No. 113 (81-14)

On the Equilibrium Concepts in a General Equilibrium Theory with Public Goods and Taxes —

Pareto Optimality and Existence

By

Sho-Ichiro Kusumoto

April 1981

Since Lindahl (1958), Samuelson (1954, 1958, 1969), and Johansen (1963) and others, the various concepts of equilibrium in an economy with public (collective) consumption goods, have been known.

Dorfman (1969) and Foley (1969, 1970) have investigated the optimality and existence of such concepts as public competitive equilibrium, Lindahl equilibrium, core with respect to the public as well as private sectors. Foley (1969) applied Brouwer's fixed point theorem to prove the existence of a public competitive equilibrium with the proportionate tax system, although he did not provide a complete proof. He also applied Debreu (1962) to prove the Lindahl equilibrium in his 1970 article. Milleron (1972) extended Foley's framework, so that it can handle the case of positive profit.

Here, first, we shall examine the results by Foley in a more general framework which dispenses the restrictive assumptions such as strict convexity of consumption set, conical production set, etc..

Secondly, we shall rigorously prove his theorems in the general reformulation, especially, prove the existence of the public competitive equilibrium under the linear income tax system, by applying Kakutani's fixed point theorem. For the latter, Negishi (1972)'s formulation for proving a competitive equilibrium of the economy without public goods, will be extended so as to incorporate a public sector. Optimality of the equilibrium prices, allocation of public and private goods, tax rate and weights of individual values in the social welfare, is confirmed in the Pareto sense.

I. Consumption and Production of Public Goods

Each consumer $i \in N$ choose a pair of public and private goods, (x, y^i) from a public consumption set $X \subset R^{\mu^1}$ and from a private consumption set $Y^i \subset R^{\mu^2}$. A vector x of public goods and a vector of private goods y^i are written; $x = (x_1, x_2, \ldots, x_{\mu^1})$ and $y^i = (y_1^i, y_2^i, \ldots, y_{\mu^2}^i)$.

Each consumer $i \in N$ chooses a pair (x, y^i) in a Cartesian product $X \times Y^i$, on which there is defined a <u>weak</u> preference ordering $\succsim_i$ and owns an initial endowment ω^i of private goods.

Debreu's assumptions are extended in order to restrict the economy incorporating a public good sector as well as private sectors.

<u>A. 1</u> (i) X, and Y^i for each $i \in N$ are closed and convex.

(ii) Y^i has for each $i \in N$ an interior point; $(Y^i)^\circ \neq \emptyset$.

<u>A. 2</u> Each consumer's preference $\succsim_i$ is continuous and convex. That is, (iii) for every $(\bar{x}, \bar{y}^i)$, $C^i(\bar{x}, \bar{y}^i) = [(x, y^i) \in X \times Y^i; (x, y^i) \succsim_i (\bar{x}, \bar{y}^i)]$ and $D^i(\bar{x}, \bar{y}^i) = [(x, y^i) \in X \times Y^i; (x, y^i) \precsim_i (\bar{x}, \bar{y}^i)]$ are closed in $X \times Y^i$, and (iv) if $(x, y^i) \succ_i (\bar{x}, \bar{y}^i)$, then, $(tx+(1-t)\bar{x}, ty^i+(1-t)\bar{y}^i) \succ_i (\bar{x}, \bar{y}^i)$, $0 < t < 1$, for all $i \in N$.

<u>A. 3</u> Each consumer's preference is monotone; $(x, y^i) \geq (\bar{x}, \bar{y}^i) \rightarrow (x, y^i) \succ_i (\bar{x}, \bar{y}^i)$, for all $i \in N$.

<u>A. 4</u> For every $i \in N$, $(x, y^i) \in X \times Y^i \rightarrow (x, \underline{y}^i) \in X \times Y^i$, $y^i \geq \underline{y}^i$.

By Assumption A. 2, (v) $(x, y^i) \succsim_i (\bar{x}, \bar{y}^i) \rightarrow (tx+(1-t)\bar{x}, ty^i+(1-t)\bar{y}^i) \succsim_i (\bar{x}, \bar{y}^i)$, $0 \leqq t \leqq 1$, $i \in N$, that is, $C^i(\bar{x}, \bar{y}^i)$ is convex for every $(\bar{x}, \bar{y}^i) \in X \times Y^i$.

<u>A. 5</u> For every $i \in N$, $(x, y^i) \in X \times Y^i \rightarrow (\underline{x}, y^{i\prime}) \in X \times Y^i$, $x > \underline{x}$.

Each producer (industry) $j \in M$ chooses a pair (x^j, y^j) in a production set $X^j x\ Y^j$, the set of all technically possibile production plans.

<u>B. 1</u> $0 \in X^j x\ Y^j$, for each $j \in M$.

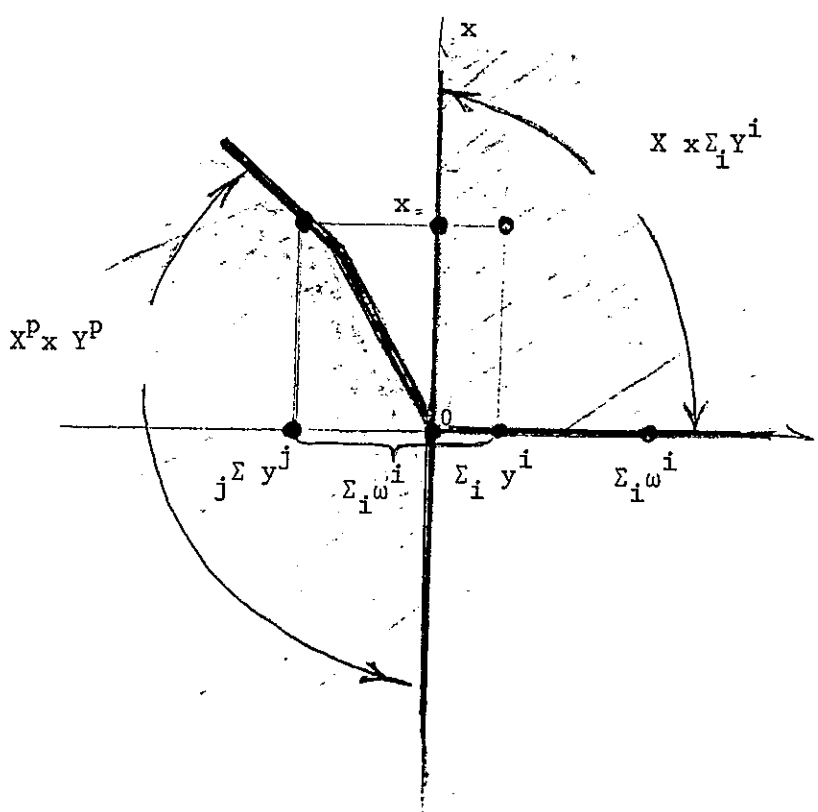
<u>B. 2</u> $X^P \times Y^P$ is closed and convex, where $X^P = \Sigma_j X^j$, $Y^P = \Sigma_j Y^j$.

<u>B. 3</u> $0 \neq (x, y) \in X^P \times Y^P \rightarrow y_h < 0$ at least one $h \in M_c$.

<u>B. 4</u> There exists $(x, y) \in X^P x\ Y^P$ with $x_h > 0$ for every $h \in M_x$.

<u>B. 5</u> $(x, y) \in X^P x\ Y^P \rightarrow (\bar{x}, y) \in X^P x\ Y^P$, where $\bar{x}_h = x_h$ if $x_h > 0$, or, 0 otherwise.

Note that M_c, M_x, M, and N are, respectively, the index set of private goods, of public goods, and the index set of producers, and of consumers. B. 3-5 may imply that $X^P x\ Y^P \cap (-(X^P x\ Y^P)) \subset \{0\}$ and $X^P x\ Y^P \supset -R^{\mu}_{+}, \mu = \mu^1 + \mu^2$.



II. Individual Economic Behaviour and Government Proposal

Take all agents as price-takers. Given price $p = (p_x, p_y)$.

Denote by θ_{ij} the share of consumer i in producer j 's profit π_j and by $\pi_i(p_x, p_y)$, the maximum total receipts of consumer i from producers. Let $\Sigma_i \theta_{ij} = 1$, $\theta_{ij} \geqq 0$, $i \in N$, $j \in M$. Then, for each price (p_x, p_y) and given θ_{ij}, $\pi_i(p_x, p_y) = \max \sum_{j \in M} \theta_{ij} \underbrace{(p_x, p_y)(x^j, y^j)}_{\pi_j}$; $(x^j, y^j) \in X^j x\, Y^j$.

Each producer j choose a more profitable production plan (x^j, y^j) than any other in his production set $X^j x\, Y^j$.

Each consumer i choose a private consumption plan $\tilde{y}^i$, which will make him better off than any other, given a government proposal (x, t^i), under the after tax budget constraint. $(x, \tilde{y}^i) \succsim_i (x, y^i)$, $p_y y^i \leqq p_y \omega^i + \pi_i(p_x, p_y) - t^i$. Government choose a public consumption and tax plan $(\tilde{x}, \tilde{t}^i; i \in N)$ with $p_x \tilde{x} = \sum_i \tilde{t}^i$, such that every individual i can choose $\tilde{y}^i$, which together with the public proposal makes him better off than any other consumption y^i, associated with an alternative proposal $(x, t^i; i \in N)$. That is; $(\tilde{x}, \tilde{y}^i) \succsim_i (x, y^i)$ $i \in N$. Here, tax includes <u>negative</u> one.

A consumption plan$(x, y^i; i \in N)$ is feasible if $x = \Sigma_j x^j$, $\Sigma_i (y^i - \omega^i) = \Sigma_j y^j$.

A government proposal $(x, t^i; i \in N)$ is feasible if $p_x x \leqq \sum_i t^i$.

III. A Public Competitive Equilibrium

We shall here introduce a fairly general concept of equilibrium which can be reduced, as a special case, to a competitive equilibrium without any public goods.

An allocation $(x, y^i; i \in N)$ is a <u>feasible</u> allocation if, for an initial endowment ω^i ; $i \in N, \{x, \sum_{i \in N} (y^i - \omega^i)\} \in X^p \times Y^p$.

A <u>public competitive equilibrium</u> is generally defined to be a feasible allocation $(\hat{x}, \hat{y}^i; i \in N)$ and a price vector $\hat{p} = (\hat{p}_x, \hat{p}_y) \geq 0$ and a vector of taxes $(\hat{t}^i; i \in N)$ with $\hat{p}_x \hat{x} = \sum_{i \in N} \hat{t}^i$ and with, for each $i \in N$, $\hat{p}_y \hat{y}^i \leqq \hat{p}_y^i \omega^i + \pi_i(\hat{p}_x, \hat{p}_y) - \hat{t}^i$,

(i) $(\hat{p}_x, \hat{p}_y)\{\hat{x}, \Sigma_i(\hat{y}^i - \omega^i)\} \geqq (\hat{p}_x, \hat{p}_y)(x, y) \quad \forall (x, y) \in X^p \times Y^p$;

(ii) $(x, y^i) \succ_i (\hat{x}, \hat{y}^i) \rightarrow t^i + \hat{p}_y y^i > \hat{t}^i + \hat{p}_y \hat{y}^i, \quad i \in N$,

with, as a special case, (iii) $(\hat{x}, y^i) \succ_i (\hat{x}, \hat{y}^i) \rightarrow \hat{p}_y y^i > \hat{p}_y \hat{y}^i$.

In stead of (i), we may use explicitly, for every $j \in M$,

(i') $(p_x, p_y)(x^j, y^j) \geqq (p_x, p_y)(x^j, y^j) \quad \forall (x^j, y^j) \in X^j \times Y^j$.

The intended interpretation for this definition is that public competitive equilibrium involves; (i) profit maximization by each producer, and, (ii) the impossibility of finding a new public good proposal with taxes to pay for it that appears to every individual to leave him better off. With (iii), preference optimization under the after tax constraint by every individual.

A <u>Pareto optimum</u> is a feasible allocation $(\tilde{x}, \tilde{y}^i; i \in N)$, such that there is no other feasible allocation $(x, y^i; i \in N)$ with

(iv) $(x, y^i) \succsim_i (\tilde{x}, \tilde{y}^i)$ for every $i \in N$,

(v) $(x, y^i) \succ_i (\tilde{x}, \tilde{y}^i)$ for some $i \in N$.

IV. Characterization of Public Competitive Equilibrium

Theorem 1 : A public competitive allocation $(\hat{x}, \hat{y}^i; i \in N)$ is a Pareto optimum.

Proof: Suppose it is not. Then, by definition, there exists an allocation $(x, y^i; i \in N)$ such that

$$\{ x, \sum_i (y^i - \omega^i)\} \in X^p \times Y^p$$

$$(x, y^i) \succsim_i (\hat{x}, \hat{y}^i) \quad \text{for every } i \in N$$

$$(x, y^i) \succ_i (\hat{x}, \hat{y}^i) \text{ for some } i \in N.$$

For this allocation (x, y^i) and $(\hat{x}, \hat{y}^i)$ $i \in N$, in view of (i),

(1) $$(\hat{p}_x, \hat{p}_y)\{\hat{x}, \Sigma_i(\hat{y}^i - \omega^i)\} \geqq (\hat{p}_x, \hat{p}_y)\{x, \sum_i(y^i - \omega^i)\}$$

which reduces to:

(2) $$\hat{p}_x \hat{x} + \hat{p}_y \sum_i \hat{y}^i \geqq \hat{p}_x x + \hat{p}_y \sum_i y^i.$$

The condition(ii) implies

(3) $$\sum_i t^i + \hat{p}_y \sum_i y^i > \sum_i \hat{t}^i + \hat{p}_y \sum_i \hat{y}^i.$$

Since $\Sigma t^i \geqq \hat{p}_x x$ and $\Sigma \hat{t}^i = \hat{p}_x \hat{x}$, it follows that

(4) $$\hat{p}_x x + \hat{p}_y \sum_i y^i > \hat{p}_x \hat{x} + \hat{p}_y \sum_i \hat{y}^i,$$

which is a contradiction. Q.E.D.

Assume A. 1–5, and B. 1–5 for the following theorem.

Theorem 2: If $(\hat{x}, \hat{y}^i;\ i \in N)$ is a Pareto optimum, then, there exists a price vector $\hat{p} = (\hat{p}_x, \hat{p}_y) \geq 0$, such that

(i) $\hat{p}_x \hat{x} + \hat{p}_y \sum_{i \in N}(\hat{y}^i - \omega^i) \geqq \hat{p}_x x + \hat{p}_y y$, for every $(x, y) \in X^P x Y^P$

(ii) $(x, y^i) \underset{i}{\succ} (\hat{x}, \hat{y}^i) \rightarrow \hat{t}_i + \hat{p}_y \hat{y}^i < t_i + \hat{p}_y y^i$

where

$$\Sigma \hat{t}_i = \hat{p}_x \hat{x}, \quad \Sigma t_i \geqq \hat{p}_x x$$

$$\hat{p}_y y^i \leqq \hat{p}_y \omega^i + \pi_i(\hat{p}_x, \hat{p}_y) - \hat{t}^i,$$

and

$$\hat{p}_y y^i \leqq \hat{p}_y \omega^i + \pi_i - t^i.$$

Proof: Define, for each $i \in N$, an upper contour set as the least preferred set $C^i(\hat{x}, \hat{y}^i)$ and a set $D(\hat{x}, \hat{y}^i; i \in N) = [(x, \Sigma_i y^i); (x, y^i) \in C^i(\hat{x}, \hat{y}^i)]$, associated with $C^i(\hat{x}, \hat{y}^i)$. Convexity of $D(\hat{x}, \hat{y}^i;\ i \in N)$ follows from that of $C^i(\hat{x}, \hat{y}^i)$ for every $i \in N$. A sum (linear combination) Z of convex sets;$D(\hat{x}, \hat{y}^i;\ i \in N)$ and $X^P x\ Y^P$ is also convex. Let $\omega = \sum_i \omega^i$. This sum does not contain a vector z, which is larger than $(0, -\omega)$ in the semi-ordering sense; $(0, -\omega) \leq z \notin Z$, since if it did, the set $D(\hat{x}, \hat{y}^i; i \in N)$ would be bounded (in fact, compact) so that it would contain a satiation point, which is (assumed to be) a Pareto optimum. A contradiction to monotonicity of preference for some i. Apply the separation theorem (Minkowski) to obtain a hyperplane with non negative coefficients $\hat{p} = (\hat{p}_x, \hat{p}_y) \geq 0$, passing through $(0, -\omega)$, by which Z and $\{(0, -\omega) + R^\mu\}$ are separated. See Berge (p.157), for example.

(1) $(\hat{p}_x, \hat{p}_y) \geq 0, \quad (\hat{p}_x, \hat{p}_y)\, u \geqq (\hat{p}_x, \hat{p}_y) z, \quad \forall z \in Z, \forall u \in \{(0, -\omega) + R^\mu\}$

and $\hat{p}(0, -\omega) \geqq (\hat{p}_x, \hat{p}_y)\, z, \forall z \in Z.$

Let $y = \sum_{i \in N}(y^i - \omega^i)$. Then, the Pareto optimum $(\hat{x}, \hat{y}^i;\ i \in N)$ satisfies $(\hat{p}_x, \hat{p}_y)(\hat{x}, \hat{y}) - (\hat{p}_x, \hat{p}_y)(\hat{x}, \sum_i \hat{y}^i) = -\hat{p}_y \omega$, since it is feasible and in $C^i(\hat{x}, \hat{y}^i)$ for every $i \in N$, so that it is in Z.

Take any $(x, y) \in X^p \times Y^p$ then, by B.1,

(2) $(\hat{p}_x, \hat{p}_y)(x, y) \leqq (\hat{p}_x, \hat{p}_y)(\hat{x}, \hat{y})$, $0 \leqq (\hat{p}_x, \hat{p}_y)(\hat{x}, \hat{y})$,

which is the condition (i).

Take any $(\hat{x}, y^i) \in C^i(\hat{x}, \hat{y}^i)$, for each i, then, (1) implies;

(3) $(\hat{p}_x, \hat{p}_y)(\hat{x}, \hat{y}) \leqq (\hat{p}_x, \hat{p}_y)\{\hat{x}, \sum_i (y^i - \omega^i)\}$.

Further, take it for each i but i' as $y^i = \hat{y}^i$, and $y^{i'} \neq \hat{y}^{i'}$, for i',(A. 3)

then, (3) implies;

(4) $\hat{p}_y \hat{y}^{i'} \leqq \hat{p}_y y^{i'}$, $(\hat{x}, y^{i'}) \in C^{i'}(\hat{x}, \hat{y}^{i'})$.

Suppose $\hat{p}_y \hat{y}^i = \hat{p}_y y^i$ and $(\hat{x}, y^i) \gtrsim_{i'} (\hat{x}, \hat{y}^{i'})$. (Use A. 2 and 4.)

Take also for i', $y^{i'}(\varepsilon) = (y^{i'}_1, .., y^{i'}_{j-1}, y^{i'}_j - \varepsilon, y^{j'}_{j+1}, .., y^{i'}_{\mu_2})$ for

$\varepsilon > 0$, $\hat{p}_j > 0$. $\lim_{\varepsilon \to 0}(\hat{x}, y^{i'}(\varepsilon)) \in C^{i'}(\hat{x}, y^{i'})$, so that $(\hat{x}, y^{i'}(\varepsilon)) \gtrsim_i$

$(\hat{x}, \hat{y}^{i'})$ for a sufficiently small $\varepsilon > 0$ by continuity of preference for i'.

Hence, $\hat{p}_y y^{i'}(\varepsilon) < \hat{p}_y y^{i'} = \hat{p}_y \hat{y}^{i'}$, which is contradicting to (4).

Thus, $(\hat{x}, y^i) \gtrsim_i (\hat{x}, \hat{y}^i) \to \hat{p}_y y^i > \hat{p}_y \hat{y}^i$, if there is j for which $\hat{p}_j > 0$.

Now take $y^i(\varepsilon)$ so that $y^{i'}_k = \hat{y}^{i'}_k$, $k \neq j,h$ and $y^{i'}_h > \hat{y}^{i'}_h$. We label this by

$\hat{y}^i(\varepsilon)$. Then, $\hat{p}_j > 0$ implies $\hat{p}_h > 0$, since, by (4), for arbitrarily chosen h,

(5) $\hat{p}_h(y^{i'}_h - \hat{y}^{i'}_h) \geqq \hat{p}_j \varepsilon_j > 0$,

where $(\hat{x}, \hat{y}^{i'}(\varepsilon)) \in C^{i'}(\hat{x}, \hat{y}^{i'})$ for a sufficiently small $\varepsilon > 0$.

Both arguments depend on that $\hat{p}_j > 0$ for some $j \in M_c$. Thus, if $\hat{p}_j > 0$ for

some j, then, $\hat{p}_y > 0$.

Take x instead of $\hat{x}$ in (3).

Suppose, for such allocation $(x, y^i; i \in N)$, $(x, y^i) \gtrsim_i (\hat{x}, \hat{y}^i)$,

(6) $t_i + \hat{p}_y y^i \leqq \hat{t}^i + \hat{p}_y \hat{y}^i$

where some $t = (t_1, t_2, ..., t_n)$, $\Sigma t_i \geqq \hat{p}_x x$.

Then, $\hat{p}_x x + \hat{p}_y \sum_i y^i \leqq \hat{p}_x \hat{x} + \hat{p}_y \sum_i \hat{y}^i$. From this and (3),

(7) $(\hat{p}_x, \hat{p}_y)(x, \sum_i y^i) = (\hat{p}_x, \hat{p}_y)(\hat{x}, \sum_i \hat{y}^i)$.

Take $y^i(\varepsilon)$ instead of y^i so that $(x, y^i(\varepsilon)) \underset{i}{>} (\hat{x}, \hat{y}^i)$ for any j and any i. We can do this by continuity of $\underset{i}{\succsim}$, as have done above.

(8) $(\hat{p}_x, \hat{p}_y)(x, \sum_i y^i) > (\hat{p}_x, \hat{p}_y)(x, \sum_i y^i(\varepsilon))$

which is contradicting (3) for x in stead of $\hat{x}$.

Thus, for each i, $(x, y^i) \underset{i}{>} (\hat{x}, \hat{y}^i) \rightarrow t^i + \hat{p}_y y^i > \hat{t}^i + \hat{p}_y \hat{y}^i$.

The proof completes if $p_y \neq 0$.

Suppose

$$(\hat{p}_x, \hat{p}_y)(\hat{x}, \hat{y}) = 0 \text{ (zero profit).}$$

Then, if $\hat{p}_y = 0$, then, $\hat{p}_x \geq 0$. (2) implies $0 = \hat{p}_x \hat{x} \geqq \hat{p}_x x, (x, y) \varepsilon X^p x Y^p$. By B. 4, $x > 0$, $(x, y) \epsilon X^p x Y^p$, $\hat{p}_x x > 0 = \hat{p}_x \hat{x}$. This is a contradiction. In fact, $-\hat{p}_y y \geqq \hat{p}_x x > 0$, implying $\hat{p}_{yj} > 0$ for some $j \varepsilon M_c$.

Suppose $(\hat{p}_x, \hat{p}_y)(\hat{x}, \hat{y}) > 0$ (positive profit), alternatively.

Then, if $\hat{p}_y = 0$, then, $\hat{p}_x \geq 0$. For $(x(\varepsilon), y^i) \underset{i}{\succ} (\hat{x}, \hat{y}^i)$ $(\hat{p}_x, \hat{p}_y)(x(\varepsilon), \Sigma y^i) \geqq (\hat{p}_x, \hat{p}_y)(\hat{x}, \Sigma \hat{y}^i)$, where $x_j(\varepsilon) = \hat{x}_j - \varepsilon$, $\varepsilon > 0$ for $j \in M_x$ such that $\hat{p}_{xj} > 0$, $y^i_h > \hat{y}^i_h$, $h \varepsilon M_c$, $y^i_k = \hat{y}^i_k, k \neq h$, and $y^{i'} = \hat{y}^{i'}$ $i' \neq i$. We can do this, by A. 3 and A. 5.

$\lim_{\varepsilon \to 0} (x(\varepsilon), y^i) \underset{i}{>} (\hat{x}, \hat{y}^i)$ and $\hat{p}_{yh}(y^i_h - \hat{y}^i_h) \geqq \hat{p}_{xj} \varepsilon > 0$, hence, $\hat{p}_{yh} > 0$ $h \varepsilon M_c$, leading a contradiction. Thus, $\hat{p}_y \neq 0$, and we can take j as such that $\hat{p}_j > 0$. Hence, $\hat{p}_y > 0$ by the above argument. Q.E.D.

Remark 1: Take, in (2) in the proof, (x, y) as such that $x_h^j = \hat{x}_h^j$, $j \neq j'$; $y_h^j = \hat{y}_h^j$, $j \neq j'$, for every h. Then, (2) implies (i').

Corollary : If $(\hat{x}, \hat{y}^i; i \in N)$ is a Pareto Optimum, then, there exists a price vector $\hat{p} = (\hat{p}_x^1, \hat{p}_x^2, \ldots, \hat{p}_x^n, \hat{p}_y) \geq 0$ such that, $\hat{p}_x = \sum_i \hat{p}_x^i$, and $\hat{t}^i = \hat{p}_y \omega^i + \pi_i(\hat{p}_x, \hat{p}_y) - \hat{p}_y \hat{y}^i$ satisfying (i) and (ii).

Proof: Let

$$F = [(x^1, x^2, x^3, \ldots, x^n, y;\ x^i = x;\ (x, y) \in X^p \times Y^p] \subset R^{n\mu^1 + \mu^2}$$

Then, F is convex and closed, since $X^p \times Y^p$ is so.

Let

$$G = [(x^1, x^2, x^3, \ldots, x^n, \sum_i y^i; (x^i, y^i) \in C^i(\hat{x}, \hat{y}^i),\ i \in N] \subset R^{n\mu^1 + \mu^2}$$

Then, this is also convex, since $C^i(\hat{x}, \hat{y}^i)$ is so.

In the proof of Theorem 2, use F instead of $X^p \times Y^p$, and G instead of D to obtain a price vector $\hat{p} = (\hat{p}_x^1, \hat{p}_x^2, \ldots, \hat{p}_x^n, \hat{p}_y) \geq 0$, such that $\hat{p}_x = \sum_i \hat{p}_x^i$, $\hat{t}^i = \hat{p}_y \omega^i + \pi_i(\hat{p}_x, \hat{p}_y) - \hat{p}_y \hat{y}^i$ satisfying (i) and (ii).

Remark 2: $t^i = p_y \omega^i + \pi_i(p_x, p_y) - p_y y^i$ may be negative.

Corollary says, corresponding to each Pareto optimum, there is a total tax on each individual, $\hat{t}^i$, which is positive or negative.

V. Existence of Lindahl Equilibrium

We shall use, instead of A. 4,

A. 4' If $(x, y^i) \in X \times Y^i$, then, there is $(x, \underline{y}^i) \in X \times Y^i$, such that $\underline{y}^i < \omega^i$.

We shall add

A. 6 $X \times Y^i$ has a lower bound for $\leqq$.

We wish to apply Debreu's result here.

Lemma (Debreu 1959): Given A. 1-3, 4', 5, 6 & B. 1-5. Then, there exist, in a private ownership economy, a feasible allocation $(\hat{y}^1, \hat{y}^2, .., \hat{y}^n)$ and a price vector $\hat{p}_y > 0$, such that,

(a) $\hat{p}_y [\Sigma_i (\hat{y}^i - \omega^i)] \geqq \hat{p}_y y, \forall y \in Y^p$

(b) for every $i \in N$, $\hat{y}^i \succsim_i y^i; \quad \hat{p}_y y^i \leqq \hat{p}_y \hat{y}^i = \hat{p}_y \omega^i + \pi_i(\hat{p}_x, \hat{p}_y)$.

A Lindahl equilibrium, with respect to intial endowment $\omega = (\omega^1, \omega^2, \omega^3, \ldots, \omega^n)$ is a feasible allocation $(\tilde{\tilde{x}}; \tilde{\tilde{y}}^1, \tilde{\tilde{y}}^2, \tilde{\tilde{y}}^3, \ldots, \tilde{\tilde{y}}^n)$ and a price system $\tilde{\tilde{p}} = (\tilde{\tilde{p}}^1_x, \tilde{\tilde{p}}^2_x, .. \tilde{\tilde{p}}^n_x, \tilde{\tilde{p}}_y) \geq 0$, such that, $\tilde{\tilde{p}}_x = \Sigma_i \tilde{\tilde{p}}^i_x$, $i = 1, 2, \ldots, n$,

(iii) $(\tilde{\tilde{p}}_x, \tilde{\tilde{p}}_y)[\tilde{\tilde{x}}, \sum_i (\tilde{\tilde{y}}^i - \omega^i)] \geq (\tilde{\tilde{p}}_x, \tilde{\tilde{p}}_y)(x, y) \forall (x, y) \in X^p \times Y^p$,

(iv) $(x^i, y^i) \succ_i (\tilde{\tilde{x}}, \tilde{\tilde{y}}^i) \rightarrow \tilde{\tilde{p}}^i_x x^i + \tilde{\tilde{p}}_y y^i > \tilde{\tilde{p}}^i_x \tilde{\tilde{x}} + \tilde{\tilde{p}}_y \tilde{\tilde{y}}^i = \tilde{\tilde{p}}_y \omega^i + \pi_i(\tilde{\tilde{p}}_x, \tilde{\tilde{p}}_y)$.

Theorem 3: There exists, under A. 1-3, 4', 5, 6 and B. 1-5, a Lindahl equilibrium allocation.

Proof: The above lemma may trivially be adapted and applied to the following. Let $\tilde{X}^i = \{0\} \times \{0\} \times \ldots \times X^i \times \ldots \times \{0\} \subset R^n$.

$$\tilde{X}^i \times \tilde{Y}^i \subset R^{n\mu^1 + \mu^2}$$

$$\tilde{C}^i(\bar{x}^i, \bar{y}^i) = [(0, 0, .., x^i, 0, .., y^i); (x^i, y^i) \in C^i(\bar{x}, \bar{y}^i)]$$

Then, if A. 1-3, 4', 5 hold, then, they all hold for $\tilde{X}^i$, $\tilde{Y}^i$, and $\tilde{C}^i$ instead of X^i, Y^i and C^i.

Let $\tilde{X}^j = \underbrace{X^j \times X^j \times \ldots \times X^j}_{n}$, $\tilde{X}^p = \underbrace{X^p \times X^p \times \ldots \times X^p}_{n}$.

Then, B. 1-5 hold for $\tilde{X}^p$ and $\tilde{Y}^p$, if they hold for X^p and Y^p.

Thus, apply the lemma to obtain a price vector $(\tilde{\tilde{p}}_x^1, \tilde{\tilde{p}}_x^2, \ldots, \tilde{\tilde{p}}_x^n, \tilde{\tilde{p}}_y) \geq 0$

(iii) and

(iv') $(\tilde{\tilde{x}}^i, \tilde{\tilde{y}}^i) \succsim_i (x^i, y^i)$, $\quad \tilde{\tilde{p}}_x^i\, x + \tilde{\tilde{p}}_y\, y^i \leqq \tilde{\tilde{p}}_x^i\, \tilde{\tilde{x}} + \tilde{\tilde{p}}_y\, \tilde{\tilde{y}}^i = \tilde{\tilde{p}}_y\, \omega^i + \pi_i(\tilde{\tilde{p}})$.

The latter is equivalent to (iv). In fact, suppose the condition of (iv) holds but the conclusion does not. Suppose $\tilde{\tilde{p}}_x^i\, x^i + \tilde{\tilde{p}}_y\, y^i = \tilde{\tilde{p}}_y\, \omega^i + \pi_i(\tilde{\tilde{p}})$, as in the proof of Theorem 2, $\tilde{\tilde{p}}_x\, x^i + \tilde{\tilde{p}}_y\, y^i(\varepsilon) < \tilde{\tilde{p}}_y\, \omega^i + \pi_i(\tilde{\tilde{p}})$, $(x^i, y^i(\varepsilon)) \succ_i (\tilde{\tilde{x}}, \tilde{\tilde{y}}^i)$, for $\tilde{\tilde{p}}_x^i \geq 0$ and $\tilde{\tilde{p}}_y > 0$, under A. 4'. This is contradicting (iv'). Q.E.D.

Remark 3: $\tilde{\tilde{t}}^i = \tilde{\tilde{p}}_x^i\, \tilde{\tilde{x}} \geq 0$. The value of public goods had by each individual is equal to the tax he pays. $\tilde{\tilde{p}}_x^i\, \tilde{\tilde{x}} + \tilde{\tilde{p}}_y\, \tilde{\tilde{y}}^i - \{\tilde{\tilde{p}}_y\, \omega^i + \pi_i(\tilde{\tilde{p}})\} = 0$; the lump-sum transfer is zero for every individual.

VI. Existence of a Public Competitive Equilibrium under the Linear Income (Wealth) Tax System

We shall investigate here whether a public competitive equilibrium does exist under a proportionate income tax system, with which the costs of public goods are financed. The tax rate itself will be found to be an equilibrium coordinate. The weights of individual values in the social welfare adjusts among individuals so that each non-satiating individual's budget constraint will be effectively satisfied. The equilibrium weights will be found, hence, a social welfare function of utilitarian type will be specfied.

We specify A. 2 (convexity) so that a continuous numerical representation u^i; $i \in N$ of preference $\succsim_i$, is concave in $X \times Y^i$. See Debreu(1959, p.73) and Negishi (1970, p.17).

1. Optimality and Maximization.

Let

(1) $Z = \{(x, y^i; i \in N); (x, y^i) \in X \times Y^i, \ (x, \sum_{i \in N} (y^i - \omega^i)) \in X^p \times Y^p\}$.

Then, Z may be said to be an attainable consumption set, which is not empty if and only if $X \times \sum_{i \in N} Y^i - X^p \times Y^p = \{0, \omega\}$.

Since B.3-5 may imply $X^p \times Y^p \supset -R^\mu_+$, $X^p \times Y^p \cap -(X^p \times Y^p) \subset \{0\}$, it follows that $X^p \times Y^p \cap R^\mu_+ = \{0\}$. By A.6, $X \times Y^i$ has a lower bound for $\leqq$, and, by B.2 $X^p \times Y^p$ is closed, by A.2 so are $X \times Y^i$, and $X \times \prod_{i \in N} Y^i$. Thus, by Debreu(1959) (pp. 76-77), the intersection of $X \times \prod_{i \in N} Y^i \times \prod_{j \in M} (X^j \times Y^j)$ and $\{(x, y^i)$ $; i \in N, (x^j, y^j) ; j \in M; (x, y^i) \in X \times Y^i, (x^j, y^j) \in X^j \times Y^j, \sum_{i \in N} (y^i - \omega^i) =$ $\sum_{j \in M} y^j; \ x = \sum_{j \in M} x^j \}$ is bounded, and so is Z. The closedness of Z is clear from A. 2 and B. 2. Z is therefore compact.

Weierstrass theorem says that there exists a z Z, which maximizes a continuous function on Z.

According to A.2 (continuity), there is, for each $i \in N$, a continuous utility function $u^i: (x, y^i) \in X^i \times Y^i \mapsto u^i(x, y^i) \in R_+$, and hence, a vector valued function $u = (u^i; i \in N)$ of $X \times \prod_{i \in N} Y^i$ in R^n. Finding a Pareto optimum is equivalent to finding a maximal element of the u(Z) $= \{u(z); z \in Z, u(z) = (u^i(x, y^i); i \in N)\}$ for the ordering $\leqq$ of R^n.

In order to obtain a maximal element of this set u(Z), it is sufficient to maximize on the the set u(Z), a continuous, increasing function from R^n to R. We shall use, as the strict monotone transformation, a class of social welfare functions, f, defined on utility space U, of the utilitarian type. That is, for each $\alpha \in S^{n-1}$, $f = \Sigma \alpha_i u^i$; $f(z, \alpha) = \Sigma \alpha_i u^i(x, y^i)$ $(x, y^i) \in X \times Y^i$; $i \in N$. This f is continuous on the space.

2. Existence of Prices

For each $\alpha \in S^{n-1}$, let

$$f(\alpha) = \max_{z \in Z} f(z, \alpha).$$

$$\bar{\nu}(\alpha) = \{ z = (x, (y^i;\ i \in N)), (x, y^i) \in X \times Y^i; i \in N,\ f(z, \alpha) \geqq f(\alpha)\}$$

$$\nu(\alpha) = \{ z \in \nu(\alpha);\ f(z, \alpha) > f(\alpha)\},$$

$$\bar{\rho}(\alpha) = \{ (x, \sum_{i \in N} y^i);\ z \in \bar{\nu}(\alpha)\}$$

and

$$\rho(\alpha) = \{ (x, \sum_{i \in N} y^i);\ z \in \nu(\alpha)\} .$$

Then, $\bar{\nu}(\alpha), \nu(\alpha)$, $\bar{\rho}(\alpha)$ and $\rho(\alpha)$ are all convex, since $u^i; i \in N$ are all concave. For example, let $z^i = (x, y^i); i \in N$ and $z, z' \in \nu(\alpha)$. Then,

$$f(tz+(1-t)z', \alpha) = \sum_i \alpha_i u^i(tz^i+(1-t)z^{i\prime}) \geqq \sum \alpha_i (tu^i(z^i)+(1-t)u^i(z^{i\prime}))$$

$$= t \sum_i \alpha_i u^i(z^i) + (1-t) \sum_i \alpha_i u^i(z^{i\prime}) = tf(z,\alpha) + (1-t)f(z', \alpha) \geqq f(\alpha).$$

Let, for each $\alpha \in S^{n-1}$,

$$\mu(\alpha) = \{z \in Z;\ f(z, \alpha) = f(\alpha)\} (\neq \emptyset).$$

Then, since f is continuous on $Z \times S^{n-1}$, and Z is continuous on S^{n-1} (in fact, $Z(\alpha) = Z$ for all $\alpha \in S^{n-1}$), it follows that $\mu(.)$ is upper semi-continuous (u.s.c.) on S^{n-1}, that is, the graph of μ is closed, and f(.) is continuous on S^{n-1}. Also note, since $\nu(\alpha)$ is convex and Z is convex so is the intersection μ of these two sets.

By the separation theorem, as we have seen in Theorem 2, there exists a semi-positive price vector $p(\alpha)$ for each $\alpha \in S^{n-1}$, μ-dimensional, such that,

$$(2)\quad p(\alpha)\ (x, \sum_{i \in N} y^i) \geqq p(\alpha)\ [(0, \omega)+(x, y)],\ (x, \sum_{i \in N} y^i) \in \rho(\alpha).$$

This is true for any element of $\bar{\rho}(\alpha)$, by continuity of f on $Z \times S^{n-1}$.

3. The graph of p(.) is closed in $S^{n-1} \times R^{\mu}$.

To see this, we first normalize the price $p(\alpha)$ for each $\alpha \in S^{n-1}$, since $p(\alpha) \geq 0$, we can do this by dividing each price p by $\sum_{j \in M} p_j > 0$. Thus, hereinafter $p \in S^{\mu-1}$, where $S^{\mu-1}$ is a μ-dimensional simplex. Take, then, a sequence $\{\alpha^{\nu}\}_{\nu=1}^{\infty} \subset S^{n-1}$, so that $u \in R^{\mu}_{+}$,

$$p(\alpha^{\nu})\ u + p(\alpha^{\nu})\ (x,\ \Sigma y^{i}) \geqq p(\alpha^{\nu})[(0,\ \omega)+(x,\ y)].$$

Since S^{n-1} is bounded, there exists a subsequence $\{\alpha^{\nu^{\lambda}}\}_{\lambda=1}^{\infty}$, such that $\lim_{\lambda\to\infty} \alpha^{\nu^{\lambda}} = \alpha$, $\lambda = 1,2,\ldots,$

$$p(\alpha^{\nu^{\lambda}})\ u + p(\alpha^{\nu^{\lambda}})(x,\ \Sigma\ y^{i})_{\alpha^{\nu^{\lambda}}} \geqq p(\alpha^{\nu^{\lambda}})\ [(0,\ \omega)+(x,\ y)].$$

Here, by Bolzano-Weierstrass theorem, we can regard the corresponding subsequence $\{p(\alpha^{\nu^{\lambda}})\}$ as convergent, hence by the uniqueness of limit the original one converges to the same limit.

$$p(\alpha)\ u + p(\alpha)\ (x,\ \Sigma y^{i})_{\alpha} \geqq p(\alpha)[(0,\ \omega)+(x,\ y)] \quad u \in R_{+}.$$

Take $u = 0$, for each $\alpha \in S^{n-1}$.

Thus, p(.) maps $\alpha \in S^{n-1}$ into $S^{\mu-1}$ and is a closed mapping; $\{\alpha,\ p(\alpha)\}$ is closed in $S^{n-1} \times S^{\mu-1}$, satisfying

(3) $p(\alpha)\ (x,\ \Sigma\ y^{i})_{\alpha} \geqq p(\alpha)[(0,\ \omega)+(x,\ y)]$, $(x,\ \Sigma\ y^{i})_{\alpha} \in \bar{\rho}(\alpha)$, $(x,\ y) \in X^{p} \times Y^{p}$.

4. Existence of a Fixed Point of Weights, Prices, Allocation and Tax Rate.

We shall use an adjustment process of weights among individuals, which has been introduced by Negishi in proving the existence of a competitive equilibrium in the economy of private ownerships.

Denote the wealth of individual i by w^i, then, $z \in X^p x Y^p$,

(4) $w^i = p_y \omega^i + \max[0, \pi_i(p, z)]$,

which is positive if $\omega^i \geq 0$ since $p_y > 0$ for each $\alpha \in S^{n-1}$.

Denote also a proportionate income tax rate by t^i for each i, and assume $t^i = t$ for all i. Then,

(5) $t = p_x x / \sum_{i \in N} w^i$,

where $\sum_{i \in N} w^i = p_y \omega + \sum_{i \in N} \max[0, \pi_i(p, z)] = p_y \omega + \max[0, \pi(p, z)]$

$\pi(p, z) = (p_x, p_y)(x, y)$.

It is clear that t continuously depends on $(p, z) \in S^{\mu-1} x Z$.

Let $w = \sum_{i \in N} w^i$. If $w - p_x x = p_y \omega + \max[0, \pi(p, z)] - p_x x$ $= p_y \sum_{i \in N} y^i \geqq 0$, then, $t \leqq 1$. $p_x x \geqq 0$ implies $0 \leqq t$.

Construct a real valued function, as an adjustment process, so that,

(6) $$\beta_i = \frac{\max\{\alpha_i + V[w^i - (tw^i + p_y y^i)], 0\}}{\sum_{h \in N} \max\{\alpha_h + V[w^h - (tw^h + p_y y^h)], 0\}}, \qquad i \in N,$$

where a real number, positive, finite V is fixed and taken so as to always make the denominator positive. Cf. Negishi (1972, p.23) for this construction. The intended interpretation for this is that the non-negative weight $\alpha_i; i \in N$, which can be said to be the inverse of marginal utility of individual i's wealth (money), of individual happiness in the social welfare adjusts so that the budget constraint may be satisfied effectively.

An n-dimensional vector valued function, $\beta = (\beta_i;\ i \in N)$, on $S^{n-1} \times S^{\mu-1} \times Z \times I$, which maps $(\alpha, p, z, t) \mapsto \beta(\alpha, p, z, t) \in S^{\mu-1}$, is continuous, since so is β_i, for each $i \in N$.

A multi-valued function, p(.) is, as well as a closed mapping, convex-valued for each α, because of the linear inequality system (3) for each α; $\mu(.)$ is also u.s.c. and convex-valued, as we have seen above.

A real valued function, t, is continuous on $S^{n-1} \times Z$, mapping into the closed unit interval $I = [0, 1]$.

Hence, the Cartesian product of mappings $p(.) \times \mu(.) \times t(.) \times \beta(.)$ maps a point of the product of convex and compact sets $S^{n-1} \times S^{\mu-1} \times Z \times I$ into $S^{n-1} \times S^{\mu-1} \times Z \times I$, with non-void convex images.

Apply Kakutani's fixed point theorem for a multi-valued mapping to get a fixed point$(\alpha°, p°, z°, t°)$, such that $(\alpha°, p°, z°, t°) \in \Phi(\alpha°, p°, z°, t°)$, where Φ is the product of the mappings. See Berge. Then, we have

$$(7) \qquad \alpha_i^\circ = \frac{\max\{\alpha_i^\circ + V[w^{i\circ} - t^\circ w^{i\circ} - p_y^\circ y^{i\circ}],\ 0\}}{\sum_{h \in N} \max\{\alpha_h^\circ + V[w^{h\circ} - t^\circ w^{h\circ} - p_y^\circ y^{h\circ}],\ 0\}}, \quad i \in N.$$

$$(8) \qquad \sum_{h \in N} (w^{h\circ} - t^\circ w^{h\circ} - p_y^\circ y^{h\circ}) = 0.$$

To see this, note that $w° = p_y^\circ \omega + \pi(p°, z°)$ and $\pi(p°, z°) = p_x^\circ x° + p_y^\circ y°$ ($\geqq 0$ by B.1) lead to the equation.

Suppose $\alpha_i^\circ = 0$. Then, $w^{i\circ} - t^\circ w^{i\circ} - p_y^\circ y^{i\circ} \leqq 0$. This inequality does not hold for every $i \in N$, since otherwise the sum contradicts the above equality (8). Suppose, further, for some $k \in N$, $\alpha_k^\circ > 0$ and $w^{k\circ} - t^\circ w^{k\circ} - p_y^\circ y^{k\circ} < 0$. This contradicts the above equality (8); $0 = w° - t°w° - p_y^\circ \Sigma_i y^{i\circ} < 0$. We shall see the last contradiction in what follows.

Here, observe the denominator of (7)

$$\sum_{h\in N}\max\{\alpha_h^\circ+V[w^{h\circ}-t^\circ w^{h\circ}-p_y^\circ y^{h\circ}],\ 0\} \geqq 1 + V \sum_{h\in N}(w^{h\circ}-t^\circ w^{h\circ}-p_y^\circ y^{h\circ})$$

and hence, for $k\varepsilon N$

$$\alpha_k^\circ \sum_{h\varepsilon N(\alpha_h^\circ > 0)}(w^{h\circ}-t^\circ w^{h\circ}-p_y^\circ y^{h\circ}) \leqq w^{k\circ}-t^\circ w^{k\circ}-p_k^\circ y^{k\circ} < 0$$

which, in turn, implies, since $\alpha_k^\circ > 0$,

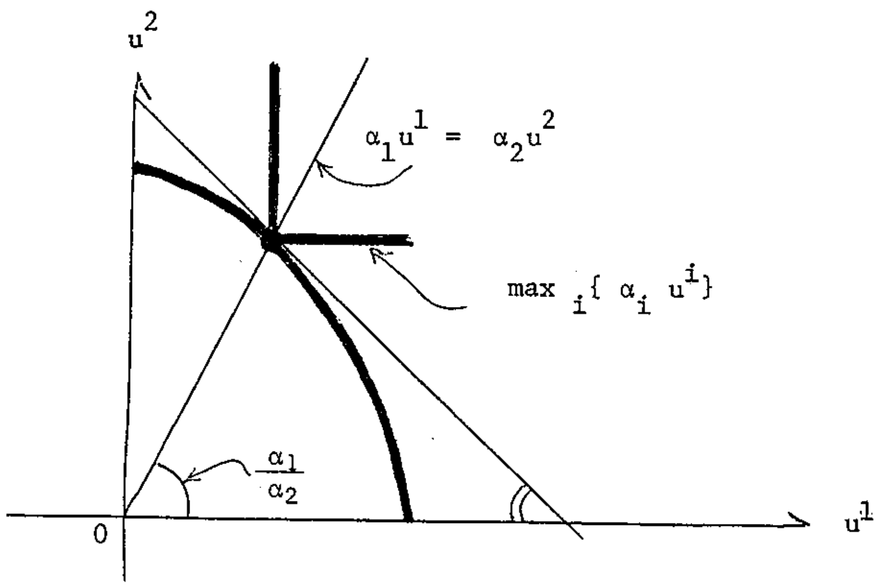
$$\sum_{h\varepsilon N(\alpha_h^\circ > 0)} (w^{h\circ}-t^\circ w^{h\circ}-p_y^\circ y^{h\circ}) < 0,$$

where $N(\alpha_h^\circ > 0) = [h\varepsilon N:\ \alpha_h^\circ > 0]$.

Thus, the supposition is negated and $\alpha_k^\circ > 0 \rightarrow w^{k\circ}-t^\circ w^{k\circ}-p_y^\circ y^{k\circ} \geqq 0$. Further, if the last inequality strictly holds for each such k, it would contradict A. 3 (monotonicity). Thus, for all $i\varepsilon N$, $\alpha_i^\circ \geqq 0 \rightarrow w^{i\circ}-t^\circ w^{i\circ}-p_y^\circ y^{i\circ} \leqq 0$, in fact, by (8), $w^{i\circ}-t^\circ w^{i\circ}-p_y^\circ y^{i\circ} = 0$.

(9) $w^{i\circ} - t^\circ w^{i\circ} - p_y^\circ y^{i\circ} = 0,\ i\varepsilon N.$

(10) $\begin{cases} t^\circ w^\circ = p_x^\circ x^\circ \geqq 0\ ;\ t^\circ = 0 \leftrightarrow x^\circ = 0 \text{ (since } p_x^\circ \geq 0)\ ;\ t^\circ = 1 \leftrightarrow \\ p_y^\circ \sum_i y^{i\circ} = 0 \text{ (since } p_y^\circ > 0) \text{ or } y^{i\circ} = 0 \text{ if } y^{i} \geqq 0). \end{cases}$



5. The Fixed Point is a Pareto Optimum and an Equilibrium

Let $z \in Z$.

We say that $\hat{z}$ is a weak Pareto optimum, if there is no $z \in Z$ such that $u(z) > u(\hat{z})$. A Pareto optimum implies a weak Pareto optimum.

Suppose u^i is positively valued for each $z \in Z$.

Suppose $\hat{z} \in Z$ satisfies the weak Pareto condition. Then, we define v^i and α_i so that $\alpha_i u^i(\hat{z}) = v^i$, $\Sigma \alpha_i = 1$, and then, $\alpha \in S^{n-1}$, $\alpha > 0$ and $\hat{z}$ is a solution to $\max_{z \in Z} \min_{i \in N} \{\alpha_i u^i(z)\}$. For, otherwise, there exists $z \in Z$ such that for some $i \in N$ $\alpha_i u^i(z) > \alpha_i u^i(\hat{z}) = v^i$. Conversely, there would exist $z \in Z$ such that $u(z) > u(\hat{z})$. This means that $\hat{z}$ is not a solution to the problem for every $\alpha \in S^{n-1}$.

For $\alpha > 0$, if $\hat{z}$ is a solution to max $f(z, \alpha)$, $z \in Z$, then, $\hat{z}$ is a Pareto optimum, since, otherwise, there exists $z \in Z$ such that $u(z) \geq u(\hat{z})$, hence, $\alpha u(z) > \alpha u(\hat{z})$; a contradiction.

Suppose that $\omega^i \geq 0$, for each $i \in N$, then, $w^i > 0$, $i \in N$.

Suppose that for each $i \in N$, $(x^\circ, y^i) \succsim_i (x, 0)$, $y^i \geq 0$.

Then, if $y^{i\circ} = 0$, there exists a better point $(x', y^{i\prime})$ by A. 5 so that $u^i(x', y^{i\prime}) \geq u^i(x^\circ, 0)$ hence, $y^{i\circ} \geq 0$. Thus, $t^\circ < 1$.

Without loss of generality we take; $u^i(x, 0) = 0 < u^i(x', y^i)$, $y^i \geq 0$.

Now, $\alpha^\circ > 0$, and from the above argument z° is a Pareto optimum.

(9) and the result that z° is a Pareto optimum imply the second condition (ii) for $z^\circ = (x^\circ, y^{i\circ}; i \in N)$ and $t^\circ w^{i\circ}$ $(= t^i)$. Thus, the fixed point $(\alpha^\circ, p^\circ, z^\circ, t^\circ)$ is a public competitive equilibrium.

Let

A. 7 (Indispensability of Private Consumption Good): There exists a consumption plan $(x', y^i) \varepsilon X \times Y^i$ for each $i \varepsilon N$ such that $(x', y^i) \succsim_i (x, 0)$, $y^i \geq 0$.

We have proved;

Theorem 4: Suppose preference $\succsim_i$ is convex in $X \times Y^i$ for each $i \varepsilon N$, so that each utility function u^i is concave in $X \times Y^i$. Suppose also $X \times Y^i \subset R_+^{\mu_1} \times R_+^{\mu_2}$, and $\omega^i \geq 0$ $i \varepsilon N$. Then, under A. 1-7, and B. 1-5, there exists a public competitive equilibrium under the linear income tax system, which is a Pareto optimum.

References

Berge, C. : Topological Spaces 1963 1st English Edition, Oliver & Boyd, Edinburgh and London p.168-, p. 115-.p. 157-.

Debreu, G.:" New Concepts and Techniques for Equilibrium Analysis," International Economic Review 3(1962) pp. 257-273.

Debreu, G. The Theory of Value 1959, 1965. New York, John Wiley.

Dorfman, R. " General Equilibrium with Public Goods " 1969 Public Economics ed. Guitton et al.

Foley, D. " Lindahl's Solution and the Core of an Economy with Public Good." Econometrica 38 (1970) pp. 66-72.

Foley, D. " Resource Allocation and the Public Sector." Yale Economic Essays 7(1967)

Johansen, L. " Some Notes on the Lindahl Theory of Determination of Public Expenditures." International Economic Review 4 (1963), pp. 346-357.

Lindahl, E. " Just Taxation - A Positive Solution." Classics in the Theory of Public Finance, R. A. Musgrave and A. T. Peacock (eds.) London, 1958.

Milleron, J-C. : " Theory of Value with Public Goods: A Survay Article " Journal of Economic Theory 5(1972) pp. 419-477.

Negishi, T. General Equilibrium Theory and International Trade (1972) Chapter 1 Part I.

Samuelson, P. A. " The Pure Theory of Public Expenditure," Review of Economics and Statistics 36(1954) pp. 387-389.

Samuelson, P. A. " Aspects of Public Expenditure Theories," Review of Economics and Statistics 40(1958) pp. 332-338.

Samuelson, P. A. " Pure Theory of Public Expenditure and Taxation," Public Economics eds by J. Margolis et al. pp. 98-123.