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A NEW VARIABLE DIMENSION ALGORITHM FOR THE FIXED POINT PROBLEM

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1. INTRODUCTION

Since van der Laan and Talman [10-13] developed a class of new restart algorithms, variable dimension algorithms, for the fixed point problem, several new variable dimension algorithms and various interpretations have been proposed (Freund [4], Kojima and Yamamoto [6,7], Reiser [14], Todd [18,19], Todd and Wright [20], and Wright [21]. See also the bibliographies in van der Laan [9] and Talman [16]). The purpose of this paper is to develop a new variable dimension algorithm for solving the system of equations (1.1):

(1.1) $f(x) = 0, \quad x \in R^n,$

where $f: R^n \longrightarrow R^n$ is a continuous mapping. The algorithm approximately solves the sequence of subproblems consisting of the first k equations of (1.1). However the dimension k does not monotonously increase.

Throughout this paper we shall call a convex polyhedral set C in some Euclidean space a cell or an m-cell if we want to clarify its dimension. We shall denote by ∂C the set of boundary points of C relative to the affine subspace spanned by C. For two cells B and C we write $B < C$ if B is a face of C. For a collection L of m-cells $|L|$ denotes the union of all m-cells of L.

The organization of this paper is as follows. In Section 2 we shall develop a basic model of the new variable dimension algorithm. This model is a generalization of the primal-dual pair of subdivided manifolds presented in Kojima and Yamamoto [6] as a unifying model of variable dimension algorithms. We also present a convergence condition. In Section 3 we shall investigate the behavior of the algorithm and give a formal description of the algorithm. Section 4 is devoted to illustration of the algorithm. Some remarks are found in Section 5.

2. BASIC MODEL OF THE ALGORITHM

Let $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)^T$ be an initial guess of a zero of $f: R^n \to R^n$. For $\alpha \in \{+1,-1\}$ and $1 \leq k \leq n$ define

$$X(k,\alpha) = \{ x \in R^n : \alpha(x_k - x_k^0) \geq 0, \ x_j - x_j^0 = 0 \quad \text{for } j > k \}$$

$$Y(k,\alpha) = \{ y \in R^n : \alpha y_k \geq 0, \quad y_j = 0 \quad \text{for } j < k \}.$$

Then it is readily seen that

$$\text{(2.1)} \qquad \begin{aligned} \partial X(k,\alpha) &= X(k-1,+1) \cup X(k-1,-1) \\ \partial Y(k,\alpha) &= Y(k+1,+1) \cup Y(k+1,-1) \end{aligned}$$

for any $\alpha \in \{+1,-1\}$. Thus we say that $X(j,\beta)$ (resp. $Y(j,\beta)$) is an pseudo face of $X(i,\alpha)$ (resp. $Y(i,\alpha)$) if either $i > j$ (resp. $i < j$) or $(i,\alpha) = (j,\beta)$ and denote

$$X(i,\alpha) \succ X(j,\beta) \quad (\ Y(i,\alpha) \succ Y(j,\beta) \ ).$$

Now define $Z(k,\alpha)$ as the Cartesian product of $X(k,\alpha)$ and $Y(k,\alpha)$:

$$\begin{aligned} Z(k,\alpha) &= X(k,\alpha) \times Y(k,\alpha) \\ &= \{ (x,y) : x \in X(k,\alpha), \ y \in Y(k,\alpha) \}, \end{aligned}$$

then we see that $Z(k,\alpha)$ is an $(n+1)$-cell. If we employ the convention that

$$X(0,\alpha) = \{x^0\}, \quad Y(n+1,\alpha) = \{ 0 \}$$

for any $\alpha \in \{+1,-1\}$, then the next lemma follows directly from the definition of $Z(k,\alpha)$ and (2.1).

Lemma 2.1. For any $\alpha \in \{+1,-1\}$ and any $1 \leq k \leq n$

$$\begin{aligned} \partial Z(k,\alpha) = &(X(k,\alpha) \times Y(k+1,+1)) \cup (X(k,\alpha) \times Y(k+1,-1)) \\ &\cup (X(k-1,+1) \times Y(k,\alpha)) \cup (X(k-1,-1) \times Y(k,\alpha)). \end{aligned}$$

Let $0 \leq i < j \leq n$, then by (2.1) and Lemma 2.1 above we obtain that

$$X(i,\alpha) \times Y(j,\beta) \subseteq \partial Z(k,\gamma)$$

if either $i < k$ or $j > k$ and

$$(2.2)\qquad \begin{aligned} &X(i,\alpha) \prec X(k,\gamma), \text{ and} \\ &Y(j,\beta) \prec Y(k,\gamma). \end{aligned}$$

Then we shall say that $X(i,\alpha) \times Y(j,\beta)$ is a pseudo face of $Z(k,\gamma)$ if (2.2) holds (we do not exclude the case where $i = j = k$) and also denote

$$X(i,\alpha) \times Y(j,\beta) \prec Z(k,\gamma).$$

For the dimension of $X(i,\alpha) \times Y(j,\beta)$ is $(n+1)-(j-i)$, we may call it an $(n+1)-(j-i)$-pseudo face to clarify the dimension. Now define L and $\bar{L}$:

$$L = \{\, Z(k,\alpha) : \alpha \in \{+1,-1\},\quad 1 \le k \le n \,\}$$

$$\bar{L} = \{\, C : \quad C \prec Z(k,\alpha) \in L \,\},$$

and we shall call a member of $\bar{L}$ a pseudo face of L.

<u>Lemma 2.2.</u> Let $Z(i,\alpha)$ and $Z(j,\beta)$ be two distinct $(n+1)$-cells of L with $Z(i,\alpha) \cap Z(j,\beta) \neq \emptyset$. Then $Z(i,\alpha) \cap Z(j,\beta)$ is either an n-pseudo face of L or the union of several pseudo faces of L whose dimensions are less than n.

proof. If $i = j$ and $\alpha \neq \beta$, it is clear that

$$Z(i,\alpha) \cap Z(j,\beta) = \bigcup \{\, X(i-1,\gamma) \times Y(i+1,\delta) : \gamma,\delta \in \{+1,-1\} \,\},$$

which is the union of pseudo faces of dimension less than n. If $i < j$, then

$$Z(i,\alpha) \cap Z(j,\beta) = X(i,\alpha) \times Y(j,\beta),$$

which is an $(n+1)-(j-i)$-pseudo face. Q.E.D.

<u>Lemma 2.3.</u> Let $X(k,\alpha) \times Y(k+1,\beta)$ be an n-pseudo face of L. Then

$$(2.3)\qquad X(k,\alpha) \times Y(k+1,\beta) \prec Z(i,\gamma) \in L$$

implies that $Z(i,\gamma)$ is either $Z(k,\alpha)$ or $Z(k+1,\beta)$.

proof. Suppose (2.3), then i must be equal to either k or $k+1$.

If $i = k$, $X(k,\alpha) \prec X(i,\gamma)$ implies that $\gamma = \alpha$, and if $i = k+1$, $Y(k+1,\beta) \prec Y(i,\gamma)$ implies that $\gamma = \beta$. Q.E.D.

Therefore we can illustrate an (n+1)-cell $Z(k,\alpha)$ surrounded by four or less (n+1)-cells of L as shown in Figure 1. Let ∂L be the collection of all n-pseudo faces each of which lies in exactly one (n+1)-cell of L. Then we obtain the following lemma.

Lemma 2.4.

(2.4) $$\partial L = \{ \{x^0\} \times Y(1,+1),\ \{x^0\} \times Y(1,-1),\ X(n,+1) \times \{0\},\ X(n,-1) \times \{0\} \},$$

(2.5) $$|\partial L| = (\{x^0\} \times R^n) \cup (R^n \times \{0\}).$$

proof. Recall that $X(0,\alpha) = \{x^0\}$ and $Y(n+1,\alpha) = \{0\}$ for any $\alpha \in \{+1,-1\}$. Then (2.4) follows from Lemma 2.3. The second assertion (2.5) is a direct consequence of (2.4) together with the fact that $Y(1,+1) \cup Y(1,-1) = X(n,+1) \cup X(n,-1) = R^n$. Q.E.D.

This lemma shows that the collection L of (n+1)-cells is very similar to the subdivided (n+1)-manifold in Kojima and Yamamoto [6, Section 5] generated by a pair of conical subdivisions of R^n to present an interpretation of the variable dimension algorithm by van der Laan and Talman [10-13].

For a triangulation T of R^n define

$$\bar{T} = \{ \tau : \tau < \sigma \text{ for some } \sigma \in T \}$$

and its restriction to a convex set $X \subseteq R^n$

$$\bar{T}|X = \{ \tau \in \bar{T} : \tau \subseteq X,\ \dim \tau = \dim X \}.$$

In the sequel we are interested in triangulations of R^n such that

(2.6) mesh size $\delta = \sup_{\sigma \in T} \sup \{ \|u - v\| : u, v \in \sigma \}$ is bounded,

(2.7) (locally finite): each point $x \in |T|$ has a neighborhood which intersects finitely many simplices of T, and

(2.8) $\bar{T}|X(k,\alpha)$ is a simplicial refinement of $X(k,\alpha)$ for each $\alpha \in \{+1,-1\}$ and $0 \le k \le n$.

Let T' be a triangulation of R^n whose restriction to each orthant provides a refinement of the orthant. Then it is readily seen that $T' + x^0 = \{ \sigma + x^0 : \sigma \in T' \}$ satisfies the condition (2.8). Thus we may employ various triangulations, for example K_1, J_1 and K', as T' (see Todd [17]).

Now define

$$M = \{ \sigma \times Y(k,\alpha): 1 \le k \le n,\ \alpha \in \{+1,-1\}, \sigma \in \bar{T}|X(k,\alpha) \},$$

then M is a locally finite refinement of L. We shall also say that $\tau \times Y(j,\beta)$ is a pseudo face of $\sigma \times Y(k,\alpha)$ and denote

$$\tau \times Y(j,\beta) \prec \sigma \times Y(k,\alpha)$$

if $\tau < \sigma$ and $Y(j,\beta) \prec Y(k,\alpha)$. We also define

$$\bar{M} = \{ C : C \prec \sigma \times Y(k,\alpha) \in M \}$$

and call each member of $\bar{M}$ a pseudo face of M. Then M inherits the properties of L in Lemma 2.1–2.4 as shown in the following lemmata, whose proofs are very similar to those of Lemma 2.1–2.4 and will be omitted.

Lemma 2.5. For $\sigma \times Y(k,\alpha) \in M$

$$\partial(\sigma \times Y(k,\alpha)) = (\sigma \times Y(k+1,+1)) \cup (\sigma \times Y(k+1,-1)) \cup \bigcup \{ \tau \times Y(k,\alpha): \tau < \sigma,\ \dim \tau = k-1\}.$$

Lemma 2.6. The intersection of two distinct (n+1)-cells of M is either an n-pseudo face of M or the union of several pseudo faces of M whose dimensions are less than n unless it is empty.

Lemma 2.7.

(2.9) Let k be an integer such that $1 \le k \le n-1$ and $\tau \times Y(k+1,\alpha)$ be an n-pseudo face of M such that $\tau \subseteq \partial X(k+1,\alpha)$. If

$$\tau \times Y(k+1,\alpha) \prec C \in M,$$

then C is either $\tau \times Y(k,\beta)$ for $\beta \in \{+1,-1\}$ such that $\tau \subseteq X(k,\beta)$ or $\sigma \times Y(k+1,\alpha)$ for the unique (k+1)-simplex $\sigma \in \bar{T}|X(k+1,\alpha)$ having τ as a face.

(2.10) Let k be an integer such that $1 \leq k \leq n-1$ and $\tau \times Y(k+1,\alpha)$ be an n-pseudo face of M such that $\tau \nsubseteq \partial X(k+1,\alpha)$. If

$$\tau \times Y(k+1,\alpha) \prec C \in M,$$

then

$$C = \sigma \times Y(k+1,\alpha)$$

for some (k+1)-simplex $\sigma \in \bar{T}|X(k+1,\alpha)$ having τ as a face.

(2.11) The n-pseudo face $X(0,\beta) \times Y(1,\alpha) = \{x^0\} \times Y(1,\alpha)$ of M lies in exactly one (n+1)-cell $\sigma \times Y(1,\alpha)$ of M, where σ is the unique 1-simplex of $\bar{T}|X(1,\alpha)$ having $\{x^0\}$ as a face.

(2.12) Let σ be an n-simplex of $\bar{T}|X(n,\alpha)$. Then the n-pseudo face $\sigma \times Y(n+1,\beta) = \sigma \times \{0\}$ lies in exactly one (n+1)-cell $\sigma \times Y(n,\alpha)$ of M.

Let ∂M be defined as the collection of all n-pseudo faces of M each of which lies in exactly one (n+1)-cell of M, then the next lemma immediately follows from Lemma 2.7.

Lemma 2.8.

(2.13) $$\partial M = \{\{x^0\} \times Y(1,+1),\ \{x^0\} \times Y(1,-1)\} \cup \{\sigma \times \{0\}:\ \sigma \in T\},$$

(2.14) $$|\partial M| = (\{x^0\} \times R^n) \cup (R^n \times \{0\}).$$

Let $F: |T| \rightarrow R^n$ be a simplicial approximation of $f: R^n \rightarrow R^n$ with respect to the triangulation T, i.e., for $x \in \sigma = co\{v^0, v^1, \cdots, v^n\} \in T$

$$F(x) = \sum_{i=0}^{n} \lambda_i\ f(v^i),$$

where $x = \sum_{i=0}^{n} \lambda_i v^i$, $\sum_{i=0}^{n} \lambda_i = 1$, $\lambda_i \geq 0$ for $i = 0,1,\cdots,n$. Our purpose is to find an approximate solution of (1.1) by solving the system of piecewise linear equations (2.15):

(2.15) $\quad F(x) = 0, \quad x \in |T|$.

Let $H: |M| \rightarrow R^n$ be a piecewise linear mapping such that

$$H(x,y) = F(x) + y$$

and consider the system of piecewise linear equations (2.16):

(2.16) $\quad H(x,y) = 0, \quad (x,y) \in |M|$.

Definition 2.9. We say that a vector $c \in R^n$ is a regular value of the piecewise linear mapping $H: |M| \rightarrow R^n$ if $B \in \bar{M}$ and $H(B) \ni c$ always imply that $\dim H(B) \geq n$.

The reader might notice that this definition of regular value is slightly different from that in Eaves [2]; it, however, creates no essential difference in further discussions. In fact by the similar argument in Eaves [2], and Eaves and Scarf [3] together with Lemma 2.5-2.7, we can see that almost every vector $c \in R^n$ is a regular value and that if c is a regular value, the solution set $\{ (x,y) \in |M|: H(x,y) = c \}$ is a disjoint union of connected subdivided 1-manifolds, each of which is either a path or a loop (see Eaves [2] for the definitions of subdivided manifold, path and loop). Furthermore each loop has no intersection with $|\partial M|$. Here let us assume that zero is a regular value of $H: |M| \rightarrow R^n$. By the definition of H and M $(x^0, y^0) = (x^0, -F(x^0))$ is a solution of (2.16). Let S be a connected component of the set of solutions to (2.16) having (x^0, y^0). Since $(x^0, y^0) \in |\partial M|$, S forms a path. If, in addition, S is bounded, S has another endpoint, say (x^1, y^1), in $|\partial M|$. By Lemma 2.8 we have either

(2.17) $(x^1, y^1) \in \{x^0\} \times R^n$

or

(2.18) $(x^1, y^1) \in R^n \times \{0\}$.

If (2.17) occurs, $(x^1, y^1) = (x^0, -F(x^0)) = (x^0, y^0)$ since (x^1, y^1) is in S. This is contrary to $(x^1, y^1) \neq (x^0, y^0)$. Thus (2.18) occurs, i.e., $y^1 = 0$. Hence x^1 is a solution of (2.15). Therefore by following the path S from the initial boundary point (x^0, y^0) we can find an approximate solution x^1 of (1.1). Replacing x^0 by x^1 and using a finer triangulation we can apply the same procedure to obtain an approximate solution of (1.1) with higher accuracy.

In order to ensure that the path S is bounded we shall assume the following condition.

Condition 2.10. There exist a bounded set $U \subseteq R^n$ containing x^0 and a positive number δ_0 such that $x \in \text{bd.}U$ always implies the existence of an index j such that

$$(x_j - x_j^0)\ f_j(v) > 0$$

for any point v such that $\|v - x\| \leq \delta_0$, where bd.U is the set of boundary points of U.

Theorem 2.11. Let δ be the mesh size of the triangulation T. If f satisfies Condition 2.10 and $\delta \leq \delta_0$, then the path S is bounded.

proof. It is sufficient to show that

$$S' = \{\ x \in R^n\colon (x,y) \in S \text{ for some } y \in R^n \}$$

is bounded. Assume on the contrary that S' is unbounded. Then there exists an $(\bar{x},\bar{y}) \in S$ such that $\bar{x} \in \text{bd.}U$. Let $\sigma = \text{co}\{v^0, v^1, \cdots, v^n\}$ be an n-simplex of T containing $\bar{x}$, then

$$F(\bar{x}) = \sum_{i=0}^{n} \lambda_i\ f(v^i)$$

$$\| v^i - \bar{x} \| \leq \delta \leq \delta_0 \quad \text{for } i = 0,1,\cdots,n,$$

where $\bar{x} = \sum_{i=0}^{n} \lambda_i v^i$, $\sum_{i=0}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 0,1,\cdots,n$.

Hence applying Condition 2.10 there exists an index j such that

$$(\bar{x}_j - x_j^0)\ f_j(v^i) > 0$$

for $i = 0,1,\cdots,n$, consequently we have that

$$(\bar{x}_j - x_j^0)\ F_j(\bar{x}) = \sum_{i=0}^{n} \lambda_i\ (\bar{x}_j - x_j^0)\ f_j(v^i) > 0.$$

Since $(\bar{x},\bar{y}) \in S \subseteq |M|$, we easily see

$$(\bar{x}_j - x_j^0)\ \bar{y}_j \geq 0.$$

Therefore

$$(\bar{x}_j - x_j^0)\ (\bar{y}_j + F_j(\bar{x})) > 0,$$

which is contrary to the fact that $(\bar{x},\bar{y}) \in S$. Q.E.D.

3. COMPUTATION OF THE PATH

In this section we shall explain the computational procedure for tracing the path S. As tracing S we alternately encounter n-pseudo faces and (n+1)-cells of M, however, it will make the behavior of the algorithm clearer to investigate n-pseudo faces of L which we may cross along the path S. Let $(x,y) \in S \cap (X(k,\alpha) \times Y(k+1,\beta))$ for some $0 \leq k \leq n$ and some $\alpha, \beta \in \{+1,-1\}$. Then by the definitions of $X(k,\alpha)$ and $Y(k+1,\beta)$ we have that

$$F_j(x) = 0 \qquad \text{for } j \leq k$$

(3.1) $$x_j - x_j^0 = 0 \qquad \text{for } j > k .$$

This means that x is an approximate solution of the k-subproblem

(3.2) $$f_j(x) = 0 \qquad \text{for } j \leq k$$

consisting of the first k equations of (1.1) under the additional constraints (3.1). In this way every time we encounter an n-pseudo face of L, we obtain an approximate solution of a k-subproblem. As Lemma 2.1 shows each (n+1)-cell $Z(k,\alpha)$ of L has four distinct n-pseudo faces: $X(k,\alpha) \times Y(k+1,+1)$, $X(k,\alpha) \times Y(k+1,-1)$, $X(k-1,+1) \times Y(k,\alpha)$ and $X(k-1,-1) \times Y(k,\alpha)$. We say that $X(k,\alpha) \times Y(k+1,+1)$ (resp. $X(k-1,+1) \times Y(k,\alpha)$) is opposite to $X(k,\alpha) \times Y(k+1,-1)$ (resp. $X(k-1,-1) \times Y(k,\alpha)$). Now suppose $S \cap Z(k,\alpha) \neq \emptyset$, then $S \cap Z(k,\alpha)$ has two endpoints in $\partial Z(k,\alpha)$. The following three cases may occur:

(3.2) two endpoints are in an n-pseudo face of $Z(k,\alpha)$,

(3.3) two endpoints are in distinct n-pseudo faces of $Z(k,\alpha)$ which are opposite to each other,

(3.4) two endpoints are in distinct n-pseudo faces of $Z(k,\alpha)$ which are not opposite to each other.

If either (3.2) or (3.3) occurs, two endpoints provide two distinct approximate solutions of a subproblem, (k-1)- or k-subproblem. If (3.4) occurs, one endpoint corresponds to an approximate solution of the (k-1)-subproblem and the other of the k-subproblem. In this way the dimension of the subproblem being solved varies as we trace the path S. The three cases above are illustrated in Figure 1.

Suppose that $(x,y) \in S \cap Z(k,\alpha)$ and let $\sigma = \mathrm{co}\{v^0, v^1, \cdots, v^k\}$ be a k-simplex such that $x \in \sigma \in \bar{T}|X(k,\alpha)$. Then there exists a solution $(\lambda,y) \in R^{(k+1)+n}$ of the system (3.5):

$$(3.5) \qquad \begin{bmatrix} e^T \\ \hline f(v^0) \cdots\cdots f(v^k) \end{bmatrix} \begin{bmatrix} \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ \hline I \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix}$$

$$\lambda \geq 0$$

$$y \in Y(k,\alpha),$$

where $e \in R^{k+1}$ is a vector of one's and I is an $n \times n$ identity matrix. Since

$$y_j = 0 \qquad \text{for } j < k$$

$$\alpha y_k \geq 0$$

and y_j is not restricted at all for any $j > k$, (3.5) has a solution if and only if the following smaller system $(3.6)_k$ has a solution $(\lambda, z_k) \in R^{(k+1)+1}$

$$(3.6)_k \qquad \left[\begin{array}{c} e^T \\ \hline f^{(k)}(v^0)\cdots f^{(k)}(v^k) \\ \\ \end{array}\right] \left[\begin{array}{c} \\ \lambda \\ \\ \end{array}\right] + \left[\begin{array}{c} \\ 0 \\ \hline \alpha \end{array}\right] z_k = \left[\begin{array}{c} 1 \\ \hline 0 \\ \\ \end{array}\right]$$

$$\lambda \geq 0$$

$$z_k \geq 0,$$

where $f^{(k)}(v^i)$ is the k-vector consisting of the first k components of $f(v^i)$. Therefore in the midway of computing the path S we have only to evaluate the first several components of the function value f at vertices of T and to handle the smaller system (3.6) instead of (3.5).

Now the variable dimension algorithm is formally stated as follows. Here for a simplex $\tau = \text{co}\{v^0, v^1, \cdots, v^k\}$ and a point $v \in R^n$ we denote by $\tau + v$ (resp. $\tau - v$) the simplex consisting of the vertex set $\{v^0, v^1, \cdots, v^k\} \cup \{v\}$ (resp. $\{v^0, v^1, \cdots, v^k\} \setminus \{v\}$).

Step 0 (initialization):

$$\tau = \{x^0\}, \quad \lambda_0 = 1$$

$$y_1 = -f_1(x^0), \quad z_1 = |y_1|, \quad \alpha = \text{sign } y_1$$

$$k = 1.$$

Step 1:

Find a vertex v^+ such that $\sigma = \tau + v^+$ is a k-simplex of $\bar{T} | X(k,\alpha)$ (Note that v^+ is uniquely determined).

Step 2:

In the system $(3.6)_k$ increase λ^+, the variable corresponding to the column $f^{(k)}(v^+)$. If z_k vanishes, go to Step 3. If some λ_j vanishes, go to Step 4.

Step 3 (dimension increasing):

If k = n, stop. Otherwise,

$$\tau = \sigma$$

$$y_{k+1} = -\{\sum_{i=0}^{k-1} \lambda_i \ f_{k+1}(v^i) + \lambda^+ \ f_{k+1}(v^+)\}, \quad z_{k+1} = |y_{k+1}|$$

$$\alpha = \text{sign } y_{k+1}, \quad k = k + 1 \text{ and go to Step 1.}$$

Step 4:

$$\tau = \sigma - v_j.$$

If $\tau \subseteq X(k-1,\beta)$ for some $\beta \in \{+1,-1\}$, then $\sigma = \tau$, $\alpha = \beta$ and go to Step 5. Otherwise, $v^- = v^j$ and go to Step 6.

Step 5 (dimension decreasing):

In the system $(3.6)_{k-1}$ increase z_{k-1} until some λ_i vanishes.

$$\tau = \sigma - v^i, \ v^- = v^i, \ k = k - 1$$

and go to Step 6.

Step 6 (replacing a vertex):

Find a vertex v^+ which can be replaced with v^- in order to obtain another k-simplex of $\bar{T}|X(k,\alpha)$ having τ as a face. $\sigma = \tau + v^+$ and go to Step 2.

In this algorithm we need inverse matrices of basis matrices of various dimensions. We shall close this section by explaining simple procedures for obtaining the inverse matrix of a new basis matrix whose dimension differs by unity from that of the old one. Suppose that z_k vanishes in Step 2, then we have the inverse matrix of the $(k+1) \times (k+1)$ basis matrix B:

$$B = \begin{bmatrix} e^T \\ \hdashline f^{(k)}(v^0) \cdots f^{(k)}(v^k) \end{bmatrix},$$

where $\sigma = \mathrm{co}\{v^0, v^1, \cdots, v^k\}$ is a k-simplex of $\bar{T}|X(k,\beta)$ for some $\beta \in \{+1,-1\}$. When we return to Step 2 after proceeding to Step 3, we need the inverse of the following $(k+2) \times (k+2)$ basis matrix $\bar{B}$:

$$\bar{B} = \left[\begin{array}{c:c} e^T & \\ \hdashline f^{(k+1)}(v^0) \cdots f^{(k+1)}(v^k) & 0 \\ \hdashline & \alpha \end{array}\right] = \left[\begin{array}{c:c} B & 0 \\ \hdashline b^T & \alpha \end{array}\right],$$

where $b^T = (f_{k+1}(v^0), \cdots, f_{k+1}(v^k))$. Since $\alpha^2 = 1$, $(\bar{B})^{-1}$ is easily obtained by

$$(\bar{B})^{-1} = \left[\begin{array}{c:c} B^{-1} & 0 \\ \hdashline -\alpha b^T B^{-1} & \alpha \end{array}\right].$$

Next consider the case where some λ_j vanishes in Step 2. Then we have the inverse matrix of the $(k+1) \times (k+1)$ matrix C at hand:

$$C = \left[\begin{array}{c|c} e^T & \\ \hline & \\ D & 0 \\ & \\ \hline & \alpha \end{array}\right] ,$$

where

$$D = \left[f^{(k)}(v^0) \cdots f^{(k)}(v^{j-1}) \;\; f^{(k)}(v^{j+1}) \cdots f^{(k)}(v^k) \right] .$$

If $\tau = \sigma - v^j$ is not contained in $X(k-1,\beta)$ for any $\beta \in \{+1,-1\}$, we proceed to Step 6 and have only to replace a vertex according to the ordinary replacing rule of $\bar{T}|X(k,\alpha)$. On the other hand if τ lies in some $X(k-1,\beta)$, the inverse matrix of the following $k \times k$ matrix $\bar{C}$ is required in Step 5:

$$\bar{C} = \left[\begin{array}{c} e^T \\ \hline \\ \bar{D} \\ \\ \end{array}\right] ,$$

where

$$\bar{D} = \left[f^{(k-1)}(v^0) \cdots f^{(k-1)}(v^{j-1}) \;\; f^{(k-1)}(v^{j+1}) \cdots f^{(k-1)}(v^k) \right] .$$

Since the matrix C is rewritten as

$$\left[\begin{array}{c|c} & \\ \bar{C} & 0 \\ & \\ \hline c^T & \alpha \end{array}\right]$$

by suitably defining the k-vector c, it is readily seen that the leading principal $k \times k$ submatrix of C^{-1} is the required inverse matrix $(\bar{C})^{-1}$.

4. EXAMPLE FOR ILLUSTRATION

Let $f: R^2 \to R^2$ be a continuous mapping and consider the system of equations:

(4.1) $\quad f(x) = 0, \quad x \in R^2.$

In Figure 2 we illustrate

$$S^1 = S_1^1 \cup S_2^1 = \{ x \in R^2 : \quad f_1(x) = 0 \}$$
$$S^2 = \{ x \in R^2 : \quad f_2(x) = 0 \}$$

and denote by +i and -i the sign of $f_i(x)$ in each region separated by S^i. Let $x^0 = 0$ be an initial guess of zero of f, then it is easily seen that f satisfies Condition 2.10. It should be noted that x^0 is hemmed in by a connected component S_1^1 of S^1.

We have seen that the algorithm approximately solves the sequence of k-subproblems of (4.1), the dimension k, however, is not necessarily monotonic. It sometimes discards an approximate solution already obtained of a subproblem and begins to move for another approximate solution of the same subproblem. We will see that this behavior of the algorithm prevents us from circling along the circumferential component S_1^1.

Now let us explain the behavior of the algorithm. Since $y_1^0 = -f_1(x^0) < 0$, we find a vertex v^+ which forms a 1-simplex of $\bar{T}|X(1,-1)$ together with x^0. Applying one pivot operation we drop x^0 and move left to this new vertex. By the ordinary replacing rule of the triangulation $\bar{T}|X(1,-1)$ we find a vertex which forms a 1-simplex of $\bar{T}|X(1,-1)$ together with v^+. Repeating this procedure several times we come up to the 1-simplex τ_1 and find that $y_1 = 0$. Then the dimension of a subproblem increases.

Since y_2 is positive, we find a vertex v^+ such that $\tau_1 + v^+$ is a 2-simplex of $\bar{T}|X(2,+1)$. By one pivot operation we also find a vertex which is replaced with v^+. Repeating this procedure we approximately follow the semicircular arch $S_1^1 \cap X(2,+1)$ and finally we have the 1-simplex $\tau_2 = co\{v^0, v^1\}$ by dropping the vertex v^-. Since τ_2 lies in $X(1,+1)$, we increase y_1 and have the vertex v^0 dropped. Now we discard an approximate solution of the 1-subproblem of (4.1) and begin to move right along x_1 coordinate axis for another approximate solution of the same subproblem. Presently we come to have the 1-simplex τ_3 and find that y_1 vanishes again. Since y_2 remains positive, we find a vertex v^+ such that $\tau_3 + v^+$ is a 2-simplex of $\bar{T}|X(2,+1)$. After repeating pivot operations and replacements of vertices we finally reach the 2-simplex σ and find that y_2 vanishes. Therefore we obtain an approximate solution of (4.1) in σ.

5. CONCLUDING REMARKS

In this paper we have confined ourselves to developing a new simplicial variable dimension algorithm; however, it is possible to develop a homotopy continuation version of the algorithm. Assume that $f: R^n \rightarrow R^n$ is continuously differentiable. Let

$$h(x,y) = f(x) + y$$

and consider the system of equations (5.1):

$$(5.1) \qquad h(x,y) = 0, \qquad (x,y) \in |L|.$$

Under an appropriate regularity condition (see Kojima [8]) the solution set of (5.1) turns out to be a disjoint union of curves which are smooth on each (n+1)-cell of L. Hence we can apply the well-known predictor-corrector procedure to trace the smooth piece of curves (see, for example, Allgower and Georg [1]). Observe an (n+1)-cell $Z(k,\alpha)$ of L, then (5.1) can be written as

$$(5.2) \qquad \begin{aligned} & f_j(x) = 0 && \text{for } j < k \\ & f_k(x) + y_k = 0 && \\ & \alpha(x_k - x_k^0),\ \alpha y_k \geq 0 && \\ & x_j - x_j^0 = 0 && \text{for } j > k. \end{aligned}$$

By eliminating the slack variable y_k and substituting x_j^0 for x_j we have

$$(5.3) \qquad f_j(x_1, \cdots, x_k, x_{k+1}^0, \cdots, x_n^0) = 0 \qquad \text{for } j < k$$

$$(5.4) \qquad \begin{aligned} & \alpha f_k(x_1, \cdots, x_k, x_{k+1}^0, \cdots, x_n^0) \leq 0 \\ & \alpha(x_k - x_k^0) \geq 0. \end{aligned}$$

Thus tracing the smooth piece of a curve of solutions of (5.1) is equivalent to tracing the path of solutions of (5.3) as long as (5.4) is satisfied. Since (5.3) is the system of k-1 equations with k variables, we have only to evaluate a $(k-1) \times k$ Jacobian matrix to obtain a predictor direction. This will contribute to the computational efficiency of the algorithm.

Recently Saigal [15] proposed the following homotopy $\bar{h}$: $R^n \times [0,1] \longrightarrow R^n$ for solving sparse and structured fixed point problems: for $0 \le k \le n + 1$ and $\max(0, (k-1)/n) \le t \le \min(1, k/n)$

$$\bar{h}_j(x,t) = \begin{cases} f_j(x) & \text{for } j < k \\ (nt-k+1)f_k(x) + (k-nt)r_k(x) & \text{for } j = k \\ r_j(x) & \text{for } j > k, \end{cases}$$

where r: $R^n \longrightarrow R^n$ is a one-to-one affine mapping. Here let us consider the homotopy $\bar{h}$ with $r(x) = x - x^0$. Suppose that $\bar{h}(x,t) = 0$ for some $x \in R^n$ and $t \in (0,1)$. Then for k such that $1 \le k \le n$ and $(k-1)/n \le t \le k/n$ we have

$$\begin{aligned} & f_j(x) = 0 && \text{for } j < k \\ (5.5) \qquad & (nt - k + 1)\, f_k(x) + (k - nt)(x_k - x_k^0) = 0 && \\ & x_j - x_j^0 = 0 && \text{for } j > k. \end{aligned}$$

Suppose further that $x_k - x_k^0 \neq 0$. Then dividing the k-th equation of (5.5) by $nt - k + 1 \neq 0$ we have

$$f_k(x) + y_k = 0$$

for some y_k such that $(\operatorname{sign}(x_k - x_k^0))\, y_k \ge 0$. Then we see that (x,y_k) satisfies (5.2) for $\alpha = \operatorname{sign}(x_k - x_k^0)$. Conversely if (x,y_k) satisfies (5.2) and $x_k - x_k^0 \neq 0$, then $y_k = \theta(x_k - x_k^0)$ for some $\theta \ge 0$. Hence x and

$$t = k/n - \theta/n(\theta+1)$$

satisfy $\bar{h}(x,t) = 0$. In this way the solution set of (5.2) corresponds to that of (5.5). Therefore the variable dimension algorithm developed in this paper also enjoys computational advantage of the homotopy $\bar{h}$, so that it will be very efficiently applied to sparse and structured systems of equations, such as the system appeared in the two-point boundary value problem.

Finally we point out that the variable dimension algorithm in this paper leads to the algorithm in van der Heyden [5] when applied to the linear complementarity problem. See Yamamoto [22] for details.

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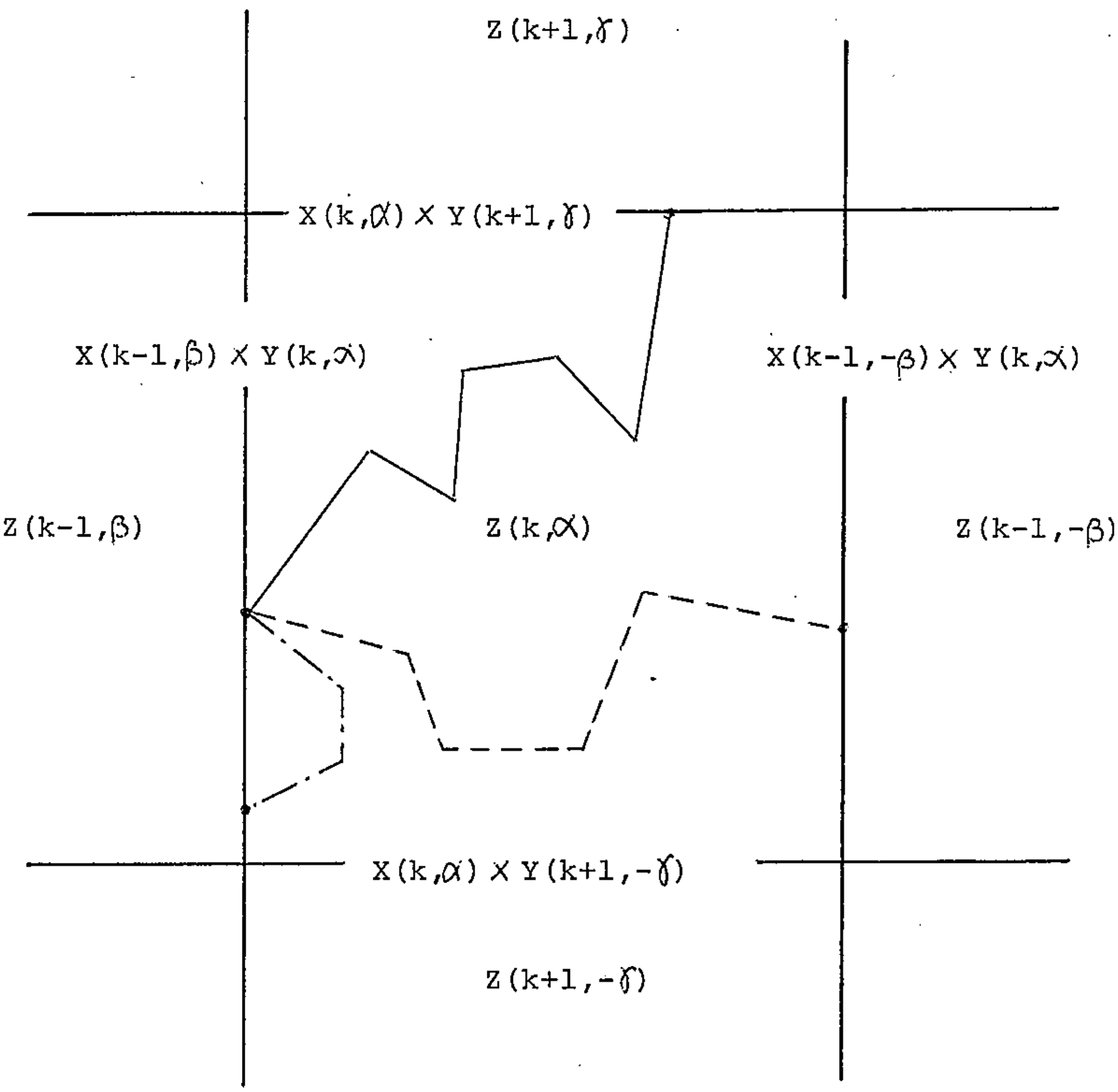

Fig. 1.

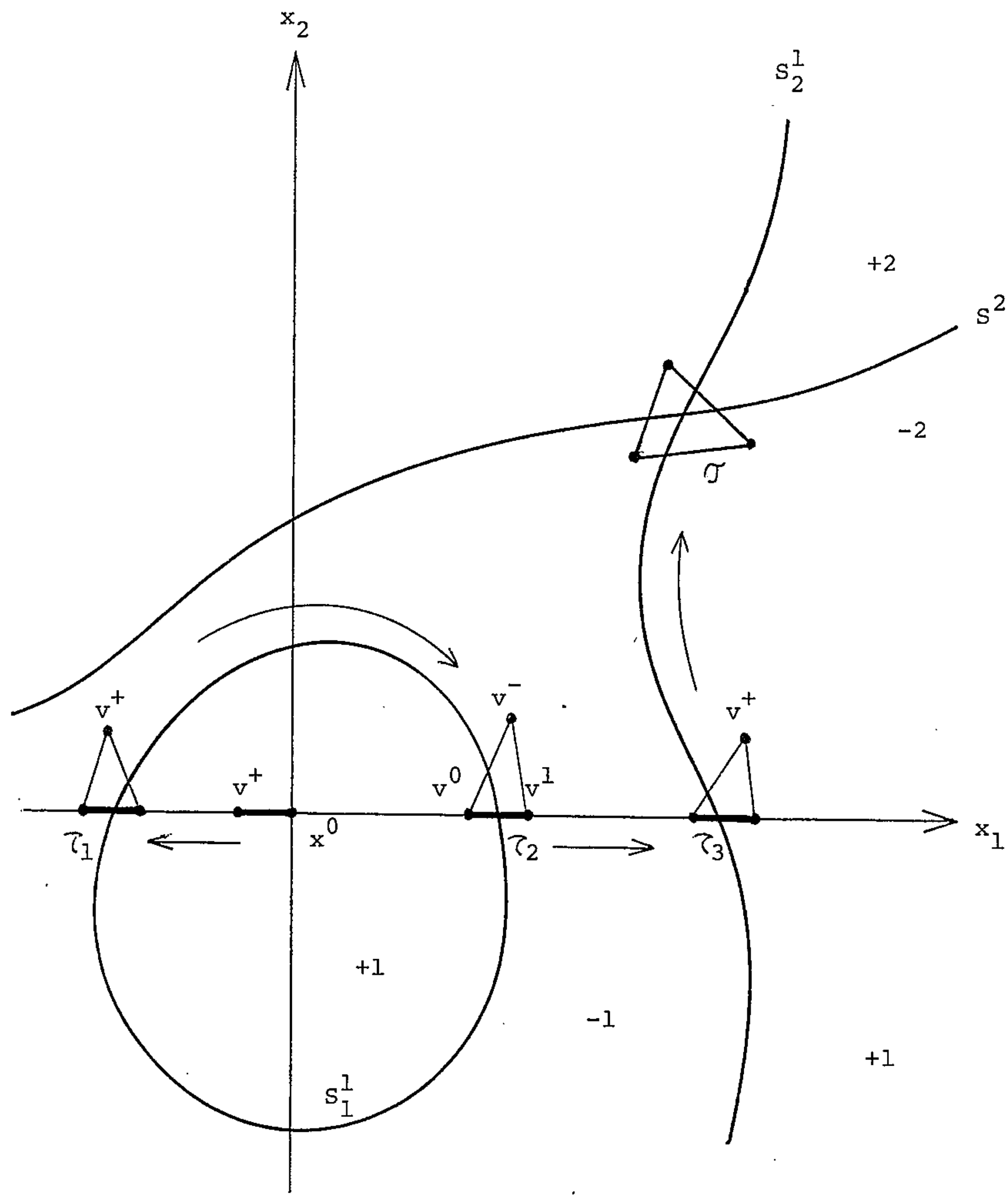

Fig. 2.

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