal{P}_j(4)$	$\mathcal{P}_j(5^-)$	$\mathcal{P}_j(6^-)$	$\mathcal{P}_j(7^-)$	$\mathcal{P}_j(8^-)$	$\mathcal{P}_j(9^-)$	$\mathcal{P}_j(10^-)$
1	$(1,2,4)$	(5)	$(2,3,4,5,6)$	$(3,4,7)$	$(4,5,8)$	$(6,8,9)$	$(7,8,10)$
2	$(1,2,3,4)$	$(4,5)$	$(2,3,4,6)$	$(3,5,6,7)$	$(5,8)$	$(6,7,8,9)$	$(7,8,9,10)$
3	(4)	$(1,2,3,4,5)$	$(2,4,6)$	$(3,5,7)$	$(5,7,8)$	$(6,7,9)$	(10)
4	$(3,4)$	$(1,2,3,5)$	$(2,4,5,6)$	$(5,7)$	$(4,8)$	$(7,9)$	$(9,10)$
5	$(1,2,3)$	$(3,4,5)$	$(2,5,6)$	$(5,6,7)$	$(4,7,8)$	$(7,8,9)$	$(6,7,8,9,10)$
6	$(2,3)$	$(3,5)$	$(1,2,5,6)$	$(4,7)$	$(5,6,7,8)$	$(8,9)$	$(6,7,8,10)$
7	$(1,3,4)$	$(2,3,5)$	(6)	$(4,6,7)$	$(5,6,8)$	(9)	$(7,9,10)$
8	$(1,4)$	$(2,3,4,5)$	$(5,6)$	$(3,4,5,7)$	$(6,8)$	$(4,5,8,9)$	$(7,10)$
9	$(2,3,4)$	$(1,4,5)$	$(4,5,6)$	$(3,4,5,6,7)$	$(6,7,8)$	$(5,8,9)$	$(8,9,10)$
10	$(2,4)$	$(1,5)$	$(4,6)$	$(6,7)$	$(3,4,5,6,7,8)$	$(5,7,8,9)$	$(8,10)$
11	(1)	$(2,4,5)$	$(3,4,6)$	(7)	$(4,5,6,7,8)$	$(5,7,9)$	$(6,8,10)$
12	$(1,2)$	$(2,5)$	$(3,4,5,6)$	$(4,5,6,7)$	$(7,8)$	$(5,6,7,9)$	$(6,8,9,10)$
13	(2)	$(1,2,5)$	$(3,5,6)$	$(4,5,7)$	(8)	$(5,6,7,8,9)$	$(6,7,10)$

Table 5: An ascending path in $\mathcal{P}_j(4)$ and $\mathcal{P}_j(h^-)$ when $n = 12$.

j	$\mathcal{P}_j(4)$	$\mathcal{P}_j(5^-)$	$\mathcal{P}_j(6^-)$	$\mathcal{P}_j(7^-)$	$\mathcal{P}_j(8^-)$
1	(1)	(5)	$(2,4,6)$	$(3,5,7)$	$(4,6,7,8)$
2	$(1,2)$	$(3,4,5)$	$(5,6)$	$(4,5,7)$	$(2,8)$
3	$(3,4)$	$(1,2,5)$	$(2,3,5,6)$	$(4,7)$	$(7,8)$
4	$(1,4)$	$(2,3,5)$	$(3,4,6)$	$(4,6,7)$	$(5,8)$
5	$(2,3)$	$(1,4,5)$	$(3,6)$	$(6,7)$	$(4,5,8)$
6	(3)	$(1,3,5)$	$(2,3,6)$	(7)	$(4,7,8)$
7	$(1,3)$	$(3,5)$	$(4,5,6)$	$(5,7)$	$(2,5,8)$
8	$(1,2,3)$	$(4,5)$	$(3,5,6)$	$(5,6,7)$	$(2,3,8)$
9	(4)	$(1,2,3,5)$	$(2,5,6)$	$(3,4,7)$	(8)
10	$(1,3,4)$	$(2,5)$	$(4,6)$	$(3,4,6,7)$	$(5,7,8)$
11	(2)	$(1,3,4,5)$	(6)	$(3,6,7)$	$(4,5,7,8)$

Table 6: An ascending path in $\mathcal{P}_j(4)$ and $\mathcal{P}_j(h^-)$ when $n = 16$.

j	$\mathcal{P}_j(4)$	$\mathcal{P}_j(5^-)$	$\mathcal{P}_j(6^-)$	$\mathcal{P}_j(7^-)$	$\mathcal{P}_j(8^-)$
1	$(1,2,3,4)$	$(2,4,5)$	$(3,4,5,6)$	$(4,5,6,7)$	$(5,6,7,8)$
2	(2)	$(1,2,3,5)$	(6)	$(3,6,7)$	$(4,6,8)$
3	$(1,3,4)$	$(3,4,5)$	$(2,4,5,6)$	$(4,5,7)$	$(5,7,8)$
4	$(1,2,3)$	$(2,5)$	$(3,5,6)$	$(5,6,7)$	$(4,5,6,7,8)$
5	$(3,4)$	$(1,3,4,5)$	$(2,3,4,6)$	$(4,7)$	$(7,8)$
6	$(1,3)$	$(3,5)$	$(2,5,6)$	$(5,7)$	$(4,5,7,8)$
7	$(2,3,4)$	$(1,2,4,5)$	$(3,4,6)$	$(4,6,7)$	$(6,7,8)$
8	(3)	$(1,3,5)$	$(2,3,6)$	(7)	$(4,7,8)$
9	$(1,2,4)$	$(2,3,4,5)$	$(4,5,6)$	$(3,4,5,6,7)$	$(5,6,8)$
10	$(2,3)$	$(1,2,5)$	$(3,6)$	$(6,7)$	$(4,6,7,8)$
11	$(1,4)$	$(4,5)$	$(2,3,4,5,6)$	$(3,4,5,7)$	$(5,8)$
12	$(1,2)$	$(2,3,5)$	$(5,6)$	$(3,5,6,7)$	$(4,5,6,8)$
13	(4)	$(1,4,5)$	$(2,4,6)$	$(3,4,7)$	(8)
14	(1)	(5)	$(2,3,5,6)$	$(3,5,7)$	$(4,5,8)$
15	$(2,4)$	$(1,2,3,4,5)$	$(4,6)$	$(3,4,6,7)$	$(6,8)$