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Structural Analysis of Two Person Game with Mixed Strategy for Public Utility Supply

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Abstract

Structural models for analyzing competitive markets characterized by homogeneous products and services such as the public utility can be traced back to 1920's. To the authors' best knowledge, the literature focuses on pure strategies and analysis for mixed strategies are largely ignored. However, the peculiarity of the public utility often allows only mixed strategies as a meaningful basis for analyzing the price competition. The purpose of this paper is to fill this gap by developing a duopoly model with two symmetric customers with mixed strategies. A necessary and sufficient condition is given for the existence of Nash equilibrium when mixed strategies are defined on a finite set of L discrete points spread in a finite interval. In addition, the Nash equilibriums are constructed explicitly when L discrete points are chosen in such a way that their reciprocals are equally distanced. The limiting strategies as $L \to \infty$ are also derived explicitly.

Keywords: OR in energy, public utility, two person game, mixed strategy, limiting strategy

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1 Introduction

While the price strategy plays a significant role in any business, it is of crucial importance to the public utility industry including electricity and gas because of several reasons. Firstly the public utility industry provides homogeneous products across different suppliers. Large-scale industrial customers are quite sensitive to prices of the public utility products. Although service quality for energy consulting, security and the like would be quite important for such industrial customers, because of the product homogeneity, the price strategy of a supplier is the key to differentiate the company from the rest and to establish its competitiveness in the market.

A second reason to emphasize the price strategy in the public utility industry can be found in that the industry has been deregulated in many advanced countries since near the end of the previous century, including the United States, EU countries and Japan. The deregulation is intended to device a variety of ways to lower barrier for new entry and the industry has been exposed to rapidly growing severe price competitions.

Lastly, it is important to realize that the public utility industry still faces certain customs for price setting which come from the public nature of the industry. Before the deregulation in Japan, for example, it is customary to offer a common price table, called the universal price table, to all customers at their site, provided that the total demand, and hourly and monthly load factors over a year are more or less the same. In addition, the universal price table cannot be altered frequently, say at most once in a few years. Such practices concerning the price strategy are still in effect to some extent even after the deregulation.

Structural models for analyzing competitive markets characterized by homogeneous products and services such as the public utility can be traced back to 1920's. The original paper by Hotelling(1929) deals with the duopoly situation where two suppliers compete over customers uniformly distributed on a finite line by choosing their locations and prices. D'Aspremont et al.(1979) show non-existence of Nash equilibrium unless the two suppliers are located relatively far apart. Economides(1986) extended the Hotelling model by introducing customers uniformly distributed on a plane. Anderson(1987) incorporates stackelberg leadership within the context of the Hotelling model. Other variations include Thisse and Vives(1988), Zhang and Teraoka (1998) and Rath(1998). Gabszewicz and Thisse(1992) provides an excellent review of the literature. More recently, for a spatially duopoly model with customers located at different nodes having separate demand functions, Matsubayashi et al.(2004) establish a necessary and sufficient condition for the existence of Nash equilibrium and develop computational algorithms for finding the equilibrium point.

The literature dicussed above focuses on pure strategies and analysis for mixed strategies has been largely ignored, to the best knowledge of the authors. Since the universal price table is still in effect to some extent and cannot be altered easily once they are set for a certain period even after the deregulation in Japan, it is of crucial importance to consider mixed strategies by reading the price strategies of competitors at the time of bidding. This means that the role of mixed strategies has been increasing its importance in analyzing the public utility industry.

The purpose of this paper is to fill this gap by developing a duopoly model with two symmetric customers and to establish a necessary and sufficient condition for the existence of Nash equilibrium when mixed strategies are defined on a finite set of L discrete points spread in a finite interval. In addition, the Nash equilibriums are constructed explicitly when L discrete points are chosen in such a way that their reciprocals are equally distanced. The limiting strategies as $L \to \infty$ are also derived explicitly.

The structure of this paper is as follows. In Secion 2, a duopoly model with two symmetric customers is introduced and a game-theoretic framework is described formally. Section 3 establishes a necessary and sufficient condition under which a Nash equilibrium within discrete mixed strategies exists. By choosing discrete pricing points in a peculiar way, the Nash equilibriums are constructed explicitly in Section 4. Section 5 is devoted to derive the limiting behavior of the strategies derived in Section 4 as $L \to \infty$.

2 Model Description

We consider a market consisting of two suppliers and two customers, where each supplier provides a homogeneous service such as city gas or electricity and each customer may represent one large industry or a group of residents in the same district. For convenience, the near customer of supplier i is defined as customer i and the distant customer as customer $3-i$, $i=1,2$ as depicted in Figure 2.1. The market is assumed to be symmetric in that a) both suppliers have the same costs c^H and c^L for providing service to the distant cus-

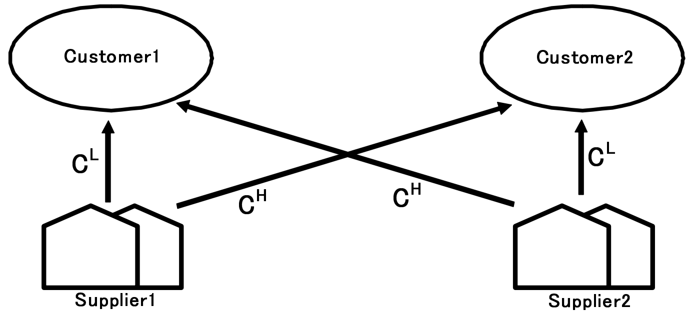

Figure 2.1: Two Supplier Two Customer Model

tomer and the near customer respectively where $c^L < c^H$; b) both customers have the same demand D; and c) each supplier has to offer a uniform price upon delivery to both of the two customers despite the cost difference. Each supplier provides its service only when it results in a positive return to do so and each customer chooses the supplier which offers the lower price. When the two suppliers happen to offer the same price to a customer, the demand of the customer is split evenly between the two suppliers. Since the service under consideration is typically a public utility service, it is also natural to assume that there exists a price upper bound. Accordingly, one has

$$\pi_i \in I = [c^H, U], \qquad i = 1, 2 \tag{2.1}$$

where π_i is the uniform price offered by supplier i. It should be noted that, if $c^L < \pi_i \leq c^H$, supplier i monopolizes its near customer and the price can be increased to c^H without losing its monopoly of the near customer. In what follows, we describe a general game structure defined on the strategy set I.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let RV be a set of random variables defined on $(\Omega, \mathcal{F}, P)$ with full support on $I = [c^H, U]$. More specifically, for $A_X(\alpha) = \{\omega | X(\omega) \leq \alpha\}$, we define

$$RV = \{X | X : \Omega \to \mathcal{R}, A_X(\alpha) \in \mathcal{F} \quad \text{for } \forall\, \alpha \in \mathcal{R}\} \tag{2.2}$$

where $\mathcal{R}$ is the set of real numbers. It should be noted that, for any $A \subset \mathcal{R}$, we write

$$P[X \in A] = \int_{\omega \in \Omega} \delta_{\{X(\omega) \in A\}} P(d\omega)$$

where $\delta_{\{X(\omega) \in A\}} = 1$ if $X(\omega) \in A$ and 0 else. In particular, it should be noted that $P[X \in I] = 1$. A mixed strategy of supplier i then corresponds to a random variable $X_i \in RV$.

Throughout the paper we assume that each supplier decides its strategy independently of the other so that X_1 and X_2 are independent, and each supplier has enough production capacity to meet customers' demands.

Given $\pi_1 = X_1(\omega_1)$ and $\pi_2 = X_2(\omega_2)$ for some $\omega_1, \omega_2 \in \Omega$, it can be readily seen that the payoff function of supplier i is given by

$$h_i(\pi_1, \pi_2) = \begin{cases} 2\Big(\pi_i - \dfrac{c^L + c^H}{2}\Big)D & \text{if} \quad \pi_i < \pi_{3-i}, \quad \pi_i, \pi_{3-i} \in (c^H, U] \\ \Big(\pi_i - \dfrac{c^L + c^H}{2}\Big)D & \text{if} \quad \pi_i = \pi_{3-i}, \quad \pi_i, \pi_{3-i} \in (c^H, U] \\ 0 & \text{if} \quad \pi_i > \pi_{3-i}, \quad \pi_i, \pi_{3-i} \in (c^H, U] \\ (c^H - c^L)D & \text{if} \quad \pi_i = c^H, \quad \pi_{3-i} \in (c^H, U] \\ (\pi_i - c^L)D & \text{if} \quad \pi_i \in (c^H, U], \ \pi_{3-i} = c^H \end{cases} . \tag{2.3}$$

If $c^H < \pi_i < \pi_{3-i} \leq U$, supplier i can monopolize the entire market with demand $2D$ at the average earning per unit of $\pi_i - (c^L + c^H)/2$. When $c^H < \pi_i = \pi_{3-i} \leq U$, the demand D of each customer is split evenly between the two suppliers and the average earning per unit is again $\pi_i - (c^L + c^H)/2$. For the case of $c^H = \pi_i < \pi_{3-i} \leq U$, supplier i can capture only the near customer with average earning per unit of $c^H - c^L$. Finally, if $c^H = \pi_{3-i} < \pi_i \leq u$, supplier i is forced to settle for the near customer with the average earning per unit of $\pi_i - c^L$. Let S_i be the strategy set of supplier i and define $S = S_1 \times S_2$. In our model, one has $S_1 = S_2 = RV$ so that $S = RV \times RV$. Given $(X_1, X_2) \in S$, let $V_i(X_1, X_2) = E[h_i(X_1, X_2)]$ be the expected payoff function of supplier i. More specifically, we define

$$V_i(X_1, X_2) = \int \int_{\omega_1 \in \Omega \ \omega_2 \in \Omega} h_i(X_1(\omega_1), X_2(\omega_2)) P(d\omega_1) P(d\omega_2), \qquad i = 1, 2. \tag{2.4}$$

The following conventional notion in game theory is employed.

Definition 2.1

a) For $i = 1, 2, X_i^*$ is a best reply against X_{3-i} if $V_1(X_1^*, X_2) = \max_{X_1 \in RV}[V_1(X_1, X_2)]$ or $V_2(X_1, X_2^*) = \max_{X_2 \in RV}[V_2(X_1, X_2)]$.

b) For $i = 1, 2, B_i(X_{3-i}) = \{X_i^* : X_i^*$ is a best reply against $X_{3-i}\}$ is called the set of best replies of supplier i against X_{3-i}.

c)The best reply correspondence $B : S \to S$ is defined as $B(X_1, X_2) = B_1(X_2) \times B_2(X_1)$.

d)(X_1^*, X_2^*) is a Nash equilibrium, denoted by $(X_1^*, X_2^*) \in \mathcal{NE}$, if and only if $(X_1^*, X_2^*) \in B(X_1^*, X_2^*)$.

It is difficult to prove the existence of a Nash equilibrium and to construct it explicitly for this model. In the following sections, we focus on discrete random variables in RV and establish a necessary and sufficient condition for the existence of a Nash equilibrium. In addition, when the discrete support points are chosen in such a way that their reciprocals are separated by equal distance, two types of Nash equilibriums can be constructed explicitly. Furthermore, it is shown that a sequence of each type of Nash equilibriums converges in law to a mixed strategy in S as the equal distance diminishes to 0.

3 Nash Equilibrium with Discrete Mixed Strategies

For a given $\underline{v} = [v_1, \cdots, v_L] \in \mathcal{R}^L$ with $v_1 = c^H < v_2 < v_3 < \cdots v_{L-1} < v_L = U$, let $DRV(\underline{v})$ be a set of discrete random variables with full support on $\{v_1, \cdots, v_L\}$, where $X \in DRV(\underline{v})$ is represented by a probability vector $\underline{q}$ with $P[X = v_m] = q_m, m \in \mathcal{L} = \{1, 2, 3 \cdots, L\}$, and we write $X \in DRV(\underline{v})$ or $\underline{q} \in DRV(\underline{v})$ interchangeably. In this section, we focus on discrete mixed strategies in $S(\underline{v}) = DRV(\underline{v}) \times DRV(\underline{v})$, where Definition 2.1 should be rewritten with RV replaced by $DRV(\underline{v})$. Let $\underline{\underline{H}}_i = [h_i(v_m, v_n)]_{m,n\in\mathcal{L}}, i = 1, 2$ with $h_i(v_m, v_n)$ as given in (2.3). From (2.4), one sees that

$$V_i(\underline{q}_1, \underline{q}_2) = \underline{q}_1^T \underline{\underline{H}}_i \, \underline{q}_2 \ , \ i = 1, 2 \qquad . \tag{3.1}$$

From the symmetric structure of (2.3), it can be seen that $h_1(\pi_1, \pi_2) = h_2(\pi_2, \pi_1)$ so that $\underline{\underline{H}}_2 = \underline{\underline{H}}_1^T$. It then follows that $V_2(\underline{q}_1, \underline{q}_2) = \underline{q}_1^T \underline{\underline{H}}_2 \, \underline{q}_2 = \underline{q}_2^T \underline{\underline{H}}_2^T \underline{q}_1 = \underline{q}_2^T \underline{\underline{H}}_1 \, \underline{q}_1$. Hence, it is possible to define $V_i(\underline{q}_1, \underline{q}_2)$ in place of (3.1) as

$$V_i(\underline{q}_1, \underline{q}_2) = \underline{q}_i^T \underline{\underline{H}} \, \underline{q}_{3-i} \ , i = 1, 2 \ , \tag{3.2}$$

$$\text{where} \qquad \underline{\underline{H}} \stackrel{def}{=} [h_1(v_m, v_n)]_{m,n\in\mathcal{L}} = \underline{\underline{H}}_1 \quad . \tag{3.3}$$

We next establish a necessary and sufficient condition under which a Nash equilibrium exists, i.e. $\mathcal{NE}(\underline{v}) \neq \emptyset$. A preliminary lemma is needed.

Lemma 3.1

Let $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\underline{v})$. For $i = 1, 2$, if $(\underline{q}_{3-i}^*)_{\hat{m}} > 0$, then $(\underline{\underline{H}} \, \underline{q}_i^*)_{\hat{m}} = \max_{m\in\mathcal{L}}[(\underline{\underline{H}} \, \underline{q}_i^*)_m]$.

Proof We prove the lemma by contraposition. Without loss of generality, we assume that $i = 1$. Suppose $(\underline{q}_2^*)_{\hat{m}} > 0$ and $(\underline{\underline{H}} \, \underline{q}_1^*)_{\hat{m}} < \max_{m\in\mathcal{L}}[(\underline{\underline{H}} \, \underline{q}_1^*)_m]$. For $\tilde{m} \in \mathcal{L}$ satisfying $(\underline{\underline{H}} \, \underline{q}_1^*)_{\tilde{m}} = \max_{m\in\mathcal{L}}[(\underline{\underline{H}} \, \underline{q}_1^*)_m]$, let $\tilde{\underline{q}}_2^*$ be defined by $\tilde{\underline{q}}_2^* = \underline{q}_2^* + (\underline{q}_2^*)_{\hat{m}}(\underline{e}_{\tilde{m}} - \underline{e}_{\hat{m}})$ where $\underline{e}_m \in \mathcal{R}^L$

is the m-th unit vector in $\mathcal{R}^L$. It is clear that $\underline{\tilde{q}}_2^* \geq 0$ and $\underline{\tilde{q}}_2^{*T}\underline{1} = 1$ so that $\underline{\tilde{q}}_2^* \in DRV(\underline{v})$, where $\underline{1}$ is a vector whose elements are all 1. It then follows from (3.2) that

$$\begin{aligned} V_2(\underline{q}_1^*, \underline{\tilde{q}}_2^*) &= \underline{\tilde{q}}_2^{*T} \underline{\underline{H}}\, \underline{q}_1^* = \underline{q}_2^{*T} \underline{\underline{H}}\, \underline{q}_1^* + (\underline{q}_2^*)_{\hat{m}} (\underline{e}_{\tilde{m}}^T - \underline{e}_{\hat{m}}^T) \underline{\underline{H}}\, \underline{q}_1^* \\ &= V_2(\underline{q}_1^*, \underline{q}_2^*) + (\underline{q}_2^*)_{\hat{m}} \{ (\underline{\underline{H}}\, \underline{q}_1^*)_{\tilde{m}} - (\underline{\underline{H}}\, \underline{q}_1^*)_{\hat{m}} \} > V_2(\underline{q}_1^*, \underline{q}_2^*) \end{aligned}$$

which contradicts to $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\underline{v})$, completing the proof. □

We are now in a position to prove the main theorem of this section.

Theorem 3.2 *For* $\underline{q} \in DRV(\underline{v})$, *let* $\underline{\epsilon}(\underline{q}) = \left[\max_{m\in\mathcal{L}}\{(\underline{\underline{H}}\,\underline{q})_m\}\right]\underline{1} - \underline{\underline{H}}\,\underline{q} \geq 0$. *Then* $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\underline{v})$ *if and only if* $\underline{q}_{3-i}^{*T}\underline{\epsilon}(\underline{q}_i^*) = 0$ *for* $i = 1, 2$.

Proof Without loss of generality, we assume that $i = 1$. Suppose $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\underline{v})$. From the definition of $\underline{\epsilon}(\underline{q})$, one sees that

$$(\underline{\epsilon}(\underline{q}_1^*))_n = \max_{m\in\mathcal{L}}\{(\underline{\underline{H}}\,\underline{q}_1^*)_m\} - (\underline{\underline{H}}\,\underline{q}_1^*)_n \geq 0 \text{ for } n \in \mathcal{L} \quad . \tag{3.4}$$

From Lemma 3.1, if $(\underline{q}_2^*)_n > 0$, then $\max_{m\in\mathcal{L}}\{(\underline{\underline{H}}\,\underline{q}_1^*)_m\} = (\underline{\underline{H}}\,\underline{q}_1^*)_n$ so that $(\underline{\epsilon}(\underline{q}_1^*))_n = 0$ from (3.4). Since $\underline{q}_2^* \geq \underline{0}$, one concludes that $\underline{\epsilon}^T(\underline{q}_1^*)\underline{q}_2^* = 0$. Conversely, suppose $\underline{\epsilon}^T(\underline{q}_1^*)\underline{q}_2^* = 0$. It can be seen from (3.4) that, for $\forall \underline{q}_2 \in DRV(\underline{v})$, one has $\underline{\underline{H}}\,\underline{q}_1^* = \left[\max_{m\in\mathcal{L}}\{(\underline{\underline{H}}\,\underline{q}_1^*)_m\}\right]\underline{1}^T - \underline{\epsilon}^T(\underline{q}_1^*)$. Since $\underline{q}_2^{*T}\underline{1} = \underline{q}_2^T\underline{1} = 1$ and $0 = \underline{q}_2^{*T}\underline{\epsilon}(\underline{q}_1^*) \leq \underline{q}_2^T\underline{\epsilon}(\underline{q}_1^*)$, it follows that

$$\begin{aligned} V_2(\underline{q}_1^*, \underline{q}_2^*) &= \underline{q}_2^{*T}\underline{\underline{H}}\,\underline{q}_1^* = \underline{q}_2^{*T}[\max_{m\in\mathcal{L}}\{(\underline{\underline{H}}\,\underline{q}_1^*)_m\}\underline{1} - \underline{\epsilon}(\underline{q}_1^*)] \\ &\geq \underline{q}_2^T[\max_{m\in\mathcal{L}}\{(\underline{\underline{H}}\,\underline{q}_1^*)_m\}\underline{1} - \underline{\epsilon}(\underline{q}_1^*)] = \underline{q}_2^T\underline{\underline{H}}\,\underline{q}_1^* = V_2(\underline{q}_1^*, \underline{q}_2) \ . \end{aligned}$$

Similarly one has $V_1(\underline{q}_1^*, \underline{q}_2^*) \geq V_1(\underline{q}_1, \underline{q}_2^*)$ for all $\underline{q}_1 \in DRV(\underline{v})$. Hence $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\underline{v})$, completing the proof. □

While Theorem 3.2 allows one to test whether or not a given $(\underline{q}_1, \underline{q}_2) \in S(\underline{v})$ is a Nash equilibrium, it does not provide a means to construct $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\underline{v})$. In the next section, a constructive proof is given for the existence of a Nash equilibrium by choosing $\underline{v}$ in a specific manner.

4 Nash Equilibriums with Specific Discrete Support

In this section, we provide a constructive proof for the existence of Nash equilibriums by choosing $\underline{v}$ in a certain manner. More specifically, let $\underline{a} = [a_1, \cdots, a_L]^T$ be such that

$$a_1 = \frac{1}{2}(c^H - c^L)D \quad ; \tag{4.1}$$

$$\frac{1}{a_m} = (L-m)\Delta + \frac{1}{a_L}, m \in \mathcal{L} \setminus \{1\} \quad ; \tag{4.2}$$

$$a_L = (U - \frac{c^L + c^H}{2})D \quad ; \text{ and}$$

$$\Delta = \frac{\frac{1}{a_1} - \frac{1}{a_L}}{L - \frac{3}{2}} \quad . \tag{4.3}$$

It should be noted that $\underline{a}$ is constructed in such a way that

$$\frac{1}{a_m} - \frac{1}{a_{m+1}} = \Delta \ , \ m \in \mathcal{L} \setminus \{1, L\} \quad ; \quad \frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{2}\Delta \quad . \tag{4.4}$$

We now define $\underline{v} = \hat{\underline{v}}_L$ in terms of $\underline{a}$ as

$$\hat{\underline{v}}_L = \frac{1}{D}\underline{a} + \frac{c^L + c^H}{2}\underline{1}_L \quad , \tag{4.5}$$

where $\underline{1}_m$ is the m-dimensional vector whose components are all 1. The decomposition of the interval $[c^H, U]$ by $\hat{\underline{v}}_L$ is rather peculiar as depicted in Figure 4.1. The following proposition

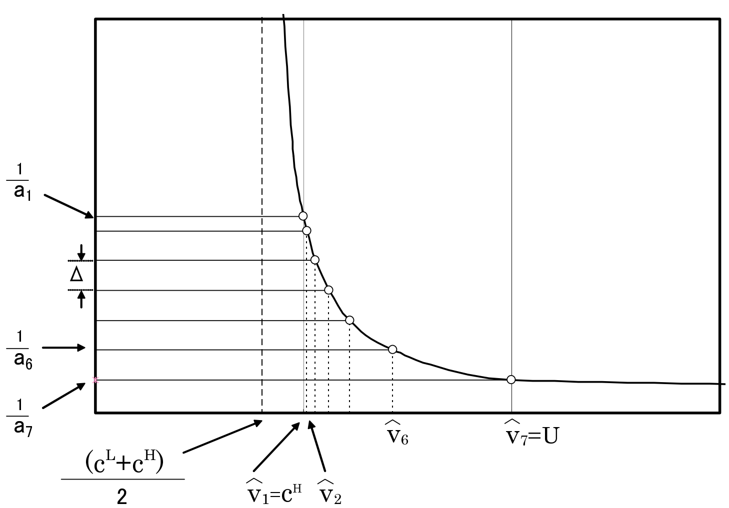

Figure 4.1: $\hat{\underline{v}}_L$ with L=7

is straightforward from (4.5) and proof is omitted.

Proposition 4.1 *$v_{m+1} - v_m$ is monotonically increasing for $m \in \mathcal{L} \setminus \{L\}$.*

We next show in a constructive manner that there exist two types of Nash equilibriums $(\underline{q}_1^*, \underline{q}_2^*), (\underline{q}_1^{**}, \underline{q}_2^{**}) \in \mathcal{NE}(\underline{\hat{v}}_L)$ where the two suppliers offer the same price with the same expected profit value in the former case, while they offer different prices but have the same expected profit value in the latter case. The first step is to prove the existence of $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\underline{\hat{v}}_L)$. A few preliminary lemmas are needed and proofs are given in Appendix.

Lemma 4.2 *Let* Δ *be as in (4.3) and define* $\underline{q}^{*T} = [\alpha_1, \alpha_2 \underline{1}_{L-1}^T] \in \mathcal{R}^L$ *where*

$$\alpha_1 = \frac{2a_1}{C_1}(\frac{2}{a_L} - \Delta), \qquad \alpha_2 = \frac{2a_1}{C_1}\Delta, \text{ and } \quad C_1 = 2(\frac{a_1}{a_L} + 1) - a_1\Delta \qquad . \tag{4.6}$$

If $L > \max(2, \frac{a_L}{2a_1} + 1)$, *then* $\underline{q}^* > \underline{0}$ *and* $\underline{q}^{*T}\underline{1}_L = 1$.

Lemma 4.3 *Let* α_1, α_2 *and* C_1 *be as in Lemma 4.2. Then one has a)* $2\alpha_2 + a_1(\alpha_1 - 2)\Delta = 0$ *and b)* $\alpha_2 + a_1(\alpha_1 - 2)\frac{1}{a_L} + \alpha_1 = 0$.

For notational convenience, the following matrices are introduced. We note that $\delta_{\{ST\}} = 1$ if the statement ST holds and $\delta_{\{ST\}} = 0$ else.

$$\underline{\underline{I}} = [\delta_{\{i=j\}}]_{i,j \in \mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)} \tag{4.7}$$

$$\underline{\underline{A}}_D = [\delta_{\{i=j\}} a_{i+1}]_{i,j \in \mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)} \tag{4.8}$$

$$\underline{\underline{L}} = [\delta_{\{i<j\}}]_{i,j \in \mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)} \tag{4.9}$$

$$\underline{\underline{L}}_1 = [\delta_{\{i+1=j\}}]_{i,j \in \mathcal{L}\setminus\{L\}} \in \mathcal{R}^{(L-1)\times(L-1)} \tag{4.10}$$

$$\underline{\underline{B}} = \underline{\underline{I}} + \underline{\underline{L}} \in \mathcal{R}^{(L-1)\times(L-1)} \tag{4.11}$$

$$\underline{\underline{C}} = \underline{\underline{I}} + 2\underline{\underline{L}} \in \mathcal{R}^{(L-1)\times(L-1)} \tag{4.12}$$

$$\underline{w}(x,y) = x\underline{1}_{L-1} + (y-x)\underline{e}_{L-1} \in \mathcal{R}^{(L-1)}, \tag{4.13}$$

where $\underline{e}_m \in \mathcal{R}^{L-1}$ is the m-th unit vector in $\mathcal{R}^{L-1}$.

Lemma 4.4 *Let* $\underline{\underline{A}}_D$ *and* $\underline{\underline{B}}$ *be as in (4.8) and (4.11). Then one has a)* $\underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} = \underline{w}(\Delta, \frac{1}{a_L})$ *and b)* $\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{1}_{L-1} = \underline{w}(2,1)$.

Lemma 4.5 *Let* $\underline{\underline{H}}$ *be as in (3.3) and define* $\underline{\hat{v}}_L^T = [\hat{v}_1, \cdots, \hat{v}_L]$ *as in (4.5). Then the* *following statements hold true.*

$$a) \quad [\underline{\underline{H}}]_{1,m} = 2a_1 \textit{ for } m \in \mathcal{L}$$

$$b) \quad [\underline{\underline{H}}]_{n,1} = a_1 + a_n \textit{ for } n \in \mathcal{L} \setminus \{1\}$$

$$c) \quad [\underline{\underline{H}}]_{m,n} = [\underline{\underline{A}}_D \underline{\underline{C}}]_{m-1,n-1} \textit{ for } m,n \in \mathcal{L} \setminus \{1\}$$

$$d) \quad \underline{\underline{H}} = \begin{bmatrix} 2a_1 & 2a_1 \underline{1}_{L-1}^T \\ (a_1 \underline{\underline{I}} + \underline{\underline{A}}_D)\underline{1}_{L-1} & \underline{\underline{A}}_D \underline{\underline{C}} \end{bmatrix}$$

$$e) \quad \underline{\underline{H}} \begin{bmatrix} x \\ y\underline{1}_{L-1} \end{bmatrix} = \begin{bmatrix} 2a_1 \\ (y\underline{\underline{A}}_D \underline{\underline{C}} + x a_1 \underline{\underline{I}} + x \underline{\underline{A}}_D)\underline{1}_{L-1} \end{bmatrix}$$
$$\textit{where } 0 < x < 1, \textit{ and } \quad y = (1-x)/(L-1)$$

The main theorem of this section can now be proven.

Theorem 4.6 *Let* $\alpha_i (i = 1, 2)$ *be as in (4.6) and define* $\underline{q}_1^* = \underline{q}_2^* = \underline{q}^*$ *where* $\underline{q}^{*T} = [\alpha_1, \alpha_2 \underline{1}_{L-1}^T]$. *If* $L > \max\{2, \frac{a_L}{2a_1} + 1\}$, *then* $(\underline{q}_1^*, \underline{q}_2^*) \in \mathcal{NE}(\hat{\underline{v}}_L)$. *Furthermore, the payoff values of the two suppliers are equal with* $V_1(\underline{q}_1^*, \underline{q}_2^*) = V_2(\underline{q}_1^*, \underline{q}_2^*) = D(c^H - c^L)$.

Proof From Lemma 4.2, one sees that $\underline{q}^* \in DRV(\hat{\underline{v}}_L)$. Let $\underline{\epsilon}(\underline{q})$ be as in Theorem 3.2. We will show that $\underline{\epsilon}(\underline{q}^*) = \underline{0}$ so that $\underline{\epsilon}(\underline{q}^*)^T \underline{q}^* = 0$. The theorem then follows from Theorem 3.2. From (4.13), one sees that $\underline{w}(x, y)$ is linear in (x, y). Lemma 4.3 then implies that $\alpha_2 \underline{w}(2,1) + a_1(\alpha_1 - 2)\underline{w}(\Delta, \frac{1}{a_L}) + \alpha_1 \underline{w}(0,1) = \underline{w}\big[2\alpha_2 + a_1(\alpha_1 - 2)\Delta, \alpha_2 + a_1(\alpha_1 - 2)\frac{1}{a_L} + \alpha_1\big] = \underline{w}(0,0) = \underline{0}$. With $\underline{w}(2,1)$ and $\underline{w}(\Delta, \frac{1}{a_L})$ in the above equation substituted by Lemma 4.4 a) and b) respectively, one sees that $\alpha_2 \underline{\underline{B}}^{-1} \underline{\underline{C}}\, \underline{1}_{L-1} + a_1(\alpha_1 - 2)\underline{\underline{B}}^{-1} \underline{\underline{A}}_D^{-1} \underline{1}_{L-1} + \alpha_1 \underline{w}(0,1) = \underline{0}$ Multiplying $\underline{\underline{A}}_D \underline{\underline{B}}$ from left, this then leads to

$$\alpha_2 \underline{\underline{A}}_D \underline{\underline{B}}\, \underline{\underline{B}}^{-1} \underline{\underline{C}}\, \underline{1}_{L-1} + a_1(\alpha_1 - 2)\underline{1}_{L-1} + \alpha_1 \underline{\underline{A}}_D \underline{\underline{B}}\, \underline{w}(0,1) = \underline{0} \ ,$$

i.e.

$$\big[\alpha_2 \underline{\underline{A}}_D \underline{\underline{C}} + a_1 \alpha_1 \underline{\underline{I}} + \alpha_1 \underline{\underline{A}}_D\big]\underline{1}_{L-1} = 2a_1 \underline{1}_{L-1} \qquad , \tag{4.14}$$

where $\underline{\underline{B}}\, \underline{w}(0,1) = \underline{1}_{L-1}$ is employed to yield (4.14). On the other hand, from Lemma 4.5 d), one sees that $\underline{\underline{H}}\, \underline{q}^* = \begin{bmatrix} 2a_1 \\ (\alpha_2 \underline{\underline{A}}_D \underline{\underline{C}} + \alpha_1 a_1 \underline{\underline{I}} + \alpha_1 \underline{\underline{A}}_D)\underline{1}_{L-1} \end{bmatrix}$. It then follows from this and (4.14) that $\underline{\underline{H}}\, \underline{q}^* = 2a_1 \underline{1}_L$. This in turn implies that $\underline{\epsilon}(\underline{q}^*) = [\max_{m \in \mathcal{L}}\{(\underline{\underline{H}}\, \underline{q}^*)_m\}]\underline{1}_L - \underline{\underline{H}}\, \underline{q}^* = 2a_1 \underline{1}_L - 2a_1 \underline{1}_L = \underline{0}$, completing the proof. □

The above theorem states that a Nash equilibrium can be achieved when the two suppliers offer the same mixed strategy $\underline{q}^{*T} = [\alpha_1, \alpha_2 \underline{1}^T] \in DRV(\hat{\underline{v}}_L)$. As can be seen from

(4.3), Δ decreases as L increases. One then sees from (4.6) that α_1 is much larger than α_2 for large values of L. In this case, the two suppliers tend to protect the near customer by assigning a higher probability of α_1 to $\hat{v}_1 = c^H$. At the same time, it is crucial to allocate a small but positive probability α_2 to all other price alternatives so that $(\underline{q}^*, \underline{q}^*) \in \mathcal{NE}(\underline{\hat{v}}_L)$ can be assured. Somewhat surprisingly, we next show that there exists a different type of Nash equilibrium $(\underline{q}_1^{**}, \underline{q}_2^{**}) \in \mathcal{NE}(\underline{\hat{v}}_L)$, where the two suppliers take different mixed strategies but share the same expected payoff which is the same as that of Theorem 4.6. As before, a few preliminary lemmas are needed and proofs are given in Appendix.

Lemma 4.7 *Let α_3 and α_4 be defined by*

$$\alpha_3 = \frac{\frac{2}{a_L}}{\frac{1}{a_1} + \frac{1}{a_L}} \; , \qquad \textit{and} \qquad \alpha_4 = \frac{2\Delta}{\frac{1}{a_1} + \frac{1}{a_L}} \; . \tag{4.15}$$

Then one has a) $\alpha_3 = a_1(2 - \alpha_3)\frac{1}{a_L}$ and b) $\alpha_4 = a_1(2 - \alpha_3)\Delta$.

In what follows, the matrices in (4.7) through (4.12) are employed.

Lemma 4.8 *Let $\underline{\underline{H}}$ be as in (3.3) and define $\underline{\hat{v}}_L$ as in (4.5). We also define $\underline{f} \in \mathcal{R}^{L-1}$ as $(\underline{f})_m = \{1 + (-1)^m\}/2, m \in \mathcal{L} \setminus \{L\}$. If L is even, then for any $0 < x < 1$ and $y = 2(1 - x)/(L - 2)$, one has*

$$\underline{\underline{H}} \begin{bmatrix} x \\ y\underline{f} \end{bmatrix} = \begin{bmatrix} 2a_1 \\ y\underline{\underline{A}}_D \underline{\underline{C}}\,\underline{f} + x a_1 \underline{1} + x \underline{\underline{A}}_D \underline{1} \end{bmatrix} \; . \tag{4.16}$$

Lemma 4.9 *Let $\underline{\underline{H}}, \underline{\hat{v}}_L$ and $\underline{w}(x, y)$ be as in (3.3),(4.5) and (4.13) respectively. Then for any $0 < y < 1$ and $x = (1 - y)/(L - 2)$, one has*

$$\underline{\underline{H}} \begin{bmatrix} 0 \\ \underline{w}(x, y) \end{bmatrix} = \begin{bmatrix} 2a_1 \\ \underline{\underline{A}}_D \underline{\underline{C}}\,\underline{w}(x, y) \end{bmatrix} \; .$$

Lemma 4.10 *Let $\underline{f}$ be as in Lemma 4.8. If L is even, then one has a) $\underline{\underline{B}}^{-1} \underline{\underline{C}}\,\underline{f} = \underline{w}(1, 0)$ and b) $\underline{\underline{B}}^{-1} \underline{1} = \underline{w}(0, 1)$.*

We are now in a position to prove the following theorem.

Theorem 4.11 *Let α_3 and α_4 be as in Lemma 4.7. For $\underline{f} \in \mathcal{R}^{L-1}$ given in Lemma 4.8, we define (q_1^{**}, q_2^{**}) as $\underline{q}_i^{**T} = \frac{4}{4-\alpha_4}[\alpha_3, \alpha_4 \underline{f}^T]$, $\underline{q}_{3-i}^{**\;T} = [0, \underline{w}^T(\alpha_5, \alpha_6)]$*

$$\textit{where} \qquad \alpha_5 = a_1 \Delta \quad \textit{and} \quad \alpha_6 = a_1 (\frac{1}{a_L} + \frac{\Delta}{2}) \; . \tag{4.17}$$

If L is even and $L > \max(2, \frac{a_L}{2a_1} + 1)$, then $(q_1^{**}, q_2^{**}) \in \mathcal{NE}(\underline{\hat{v}}_L)$. The payoff values of the two suppliers at this equilibrium are equal with $V_1(q_1^{**}, q_2^{**}) = V_2(q_1^{**}, q_2^{**}) = D(c^H - c^L)$.

Proof Without loss of generality we assume $i = 1$. First we show that $\underline{q}_1^{**}, \underline{q}_2^{**} \in DRV(\hat{\underline{v}}_L)$. It can be seen from (4.15) that $\alpha_3 + \frac{L-2}{2}\alpha_4 = (\alpha_3 + \frac{L-\frac{3}{2}}{2}\alpha_4) - \frac{1}{4}\alpha_4 = 1 - \frac{1}{4}\alpha_4$, so that $\underline{q}_1^{**T}\underline{1} = \frac{4}{4-\alpha_4}(\alpha_3 + \alpha_4 \underline{f}^T \underline{1}_{L-1}) = \frac{1}{1-\frac{1}{4}\alpha_4}(\alpha_3 + \frac{L-2}{2}\alpha_4) = 1$. From (4.13) and the definition of q_2^{**}, one sees that $q_2^{**T}\underline{1} = \underline{w}^T(\alpha_5, \alpha_6)\underline{1}_{L-1} = (L-1)\alpha_5 + (\alpha_6 - \alpha_5) = (L-2)\alpha_5 + \alpha_6$. It then follows from (4.3) and (4.17) that $q_2^{**T}\underline{1} = (L - \frac{3}{2})a_1\Delta + \frac{a_1}{a_L} = (\frac{1}{a_1} - \frac{1}{a_L})a_1 + \frac{a_1}{a_L} = 1$. One sees from (4.3), (4.15) and the condition $L > \max\{2, \frac{a_L}{2a_1} + 1\}$ that

$$\alpha_4 = \frac{2\Delta}{\frac{1}{a_1} + \frac{1}{a_L}} = \frac{2}{L - \frac{3}{2}} \frac{a_L - a_1}{a_L + a_1} < \frac{2}{\frac{a_L}{2a_1} + 1 - \frac{3}{2}} \frac{a_L - a_1}{a_L + a_1}$$
$$= \frac{4a_1}{a_L - a_1} \frac{a_L - a_1}{a_L + a_1} = 4 \frac{1}{1 + \frac{a_L}{a_1}} < 4 \quad .$$

Hence $\underline{q}_i^{**} \geq \underline{0}, i = 1, 2$ and $q_1^{**}, q_2^{**} \in DRV(\hat{\underline{v}}_L)$. We next show that $\underline{\epsilon}^T(\underline{q}_1^{**})\underline{q}_2^{**} = 0$. From Lemma 4.10 together with (4.13), one easily sees that $\alpha_4 \underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{f} + \alpha_3 \underline{\underline{B}}^{-1}\underline{1}_{L-1} = \alpha_4 \underline{w}(1,0) + \alpha_3 \underline{w}(0,1) = \underline{w}(\alpha_4, \alpha_3)$. By Lemma 4.7 this then leads to

$$\alpha_4 \underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{f} + \alpha_3 \underline{\underline{B}}^{-1}\underline{1}_{L-1} = a_1(2-\alpha_3)\underline{w}(\Delta, \frac{1}{a_L}) = a_1(2-\alpha_3)\underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} \ ,$$

where Lemma 4.4 a) is employed to yield the last equality. By multiplying $\underline{\underline{A}}_D\underline{\underline{B}}$ from left to the above equation, it follows that $\alpha_4 \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} + \alpha_3 \underline{\underline{A}}_D\,\underline{1}_{L-1} = a_1(2-\alpha_3)\underline{1}_{L-1}$, and one has

$$\alpha_4 \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} + \alpha_3 \underline{\underline{A}}_D\underline{1}_{L-1} + \alpha_3 a_1 \underline{1}_{L-1} = 2a_1\underline{1}_{L-1} \quad . \tag{4.18}$$

Let $x = \frac{4\alpha_3}{4-\alpha_4}$ and $y = \frac{4\alpha_4}{4-\alpha_4}$. One sees that $(L-2)(4-\alpha_4)y = (L - \frac{3}{2} - \frac{1}{2})4\frac{2\Delta}{\frac{1}{a_1}+\frac{1}{a_L}} = 8\frac{\frac{1}{a_1}-\frac{1}{a_L}}{\frac{1}{a_1}+\frac{1}{a_L}} - 2\frac{2\Delta}{\frac{1}{a_1}+\frac{1}{a_L}} = 8 - 8\frac{\frac{2}{a_L}}{\frac{1}{a_1}+\frac{1}{a_L}} - 2\alpha_4 = 8 - 8\alpha_3 - 2\alpha_4 = 2(4-\alpha_4)(1-x)$, so that $y = \frac{2(1-x)}{L-2}$. Since $\underline{q}_1^{T**} \in DRV(\hat{\underline{v}}_L)$ and the first component of $\underline{q}_1^{T**}$ is x, one has $0 < x < 1$. Applying these x and y to Lemma 4.8 and using (4.18), one sees that $\underline{\underline{H}}\,\underline{q}_1^{**} = \begin{bmatrix} 2a_1 \\ \frac{4}{4-\alpha_4}2a_1\underline{1}_{L-1} \end{bmatrix}$. Substituting this into the definition of $\underline{\epsilon}(\underline{q}_1^{**})$ yields $\underline{\epsilon}(\underline{q}_1^{**})^T = \max_m(\underline{\underline{H}}\,\underline{q}_1^{**})_m\underline{1}^T - (\underline{\underline{H}}\,\underline{q}_1^{**})^T = \frac{4}{4-\alpha_4}2a_1\underline{1}_L^T - [2a_1, \frac{4}{4-\alpha_4}2a_1\underline{1}_{L-1}^T] = [\frac{\alpha_4}{4-\alpha_4}2a_1, \underline{0}_{L-1}^T]$. This in turn implies that $\underline{\epsilon}(\underline{q}_1^{**})^T\underline{q}_2^{**} = (\frac{\alpha_4}{4-\alpha_4}2a_1, \underline{0}_{L-1}^T)[0, \underline{w}(\alpha_5, \alpha_6)]^T = 0$. We also need to show $\underline{\epsilon}(\underline{q}_2^{**})^T\underline{q}_1^{**} = 0$. From (4.17) together with (4.13), one sees that

$$\alpha_5 \underline{w}(2,1) + \underline{w}(0, 2(\alpha_6 - \alpha_5)) = \underline{w}(2\alpha_5, 2\alpha_6 - \alpha_5) = 2a_1\underline{w}(\Delta, \frac{1}{a_L}) \quad . \tag{4.19}$$

Since $\underline{\underline{B}}^{-1}(\underline{\underline{C}} + \underline{\underline{I}}) = (\underline{\underline{I}} + \underline{\underline{L}})^{-1}(\underline{\underline{I}} + 2\underline{\underline{L}} + \underline{\underline{I}}) = 2\underline{\underline{I}}$, using Lemma 4.4 a) b), (4.19) leads to

$$\alpha_5 \underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{1}_{L-1} + \underline{\underline{B}}^{-1}(\underline{\underline{C}} + \underline{\underline{I}})\underline{w}(0, \alpha_6 - \alpha_5) = 2a_1\underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} \quad . \tag{4.20}$$

Multiplying $\underline{\underline{A}}_D\underline{\underline{B}}$ from left in (4.20), one obtains $\alpha_5\,\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{1}_{L-1}+\underline{\underline{A}}_D\,(\underline{\underline{C}}+\underline{\underline{I}})\,\underline{w}(0,\alpha_6-\alpha_5) = 2a_1\underline{1}_{L-1}$. From the linearity of $\underline{w}(x,y)$ in (4.13) and (4.8), this then leads to

$$\begin{aligned}\underline{\underline{A}}_D\,\underline{\underline{C}}\,\underline{w}(\alpha_5,\alpha_6) \;+\; & \underline{w}(0,a_L(\alpha_6-\alpha_5)) \\ = \; & \underline{\underline{A}}_D\,\underline{\underline{C}}\,\underline{w}(\alpha_5,\alpha_5)+\underline{\underline{A}}_D\,\underline{\underline{C}}\,\underline{w}(0,\alpha_6-\alpha_5)+\underline{w}(0,a_L(\alpha_6-\alpha_5)) \\ = \; & \underline{\underline{A}}_D\,\underline{\underline{C}}\,\underline{w}(\alpha_5,\alpha_5)+\underline{\underline{A}}_D\,\underline{\underline{C}}\,\underline{w}(0,\alpha_6-\alpha_5)+\underline{\underline{A}}_D\,\underline{w}(0,(\alpha_6-\alpha_5)) \\ = \; & \alpha_5\,\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{1}_{L-1}+\underline{\underline{A}}_D\,(\underline{\underline{C}}+\underline{\underline{I}})\,\underline{w}(0,\alpha_6-\alpha_5) = 2a_1\underline{1}_{L-1} \qquad ,\end{aligned}$$

that is,

$$\underline{\underline{A}}_D\,\underline{\underline{C}}\,\underline{w}(\alpha_5,\alpha_6) = 2a_1\underline{1}_{L-1} - \underline{w}(0,a_L(\alpha_6-\alpha_5)) \qquad . \tag{4.21}$$

Let $x = \alpha_5$ and $y = \alpha_6$, then one has $x(L-2) = a_1\Delta(L-2) = a_1\Delta(L-\frac{3}{2}-\frac{1}{2}) = (1-\frac{a_1}{a_L}) - a_1\frac{\Delta}{2} = 1-\alpha_6 = 1-y$, so that $x = (1-y)/(L-2)$. From (4.17) with (4.3) and the condition $L > 2$, one has

$$y = \alpha_6 = a_1(\frac{1}{a_L}+\frac{\Delta}{2}) \;=\; \frac{a_1}{a_L}+\frac{a_1}{2}\,\frac{\frac{1}{a_1}-\frac{1}{a_L}}{L-\frac{3}{2}} < \frac{a_1}{a_L}+\frac{1-\frac{a_1}{a_L}}{2(2-\frac{3}{2})} = 1 \qquad .$$

Hence with x and y above, Lemma 4.9 can be applied, yielding

$$\underline{\underline{H}}\,\underline{q}_2^{**} = \underline{\underline{H}}\begin{bmatrix} 0 \\ \underline{w}(\alpha_5,\alpha_6)\end{bmatrix} = \begin{bmatrix} 2a_1 \\ \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{w}(\alpha_5,\alpha_6)\end{bmatrix} = \begin{bmatrix} 2a_1 \\ 2a_1\underline{1}_{L-1} - \underline{w}(0,a_L(\alpha_6-\alpha_5))\end{bmatrix} \qquad .$$

It should be noted that from the condition $L > \dfrac{a_L}{2a_1}+1$, one has $\alpha_6-\alpha_5 = \frac{a_1}{a_L} - \frac{a_1\Delta}{2} = \frac{a_1}{a_L} - \frac{1-\frac{a_1}{a_L}}{2(L-\frac{3}{2})} > \frac{a_1}{a_L} - \frac{1-\frac{a_1}{a_L}}{2(\frac{a_L}{2a_1}+1-\frac{3}{2})} = \frac{a_1}{a_L} - \frac{1-\frac{a_1}{a_L}}{\frac{a_L}{a_1}-1} = 0$, so that $\max_m\{(\underline{\underline{H}}\,\underline{q}_2^{**})_m\} = 2a_1$. Thus we obtain

$$\begin{aligned}\underline{q}_1^{**T}\underline{\epsilon}(\underline{q}_2^{**}) \;&=\; \underline{q}_1^{**T}\left[\max_m\{(\underline{\underline{H}}\,\underline{q}_2^{**})_m\}\underline{1}_L - \underline{\underline{H}}\,\underline{q}_2^{**}\right] \\ &=\; \underline{q}_1^{**T}\left[2a_1\underline{1}_L - \begin{bmatrix} 2a_1 \\ 2a_1\underline{1}_{L-1} - \underline{w}(0,a_L(\alpha_6-\alpha_5))\end{bmatrix}\right] \\ &=\; \frac{4}{4-\alpha_4}[\alpha_3,\alpha_4\underline{f}^T]\begin{bmatrix} 0 \\ \underline{w}(0,a_L(\alpha_6-\alpha_5))\end{bmatrix} \qquad .\end{aligned}$$

Since $\underline{w}(0,a_L(\alpha_6-\alpha_5)) = [0,\cdots,0,a_L(\alpha_6-\alpha_5)]^T$ from (4.13), and the last component of $\underline{f}^T$ as defined in Lemma 4.8 is 0 when L is even, one then concludes that $\underline{\varepsilon}^T(\underline{q}_2^{**})\underline{q}_1^{**} = 0$. The theorem now follows from Theorem 3.2. □

It should be noted that the strategies of the two suppliers at Nash equilibrium in Theorem 4.11 can be written as $\underline{q}_1^{**T} = \frac{4}{4-\alpha_4}\left[\alpha_3,\alpha_4\underline{f}^T\right]$ and $\underline{q}_2^{**T} = [0,\alpha_5,\cdots,\alpha_5,\alpha_6]$ while those in

Theorem 4.6 are $\underline{q}_1^{*T} = \underline{q}_2^{*T} = \left[\alpha_1, \alpha_2 \underline{1}_{L-1}^T\right]$. As we will see, one has $\lim_{L\to\infty} \underline{q}_1^* = \lim_{L\to\infty} \underline{q}_1^{**}$, while $\lim_{L\to\infty} \underline{q}_2^{**}$ is quite different. The supplier with $\underline{q}_1^{**}$ is risk-aversive with tendency to protect the near customer by offering lower prices with higher probabilities, while the supplier with $\underline{q}_2^{**}$ is risk-taking, by offering higher prices with higher probabilities.

5 Limit Theorems of Nash Equilibriums with Specific Discrete Support

In the previous section, two Nash equilibriums (q_1^*, q_2^*) and (q_1^{**}, q_2^{**}) are constructed explicitly, when the strategy set consists of L discrete supporting points for pricing with $\underline{\hat{v}}_L = [\hat{v}_{L:1}, \cdots, \hat{v}_{L:L}]$ as given in (4.5). While $\underline{\hat{v}}_L$ partitions the strategy set $I = [c^H, U]$ of the original problem in a rather peculiar way as shown in Figure 4.1, the set $\{\underline{\hat{v}}_L : L = L_0, L_0+1, \cdots\}$ with $L_0 > 2$ becomes dense in $I = [c^H, U]$ as we will see. It is then of interest to see whether or not $(X_1^*(L), X_2^*(L))$ and $(X_1^{**}(L), X_2^{**}(L))$ in $S(\underline{\hat{v}}_L) = DRV(\underline{\hat{v}}_L) \times DRV(\underline{\hat{v}}_L)$ converge to any mixed strategies (X_1^*, X_2^*) and (X_1^{**}, X_2^{**}) in $S = RV \times RV$ of the original problem as $L \to \infty$, where $X_i^*(L)$ and $X_i^{**}(L)$ are discrete random variables associated with q_i^* and q_i^{**} respectively, $i = 1, 2$.

In order to understand such limiting behaviors, we first define the partition of $I = [c^H, U]$ based on $\underline{\hat{v}}_L = [\hat{v}_{L:1}, \cdots, \hat{v}_{L:L}]$ as $PT(\underline{\hat{v}}_L) = \{[\hat{v}_{L:m}, \hat{v}_{L:m+1}) : m = 1, 2, \cdots, L-1\}$. The partition of $I = [c^H, U]$ with N intervals of equal distance is defined similarly as $PT(\underline{u}_N) = \{[u_{N;r}, u_{N:r+1}) : r = 1, 2, \cdots, N-1\}$, where $\underline{u}_N = [u_{N:1}, \cdots, u_{N:N}]$; $\quad u_{N:r} = c^H + \frac{(r-1)(U-c^H)}{N-1}$, $\quad 1 \le r \le N$. For later use, we also define

$$\underline{\tau}_N = [\tau_{N:1}, \cdots, \tau_{N:N}] = D\Big(\underline{u}_N - \frac{c^L + c^H}{2} \underline{1}_N\Big) \ . \tag{5.1}$$

From (4.5) and (5.1), it should be noted that

$$\underline{u}_N = \frac{1}{D}\underline{\tau}_N + \frac{c^L + c^H}{2}\underline{1}_N \ ; \ \text{and} \quad \underline{\hat{v}}_L = \frac{1}{D}\underline{a} + \frac{c^L + c^H}{2}\underline{1}_L \qquad . \tag{5.2}$$

Hence the basic question to be answered is how many components of $\underline{\hat{v}}_L$ are contained in each interval of $PT(\underline{u}_N)$, and the limit of the result as $L \to \infty$. For describing this problem more precisely, the following notation and definitions are introduced.

Definition 5.1 *Let $\underline{a}_L = [a_{L:1}, \cdots, a_{L:L}]^T$ be as in (4.1) through (4.3) where the index L is attached to emphasize $\underline{a}_L \in R^L$. Then we define* $\quad a) K = \frac{1}{a_{L:1}} - \frac{1}{a_{L:L}} \quad$ *and* $\quad b) \Delta(L) = \frac{1}{L-\frac{3}{2}} K.$

From (4.2), one sees that $\frac{1}{a_{L:m}} = (L-m)\Delta(L) + \frac{1}{a_{L:L}}$ for $m \in \mathcal{L} \setminus \{1\}$. This then implies

$$a_{L:m} = \frac{a_{L:L}}{(L-m)\Delta(L)a_{L:L} + 1} \quad \text{for} \quad m \in \mathcal{L} \setminus \{1\} \quad . \tag{5.3}$$

We also note from (4.4) that

$$\frac{1}{a_{L:1}} - \frac{1}{a_{L:2}} = \frac{1}{2}\Delta(L), \quad \frac{1}{a_{L:m}} - \frac{1}{a_{L:m+1}} = \Delta(L) \quad \text{for} \quad m \in \mathcal{L} \setminus \{1, L\} \quad . \tag{5.4}$$

It can be readily seen from Definition 5.1 b) that

$$\lim_{L\to\infty} L\Delta(L) = K \quad . \tag{5.5}$$

Definition 5.2 *For $1 \le r \le N-1$, we define:*

$$J(r, L, N) = \{m : \hat{v}_{L:m} \in [u_{N:r}, u_{N:r+1})\}; \tag{5.6}$$

$$Z(r, L, N) = |J(r, L, N)|; \tag{5.7}$$

$$m_{min}(r, L, N) = \min\{m : m \in J(r, L, N)\}; \tag{5.8}$$

$$m_{max}(r, L, N) = \max\{m : m \in J(r, L, N)\}; \tag{5.9}$$

$$\epsilon_{min}(r, L, N) = \hat{v}_{L:m_{min}(r,L,N)} - u_{N:r} \quad ; and \tag{5.10}$$

$$\epsilon_{max}(r, L, N) = u_{N:r+1} - \hat{v}_{L:m_{max}(r,L,N)} \quad . \tag{5.11}$$

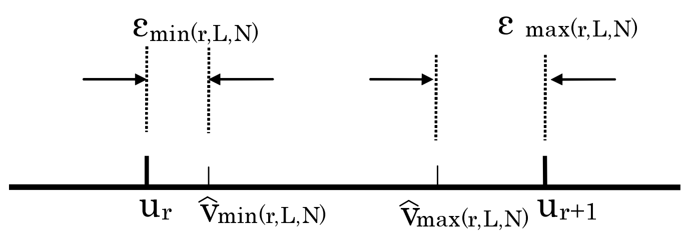

Figure 5.1: The r-th Interval Generated by $PT(\underline{u}_N)$

As can be seen in Figure 5.1, given $PT(\underline{\hat{v}}_L)$ and $PT(\underline{u}_L)$, $J(r, L, N)$ is the index set for the components of $\underline{\hat{v}}_L$ contained in the r-th interval generated by $PT(\underline{u}_L)$, and $Z(r, L, N)$ is the cardinality of $J(r, L, N)$. Since each interval generated by $PT(\underline{u}_L)$ is defined as a left-closed and right-open interval, the last point $\hat{v}_L = U = u_{N:N}$ never belongs to any interval. Accordingly, the qustion to be answered is to find $Z(r, L, N)/(L-1)$ and its limiting behavior as $L \to \infty$. Before proving this, we first show that, given $PT(\underline{u}_N)$, one can make each interval in $PT(\underline{u}_N)$ contain an arbitrary number of components of $\underline{\hat{v}}_L$ by taking L sufficiently large.

Proposition 5.3 *Let N_0 and N be any positive intergers greater than or equal to 2. Then there exists a positive integer $L(N_0, N)$ such that for any $L > L(N_0, N)$, one has $Z(r, L, N) \geq N_0$ for all $r = 1, \cdots, N-1$.*

Proof From (5.2) and (5.4) together with Proposition 4.1, one sees that $\max_{1\leq m<L}\{\hat{v}_{L:m+1} - \hat{v}_{L:m}\} = \hat{v}_{L:L} - \hat{v}_{L:L-1} = \frac{a_{L:L}a_{L:L-1}}{D}\left(\frac{1}{a_{L:L-1}} - \frac{1}{a_{L:L}}\right) < \frac{a_{L:L}^2}{D}\Delta(L) = D\left(U - \frac{c^L+c^H}{2}\right)^2\Delta(L)$. Since $\Delta(L) \to 0$ as $L \to \infty$ from Definition 5.1 b), this then implies that

$$\max_{1\leq m<L}\{\hat{v}_{L:m+1} - \hat{v}_{L:m}\} = \hat{v}_{L:L} - \hat{v}_{L:L-1} \to 0 \quad \text{as} \quad L \to \infty \ . \tag{5.12}$$

Consequently, for any $\epsilon > 0$, there exists a positive integer $L(\epsilon)$ such that, for any $L > L(\epsilon)$, one has $|\hat{v}_{L:m+1} - \hat{v}_{L:m}| < \epsilon$ for all $m = 1, \cdots, L-1$. For given N_0 and N, choose $\epsilon(N_0, N)$ so that $0 < \epsilon(N_0, N) < (U - c^H)/(N_0(N-1))$. Then for any $L > L(N_0, N) = L(\epsilon(N_0, N))$, one has $Z(r, L, N) \geq N_0$, completing the proof. □

In what follows, we assume that $L > L(N_0, N)$ for some $N_0 \geq 2$. Three more lemmas are needed before proving the first main theorem of this section and proofs are given in Appendix.

Lemma 5.4 *For $1 \leq r \leq N-1$, one has*

$$\lim_{L\to\infty} \epsilon_{max}(r, L, N) = 0 \qquad \textit{and} \qquad \lim_{L\to\infty} \epsilon_{min}(r, L, N) = 0 \ .$$

In what follows, the arguments r, L and N are omitted whenever there is no ambiguity.

Lemma 5.5 *Let $L(N_0, N)$ be as in Proposition 5.3. Then the following equations hold for all $L > L(N_0, N)$.*

a) $m_{min}(1, L, N) = 1$; *and* (5.13)

$$m_{min}(r, L, N) = L - \frac{1}{\Delta(L)}\left(\frac{1}{\tau_{N:r} + D\epsilon_{min}(r, L, N)} - \frac{1}{a_{L:L}}\right) \quad \textit{for} \quad 2 \leq r \leq N-1 \tag{5.14}$$

b) $$m_{max}(r, L, N) = L - \frac{1}{\Delta(L)}\left(\frac{1}{\tau_{N:r+1} - D\epsilon_{max}(r, L, N)} - \frac{1}{a_{L:L}}\right) \textit{for } 1 \leq r \leq N-1 \tag{5.15}$$

Lemma 5.6 *Let $L(N_0, N)$ be as in Proposition 5.3. Then for all $L > L(N_0, N)$, one has:*

$$Z(1, L, N) = L - \frac{1}{\Delta(L)}\left(\frac{1}{\tau_{N:2} - D\epsilon_{max}(1, L, N)} - \frac{1}{a_{L:L}}\right) \quad \text{and} \tag{5.16}$$

$$Z(r, L, N) = \frac{1}{\Delta(L)}\left(\frac{1}{\tau_{N:r} - D\epsilon_{max}(r-1, L, N)} - \frac{1}{\tau_{N:r+1} - D\epsilon_{max}(r, L, N)}\right), \; 2 \le r \le N-1 . \tag{5.17}$$

We are now in a position to prove the first main theorem of this section.

Theorem 5.7 *Let $g(r, L, N)$ be defined by*

$$g(r, L, N) = \frac{Z(r, L, N)}{L-1} \qquad \text{for all} \quad 1 \le r \le N-1 \quad . \tag{5.18}$$

One then has, for $1 \le r \le N-1$,

$$g(r, N) \overset{def}{=} \lim_{L\to\infty} g(r, L, N) = \frac{1}{K}\left(\frac{1}{\tau_{N:r}} - \frac{1}{\tau_{N:r+1}}\right) \quad , \tag{5.19}$$

where $\tau_{N:r}$ and K are as given in (5.1) and Definition 5.1 a) respectively.

Proof Form Lemma 5.6, one sees that $\frac{Z(1,L,N)}{L-1} = \frac{L}{L-1}\left\{1 - \frac{1}{L\Delta(L)}\left(\frac{1}{\tau_{N:2} - D\epsilon_{max}(1,L,N)} - \frac{1}{a_{L:L}}\right)\right\}$. By letting $L \to \infty$, it then follows from Lemma 5.4 and (5.5) that $g(1, N) = \frac{1}{K}\left(K - \frac{1}{\tau_{N:2}} + \frac{1}{a_{L:L}}\right)$. From Definition 5.1 a) and noting $\tau_{N:1} = a_{L:1}$, this then leads to $g(1, N) = \frac{1}{K}\left(\frac{1}{\tau_{N:1}} - \frac{1}{\tau_{N:2}}\right)$. For $2 \le r \le N-1$, the theorem follows similarly. □

From Theorem 5.7, one realizes that $\sum_{r=1}^{N-1} g(r, N) = \frac{1}{K}(\frac{1}{\tau_{N:1}} - \frac{1}{\tau_{N:2}} + \frac{1}{\tau_{N:2}} - \frac{1}{\tau_{N:3}} + \cdots + \frac{1}{\tau_{N:N-1}} - \frac{1}{\tau_{N:N}}) = \frac{1}{K}(\frac{1}{\tau_{N:1}} - \frac{1}{\tau_{N:N}}) = \frac{1}{K}(\frac{1}{a_{L:1}} - \frac{1}{a_{L:L}}) = \frac{K}{K} = 1$. Since $g(r, N) > 0$ for $1 \le r \le N-1$, one can associate $\{g(r, N)\}_{r=1}^{N-1}$ with a random variable in the following manner. Let $X(L) \in DRV(\underline{\hat{v}}_L)$ having a probability vector $\underline{q}_L = [q_{L:1}, \cdots, q_{L:L}]$ given by

$$\underline{q}_L^T = \frac{1}{(L-1)}[0, \underline{1}_{L-1}^T] \in \mathcal{R}^L . \tag{5.20}$$

Since $q_{L:m} = P[X(L) = \hat{v}_m] = 1/(L-1)$ for $2 \le m \le L$, one sees from (5.18) that $g(1, L, N) = P[u_{N:1} \le X(L) < u_{N:2}] + \frac{1}{L-1}$, and for $2 \le r \le L-1$ $g(r, L, N) = P[u_{N:r} \le X(L) < u_{N:r+1}]$. Consequently, it follows for $1 \le r \le L-1$ that

$$g(r, N) = \lim_{L\to\infty} P[u_{N:r} \le X(L) < u_{r+1}] \quad . \tag{5.21}$$

Based on (5.21), it is possible to identify the limiting distribution of $X(L)$. More specifically, let $X(\infty)$ be a random variable satisfying $X(L) \xrightarrow{d} X(\infty)$ where "$\xrightarrow{d}$" denotes the convergence in law, i.e., if one defines

$$F_L(x) = P[X(L) \le x]; \quad F_\infty(x) = P[X(\infty) \le x], \; x \in \mathcal{R} \quad , \tag{5.22}$$

then $\lim_{L\to\infty} F_L(x) = F_\infty(x)$ for all continuity points $x \in \mathcal{R}$ of $F_\infty(x)$. Of interest is then to find $F_\infty(x)$. A preliminary lemma is needed.

Lemma 5.8 *For $I = [c^H, U]$ in (2.1), let $S = U - c^H$. Then, for any $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that one has, for all $N > N(\epsilon)$,*

$$\left| \frac{g(r,N)}{\frac{S}{N-1}} - \frac{D}{K\tau_{N:r+1}^2} \right| < \epsilon \ , r \in \mathcal{N} \setminus \{N\} \quad . \tag{5.23}$$

Proof It should be noted from (5.1) that $\tau_{N:r+1} - \tau_{N:r} = D(u_{N:r+1} - u_{N:r}) = DS/(N-1), r \in \mathcal{N} \setminus \{N\}$. Using this and (5.19), one sees that

$$\begin{aligned} \left| \frac{g(r,N)}{\frac{S}{N-1}} - \frac{D}{K\tau_{N:r+1}^2} \right| &= \frac{1}{K} \left| \frac{N-1}{S} \left(\frac{1}{\tau_{N:r}} - \frac{1}{\tau_{N:r+1}} \right) - \frac{D}{\tau_{N:r+1}^2} \right| \\ = \frac{1}{K} \left| \frac{D}{\tau_{N:r}\tau_{N:r+1}} - \frac{D}{\tau_{N:r+1}^2} \right| &= \frac{D}{K} \left| \frac{\tau_{N:r+1} - \tau_{N:r}}{\tau_{N:r}\tau_{N:r+1}^2} \right| \le \frac{D}{K} \frac{D\, S}{(N-1)} \frac{1}{\tau_{N:1}^3} \quad . \end{aligned} \tag{5.24}$$

where $\tau_{N:r} \ge \tau_{N:1}, r \in \mathcal{N}$ is employed to yield the last inequality. The right most side of (5.24) does not depend on r. With $[z]$ being the smallest integer which is greater than or equal to z, let

$$N(\epsilon) = \left[\frac{D^2 S}{K\epsilon\tau_{N:1}^3} \right] + 1 \quad , \tag{5.25}$$

which is independent of r. Inequality (5.23) then holds for all $r \in \mathcal{N} \setminus \{N\}$, completing the proof. □

Now we give the proof of the main theorem of this section.

Theorem 5.9 *Let $X(L)$ and $X(\infty)$ be as given in (5.22). Then $X(\infty)$ is absolutely continuous with probability density function $f_\infty(x) = \frac{d}{dx}F_\infty(x)$ given by*

$$f_\infty(x) = \begin{cases} \frac{1}{KD}\left(x - \frac{c^L + c^H}{2}\right)^{-2} & , \quad x \in I = [c^H, U] \\ 0 & , \quad else \end{cases} \quad . \tag{5.26}$$

Proof Let $\tilde{r}(x,N) = \arg\min_{1 \le r \le N-1}\{x < u_{N:r+1}\}$, so that $x \in [u_{N:\tilde{r}(x,N)}, u_{N:\tilde{r}(x,N)+1})$. It then follows that

$$\Pr\{X(L) < x\} \le \Pr\{X(L) < u_{N:\tilde{r}(x,N)+1}\} = \sum_{r=1}^{\tilde{r}(x,N)} \Pr\{u_{N:r} \le X(L) < u_{N:r+1}\} \tag{5.27}$$

and, for $\tilde{r}(x,N) \ge 2$,

$$\Pr\{X(L) < x\} \ge \Pr\{X(L) < u_{N:\tilde{r}(x,N)}\} = \sum_{r=1}^{\tilde{r}(x,N)-1} \Pr\{u_{N:r} \le X(L) < u_{N:r+1}\} \quad . \tag{5.28}$$

For sufficiently large N, one has $\tilde{r}(x,N) \geq 2$. Combining (5.27) and (5.28) with (5.21), one has

$$\sum_{r=1}^{\tilde{r}(x,N)-1} g(r,N) \leq \lim_{L\to\infty} \sum_{r=1}^{\tilde{r}(x,N)-1} \Pr\{u_{N:r} \leq X(L) < u_{N:r+1}\} \leq \lim_{L\to\infty} \Pr\{X(L) < x\}$$
$$\leq \lim_{L\to\infty} \sum_{r=1}^{\tilde{r}(x,N)} \Pr\{u_{N:r} \leq X(L) < u_{N:r+1}\} \leq \sum_{r=1}^{\tilde{r}(x,N)} g(r,N) \qquad (5.29)$$

From (5.19), one sees that $\lim_{N\to\infty} g(r,N) = 0$ for $1 \leq r \leq N-1$ and this convergence is uniform in terms of r. This then leads to $\lim_{N\to\infty} g(\tilde{r}(x,N),N) = 0$ for $c^H \leq x \leq U$. Hence one has

$$\lim_{N\to\infty} \sum_{r=1}^{\tilde{r}(x,N)} g(r,N) = \lim_{N\to\infty} \sum_{r=1}^{\tilde{r}(x,N)} g(r,N) - \lim_{N\to\infty} g(\tilde{r}(x,N),N) = \lim_{N\to\infty} \sum_{r=1}^{\tilde{r}(x,N)-1} g(r,N) \qquad (5.30)$$

From (5.29), this then implies that

$$\lim_{N\to\infty} \sum_{r=1}^{\tilde{r}(x,N)} g(r,N) = \lim_{L\to\infty} \Pr\{X(L) < x\} \quad . \qquad (5.31)$$

Let $f_\infty(x)$ be as in (5.26). Then it should be noted that, for all $x \in [c^H, U]$, $\sum_{r=1}^{\tilde{r}(x,N)} g(r,N)$ can be expressed as

$$\sum_{r=1}^{\tilde{r}(x,N)} g(r,N) = \frac{S}{N-1} \sum_{r=1}^{\tilde{r}(x,N)} \left[\frac{g(r,N)}{\frac{S}{N-1}} - f_\infty(u_{N:r+1}) \right]$$
$$+ \left[\sum_{r=1}^{\tilde{r}(x,N)} f_\infty(u_{N:r+1}) \frac{S}{N-1} - \int_{c^H}^{x} f_\infty(y)dy \right] + \int_{c^H}^{x} f_\infty(y)dy \quad . \qquad (5.32)$$

From (5.1) and (5.26) one sees that

$$f_\infty(u_{N:r+1}) = \frac{D}{KD^2} \left(u_{N:r+1} - \frac{c^L + c^L}{2} \right)^{-2} = \frac{D}{K\tau_{N:r+1}^2} \quad . \qquad (5.33)$$

For any $\epsilon > 0$, let $N(\epsilon)$ be as in (5.25). From Lemma 5.8, it follows that, for all $N > N(\epsilon)$,

$$\frac{S}{N-1} \sum_{r=1}^{\tilde{r}(x,N)} \left| \frac{g(r,N)}{\frac{S}{N-1}} - f_\infty(u_{N:r+1}) \right| \leq \frac{S}{N-1} \sum_{r=1}^{\tilde{r}(x,N)} \epsilon = \frac{S\tilde{r}(x,N)}{N-1} \epsilon \leq S\epsilon \quad . \qquad (5.34)$$

Hence the first term of (5.32) goes to 0 as $N \to \infty$. Since $f_\infty(x)$ in (5.26) is absolutely continuos on $[c^H, U]$, the second term of (5.32) also goes to 0 as $N \to \infty$. Combining these observations with (5.32), one finally concludes that $\lim_{N\to\infty} \sum_{r=1}^{\tilde{r}(x,N)} g(r,N) = \int_{c^H}^{x} f_\infty(y)dy$, and the theorem follows from (5.31). □

Theorem 5.9 states that $X(L)$ specified by probability vector $\underline{q}_L$ in (5.20) converges in law to $X(\infty)$ as $L \to \infty$ where $X(\infty)$ is absolutely continuous with probability density function $f_\infty(x)$ given in (5.26). Using this theorem, we next show that both $(X_1^*(L), X_2^*(L)) \in \mathcal{NE}(\hat{\underline{v}}_L)$ of Theorem 4.6 and $(X_1^{**}(L), X_2^{**}(L)) \in \mathcal{NE}(\hat{\underline{v}}_L)$ of Theorem 4.11 defined on $S(\hat{\underline{v}}_L) = DRV(\hat{\underline{v}}_L) \times DRV(\hat{\underline{v}}_L)$ converge in law to (X_1^*, X_2^*) and (X_1^{**}, X_2^{**}) defined on $S = RV \times RV$ respectively as $L \to \infty$.

Theorem 5.10 *Let $X^*(L) = X_1^*(L) = X_2^*(L) \in DRV(\hat{\underline{v}}_L)$ be associated with $\underline{q}^* = \underline{q}_1^* = \underline{q}_2^*$ as in Theorem 4.6 and define $B(p)$ as a Bernoulli random variable with $P[B(p) = 1] = p$ for $0 < p < 1$. Then, one has*

$$X^*(L) \xrightarrow{d} B\Big(\frac{2a_1}{a_1 + a_L}\Big)c_H + \Big(1 - B\Big(\frac{2a_1}{a_1 + a_L}\Big)\Big)X(\infty) \in RV \tag{5.35}$$

as $L \to \infty$, where $X(\infty)$ is as in Theorem 5.9, and $B(\frac{2a_1}{a_1+a_L})$ is independent of $X(\infty)$.

Proof From Lemma 4.2, one has $(L-1)\alpha_2 + \alpha_1 = 1$, so that $\underline{q}^{*T} = [\alpha_1, \alpha_2 \underline{1}_{L-1}^T] = \alpha_1[1, \underline{o}_{L-1}^T] + (1-\alpha_1)[0, \frac{1}{L-1}\underline{1}_{L-1}^T]$. This then implies that $X^*(L)$ can be written as

$$X^*(L) = B(\alpha_1)c^H + (1 - B(\alpha_1))X(L) \qquad , \tag{5.36}$$

where $B(\alpha_1)$ is independent of $X(L)$. The theorem then follows sinc $\alpha_1 \to \frac{2a_1}{a_1+a_L}$ as $L \to \infty$ from (4.3) and (4.6). □

Next we prove the limit thoerem of the equilibrium $(X_1^{**}(L), X_2^{**}(L))$ given by Theorem 4.11 as $L \to \infty$.

Theorem 5.11 *Let $B(p)$ be as in Theorem 5.10 and let $X'(L)$ be the random variable represented by probability vector $\underline{q}_L^{'T} \stackrel{def}{=} \frac{2}{L-2}[0, \underline{f}^T]$ where $\underline{f}$ is as in Lemma 4.8. For $i = 1, 2$, let $X_1^{**}(L)$ and $X_2^{**}(L)$ be the random variables associated with $\underline{q}_1^{**}$ and $\underline{q}_2^{**}$ given in Theorem 4.11 respectively. Then, for $i = 1, 2$, one has*

$$X_i^{**}(L) \;\xrightarrow{d}\; B\Big(\frac{2a_1}{a_1 + a_L}\Big)c^H + \Big(1 - B\Big(\frac{2a_1}{a_1 + a_L}\Big)\Big)X(\infty) \in RV \;;\text{and} \tag{5.37}$$

$$X_{3-i}^{**}(L) \;\xrightarrow{d}\; B\Big(\frac{a_1}{a_L}\Big)\,U \;+\Big(1 - B\Big(\frac{a_1}{a_L}\Big)\Big)X(\infty) \in RV \tag{5.38}$$

as $L \to \infty$, where $X(\infty)$ is as in Theorem 5.9, and $B(\frac{2a_1}{a_1+a_L})$ and $B(\frac{a_1}{a_L})$ are independent of $X(\infty)$ respectively.

Proof Without loss of generality we assume $i = 1$. Since $\underline{q}_1^{**T}\underline{1}_L = 1$, one has $\frac{4\alpha_3}{4-\alpha_4} + \frac{4(L-2)\alpha_4}{4-\alpha_4} = 1$, so that $\underline{q}_1^{**T} = \frac{4}{4-\alpha_4}[\alpha_3, \alpha_4\underline{f}^T] = \frac{4\alpha_3}{4-\alpha_4}[1, \underline{0}_{L-1}^T] + \frac{4\alpha_4}{4-\alpha_4}[0, \underline{f}^T] = \frac{4\alpha_3}{4-\alpha_4}[1, \underline{0}_{L-1}^T] + (1 - \frac{4\alpha_3}{4-\alpha_4})[0, \frac{1}{L-2}\underline{f}^T]$. This then implies that $X_1^{**}(L)$ can be written as

$$X_1^{**}(L) = B\Big(\frac{4\alpha_3}{4-\alpha_4}\Big)c^H + \Big(1 - B\Big(\frac{4\alpha_3}{4-\alpha_4}\Big)\Big)X'(L) \quad , \tag{5.39}$$

where $B(\frac{4\alpha_3}{4-\alpha_4})$ is independent of $X'(L)$. Similarly, since $\underline{q}_2^{**T}\underline{1}_L = 1$ one has $(L-2)\alpha_5 + \alpha_6 = 1$, so that $\underline{q}_2^{**T} = [0, \underline{w}^T(\alpha_5, \alpha_6)] = [0, \underline{w}^T(\alpha_5, \alpha_5)] + [0, \underline{w}^T(0, \alpha_6 - \alpha_5)] = \{1 - (\alpha_6 - \alpha_5)\}[0, \frac{1}{L-1}\underline{1}_{L-1}] + (\alpha_6 - \alpha_5)[0, \underline{w}^T(0, 1)]$. Hence $X_2^{**}(L)$ can be written as

$$X_2^{**}(L) = (1 - B(\alpha_6 - \alpha_5))X(L) + B(\alpha_6 - \alpha_5)\,U \quad , \tag{5.40}$$

where $B(\alpha_6 - \alpha_5)$ are also independent of $X(L)$. Since $\|\underline{q}'_L - \underline{q}_L\|^2 \to 0$ as $L \to \infty$, one has $X'(L) \xrightarrow{d} X(\infty)$ as $L \to \infty$. The theorem then follows since $\frac{4\alpha_3}{4-\alpha_4}$ and $\alpha_6 - \alpha_5$ go to $\frac{2a_1}{a_1+a_L}$ and $\frac{a_1}{a_L}$ as $L \to \infty$ respectively from (4.4), (4.15) and (4.17). □

It is worth noting that the limit of $X^*(L)$ in Theorem 5.10 has the mass $m(c^H) = 2a_1/(a_1 + a_L)$ at c^H. Let $U = c^H + d$. From (4.1) and (4.2), one then sees that

$$m(c^H) = \frac{c^H - c^L}{c^H - c^L + d} \quad . \tag{5.41}$$

Adopting the lowest possible price at c^H is the risk aversive strategy in that the supplier secures the near customer while giving up the distant customer. Equation (5.41) states that the mass assigned to this strategy at the limit is the ratio of the unit profit expected from the near customer under this strategy against that obtained by offering the highest possible price $U = c^H + d$. Clearly, the mass $m(c^H)$ vanishes as $U \to \infty$ and the associated limiting distribution becomes absolutely continuous on $[c^H, \infty)$ having the probability density function given by

$$f_{\infty:U=\infty}(x) = \frac{c^H - c^L}{2}\left(x - \frac{c^L + c^H}{2}\right)^{-2} \quad . \tag{5.42}$$

The interpretation for Theorem 5.11 can be stated as supplier i takes the risk aversive strategy by placing the mass $m_i(c^H)$ as given in (5.41), while supplier $3-i$ adopts the risk taking strategy by placing the mass $m_{3-i}(U)$ at the highest possible price U where

$$m_{3-i}(U) = \frac{c^H - c^L}{c^H - c^L + 2d} \quad . \tag{5.43}$$

Both $m_i(c^H)$ and $m_{3-i}(U)$ diminish to zero as $U \to \infty$ and one observes again that both suppliers have the same associated limiting strategy specified by (5.42). One may then expect that there exists the unique Nash equilibrium specified by (5.42) with the strategy space $S = RV \times RV$ where RV is the set of all random variables defined on $[c^H, \infty)$. This conjecture is currently under study and will be reported elsewhere.

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Appendix

Proof of Lemma 4.2: Since $L > 2$, one has $0 < \frac{2}{a_L}(L-1) + \frac{2}{a_1}(L-2) = 2(\frac{1}{a_L}+\frac{1}{a_1})(L-\frac{3}{2}) - \left(\frac{1}{a_1}-\frac{1}{a_L}\right) = \frac{L-\frac{3}{2}}{a_1}\left[2(\frac{a_1}{a_L}+1) - a_1\Delta\right] = \frac{L-\frac{3}{2}}{a_1}C_1$, so that $C_1 > 0$. Similarly, since $L > \dfrac{a_L}{2a_1} + 1$, it can be seen that $\frac{2}{a_L} - \Delta = \frac{2}{a_L} - \frac{\frac{1}{a_1}-\frac{1}{a_L}}{(L-\frac{3}{2})} = \frac{1}{L-\frac{3}{2}}\left\{\frac{1}{a_L}(2L-3) - \frac{1}{a_1} + \frac{1}{a_L}\right\} = \frac{1}{L-\frac{3}{2}}\left\{\frac{2}{a_L}(L-1) - \frac{1}{a_1}\right\} = \frac{2}{a_L(L-\frac{3}{2})}\left(L-1-\frac{a_L}{2a_1}\right) > 0$. It then follows that $\alpha_1 > 0$ and $\alpha_2 > 0$. Furthermore, one has $\underline{q}^{*T}\underline{1}_L = \alpha_1 + (L-1)\alpha_2 = \frac{2a_1}{C_1}\left\{\frac{2}{a_L} - \Delta + (L-1)\Delta\right\} = \frac{2a_1}{C_1}\left\{\frac{2}{a_L} - \frac{\Delta}{2} + (L-\frac{3}{2})\Delta\right\} = \frac{2a_1}{C_1}\left\{\frac{2}{a_L} - \frac{\Delta}{2} + \frac{1}{a_1} - \frac{1}{a_L}\right\} = \frac{1}{C_1}\left(\frac{2a_1}{a_L} + 2 - a_1\Delta\right) = 1.$

Proof of Lemma 4.3: From the definition of $\Delta, \alpha_1, \alpha_2$ and C_1, one sees that $2\alpha_2 + a_1(\alpha_1 - 2)\Delta = \frac{2a_1}{C_1}\left[2\Delta + \frac{C_1}{2}\left\{\frac{2a_1}{C_1}\left(\frac{2}{a_L} - \Delta\right) - 2\right\}\Delta\right] = \frac{2a_1}{C_1}\Delta\left(2 + \frac{2a_1}{a_L} - a_1\Delta - C_1\right)$. Substituting $-a_1\Delta = C_1 - \frac{2a_1}{a_L} - 2$ into the last term then yields $\frac{2a_1}{C_1}\Delta\left(2 + \frac{2a_1}{a_L} + C_1 - \frac{2a_1}{a_L} - 2 - C_1\right) = 0$, proving a). For part b), we first note from a) and the definition of α_2 that $a_1(\alpha_1 - 2) = -\frac{2\alpha_2}{\Delta} = -\frac{4\alpha_1}{C_1}$. We also note that $\alpha_1 + \alpha_2 = \frac{2a_1}{C_1}(\frac{2}{a_L} - \Delta + \Delta) = \frac{4a_1}{C_1}\frac{1}{a_L}$. It then follows that $\alpha_2 + a_1(\alpha_1 - 2)\frac{1}{a_L} + \alpha_1 = -\frac{4\alpha_1}{C_1}\frac{1}{a_L} + \frac{4\alpha_1}{C_1}\frac{1}{a_L} = 0$, completing the proof.

Proof of Lemma 4.4: We first note that $\underline{\underline{L}} - \underline{\underline{L}}_1 = \underline{\underline{L}}_1\underline{\underline{L}}$ so that $(\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{\underline{B}} = \underline{\underline{I}} - \underline{\underline{L}}_1 + \underline{\underline{L}} - \underline{\underline{L}}_1\underline{\underline{L}} = \underline{\underline{I}}$, and hence $\underline{\underline{B}}^{-1} = \underline{\underline{I}} - \underline{\underline{L}}_1$. From (4.4)(4.8) and (4.10), one then sees that

$$
\begin{aligned}
\underline{\underline{B}}^{-1}\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} &= (\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} = \underline{\underline{A}}_D^{-1}\underline{1}_{L-1} - \underline{\underline{L}}_1\underline{\underline{A}}_D^{-1}\underline{1}_{L-1} \\
&= \begin{bmatrix} \frac{1}{a_2} & & & & \\ & \frac{1}{a_3} & & \underline{\underline{0}} & \\ & & \ddots & & \\ & \underline{\underline{0}} & & \frac{1}{a_{L-1}} & \\ & & & & \frac{1}{a_L} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{a_3} & & & \\ 0 & & \frac{1}{a_4} & & \\ 0 & & & \ddots & \\ 0 & & & & \frac{1}{a_L} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{a_2} \\ \frac{1}{a_3} \\ \vdots \\ \frac{1}{a_{L-1}} \\ \frac{1}{a_L} \end{bmatrix} - \begin{bmatrix} \frac{1}{a_3} \\ \frac{1}{a_4} \\ \vdots \\ \frac{1}{a_L} \\ 0 \end{bmatrix} = \begin{bmatrix} \Delta \\ \Delta \\ \vdots \\ \Delta \\ \frac{1}{a_L} \end{bmatrix} = \Delta\underline{1}_{L-1} + \left(\frac{1}{a_L} - \Delta\right)\underline{e}_{L-1} \qquad ,
\end{aligned}
$$

where Δ is as in (4.3), proving a). For part b), since $\underline{\underline{B}}^{-1} = \underline{\underline{I}} - \underline{\underline{L}}_1$ and $(\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{\underline{L}} = \underline{\underline{L}}_1$, it can be seen that $\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{1}_{L-1} = (\underline{\underline{I}} - \underline{\underline{L}}_1)(\underline{\underline{I}} + 2\underline{\underline{L}})\underline{1}_{L-1} = \{(\underline{\underline{I}} - \underline{\underline{L}}_1) + (2\underline{\underline{L}} - 2\underline{\underline{L}}_1\underline{\underline{L}})\}\underline{1}_{L-1} = \underline{\underline{I}}\,\underline{1}_{L-1} + \underline{\underline{L}}_1\underline{1}_{L-1} = \underline{w}(1,1) + \underline{w}(1,0) = \underline{w}(2,1)$ where $\underline{\underline{I}}\,\underline{1}_{L-1} = \underline{w}(1,1)$ and $\underline{\underline{L}}_1\underline{1}_{L-1} = \underline{w}(1,0)$ are employed to yield the last eauality, proving the lemma.

Proof of Lemma 4.5: In what follows, since $\underline{\underline{H}} = \underline{\underline{H}}_1$ as in (3.3), any reference to (2.3) assumes $i = 1$. We first note from (4.1) and (4.5) that $\hat{v}_1 = \frac{a_1}{D} + \frac{c^L + c^H}{2} = \frac{c^H - c^L}{2} + \frac{c^L + c^H}{2} = c^H$.

Hence from (2.3) and (4.1), one has $[\underline{\underline{H}}]_{1,m} = h_1(\hat{v}_1, \hat{v}_m) = h_1(c^H, \hat{v}_m) = (c^H - c^L)D = 2a_1$, proving a). For part b), one sees from (2.3) that $[\underline{\underline{H}}]_{n,1} = h_1(\hat{v}_n, \hat{v}_1) = h_1(\hat{v}_n, c^H) = (\hat{v}_n - c^L)D$. Substituting $\hat{v}_n = \frac{a_n}{D} + \frac{c^L+c^H}{2}$ from (4.5) into the last term and using (4.1), we obtain $(\hat{v}_n - c^L)D = a_n + \frac{c^H-c^L}{2}D = a_n + a_1$. In order to prove part c), we consider the following three cases:

Case:1 $1 < m < n \leq L$

For this case, one has $\hat{v}_m < \hat{v}_n$ from (4.4) and (4.5) so that it follows from (2.3) that $[\underline{\underline{H}}]_{m,n} = h(\hat{v}_m, \hat{v}_n) = 2(\hat{v}_m - \frac{c^L+c^H}{2})D = 2(\frac{a_m}{D} + \frac{c^L+c^H}{2} - \frac{c^L+c^H}{2})D = 2a_m$.

Case:2 $m = n \leq L$

Similarly, for $m = n$, one has $[\underline{\underline{H}}]_{m,n} = h(\hat{v}_m, \hat{v}_n) = (\hat{v}_m - \frac{c^L+c^H}{2})D = a_m$ for $m \in \mathcal{L} \setminus \{1\}$.

Case:3 $L \geq m > n > 1$

In this case, one has $\hat{v}_m > \hat{v}_n$ and from (2.3) $[\underline{\underline{H}}]_{m,n} = 0$.

We note from (4.8) and (4.12) that

$$\underline{\underline{A}}_D\underline{\underline{C}} = \begin{bmatrix} a_2 & 2a_2 & 2a_2 & \cdots & 2a_2 \\ & a_3 & 2a_3 & \cdots & 2a_3 \\ & & a_4 & \cdots & 2a_4 \\ & \underline{\underline{0}} & & \ddots & \vdots \\ & & & & a_L \end{bmatrix}$$

and part c) follows. Part d) is immediate from a), b), and c). Finally we prove part e). Using the result of d), one sees that

$$\underline{\underline{H}}\begin{bmatrix} x \\ y\underline{1}_{L-1} \end{bmatrix} = \begin{bmatrix} 2a_1 & 2a_1\underline{1}_{L-1}^T \\ (a_1\underline{\underline{I}} + \underline{\underline{A}}_D)\underline{1}_{L-1} & \underline{\underline{A}}_D\underline{\underline{C}} \end{bmatrix}\begin{bmatrix} x \\ y\underline{1}_{L-1} \end{bmatrix}$$
$$= \begin{bmatrix} 2a_1\{x + y(L-1)\} \\ \{a_1x\underline{\underline{I}} + x\underline{\underline{A}}_D + y\underline{\underline{A}}_D\underline{\underline{C}}\}\underline{1}_{L-1} \end{bmatrix} = \begin{bmatrix} 2a_1 \\ \{y\underline{\underline{A}}_D\underline{\underline{C}} + xa_1\underline{\underline{I}} + x\underline{\underline{A}}_D\}\underline{1}_{L-1} \end{bmatrix} .$$

Proof of Lemma 4.7: By the definition of α_3 in (4.15), one sees that $\alpha_3(1 + \frac{a_1}{a_L}) = 2\frac{a_1}{a_L}$, so that $\alpha_3 = 2\frac{a_1}{a_L} - \frac{a_1}{a_L}\alpha_3 = a_1(2 - \alpha_3)\frac{1}{a_L}$, proving a). For part b), we first note that $2 - \alpha_3 = 2 - \frac{2\frac{a_1}{a_L}}{1+\frac{a_1}{a_L}} = \frac{2}{1+\frac{a_1}{a_L}}$. Hence from the definition of α_4 in (4.15), one sees that $\alpha_4 = a_1\frac{2}{1+\frac{a_1}{a_L}}\Delta = a_1(2 - \alpha_3)\Delta$, completing the proof.

Proof of Lemma 4.8: From Lemma 4.5 d), one sees that

$$\underline{\underline{H}}\begin{bmatrix} x \\ y\underline{f} \end{bmatrix} = \begin{bmatrix} 2a_1 & 2a_1\underline{1}_{L-1}^T \\ (a_1\underline{\underline{I}} + \underline{\underline{A}}_D)\underline{1}_{L-1} & \underline{\underline{A}}_D\underline{\underline{C}} \end{bmatrix}\begin{bmatrix} x \\ y\underline{f} \end{bmatrix} = \begin{bmatrix} 2a_1(x + y\frac{L-2}{2}) \\ a_1x\underline{\underline{I}}\,\underline{1}_{L-1} + x\underline{\underline{A}}_D\underline{1}_{L-1} + y\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} \end{bmatrix}$$
$$= \begin{bmatrix} 2a_1 \\ y\underline{\underline{A}}_D\underline{\underline{C}}\,\underline{f} + a_1x\underline{1}_{L-1} + x\underline{\underline{A}}_D\underline{1}_{L-1} \end{bmatrix},$$ completing the proof.

Proof of Lemma 4.9: From Lemma 4.5 d), one sees that $\underline{\underline{H}} \begin{bmatrix} 0 \\ \underline{w}(x,y) \end{bmatrix}$

$$= \begin{bmatrix} 2a_1 & 2a_1 \underline{1}_{L-1}^T \\ (a_1\underline{\underline{I}} + \underline{\underline{A}}_D)\underline{1}_{L-1} & \underline{\underline{A}}_D\underline{\underline{C}} \end{bmatrix} \begin{bmatrix} 0 \\ \underline{w}(x,y) \end{bmatrix} = \begin{bmatrix} 2a_1\underline{1}_{L-1}^T\underline{w}(x,y) \\ \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{w}(x,y) \end{bmatrix} = \begin{bmatrix} 2a_1 \\ \underline{\underline{A}}_D\underline{\underline{C}}\,\underline{w}(x,y) \end{bmatrix}$$

where $\underline{1}_{L-1}^T\underline{w}(x,y) = (L-2)x + y = 1$ is employed to yield the last equality.

Proof of Lemma 4.10: We first note that $(\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{\underline{L}} = \underline{\underline{L}}_1$ and $\underline{\underline{B}}^{-1} = \underline{\underline{I}} - \underline{\underline{L}}_1$ so that $\underline{\underline{B}}^{-1}\underline{\underline{C}}\,\underline{f} = (\underline{\underline{I}} - \underline{\underline{L}}_1)(\underline{\underline{I}} + 2\underline{\underline{L}})\underline{f} = (\underline{\underline{I}} - \underline{\underline{L}}_1 + 2\underline{\underline{L}}_1)\underline{f} = (\underline{\underline{I}} + \underline{\underline{L}}_1)\underline{f} = \underline{w}(1,0)$, proving part a). For part b), one sees that $\underline{\underline{B}}^{-1}\underline{1}_{L-1} = (\underline{\underline{I}} - \underline{\underline{L}}_1)\underline{1}_{L-1} = \underline{1}_{L-1} - \underline{w}(1,0) = \underline{w}(0,1)$, completing the proof.

Proof of Lemma 5.4: For $r = 1$, we first note from (5.8) that $m_{min}(1, L, N) = 1$ and hence from (5.10) $\epsilon_{min}(1, L, N) = v_{L:1} - u_{N:1} = c^H - c^H = 0$. For $r = N - 1$, it should be noted from Proposition 5.3 that $m_{max}(N-1, L, N) = L - 1$ for any $L > L(2, N)$. It then follows from (5.11) that $\epsilon_{max}(N-1, L, N) = u_{N:N} - \hat{v}_{L:L-1} = \hat{v}_{L:L} - \hat{v}_{L:L-1}$ which goes to 0 as $L \to \infty$ from (5.12). In general, for $2 \le r \le N-1$, one sees from Proposition 4.1 that $|\hat{v}_{L:L} - \hat{v}_{L:L-1}| > |\hat{v}_{L:m_{min}(r,L,N)} - \hat{v}_{L:m_{max}(r-1,L,N)}| = |\{\epsilon_{min}(r, L, N) + u_{N:r}\} - \{u_{N:r} - \epsilon_{max}(r - 1, L, N)\}| = |\epsilon_{min}(r, L, N) + \epsilon_{max}(r-1, L, N)|$, and the first term goes to 0 as $L \to \infty$ from (5.12), proving the lemma.

Proof of Lemma 5.5: Since $\hat{v}_{L:1} = u_{N:1}$, one has $m_{min}(1, L, N) = 1$. From (5.10), one sees for $2 \le r \le N-1$ that $\epsilon_{min}(r, L, N) = \hat{v}_{L:m_{min}(r,L,N)} - u_{N:r}$. Substituting (5.2) into this, one has

$$\epsilon_{min}(r, L, N) = \frac{1}{D}(a_{L:m_{min}(r,L,N)} - \tau_{N:r}) \quad . \tag{A.1}$$

From (5.3), one sees that $a_{L:m_{min}(r,L,N)} - \tau_{N:r} = \dfrac{a_{L:L}}{(L - m_{min}(r, L, N))\Delta(L)a_{L:L} + 1} - \tau_{N:r}$. Part a) now follows by substituting this into (A.1) and solving for $m_{min}(r, L, N)$. The proof for part b) is similar to that for (5.14), completing the proof.

Proof of Lemma 5.6: We note that $Z(1, L, N) = m_{max}(1, L, N) - m_{min}(1, L, N) + 1$ and for $2 \le r \le N-1$, $Z(r, L, N) = m_{max}(r, L, N) - m_{max}(r-1, L, N)$. The lemma then follows from Lemma 5.5.