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Abstract

This paper focuses on the problem of selling an asset by a specified deadline in which the decision is made between 1) proposing a selling price up front to an appearing buyer or 2) concealing the price and forcing the buyer to make an offer. Our analysis indicates that under certain conditions there emerges a time threshold after which the seller switches from concealing his idea of the selling price to proposing the price, or vice versa. For this reason we call this problem the *switching problem* of asset selling.

Keywords: Dynamic programming; Optimal stopping problem; Optimal pricing problem

1   Introduction

Let us consider the problem of selling an asset up to a specified date in the future, the deadline, one example would be selling a car by a deadline. From the standpoint of the seller, there are two facets to the problem. On one hand, it is assumed that an appearing buyer offers a buying price to the seller. The seller then decides whether or not to sell the asset for the price offered. In this case, the seller faces the *optimal stopping problem*, i.e., his main concern is to determine up to when to continue the search for a desirable price offered [5] [6] [7]. On the other hand, when it is assumed that the seller proposes the selling price, the appearing buyer then decides whether or not to buy the asset for the price offered. In this case, the seller faces the *optimal pricing problem*, i.e., he has to determine the price to propose to each buyer appearing up to the deadline [1] [2] [3] [8].

The above situation gives rise to the following dilemma. The seller must decide at each point in time whether to propose a price to an appearing buyer or to conceal the price and wait for the buyer to make an offer for the asset. In this paper, we take into consideration the seller's decision of whether to conceal or propose a selling price. We can consider the following two scenarios:

1. Intuitively, if the deadline is still some distance away, the seller may be more willing to conceal his selling price and wait for the appearing buyer to offer a price. This is because if the seller proposes a price to the buyer, he faces the risk that this price may be substantial lower than the price the buyer has in mind. Therefore, the seller may forgo the opportunity of earning a greater profit.

2. As the deadline draws near, the seller is more compelled to propose a selling price to the buyer. This is because if he finds himself unable to sell the asset up to the deadline, he may be resigned to sell it to a salvage dealer at a giveaway price or may even need to pay some costs to dispose of it.

However, we have to ask, does this conjecture always hold true ? In other words, can we always assert that the seller should propose the selling price if the selling period is long and conceal the price if the selling period is short ? Through discussion in the following sections, we will show that this conjecture

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does not always hold. In fact, in Section 7, we present some examples of scenarios which run counter to this conjecture.

The above suggests that there may exist a time threshold(s) after which the seller may *switch* his action from concealing the selling price to proposing the price, or vice versa. We shall call the above problem the *switching problem* of asset selling and the model for this problem, the *switching model*. A great deal of research has been done separately on the optimal stopping problem and the optimal pricing problem. To the best of our knowledge, there is no work that links both problems. The purpose of this paper is to propose a basic model on the above mentioned asset selling problem by taking the concept of switching into consideration and to clarify the properties of its optimal decision rule.

Section 2 provides a strict definition of the model examined in the paper. Section 3 defines several functions used to describe the optimal equation of the model derived in Section 4 and examines properties of these functions. Section 5 describes the optimal decision rule of the model and Section 6 analyzes their properties. In Section 7 we provide some numerical experiments and in Section 8 we present conclusions of our research and suggestions for further work.

2   Model

The model discussed in this paper is defined on the seven assumptions below:

A1. Consider the following discrete-time sequential decision problem with a finite planning horizon. The points in time are numbered backward from the final point in time of the planning horizon, time 0, the deadline, as $0, 1, \cdots$ and so on. Accordingly, if a time t is the present point in time, the two adjacent times $t + 1$ and $t - 1$ are the previous and next points in time, respectively. Further, let the time interval between times t and $t - 1$ be called the *period t*.

A2. A seller must sell an asset up to the deadline, i.e., time 0.

A3. An asset remaining unsold at time 0, can be sold at the salvage price $\rho \in (-\infty, \infty)$. Here, $\rho < 0$ implies the disposal cost to discard an unsold asset.

A4. A buyer who requests an asset appears at each point in time with a probability λ $(0 < \lambda < 1)$.

A5. When a buyer appears, the seller has to make a decision between two alternatives: 1) proposing a selling price or 2) concealing the selling price and making the buyer offer a price.

   1) If the seller chooses the first alternative, he proposes a price to a buyer. The buyer then decides whether or not to buy the item based on this price. By w let us denote the reservation price of a buyer, implying that the buyer is willing to buy an asset if and only if the selling price z proposed is lower than or equal to w, i.e., $z \leq w$. Here, assume that subsequent buyers' reservation prices $w, w', \cdots$ are independent identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ. Also, let $f(w)$ denote its probability density function, which is truncated on both sides (see Figure 2.1). Hence, for certain given numbers a and b such that $0 < a < b < \infty$ assume

$$F(w) = 0, \;\; w \leq a, \qquad 0 < F(w) < 1, \;\; a < w < b, \qquad F(w) = 1, \;\; b \leq w.$$

   where

$$f(w) = 0, \;\; w < a, \qquad f(w) > 0, \;\; a \leq w \leq b, \qquad f(w) = 0, \;\; b < w.$$

2

Then clearly $0 < a < \mu < b$. Thus, the probability of an appearing buyer buying the asset, provided that a price z is offered by the seller, is given by

$$p(z) = \Pr\{z \leq w\},$$

where $0 \leq p(z) \leq 1$. Then it can be easily seen that

$$p(z) \begin{cases} = 1, & z \leq a \quad \cdots(1), \\ < 1, & a < z \quad \cdots(2), \end{cases} \qquad p(z) \begin{cases} > 0, & z < b \quad \cdots(3), \\ = 0, & b \leq z \quad \cdots(4). \end{cases} \tag{2.1}$$
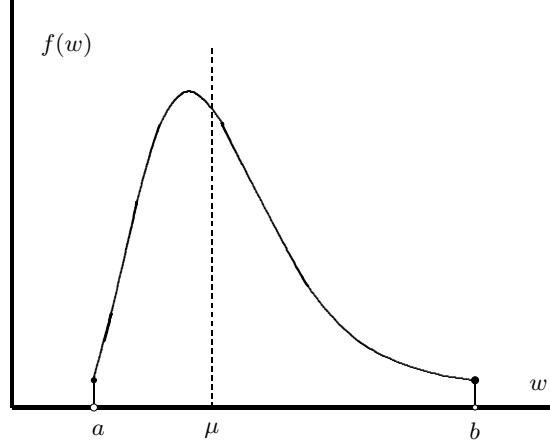

Figure 2.1: Probability density function $f(w)$.

2) If the seller chooses the second alternative, he will decide whether or not to sell the item judging from the price being offered by the buyer. In this case, it is assumed that the buyer offers a price $\xi = \alpha w$ $(0 < \alpha \leq 1)$, less than or equals to his reservation price w. Here, let us call α the *price offering ratio*, which measures the degree of a buyer's desirability for an item. Thus, the greater (lower) the buyer's desirability is, the closer is α to 1 (0). Also, we assume that subsequent buyers' price offering ratios, $\alpha, \alpha', \cdots$, are independent identically distributed random variables having a known distribution function $Q(\alpha)$ with a finite expectation $\mu_\alpha > 0$. In addition, we also assume that α and w are stochastically independent. Thus we have

$$\mathbf{E}[\xi] = \mu_\alpha \mu. \tag{2.2}$$

Further, let $G(\xi)$ denotes the distribution function of $\xi = \alpha w$, i.e.,

$$G(x) = \Pr\{\xi \leq x\} = \Pr\{\alpha w \leq x\} = \Pr\{w \leq x/\alpha\} = \mathbf{E}_\alpha[\int_0^{x/\alpha} f(w)dw].$$

Accordingly, the probability density function of $G(x)$, $g(x)$, is given by

$$g(x) = \mathbf{E}_\alpha[1/\alpha f(x/\alpha)]. \tag{2.3}$$

A6. Let $h \geq 0$ be the holding cost of the asset remaining unsold for a period.

A7. By β $(0 < \beta \leq 1)$ let us denote the discount factor, implying that the monetary value of one unit a period hence is equivalent to that of β units at the present point in time.

Throughout the paper, for explanatory convenience, let us define the following notations.

1. Symbol P means the decision of proposing a selling price to a buyer.

3

2. Symbol C means the decision of concealing the selling price and forcing the buyer to offer a price.

3 Underlying Functions

This section defines the functions that will be used to describe the optimal equation of the model examined in the paper. The properties of the functions verified in this section will be utilized in the analysis of the model developed in Section 4.

3.1 Definitions

For any x let us define the following two functions [4]:

$$T_{\mathrm{s}}(x) = \int_x^\infty \max\{w - x, 0\} f(w) dw, \tag{3.1}$$

$$T_{\mathrm{p}}(x) = \max_z p(z)(z - x); \tag{3.2}$$

the former is called the *T-function of Type-$\mathbb{S}$* and the latter the *T-function of Type-$\mathbb{P}$*. Further, by $z(x)$ let us designate the z attaining the maximum of the right hand side of Eq. (3.2) if it exists, i.e.,

$$T_{\mathrm{p}}(x) = p(z(x))(z(x) - x). \tag{3.3}$$

Further, for any x we shall define the following functions.

$$\mathcal{T}_{\mathrm{s}}(x) = \mathbf{E}_\alpha[\alpha T_{\mathrm{s}}(x/\alpha)], \tag{3.4}$$

$$J(x) = \mathcal{T}_{\mathrm{s}}(x) - T_{\mathrm{p}}(x), \tag{3.5}$$

$$B(x) = \lambda\beta \max\{J(x), 0\} + \lambda\beta T_{\mathrm{p}}(x) - (1 - \beta)x - h \tag{3.6}$$

$$= \lambda\beta \max\{\mathcal{T}_{\mathrm{s}}(x), T_{\mathrm{p}}(x)\} - (1 - \beta)x - h. \tag{3.7}$$

Here let us define $x^\star$, $a^\star$, a°, b°, and x° as follows, if they exist.

$$x^\star = \inf\{x \mid z(x) > a\}, \qquad a^\star = \inf\{x \mid T_{\mathrm{p}}(x) > a - x\}, \qquad \text{(see [4])} \tag{3.8}$$

$$a^\circ = \max\{x \mid \mathcal{T}_{\mathrm{s}}(x) = \mu_\alpha\mu - x\}, \qquad b^\circ = \sup\{x \mid \mathcal{T}_{\mathrm{s}}(x) > 0\}, \tag{3.9}$$

$$x^\circ = \max\{\hat{x} \mid J(x) = 0 \quad for \ all \ x \le \hat{x} < b\}. \tag{3.10}$$

Now by x_B and x_J let us denote the solutions of $B(x) = 0$ and $J(x) = 0$, respectively, if they exist, i.e.,

$$B(x_B) = 0, \qquad J(x_J) = 0.$$

If $B(x) = 0$ has multiple solutions, let us newly define the *minimum* of them by x_B. If $J(x) \ne 0$ on $(-\infty, b)$, then let $x^\circ = -\infty$. If $J(x) = 0$ has multiple solutions on (x°, b), then let X_n, $1 \le n \le N$, be the subintervals on (x°, b) such that $J(x) = 0$, and let $x_J^n = \min X_n$, $1 \le n \le N$. Here without loss of generality let $x_J^1 < x_J^2 < \cdots < x_J^N$.

3.2 Properties

Proposition 3.1 *In general, let $r(x)$ be a continuous function which is nonincreasing on $(-\infty, \infty)$. Then if $r(x)$ is strictly decreasing on $(-\infty, A)$ or (B, ∞) for certain given finite A and B, so also on $(-\infty, A]$ or $[B, \infty)$, respectively.*

Proof.   See [4].   ∎

Lemma 3.1

(a)   *For any x and y we have*

$$(x - y)F(y) \ \leq \ T_\mathrm{s}(x) + x - T_\mathrm{s}(y) - y \leq (x - y)F(x). \tag{3.11}$$

(b) $T_\mathrm{s}(x) > 0$ *on* $(-\infty, b)$ *and* $T_\mathrm{s}(x) = 0$ *on* $[b, \infty)$.

(c) $T_\mathrm{s}(x)$ *is continuous, nonincreasing, and convex on* $(-\infty, \infty)$.

(d) $T_\mathrm{s}(x) = \mu - x$ *on* $(-\infty, a]$ *and* $T_\mathrm{s}(x) > \mu - x$ *on* (a, ∞).

(e) $|T_\mathrm{s}(x) - T_\mathrm{s}(y)| \leq |x - y|$ *for any x and y.*

Proof.   See [4].   ∎

Lemma 3.2

(a) $T_\mathrm{p}(x) > 0$ *on* $(-\infty, b)$ *and* $T_\mathrm{p}(x) = 0$ *on* $[b, \infty)$.

(b) $T_\mathrm{p}(x)$ *is strictly decreasing on* $(-\infty, b]$.

(c) $T_\mathrm{p}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.

(d) $\lambda T_\mathrm{p}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$.

(e) $\lim_{x \to -\infty} T_\mathrm{p}(x) = \infty$.

(f) $|T_\mathrm{p}(x) - T_\mathrm{p}(y)| \leq |x - y|$ *for any x and y.*

(g) $x^\star \leq a^\star < a$.

(h) $T_\mathrm{p}(x) = a - x$ *on* $(-\infty, a^\star]$ *and* $T_\mathrm{p}(x) > a - x$ *on* $(a^\star, \infty)$.

(i) $z(x) \geq a$ *for any x.*

(j) $z(x)$ *is nondecreasing on* $(-\infty, \infty)$.

(k)   *If $x > (<) \ x^\star$, then $z(x) > (=) \ a^\dagger$.*

Proof.   See [4].   ∎

Lemma 3.3

(a) $\mathcal{T}_\mathrm{s}(x)$ *is continuous, nonincreasing, and convex on* $(-\infty, \infty)$.

(b) $\mathcal{T}_\mathrm{s}(x) > 0$ *on* $(-\infty, b^\circ)$ *and* $\mathcal{T}_\mathrm{s}(x) = 0$ *on* $[b^\circ, \infty)$ *where* $b^\circ \leq b$.

(c) $\mathcal{T}_\mathrm{s}(x)$ *is strictly decreasing on* $(-\infty, b^\circ]$.

(d) $\mathcal{T}_\mathrm{s}(x) + x$ *is nondecreasing on* $(-\infty, \infty)$.

(e) $\lambda \mathcal{T}_\mathrm{s}(x) + x$ *is strictly increasing on* $(-\infty, \infty)$.

(f) $\mathcal{T}_\mathrm{s}(x) = \mu_\alpha \mu - x$ *on* $(-\infty, a^\circ]$ *and* $\mathcal{T}_\mathrm{s}(x) > \mu_\alpha \mu - x$ *on* (a°, ∞).

(g) $0 \leq a^\circ \leq a$ *and* $a^\circ \leq \mu_\alpha \mu \leq b^\circ$.

(h) $\lim_{x \to -\infty} \mathcal{T}_\mathrm{s}(x) = \infty$.

(i) $|\mathcal{T}_\mathrm{s}(x) - \mathcal{T}_\mathrm{s}(y)| \leq |x - y|$ *for any x and y..*

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†Any one of $z(x^\star) = a$ and $z(x^\star) > a$ may occur due to the fact that $z(x)$ might be a discontinuous function of x as stated in Remark 6.1 of [4]. Fortunately, this fact does not directly relate to the discussions in this paper.

Proof.   See Appendix A.   ∎

Lemma 3.4

(a) $B(x)$ *is strictly decreasing on* $(-\infty, b]$.

(b) $\lim_{x \to -\infty} B(x) = \infty$.

(c)   *Let* $(1-\beta)^2 + h^2 = 0$.

    1. $x_B = b$.

    2. $x < (\geq) \, x_B \Leftrightarrow B(x) > (=) \, 0$.

(d)   *Let* $(1-\beta)^2 + h^2 \neq 0$.

    1. x_B *uniquely exists with* $x_B < b$.

    2. $x < (=) > x_B \Leftrightarrow B(x) > (= (<)) \, 0$.

Proof.   See Appendix B.   ∎

Lemma 3.5

(a)   *Let* $x \leq \min\{x^\star, a^\circ\}$. *Then* $J(x) = \mu_\alpha \mu - a$ *and* $\min\{x^\star, a^\circ\} \leq b^\circ$.

(b)   *Let* $b^\circ < b$. *Then* $J(x)$ *is strictly increasing on* $[b^\circ, b)$ *where* $J(x) < 0$ *for* $b^\circ \leq x < b$.

(c) $J(x) = 0$ *on* $[b, \infty)$.

Proof.   See Appendix C.   ∎

3.3   The shape of $J(x)$

Lemma 3.5 partially specifies the shape of the function $J(x)$; its shape on the interval $(\min\{x^\star, a^\circ\}, b^\circ)$ cannot be easily determined. In Section 6.3 we will see that the existence of the solution of $J(x) = 0$, denoted by x_J, on the interval $(\min\{x^\star, a^\circ\}, b)$ plays a key role in clarifying the properties of the optimal decision rule. Below, using two examples of density functions $f(w)$, we depict some shapes of the function $J(x)$ obtained through numerical integration by the trapezoidal rule.

1.   Let $F(w)$ be the uniform distribution function on $[1.5, 2.5]$, i.e., $a = 1.5$ and $b = 2.5$. For $Q(\alpha)$, we consider the uniform distribution functions on $[0.1, 0.4]$ and on $[0.7, 0.9]$. For the former case, the function $J(x)$ has no solution on the interval $(\min\{x^\star, a^\circ\}, b)$ (see Figure 3.2(I)), and for the latter case it has a unique solution, x_J (see Figure 3.2(II)).

2.   Let the probability density function of $F(w)$, $f(w)$, be such that

$$f(w) \begin{cases} = 0.0570119511, & \text{on } [0.1, 0.599], \\ = triangle, & \text{on } [0.599, 0.7] \text{ with its maximum at } w = 0.6, \\ = 0.06981913043, & \text{on } [0.7, 3.0], \end{cases} \left. \right\} \quad (see \text{ Figure 3.3(I))} \quad (3.12)$$

and let $Q(\alpha)$ be a uniform distribution on $[0.64, 0.74]$. Then the equation $J(x) = 0$ has three solutions on the interval $(\min\{x^\star, a^\circ\}, b)$ as shown in Figure 3.3(II).
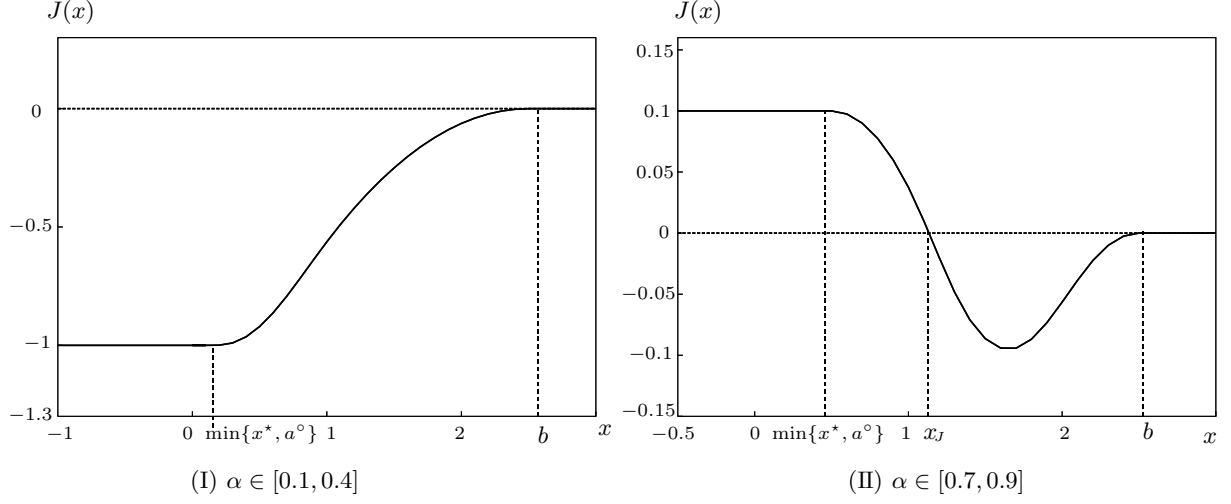
(I) $\alpha \in [0.1, 0.4]$ (II) $\alpha \in [0.7, 0.9]$

Figure 3.2: The shape of $J(x)$

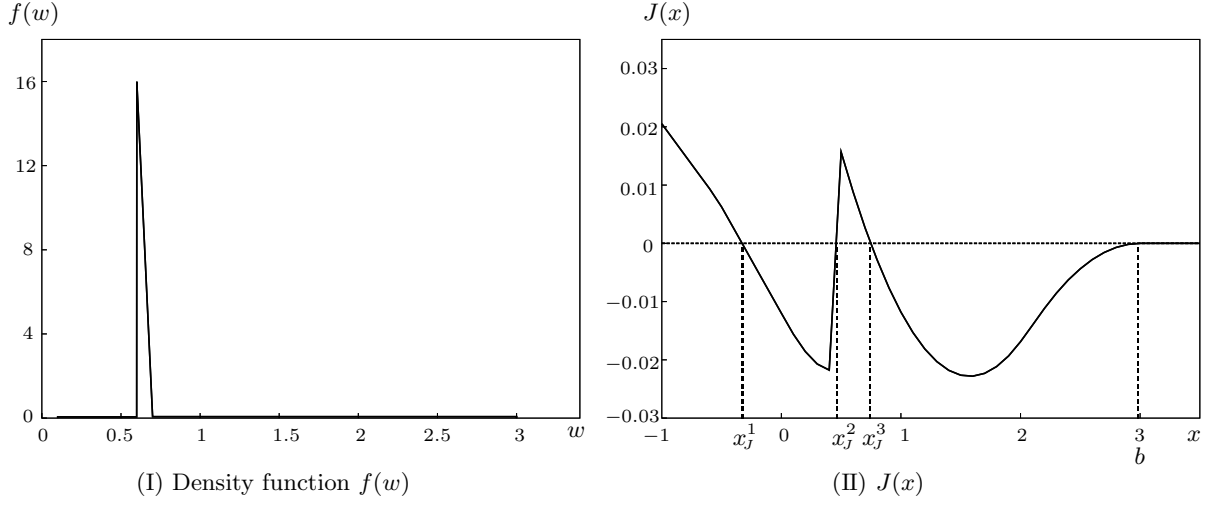(I) Density function $f(w)$ (II) $J(x)$

Figure 3.3: $f(w)$ and $J(x)$ where $\min\{x^\star, a^\circ\} < -1$

4   Optimal Equation

Suppose that an item purchased at a certain past point in time remains unsold at a present time t. Let $v_t(0)$ and $v_t(1)$ be the maximums of the total expected present discounted profit, respectively, with no buyer and with a buyer. Then we have:

$$v_0(0) = \rho, \tag{4.1}$$

$$v_t(0) = \beta(\lambda v_{t-1}(1) + (1-\lambda)v_{t-1}(0)) - h, \qquad t \geq 1, \tag{4.2}$$

$$v_t(1) = \max \begin{cases} \texttt{C} : \int_0^\infty \max\{\xi, v_t(0)\}g(\xi)d\xi & \cdots(1), \\ \texttt{P} : \max_z \{p(z)z + (1-p(z))v_t(0)\} \cdots(2), \end{cases} \quad t \geq 0. \tag{4.3}$$

Since we can rewrite Eq. (4.3 (2)) as $\max_z p(z)(z - v_t(0)) + v_t(0)$, using the T-function of Type-$\mathbb{P}$, we can express Eq. (4.3) as follows.

7

$$v_t(1) = \max\{\int_0^\infty \max\{\xi, v_t(0)\}g(\xi)d\xi, T_{\mathrm{p}}(v_t(0)) + v_t(0)\}, \qquad t \ge 0. \tag{4.4}$$

Further, from Eq. (2.3) we have

$$\int_0^\infty \max\{\xi, v_t(0)\}g(\xi)d\xi = \int_0^\infty \max\{\xi, v_t(0)\}\, \mathop{\mathbf{E}}_\alpha[1/\alpha f(\xi/\alpha)]d\xi$$

$$= \mathop{\mathbf{E}}_\alpha[1/\alpha\big(\int_0^\infty \max\{\xi, v_t(0)\}f(\xi/\alpha)d\xi\big)], \qquad t \ge 0.$$

Let $\eta = \xi/\alpha$. Then using the T-function of Type-$\mathbb{S}$ and Eq. (3.4), we get

$$\int_0^\infty \max\{\xi, v_t(0)\}g(\xi)d\xi = \mathop{\mathbf{E}}_\alpha[1/\alpha\big(\int_0^\infty \max\{\alpha\eta, v_t(0)\}f(\eta)\alpha d\eta\big)]$$

$$= \mathop{\mathbf{E}}_\alpha[\alpha\big(\int_0^\infty \max\{\eta, v_t(0)/\alpha\}f(\eta)d\eta\big)]$$

$$= \mathop{\mathbf{E}}_\alpha[\alpha\big(\int_0^\infty \big(\max\{\eta - v_t(0)/\alpha, 0\} + v_t(0)/\alpha\big)f(\eta)d\eta\big)]$$

$$= \mathop{\mathbf{E}}_\alpha[\alpha\big(T_{\mathrm{s}}(v_t(0)/\alpha) + v_t(0)/\alpha\big)]$$

$$= \mathop{\mathbf{E}}_\alpha[\alpha T_{\mathrm{s}}(v_t(0)/\alpha)] + v_t(0) = \mathcal{T}_{\mathrm{s}}(v_t(0)) + v_t(0), \qquad t \ge 0.$$

Hence Eq. (4.4) can be expressed as

$$v_t(1) = \max\{\mathcal{T}_{\mathrm{s}}(v_t(0)), T_{\mathrm{p}}(v_t(0))\} + v_t(0), \qquad t \ge 0. \tag{4.5}$$

Substituting Eq. (4.5) into Eq. (4.2) produces

$$v_t(0) = \lambda\beta \max\{\mathcal{T}_{\mathrm{s}}(v_{t-1}(0)), T_{\mathrm{p}}(v_{t-1}(0))\} + \beta v_{t-1}(0) - h, \qquad t \ge 1. \tag{4.6}$$

Below, for expressional simplicity, let us denote

$$v_t = v_t(0), \qquad t \ge 0.$$

Then using Eq. (3.5), we can express Eq. (4.6) as follows.

$$v_t = \lambda\beta \max\{\mathcal{T}_{\mathrm{s}}(v_{t-1}), T_{\mathrm{p}}(v_{t-1})\} + \beta v_{t-1} - h \tag{4.7}$$

$$= \lambda\beta \max\{\mathcal{T}_{\mathrm{s}}(v_{t-1}) - T_{\mathrm{p}}(v_{t-1}), 0\} + \lambda\beta T_{\mathrm{p}}(v_{t-1}) + \beta v_{t-1} - h \tag{4.8}$$

$$= \lambda\beta \max\{J(v_{t-1}), 0\} + \lambda\beta T_{\mathrm{p}}(v_{t-1}) + \beta v_{t-1} - h, \qquad t \ge 1. \tag{4.9}$$

Noting Eq. (4.1), from Eq. (4.9) we obtain

$$v_1 = \lambda\beta \max\{J(\rho), 0\} + \lambda\beta T_{\mathrm{p}}(\rho) + \beta\rho - h. \tag{4.10}$$

Then from Eqs. (4.1) and (3.6) we get

$$v_1 - v_0 = B(\rho). \tag{4.11}$$

Finally, define $v = \lim_{t\to\infty} v_t$ if it exists.

5   Optimal Decision Rule

From Eqs. (4.9), (4.1), and (3.5) the optimal decision rule can be prescribed for a given $t \ge 0$ as follows.

(a) If $J(v_t) \ge 0$, conceal the selling price and wait for the appearing buyer to offer a price. Accordingly,

8

for a price ξ offered by the buyer appearing at that time, if $\xi \geq v_t$, then sell the item, or else do not; in other words, v_t becomes the *reservation price* of a seller.

(b) If $J(v_t) \leq 0$, propose a price to the appearing buyer. Then the optimal selling price for an item remaining unsold at time $t \geq 0$ is given by the z attaining the maximum of the right hand side of Eq. (4.3 (2)) if it exists, denoted by z_t. In addition, the z_t for $t \geq 0$ is given by the z attaining the maximum of $p(z)(z - v_t)$ in $T_\mathrm{p}(v_t)$ over $(-\infty, \infty)$; in other words, $z_t = z(v_t)$ due to the definition of $z(x)$ (see Eq. (3.3)).

6 Analysis

6.1 Monotonicity

Lemma 6.1 *If $\rho < (\geq) x_\scriptscriptstyle B$, then v_t is strictly increasing (nonincreasing) in $t \geq 0$.*

Proof. Let $\rho < (\geq) x_\scriptscriptstyle B$. Then $B(\rho) > (\leq) 0$ from Lemma 3.4(c2,d2). Accordingly, we have $v_1 > (\leq) v_0$ from Eq. (4.11). Suppose $v_{t-1} > (\leq) v_{t-2}$. Then from Eq. (4.7), Lemmas 3.3(e), and 3.2(d) we have, for $t \geq 1$,

$$
\begin{aligned}
v_t &= \beta \max\{\lambda \mathcal{T}_\mathrm{s}(v_{t-1}) + v_{t-1}, \lambda T_\mathrm{p}(v_{t-1}) + v_{t-1}\} - h \\
&> (\leq) \ \beta \max\{\lambda \mathcal{T}_\mathrm{s}(v_{t-2}) + v_{t-2}, \lambda T_\mathrm{p}(v_{t-2}) + v_{t-2}\} - h = v_{t-1}.
\end{aligned}
$$

Accordingly, by induction the assertion holds. ∎

Lemma 6.2
(a) v_t *is bounded in t.*
(b) v_t *converges to a finite v as $t \to \infty$ where $B(v) = 0$.*

Proof. (a) Let M be a sufficiently large number greater than b such that $\rho < M$ and $T_\mathrm{p}(M) = 0$ (see Lemma 3.2(a)). First, note that $v_0 = \rho < M$ from Eq. (4.1). Since $\xi = \alpha w$, for $a < w < b$ we have $0 < \alpha a < \alpha w < \alpha b \leq b < M$, hence $\int_0^\infty \max\{\xi, M\} dG(\xi) = \int_0^\infty M dG(\xi) = M$. Further, from Eqs. (4.4), (4.1), and Lemma 3.2(c) we have

$$
v_0(1) \ \leq \ \max\Big\{ \int_0^\infty \max\{\xi, M\} g(\xi) d\xi, T_\mathrm{p}(M) + M \Big\} = \max\{M, M\} = M.
$$

Suppose $v_{t-1} \leq M$ and $v_{t-1}(1) \leq M$. From Eq. (4.2) we get $v_t \leq \beta(\lambda M + (1 - \lambda)M) - h = \beta M - h = \beta M \leq M$ for $t \geq 1$. Then from Eq. (4.4) and Lemma 3.2(c) we obtain

$$
v_t(1) \ \leq \ \max\Big\{ \int_0^\infty \max\{\xi, M\} g(\xi) d\xi, T_\mathrm{p}(M) + M \Big\} \leq \max\{M, M\} = M, \qquad t \geq 0,
$$

hence $v_t \leq M$ and $v_t(1) \leq M$ for $t \geq 0$ by induction, implying that v_t is upper bounded in t. Further, noting Eq. (2.2), Lemma 3.2(i), and the fact that $p(a) = 1$ due to Eq. (2.1 (1)), from Eq. (4.3) we get

$$
v_t(1) \ \geq \ \max\Big\{ \int_0^\infty \xi g(\xi) d\xi, p(a)a + (1 - p(a))v_t(0) \Big\} = \max\{\mu_\alpha \mu, a\} \geq a > 0, \qquad t \geq 0.
$$

Then from Eq. (4.2) we have $v_t \geq \beta(1 - \lambda)v_{t-1} - h$ for $t \geq 1$. For expressional simplicity, below let $\eta = \beta(1 - \lambda)$ where $0 < \eta < 1$ due to the assumption of $0 < \lambda < 1$. Then $v_t \geq \eta v_{t-1} - h$ for $t \geq 1$. Since

$v_0 = \rho$ from Eq. (4.1), from the above inequality we immediately obtain $v_t \geq \eta^t \rho - (1 + \eta + \cdots + \eta^{t-1})h > \eta^t \rho - h/(1-\eta)$ for $t \geq 1$. Accordingly, the inequalities $\rho \geq 0$ and $\rho < 0$ lead to, respectively, $v_t \geq -h/(1-\eta)$ and $v_t \geq \rho - h/(1-\eta)$; in other words, v_t is lower bounded in t whether $\rho \geq 0$ or $\rho < 0$. Hence, the assertion holds.

(b)　Since v_t is bounded in t from (a) and monotone in $t \geq 0$ from Lemma 6.1, it converges to a finite v as $t \to \infty$. Now, from Eq. (3.5), Lemmas 3.3(i), and 3.2(f) we have

$$
\begin{aligned}
|J(v_t) - J(v)| &= |\mathcal{T}_s(v_t) - T_p(v_t) - \mathcal{T}_s(v) + T_p(v)| \\
&\leq |\mathcal{T}_s(v_t) - \mathcal{T}_s(v)| + |T_p(v_t) - T_p(v)| \\
&\leq |v_t - v| + |v_t - v| = 2|v_t - v| \cdots (1^*).
\end{aligned}
$$

Further, from Eqs. (4.9), (1^*), and Lemma 3.2(f) we obtain

$$
\begin{aligned}
&|v_t - \lambda\beta \max\{J(v), 0\} - \lambda\beta T_p(v) - \beta v + h| \\
&= |\lambda\beta \max\{J(v_{t-1}), 0\} + \lambda\beta T_p(v_{t-1}) + \beta v_{t-1} - \lambda\beta \max\{J(v), 0\} - \lambda\beta T_p(v) - \beta v| \\
&\leq \lambda\beta \max\{|J(v_{t-1}) - J(v)|, 0\} + \lambda\beta |T_p(v_{t-1}) - T_p(v)| + \beta|v_{t-1} - v| \\
&= 2\lambda\beta|v_{t-1} - v| + \lambda\beta|v_{t-1} - v| + \beta|v_{t-1} - v| = \beta(3\lambda + 1)|v_{t-1} - v|,
\end{aligned}
$$

which converges to 0 as $t \to \infty$. Noting Eq. (3.6), we can express $v = \lambda\beta \max\{J(v), 0\} + \lambda\beta T_p(v) + \beta v - h$ as $B(v) = 0$. ∎

From Lemma 6.2(b) and Eq. (3.6) we obtain

$$
0 = B(v) = \lambda\beta \max\{J(v), 0\} + \lambda\beta T_p(v) - (1-\beta)v - h. \tag{6.1}
$$

Lemma 6.3

(a)　*Let* $(1-\beta)^2 + h^2 = 0$. *Then* $v \geq b = x_B$.

(b)　*Let* $(1-\beta)^2 + h^2 \neq 0$. *Then* $v = x_B < b$.

Proof.　(a) Let $(1-\beta)^2 + h^2 = 0$. Then from Eq. (6.1) we obtain $0 = B(v) = \lambda \max\{J(v), 0\} + \lambda T_p(v) = \lambda \max\{\mathcal{T}_s(v), T_p(v)\}$. If $v < b$, then $T_p(v) > 0$ from Lemma 3.2(a), leading to the contradiction of $B(v) > 0$. Hence, it must be that $v \geq b$. In addition, from Lemma 3.4(c1) we have $x_B = b$.

(b) Let $(1-\beta)^2 + h^2 \neq 0$. Then the assertion is clear from Lemma 3.4(d1) and Eq. (6.1). ∎

Lemma 6.4

(a)　*If* $v_t > (<) x^\star$, *then* $z_t > (=)a$　*for* $t \geq 0$.

(b)　*If* $\rho < (\geq) x_B$, *then* z_t *is nondecreasing (nonincreasing) in* $t \geq 0$.

Proof.　Note Lemma 3.2(j) and the definition of z_t (see Section 5).

(a) Immediate from Lemma 3.2(k).

(b) Evident from Lemma 6.1 and (a). ∎

6.2　Switching Property

In this subsection we shall provide the definition of the switching property.

Definition 6.1   *If there exists a time threshold t^* at which the optimal decision rule switches from proposing a selling price to concealing the selling price or from concealing the selling price to proposing the selling price, it is said to have the switching property.*

Definition 6.2   *If there exist time thresholds $t_1^* < t_2^* < \cdots < t_N^*$ for $N \geq 1$, then the optimal decision rule is said to have the N-Switching property. If $N = 1$ (> 1), then it is said to have the single (multiple) switching property.*

Furthermore, for convenience in the later discussions, by 0-switching property we mean that the optimal decision rule does not possess the switching property.

6.3   Optimal Decision Rule

In this subsection we describe the conditions under which the optimal decision rule possesses the switching property.

Lemma 6.5

(a)   *If $J(v_t) \leq (\geq) 0$ for $t \geq 0$, the optimal decision rule has 0-switching property.*

(b)   *If $J(x) \leq (\geq) 0$ on $(-\infty, \infty)$, the optimal decision rule has 0-switching property.*

Proof.   (a)   Evident from Eq. (3.5).

(b)   Clear from (a).   ∎

Below, we provide a lemma that describes the optimal decision rule for the case where the equation $J(x) = 0$ has a solution, x_J, on the interval $(\min\{x^\star, a^\circ\}, b)$, i.e., $\min\{x^\star, a^\circ\} < x_J < b$. Note that if $J(x) = 0$, then P is indifferent to C. In order to present the optimal decision rule in a neat form, throughout the remainder of the paper, we will assign $J(x) = 0$ to $J(x) \geq 0$ or $J(x) \leq 0$ appropriately.

Lemma 6.6   *Suppose the equation $J(x) = 0$ has a solution, x_J, on the interval $(\min\{x^\star, a^\circ\}, b)$, i.e., $\min\{x^\star, a^\circ\} < x_J < b$.*

(a)   *Let $(1 - \beta)^2 + h^2 = 0$.*

   1.   *Let $\rho < x_J$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

   2.   *Let $\rho \geq x_J$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

(b)   *Let $(1 - \beta)^2 + h^2 \neq 0$.*

   1.   *Let $x_J < x_B$.*

      i.   *Let $\rho < x_J$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

      ii.   *Let $\rho \geq x_J$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

   2.   *Let $x_B = x_J$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

   3.   *Let $x_B < x_J$.*

      i.   *Let $\rho < x_J$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

      ii.   *Let $\rho \geq x_J$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

Proof. Noting the definition of x_J and the optimal decision rule in Section 5, we shall draw attention to the changes in the sign of the function $J(x)$ at $x = x_J$. In addition, note that $v_0 = \rho$ from Eq. (4.1).

(a) Let $(1 - \beta)^2 + h^2 = 0$. Then $v \geq b = x_B$ due to Lemma 6.3(a), hence $x_J < b = x_B \leq v$ by assumption.

(a1) Let $\rho < x_J$. Then we get $v_0 = \rho < x_J < b = x_B \leq v$. From Lemmas 6.1 and 6.2(b) we see that v_t passes through x_J as t increases from 0 to ∞. Hence there exists a $t^* \geq 0$ such that $v_t < x_J$ if $0 \leq t \leq t^*$, or else $v_t \geq x_J$. Thus the assertion holds from the changes in the sign of $J(x)$ at $x = x_J$ (see Note 6.1).

(a2) Let $\rho \geq x_J$.

1. Suppose $\rho < x_B$. Then $x_J \leq v_0 = \rho < x_B = b \leq v$. Hence $x_J \leq v_t$ for $t \geq 0$ due to Lemma 6.1, implying that $J(v_t) \leq 0$ or $J(v_t) \geq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

2. Suppose $\rho \geq x_B$. Then we have $v_0 = \rho \geq x_B$, so $v_t \geq v \geq x_B = b > x_J$ for $t \geq 0$ from Lemmas 6.1 and 6.2(b), implying that $J(v_t) \leq 0$ or $J(v_t) \geq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

(b) Let $(1 - \beta)^2 + h^2 \neq 0$. Then $v = x_B < b$ due to Lemma 6.3(b).

(b1) Let $x_J < x_B$.

(b1i) Let $\rho < x_J$. Then we get $v_0 = \rho < x_J < x_B = v$. From Lemmas 6.1 and 6.2(b) we see that v_t passes through x_J as t increases from 0 to ∞. Hence there exists a $t^* \geq 0$ such that $v_t < x_J$ if $0 \leq t \leq t^*$, or else $v_t \geq x_J$. Thus the assertion holds from the changes in the sign of $J(x)$ at $x = x_J$.

(b1ii) Let $\rho \geq x_J$.

1. Suppose $\rho < x_B$. Then we get $x_J \leq v_0 = \rho < x_B = v$. Hence $x_J \leq v_t$ for $t \geq 0$ due to Lemma 6.1, implying that $J(v_t) \leq 0$ or $J(v_t) \geq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

2. Suppose $\rho \geq x_B$. Then we obtain $v_0 = \rho \geq x_B = v$. Hence we have $v_t \geq v = x_B > x_J$ due to Lemmas 6.1 and 6.2(b), implying that $J(v_t) \leq 0$ or $J(v_t) \geq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

(b2) Let $x_B = x_J$. Then we have $x_J = x_B = v < b$. If $\rho < (\geq) x_J$, then $v_0 < (\geq) x_J = x_B = v$, hence $v_t < (\geq) x_J$ for $t \geq 0$ due to Lemmas 6.1 and 6.2(b), implying that $J(v_t) \leq 0$ or $J(v_t) \geq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

(b3) Let $x_J > x_B$. Then we have $x_J > x_B = v$.

(b3i) Let $\rho \leq x_J$.

1. Suppose $\rho < x_B$. Then $v_0 = \rho < x_B = v < x_J$. Hence $v_t < x_J$ for $t \geq 0$ due to Lemma 6.1, implying that $J(v_t) \leq 0$ or $J(v_t) \geq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

2. Suppose $\rho \geq x_B$. Then we get $x_J \geq v_0 = \rho \geq x_B = v$. Hence we have $x_J \geq v_t \geq v = x_B$ due to Lemmas 6.1 and 6.2(b), implying that $J(v_t) \leq 0$ or $J(v_t) \geq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

(b3ii) Let $\rho > x_J$. Then we get $v_0 = \rho > x_J > x_B = v$. From Lemmas 6.1 and 6.2(b), we see that v_t passes through x_J as t increases from 0 to ∞. Hence there exists a $t^* \geq 0$ such that $v_t > x_J$ if $0 \leq t \leq t^*$, or else $v_t \leq x_J$. Thus the assertion holds from the changes in the sign of $J(x)$ at $x = x_J$. ∎

Note 6.1   Suppose there exists a $t^* \geq 0$ such that $v_t > (<) \; x_J$ if $0 \leq t \leq t^*$, or else $v_t \leq (\geq) \; x_J$. Then we have the following cases:

1. If $0 \leq t \leq t^*$, then $J(v_t) > (<) \; 0$, or else $J(v_t) \leq (\geq) \; 0$.

2. If $0 \leq t \leq t^*$, then $J(v_t) \geq (\leq) \; 0$, or else $J(v_t) < (>) \; 0$.

3. If $0 \leq t \leq t^*$, then $J(v_t) \geq (\leq) \; 0$, or else $J(v_t) \leq (\geq) \; 0$.   ⬚

Lemma 6.7   *Suppose the equation $J(x) = 0$ has 2 solutions, x_J^1 and x_J^2, on the interval $(\min\{x^\star, a^\circ\}, b)$, i.e., $\min\{x^\star, a^\circ\} < x_J^1 < x_J^2 < b$.*

(a)   *Let $(1 - \beta)^2 + h^2 = 0$.*

   1. *Let $\rho < x_J^1$. Then the optimal decision rule has 2-switching property for $t \geq 0$.*

   2. *Let $x_J^1 \leq \rho < x_J^2$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

   3. *Let $\rho \geq x_J^2$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

(b)   *Let $(1 - \beta)^2 + h^2 \neq 0$.*

   1. *Let $x_J^2 < x_B$.*

      i. *Let $\rho < x_J^1$. Then the optimal decision rule has 2-switching property for $t \geq 0$.*

      ii. *Let $x_J^1 \leq \rho < x_J^2$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

      iii. *Let $\rho \geq x_J^2$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

   2. *Let $x_B = x_J^2$.*

      i. *Let $\rho < x_J^1$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

      ii. *Let $x_J^1 \leq \rho$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

   3. *Let $x_J^1 < x_B < x_J^2$.*

      i. *Let $\rho < x_J^1$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

      ii. *Let $x_J^1 \leq \rho \leq x_J^2$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

      iii. *Let $\rho > x_J^2$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

   4. *Let $x_B = x_J^1$.*

      i. *Let $\rho \leq x_J^2$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

      ii. *Let $x_J^2 < \rho$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

   5. *Let $x_B < x_J^1$.*

      i. *Let $\rho \leq x_J^1$. Then the optimal decision rule has 0-switching property for $t \geq 0$.*

      ii. *Let $x_J^1 < \rho \leq x_J^2$. Then the optimal decision rule has 1-switching property for $t \geq 0$.*

      iii. *Let $\rho > x_J^2$. Then the optimal decision rule has 2-switching property for $t \geq 0$.*

Proof.   Noting the definition of x_J and the optimal decision rule in Section 5, we shall draw attention to the changes in the sign of the function $J(x)$ at $x = x_J^1$ and $x = x_J^2$. In addition, note that $v_0 = \rho$ from Eq. (4.1).

(a) Let $(1 - \beta)^2 + h^2 = 0$. Then $v \geq b = x_B$ due to Lemma 6.3(a), hence $x_J^1 < x_J^2 < b = x_B \leq v$ by assumption.

(a1) Let $\rho < x_J^1$. Then we get $v_0 = \rho < x_J^1 < x_J^2 < b = x_B \leq v$. From Lemmas 6.1 and 6.2(b) we see that v_t passes through x_J^1 and x_J^2 sequentially as t increases from 0 to ∞. Hence there exist t_1^* and t_2^* $(0 \leq t_1^* < t_2^* < \infty)$ such that $v_t < x_J^1$ if $0 \leq t \leq t_1^*$, $x_J^1 \leq v_t < x_J^2$ if $t_1^* < t \leq t_2^*$, and $x_J^2 \leq v_t$ if $t_2^* < t$. Thus the assertion holds from the changes in the sign of $J(x)$ at $x = x_J^1$ and $x = x_J^2$.

(a2) Let $x_J^1 \leq \rho < x_J^2$. Then we have $x_J^1 \leq v_0 = \rho < x_J^2 < b = x_B \leq v$. From Lemmas 6.1 and 6.2(b) we see that v_t passes through x_J^2 as t increases from 0 to ∞. Hence there exists a $t^* \geq 0$ such that $v_t < x_J^2$ if $0 \leq t \leq t^*$, or else $v_t \geq x_J^2$. Thus the assertion holds from the changes in the sign of $J(x)$ at $x = x_J^2$.

(a3) Let $\rho \geq x_J^2$.

1. Suppose $\rho < x_B$. Then $x_J^2 \leq v_0 = \rho < b = x_B \leq v$. Hence $x_J^2 \leq v_t$ for $t \geq 0$ due to Lemma 6.1, implying that $J(v_t) \geq 0$ or $J(v_t) \leq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

2. Suppose $\rho \geq x_B$. Then $v_0 = \rho \geq x_B$. Hence we have $v_t \geq v \geq x_B = b > x_J^2$ for $t \geq 0$ due to Lemmas 6.1 and 6.2(b), implying that $J(v_t) \geq 0$ or $J(v_t) \leq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$ from Lemma 6.5(a).

(b) Almost the same as the proof of (a). ∎

If the equation $J(x) = 0$ has $N \geq 3$ solutions on the interval $(\min\{x^\star, a^\circ\}, b)$, $x_J^1, x_J^2, \cdots, x_J^N$, such that $\min\{x^\star, a^\circ\} < x_J^1 < x_J^2 < \cdots < x_J^N < b$, we can immediately generalize Lemma 6.7. Since this generalization involves merely expansion of Lemma 6.7, we can easily prescribe the optimal decision rule for $N \geq 3$ solutions in almost the same way as the above.

7 Numerical Experiments

Using some numerical examples, we shall demonstrate the optimal decision rules for Lemma 6.6 and for the cases in which $J(x) = 0$ has multiple solutions.

1. Let $\beta = 0.99$, $\lambda = 0.5$, and let $F(w)$ be the uniform distribution on $[1.5, 2.5]$, i.e., $a = 1.5$ and $b = 2.5$. Hence the condition $(1 - \beta)^2 + h^2 \neq 0$ in Lemma 6.6(b) is satisfied. In addition, let $Q(\alpha)$ be a uniform distribution on $[0.7, 0.9]$. Then clearly we have

$$p(z) = \begin{cases} 1, & z \leq 1.5, \\ 2.5 - z, & 1.5 \leq z \leq 2.5, \\ 0, & 2.5 \leq z, \end{cases}$$

from which we easily obtain

$$T_p(x) = \begin{cases} 1.5 - x, & x \leq 0.5, \\ 0.25(2.5 - x)^2, & 0.5 \leq x \leq 2.5, \\ 0, & 2.5 \leq x, \end{cases} \qquad T_s(x) = \begin{cases} 2.0 - x, & x \leq 1.5, \\ 0.5(2.5 - x)^2, & 1.5 \leq x \leq 2.5, \\ 0, & 2.5 \leq x. \end{cases}$$

The solution of $J(x) = 0$, denoted by x_J, can be easily obtained by numerical calculation using Eq. (3.5) and numerical integration (trapezoidal rule). In this case, we see that only one x_J exists,

where $x_J \simeq 1.1339$. Figure 7.4 depicts two graphs of $J(x)$ and v_t with $h = 0.05$ and $h = 0.4$, respectively.
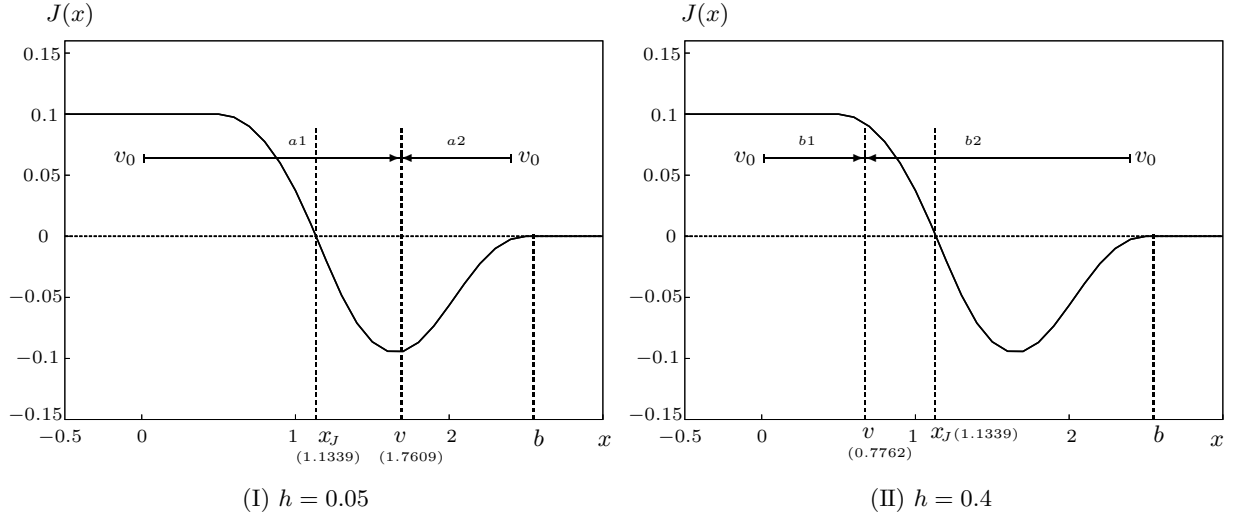

Figure 7.4: Optimal decision rule

(a) Suppose $h = 0.05$. Then we have $x_B = v \simeq 1.7609 > 1.1339 \simeq x_J$ from Eq. (3.6) by numerical calculation.

1. Let $\rho = 0.1$. Then $v_0 = 0.1 < 1.1339 \simeq x_J$, i.e., the condition in Lemma 6.6(b1i) is satisfied. Figure 7.4(I) shows that v_t is strictly increasing in t with $v > x_J$ and that there exists a $t^* = 2$ such that $v_t < x_J$ if $0 \leq t \leq t^*$ or else $v_t \geq x_J$. Therefore, the optimal decision rule has 1-switching property for $t \geq 0$.

2. Let $\rho = 2.5$. Then $v_0 = 2.5 > 1.1339 \simeq x_J$, i.e., the condition in Lemma 6.6(b1ii) is satisfied. Figure 7.4(I) demonstrates that v_t is strictly decreasing in t with $v_t > x_J$ for $t \geq 0$, i.e., $J(v_t) \leq 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$.

(b) Suppose $h = 0.4$. Then we have $x_B = v \simeq 0.7762 < 1.1339 \simeq x_J$ from Eq. (3.6).

1. Let $\rho = 0.1$. Then $v_0 = 0.1 < 1.1339 \simeq x_J$, i.e., the condition in Lemma 6.6(b3i) is satisfied. Figure 7.4(II) shows that v_t is strictly increasing in t with $v_t < x_J$ for $t \geq 0$, i.e., $J(v_t) > 0$ for $t \geq 0$. Thus the optimal decision rule has 0-switching property for $t \geq 0$.

2. Let $\rho = 2.5$. Then $v_0 = 2.5 > 1.1339 \simeq x_J$, i.e., the condition in Lemma 6.6(b3ii) is satisfied. Figure 7.4(II) demonstrates that v_t is strictly decreasing in t with $v < x_J$ and that there exists a $t^* = 3$ such that $v_t > x_J$ if $0 \leq t \leq t^*$ or else $v_t \geq x_J$. Therefore, the optimal decision rule has 1-switching property for $t \geq 0$.

2. Let $f(w)$ be the density function as shown in Figure 3.3 (I). Also, let $\beta = 0.99$, $\lambda = 0.5$, and $Q(\alpha)$ be a uniform distribution on $[0.64, 0.74]$. Then we obtained $x_J^1 \simeq -0.3288$, $x_J^2 \simeq 0.4625$, and $x_J^3 \simeq 0.7471$ using Eq. (3.5).

(a) Let $h = 0.01$ and $\rho = -3$. Then we obtained $x_B \simeq 1.3506 > x_J^3 > x_J^2 > x_J^1 > -3 = \rho = v_0$ due to Eq. (4.1). Figure 7.5(I) shows that v_t is strictly increasing in t and passes through x_J^1, x_J^2, and x_J^3 sequentially. From Figure 7.5(I) we see that there exist $t_1^* = 2$, $t_2^* = 5$, and $t_3^* = 12$ such that $v_t < x_J^1$ if $0 \leq t \leq t_1^*$, $x_J^1 \leq v_t < x_J^2$ if $t_1^* < t \leq t_2^*$, $x_J^2 \leq v_t < x_J^3$ if $t_2^* < t \leq t_3^*$, and $x_J^3 \leq v_t$ if $t_3^* < t$. Therefore, the optimal decision rule has 3-switching property for $t \geq 0$.

15

(b) Let $h = 0.5$ and $\rho = 3$. Then we obtained $x_B \simeq -0.4352 < x_J^1 < x_J^2 < x_J^3 < 3 = \rho = v_0$ due to Eq. (4.1). Figure 7.5(II) demonstrates that v_t is strictly decreasing in t and passes through x_J^3, x_J^2, and x_J^1 sequentially. From Figure 7.5(II) we see that there exist $t_3^* = 4$, $t_2^* = 5$, and $t_1^* = 8$ such that $x_J^3 < v_t$ if $0 \le t \le t_3^*$, $x_J^2 < v_t \le x_J^3$ if $t_3^* < t \le t_2^*$, $x_J^1 < v_t \le x_J^2$ if $t_2^* < t \le t_1^*$, and $v_t \le x_J^1$ if $t_1^* < t$. Therefore, the optimal decision rule has 3-switching property for $t \ge 0$.

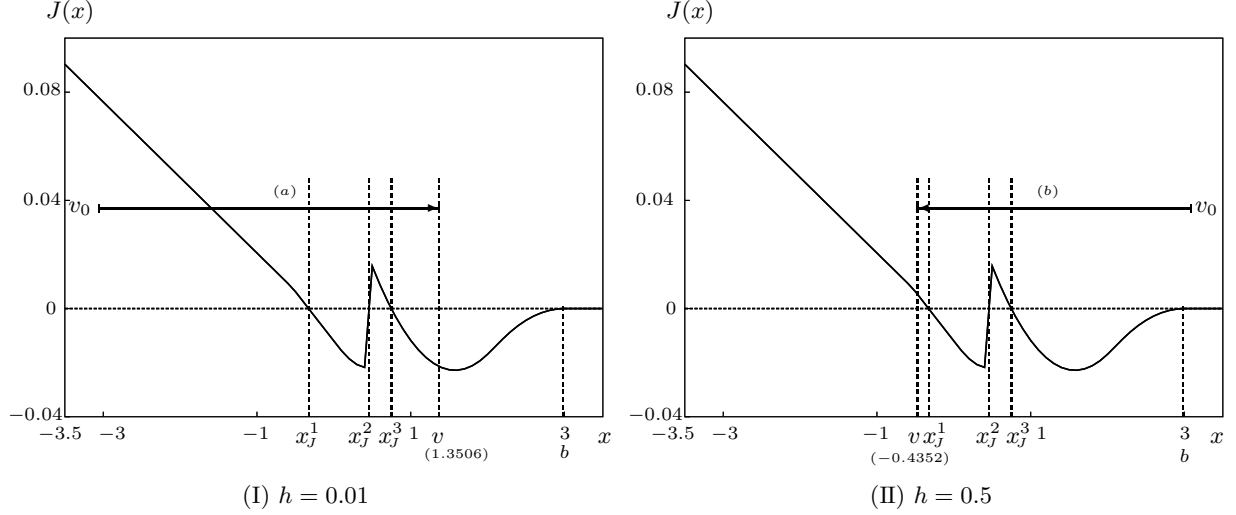| (I) $h = 0.01$ | (II) $h = 0.5$ |

Figure 7.5: Optimal decision rule

8 Conclusions and Suggested Future Studies

In this paper we proposed a basic model of an asset selling problem where the seller can switch between proposing a selling price to the appearing buyer and concealing the price. From our analysis, we obtained some conditions that guarantee the existence of the switching property. Below, we shall state the two distinctive points derived from our analysis.

1. From Lemmas 6.6 and 6.7 we see that the optimal decision rule possesses the switching property even for a simple case likes $\beta = 1$ and $h = 0$.

2. It is possible that the conjecture stated in Section 1 fails to hold. In fact, from Figure 7.4(I), we see that there exists a $t^* \ge 0$ such that it is optimal to conceal the selling price for $0 \le t \le t^*$ and to switch from concealing the price to proposing the price for $t^* < t$. This result runs counter to our conjecture. In addition, from the Figure 7.5(I), we also see that there exist t_1^*, t_2^*, and t_3^* $(0 \le t_1^* < t_2^* < t_3^* < \infty)$ such that it is optimal to conceal the selling price for $0 \le t \le t_1^*$, to switch from concealing the price to proposing the price for $t_1^* < t \le t_2^*$, to switch from proposing the price to concealing the price for $t_2^* < t \le t_3^*$, and again to switch from concealing the price to proposing the price for $t_3^* < t$.

Finally, in order to make our approach more practical, the following should be considered for future studies:

1. A general case where multiple items are to be sold.

2. A seller may have the opportunity of selling a product to a salvage dealer not only on the deadline but also at any point in time before the deadline if he wishes.

16

3. A seller can advertise to attract customers by paying some cost, called the search cost. The introduction of a search cost inevitably yields the option whether to conduct or to skip the search.

4. α and w are stochastically dependent.

Appendices : Proofs

A. Lemma 3.3

First note that since $T_s(0) = \mu$ from Eq. (3.1), we have $\mathcal{T}_s(0) = \mathbf{E}[\alpha T_s(0)] = \mathbf{E}[\alpha \mu] = \mu_\alpha \mu > 0$ by assumptions.

(a) Evident from Lemma 3.1(c) and Eq. (3.4).

(b) If $x \geq b$, then $x/\alpha \geq b$ due to the assumption of $0 < \alpha \leq 1$, hence $\mathcal{T}_s(x) = \mathbf{E}[\alpha T_s(x/\alpha)] = 0$ from Lemma 3.1(b). From this result, (a), and the fact that $\mathcal{T}_s(0) > 0$, there exists a supremum of x such that $\mathcal{T}_s(x) > 0$; the supremum is given by b° from Eq. (3.9). Accordingly, $\mathcal{T}_s(x) > 0$ for $x < b^\circ$ and $\mathcal{T}_s(x) = 0$ for $x \geq b^\circ$ where clearly $b^\circ \leq b$.

(c) Let $x < b^\circ$, hence $\mathcal{T}_s(x) > 0$ due to (b). Here, note that $\mathcal{T}_s(x)$ is nonincreasing on $(-\infty, \infty)$ from (a). Now, suppose there exist x' and x'' with $x' < x'' < b^\circ$ such that $\mathcal{T}_s(x)$ is a positive constant on $[x', x'']$ and strictly decreasing on $[x'', x'' + \epsilon]$ with $x'' + \epsilon < b^\circ$ for an infinitesimal $\epsilon > 0$ such that $x' < x'' - \epsilon < x'' < x'' + \epsilon < b^\circ$. Since $\mathcal{T}_s(x'') = \mathcal{T}_s(x'' - \epsilon)$ and $\mathcal{T}_s(x'' + \epsilon) < \mathcal{T}_s(x'')$ by assumption, we get $\big(\mathcal{T}_s(x'' + \epsilon) - \mathcal{T}_s(x'')\big) - \big(\mathcal{T}_s(x'') - \mathcal{T}_s(x'' - \epsilon)\big) = \mathcal{T}_s(x'' + \epsilon) - \mathcal{T}_s(x'') < 0$, i.e., $\mathcal{T}_s(x'' + \epsilon) - \mathcal{T}_s(x'') < \mathcal{T}_s(x'') - \mathcal{T}_s(x'' - \epsilon)$, which contradicts the fact that $\mathcal{T}_s(x)$ is convex on $(-\infty, \infty)$ due to (a). Therefore, it follows that there do not exist x' and x'' such as defined above; hence, it must be that $\mathcal{T}_s(x)$ is strictly decreasing in $x < b^\circ$, thus $x \leq b^\circ$ due to Proposition 3.1.

(d) Let $y < x$. Then noting Eqs. (3.4) and (3.11), we get

$$\mathcal{T}_s(x) + x - \mathcal{T}_s(y) - y = \mathbf{E}[\alpha\big(T_s(x/\alpha) - T_s(y/\alpha) + x/\alpha - y/\alpha\big)]$$
$$\geq \mathbf{E}[\alpha(x/\alpha - y/\alpha)F(y/\alpha)] = (x - y)\mathbf{E}[F(y/\alpha)] \geq 0,$$

so that $\mathcal{T}_s(y) + y \leq \mathcal{T}_s(x) + x$. Thus $\mathcal{T}_s(x) + x$ is nondecreasing on $(-\infty, \infty)$.

(e) Immediate from (d) and the fact that $\lambda \mathcal{T}_s(x) + x = \lambda(\mathcal{T}_s(x) + x) + (1 - \lambda)x$ where $\lambda < 1$ by assumption.

(f) Let $A(x) = \mathcal{T}_s(x) + x - \mu_\alpha \mu$. First, note that $\mathcal{T}_s(0) = \mu_\alpha \mu$, or equivalently, $A(0) = 0 \cdots (1^*)$. Next, since $\mu > a$ by assumption, we have $\mu/\alpha > a$ for all $\alpha \in (0, 1]$, hence $T_s(\mu/\alpha) > \mu - \mu/\alpha$ from Lemma 3.1(d), so $\alpha T_s(\mu/\alpha) > \alpha\mu - \mu$. Thus we obtain $\mathcal{T}_s(\mu) = \mathbf{E}[\alpha T_s(\mu/\alpha)] > \mathbf{E}[\alpha\mu - \mu] = \mu_\alpha\mu - \mu$, so $A(\mu) > 0 \cdots (2^*)$. In addition, since $A(x)$ is nondecreasing on $(-\infty, \infty)$ from (d), noting Eq. (1^*) and (2^*), we see that there exists a maximum x such that $A(x) = 0$, i.e., $\mathcal{T}_s(x) = \mu_\alpha\mu - x$; the maximum is given by a° from Eq. (3.9). Therefore, $A(x) > 0$, hence $\mathcal{T}_s(x) > \mu_\alpha\mu - x$ for $x > a^\circ$ and $A(x) = 0$, hence $\mathcal{T}_s(x) = \mu_\alpha\mu - x$ for $x \leq a^\circ$, so $A(a^\circ) = 0 \cdots (3^*)$.

(g) Since $T_s(a/\alpha) \geq \mu - a/\alpha$ from Lemma 3.1(d), we have $\alpha T_s(a/\alpha) \geq \alpha\mu - a$. Thus we obtain $\mathcal{T}_s(a) = \mathbf{E}[\alpha T_s(a/\alpha)] \geq \mathbf{E}[\alpha\mu - a] = \mu_\alpha\mu - a$, so $A(a) = \mathcal{T}_s(a) + a - \mu_\alpha\mu \geq 0 = A(a^\circ)$ due to (3^*). Since $A(x)$ is nondecreasing on $(-\infty, \infty)$ from the proof of (f), we have $a \geq a^\circ$. Further, since $A(0) = 0$ from (1^*), we have $\mathcal{T}_s(0) = \mu_\alpha\mu - 0$, hence $a^\circ \geq 0$ due to Eq. (3.9). Accordingly, the former half of the

17

assertion holds. Suppose $\mu_\alpha\mu < a^\circ$. Then from (f) we get $\mathcal{T}_s(a^\circ) = \mu_\alpha\mu - a^\circ < 0$, which contradicts the fact that $\mathcal{T}_s(x) \geq 0$ from (b). Hence, it must be that $\mu_\alpha\mu \geq a^\circ$. In addition, since $0 = \mathcal{T}_s(b^\circ) \geq \mu_\alpha\mu - b^\circ$ from (f,b), we get $b^\circ \geq \mu_\alpha\mu$.

(h)   Immediate from (f).

(i)   From Eq. (3.4) and Lemma 3.1(e) we get, for any x and y,

$$
\begin{aligned}
|\mathcal{T}_s(x) - \mathcal{T}_s(y)| &= |\mathbf{E}[\alpha\mathcal{T}_s(x/\alpha)] - \mathbf{E}[\alpha\mathcal{T}_s(y/\alpha)]| \\
&= \mathbf{E}[\alpha|(\mathcal{T}_s(x/\alpha) - \mathcal{T}_s(y/\alpha)|] \leq \mathbf{E}[\alpha|x/\alpha - y/\alpha|] = |x - y|. \quad \blacksquare
\end{aligned}
$$

B.   Lemma 3.4

(a)   Let $x \leq b$. Then $T_p(x)$ is strictly decreasing in $x \leq b$ from Lemma 3.2(b) and $\mathcal{T}_s(x)$ is strictly decreasing in $x \leq b^\circ$ from Lemma 3.3(c). Since $b^\circ \leq b$ from Lemma 3.3(b), clearly $B(x)$ is strictly decreasing in $x \leq b^\circ$. If $b^\circ = b$, then clearly $B(x)$ is strictly decreasing in $x \leq b$. Suppose $b^\circ < b$. Let $b^\circ < x \leq b$. Since $\mathcal{T}_s(x) = 0$ due to Lemma 3.3(b) and $T_p(x) \geq 0$ due to Lemma 3.2(a), we obtain $\mathcal{T}_s(x) \leq T_p(x)$. Hence we have $B(x) = \lambda\beta T_p(x) - (1 - \beta)x - h$, which is strictly decreasing on $(b^\circ, b]$ due to Lemma 3.2(b). Thus it follows that $B(x)$ is strictly decreasing on $(-\infty, b]$.

(b)   Obvious from Eq. (3.7), Lemmas 3.3(h), and 3.2(e).

(c)   Let $(1 - \beta)^2 + h^2 = 0$. Then $B(x) = \lambda \max\{\mathcal{T}_s(x), T_p(x)\}$.

(c1)   If $x \geq b$, then $B(x) = 0$ due to $\mathcal{T}_s(x) = 0$ from Lemma 3.3(b) and $T_p(x) = 0$ from Lemma 3.2(a). If $x < b$, then $T_p(x) > 0$ from Lemma 3.2(a), hence $B(x) \geq \lambda T_p(x) > 0$. Thus $x_B = b$ by the definition of x_B.

(c2)   Evident from the proof of (c1)

(d)   Let $(1 - \beta)^2 + h^2 \neq 0$.

(d1)   Note that $B(b) = -(1 - \beta)b - h < 0$ from Eq. (3.7), Lemmas 3.3(b), and 3.2(a). From this result and (a,b) it follows that x_B uniquely exists. The inequality $x_B < b$ is immediate from $B(b) < 0$ and (a).

(d2) Evident from (d1,a). $\blacksquare$

C.   Lemma 3.5

(a)   Let $x \leq \min\{x^\star, a^\circ\}$. Then since $x \leq x^\star \leq a^\star$ from Lemma 3.2(g), we have $T_p(x) = a - x$ from Lemma 3.2(h). In addition, since $x \leq a^\circ \leq a$ due to Lemma 3.3(g), we get $\mathcal{T}_s(x) = \mu_\alpha\mu - x$ from Lemma 3.3(f). Therefore, from Eq. (3.5) we obtain $J(x) = \mu_\alpha\mu - x - (a - x) = \mu_\alpha\mu - a$. Further, if $a^\circ \geq x^\star$, then $\min\{x^\star, a^\circ\} = x^\star \leq a^\circ \leq b^\circ$ from Lemma 3.3(g). If $a^\circ < x^\star$, then $\min\{x^\star, a^\circ\} = a^\circ \leq b^\circ$ from Lemma 3.3(g). Thus we have $\min\{x^\star, a^\circ\} \leq b^\circ$ whether $a^\circ \geq x^\star$ or $a^\circ < x^\star$.

(b)   Let $b^\circ < b$. Then from Eq. (3.5), Lemmas 3.3(b), and 3.2(a) we obtain $J(x) = -T_p(x) < 0$ for $b^\circ \leq x < b$. Since $T_p(x)$ is strictly decreasing on $[b^\circ, b)$ due to Lemma 3.2(b), it follows that $J(x)$ is strictly increasing on $[b^\circ, b)$.

(c)   Let $x \geq b$, hence $x/\alpha \geq b$. Then $J(x) = 0$ for $x \geq b$ due to $T_p(x) = 0$ from Lemma 3.2(a) and

18

$\mathcal{T}_s(x) = 0$ from Lemma 3.3(b). ∎

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