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D.C. OPTIMIZATION METHODS FOR THE MINIMUM MAXIMAL FLOW PROBLEM

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Abstract. This paper is concerned with the minimum maximal flow problem, i.e., a problem of minimizing the flow value attained by a maximal flow for a given network. The optimal value indicates how inefficiently the network can be utilized under restricted controllability. We discuss the extension of the gap function defining the set of all maximal flows as well as the regularity of this problem, and then formulate the problem as a D.C. optimization problem. Based on this formulation, we propose two algorithms: the cut-and-split method and the outer approximation method combined with a local search technique, and discuss their finite convergence.

1. Introduction

We consider a network $(V, s, t, E, \partial^+, \partial^-, \boldsymbol{c})$, where V is the set of $m+2$ nodes containing the source node s and the sink node t, E is the set of n arcs, ∂^+ and ∂^- are incidence functions, and $\boldsymbol{c}$ is the vector of dimension n whose hth element c_h is the capacity of arc h. We assume that each element of $\boldsymbol{c}$ is a positive integer throughout this paper. A vector $\boldsymbol{x}$ of dimension n is said to be a *feasible flow* if it satisfies the inequality $\mathbf{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c}$, called the *capacity constraints*, and the equality $A\boldsymbol{x} = \mathbf{0}$, called the *conservation constraints*, where the $m \times n$ matrix A is the incidence matrix whose (v, h) element a_{vh} is given by

$$a_{vh} = \begin{cases} +1 & \text{if } \partial^+ h = v \\ -1 & \text{if } \partial^- h = v \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Let X denote the set of all feasible flows, i.e.,

$$X = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \mathbf{0},\ \mathbf{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}, \tag{1.2}$$

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and X_V denote the set of all vertices of X. Note that each vertex $\boldsymbol{v} \in X_V$ is an integer vector by the total unimodularity of A and the integrality of the capacity vector $\boldsymbol{c}$. For a feasible flow $\boldsymbol{x}$, the *flow value of* $\boldsymbol{x}$, denoted by $\phi(\boldsymbol{x})$, is given by

$$\phi(\boldsymbol{x}) = \sum_{\partial^+ h = s} x_h - \sum_{\partial^- h = s} x_h. \tag{1.3}$$

Using the row vector $\boldsymbol{d}$ of dimension n whose element d_h is

$$d_h = \begin{cases} +1 & \text{if } \partial^+ h = s \\ -1 & \text{if } \partial^- h = s \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

we can simply write $\phi(\boldsymbol{x}) = \boldsymbol{dx}$. We assume that a given network has no t-s-path, which ensures that $\boldsymbol{dx} \geqq 0$ for all $\boldsymbol{x} \in X$. A feasible flow $\boldsymbol{x}$ is said to be a *maximal flow* if there is no feasible flow $\boldsymbol{y}$ such that $\boldsymbol{y} \geqq \boldsymbol{x}$ and $\boldsymbol{y} \neq \boldsymbol{x}$. We denote the set of all maximal flows by X_M, i.e.,

$$X_M = \{\, \boldsymbol{x} \in X \mid \nexists \boldsymbol{y} \in X \ : \ \boldsymbol{y} \geqq \boldsymbol{x},\ \boldsymbol{y} \neq \boldsymbol{x} \,\}. \tag{1.5}$$

We assume that X_M is nonempty. The problem of minimizing flow value among the maximal flows, called the *minimum maximal flow problem* and abbreviated by (mmF), is written as

$$(mmF) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \phi(\boldsymbol{x}) = \boldsymbol{dx} \\ \text{s.t.} & \boldsymbol{x} \in X_M. \end{array} \right.$$

The difficulty of the problem comes from the nonconvexity of X_M, and hence sophisticated algorithms are required. Note that (mmF) embraces the minimum maximal matching problem, which is an NP-hard problem (see e.g. Garay-Johnson [12]).

In the field of network flow theory such as maximum flow problem and minimum cost flow problem, we usually take it for granted that we can control each arc flow, namely we can freely increase and decrease each arc flow as long as the feasibility is met. However, when we are not allowed to decrease arc flows, the solution to be obtained depends on a given initial flow. Under the condition of such uncontrollability, it is meaningful to know the minimum flow value among the maximal flows since this value indicates how inefficiently a network can be utilized. The concept of uncontrollable flow was first raised by Iri [17, 18], which is closely related to but different from the maximal flow. Shi-Yamamoto first raised (mmF) and proposed an algorithm in [26]. After this several algorithms for (mmF) combining local search and global optimization technique have been proposed in e.g. Gotoh-Thoai-Yamamoto [13], Shigeno-Takahashi-Yamamoto [27]. As will be seen

in the next section, (mmF) is a special case of linear optimization problems over the efficient set. The algorithms for (mmF) mentioned above are mainly based on the algorithms for the linear optimization problem over the efficient set. However, we have neither theoretical evidence that these algorithms are efficient, nor comparative study from the viewpoint of computational time.

The purpose of this paper is to propose an algorithm for the minimum maximal flow problem within the framework of D.C. optimization. The algorithm is based on local search and global optimization technique. In the part of the global optimization technique, we apply either the cut-and-split method or the outer approximation method.

In Section 2 we briefly review some fundamental theorems of the linear optimization problem over the efficient set, and then we reformulate the problem by the gap function. We also mention the connectedness of the efficient set. Section 3 is devoted to review of D.C. optimization. We explain two methods: the cut-and-split method and the outer approximation method. In Section 4 we extend the gap function and apply D.C. optimization methods to (mmF). The convergence of each algorithm is discussed. Finally, some conclusion and further works will be discussed in the last section.

Throughout this paper we use the following notations: $\mathbb{R}^n$ denotes the set of all real column vectors of dimension n. Let $\mathbb{R}^n_+ = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geqq \boldsymbol{0} \,\}$ and $\mathbb{R}^n_{++} = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} > \boldsymbol{0} \,\}$. Let $\mathbb{R}_n$ denote the set of all real row vectors of dimension n, $\mathbb{R}_{n+}$ and $\mathbb{R}_{n++}$ are defined in the similar way. We use $\boldsymbol{e}$ to denote the row vector of ones, $\boldsymbol{1}$ to denote the column vector of ones, and $\boldsymbol{e}_i$ to denote the ith unit row or column vector of an appropriate dimension. Let I denote the identity matrix of an appropriate size. We use $\boldsymbol{a}^\top$ and $A^\top$ to denote the transposed vector of $\boldsymbol{a}$ and the transposed matrix of A, respectively. For a set S, we denote the interior of S by $\operatorname{int} S$, the closure of S by $\operatorname{cl} S$, and the relative boundary of S by ∂S. We use P_V to denote the set of all vertices of a polyhedron P. For two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ of dimension n, let $[\boldsymbol{v}, \boldsymbol{w}]$ denote the line segment with endpoints $\boldsymbol{v}$ and $\boldsymbol{w}$, and let $(\boldsymbol{v}, \boldsymbol{w}] = [\boldsymbol{v}, \boldsymbol{w}] \backslash \{\boldsymbol{v}\}$. Also $[\boldsymbol{v}, \boldsymbol{w})$ and $(\boldsymbol{v}, \boldsymbol{w})$ are defined in the similar way.

2. Linear Optimization Problem over the Efficient Set

Given a polyhedron $D = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid B\boldsymbol{x} \leqq \boldsymbol{z},\ \boldsymbol{x} \geqq \boldsymbol{0} \,\}$ with $B \in \mathbb{R}^{m \times n}$ and $\boldsymbol{z} \in \mathbb{R}^m$, and a criterion matrix $C \in \mathbb{R}^{p \times n}$ with $p \geqq 2$, the *linear multicriteria problem* is

$$(MC) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & C\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D. \end{array} \right.$$

The point $\boldsymbol{x} \in D$ is said to be an *efficient point for* (MC) if there is no point $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$ and $C\boldsymbol{y} \neq C\boldsymbol{x}$. The *efficient set for* (MC), denoted by D_E, is the set of all efficient points for (MC), i.e.,

$$D_E = \{\, \boldsymbol{x} \in D \mid \nexists \boldsymbol{y} \in D : C\boldsymbol{y} \geqq C\boldsymbol{x},\ C\boldsymbol{y} \neq C\boldsymbol{x} \,\}. \tag{2.1}$$

We assume that D is bounded and D_E is nonempty for simplicity. The *linear optimization problem over the efficient set* is

$$(P_E) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D_E, \end{array} \right.$$

where $\boldsymbol{p} \in \mathbb{R}_n$. Note that (mmF) is (P_E) with $D = X$ and $C = I$, hence $D_E = X_M$, and $\boldsymbol{p} = \boldsymbol{d}$. Figure 2.1 shows a two-dimensional example of the problem (P_E), where $\boldsymbol{c}^i$ is the ith row of C for $i = 1, 2$. The efficient set D_E is depicted by bold lines.

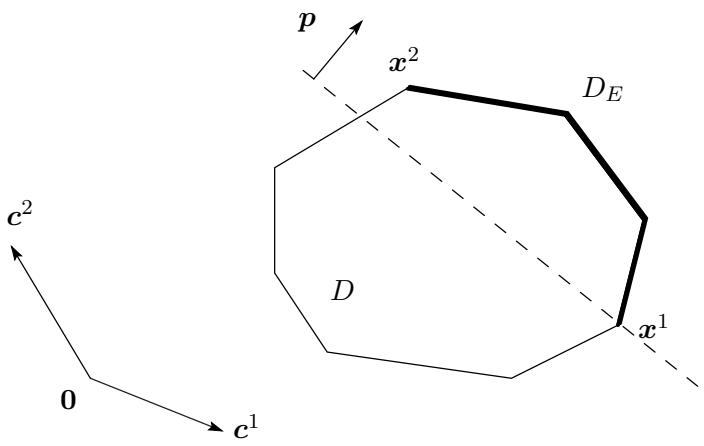


FIGURE 2.1. A two-dimensional example of (P_E)

Since Philip first considered (P_E) and proposed an algorithm based on local search and cutting plane technique in [22], a number of papers followed his work. The overview about the efficient set and several algorithms for (P_E) can be found in Yamamoto [34]. For the details about (P_E), the reader should refer to White [33], Sawaragi-Nakayama-Tanino [24], Steuer [28]. The mathematical structure of the efficient set is studied in Naccache [21], Benson [5] and Hu-Sun [16]. The method enumerating the efficient vertices for (MC) can be found in Ecker-Kouada [8, 9]. For solution methods for (P_E), see Benson [2–4], Bolintineanu [6], Ecker-Song [10],

Fülöp [19], Dauer-Fosnaugh [7], Thach-Konno-Yokota [29], Sayin [25], Phong-Tuyen [23], Thoai [30] and An-Tao-Thoai [1].

2.1. **Gap Function and Some Fundamental Theorems.**

We define the *gap function* $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ by

$$g(\boldsymbol{x}) = \max\{\, \boldsymbol{e}C\boldsymbol{y} \mid \boldsymbol{y} \in D,\ C\boldsymbol{y} \geqq C\boldsymbol{x} \,\} - \boldsymbol{e}C\boldsymbol{x}. \tag{2.2}$$

If there is no point $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$ then $g(\boldsymbol{x}) = -\infty$. Note that $g(\boldsymbol{x}) \geqq 0$ for all $\boldsymbol{x} \in D$ and g is a concave and piece-wise linear function (see Figure 2.2). It is easily seen that

$$D_E = \{\, \boldsymbol{x} \in D \mid g(\boldsymbol{x}) \leqq 0 \,\}. \tag{2.3}$$

Indeed, a point $\boldsymbol{x} \in D$ is in D_E if and only if $C\boldsymbol{y} = C\boldsymbol{x}$ for all $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$, that is equivalent to $g(\boldsymbol{x}) \leqq 0$. Then the alternative form of (P_E) is

$$(P_E) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D,\ -g(\boldsymbol{x}) \geqq 0. \end{array} \right.$$

Since $-g$ is a convex function, the inequality constraint $-g(\boldsymbol{x}) \geqq 0$ is a *reverse convex constraint*, and hence (P_E) is a *linear reverse convex problem*, which is one of D.C. optimization problems (See Tuy [32] and Horst-Tuy [15]).

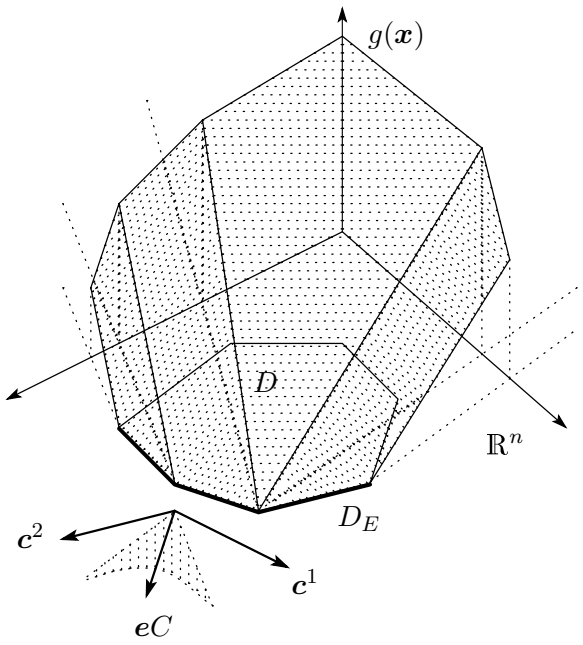


FIGURE 2.2. An example of the gap function g

We introduce some important theorems about (P_E), whose proofs can be found in e.g. Steuer [28] and White [33]. We will outline some of the proofs to make this paper self-contained.

Theorem 2.1. *The point $\bar{\boldsymbol{x}} \in D$ is an efficient point for (MC) if and only if there exists $\boldsymbol{\lambda} \in \mathbb{R}_{p++}$ such that $\bar{\boldsymbol{x}}$ is an optimal solution of the single criterion problem*

$$(SC(\boldsymbol{\lambda})) \qquad \left| \begin{array}{ll} \max\limits_{\boldsymbol{x}} & \boldsymbol{\lambda} C \boldsymbol{x} \\ s.t. & \boldsymbol{x} \in D. \end{array} \right.$$

Furthermore, there exists $M > 0$ such that we can replace the above $\mathbb{R}_{p++}$ with

$$\Lambda = \{\, \boldsymbol{\lambda} \in \mathbb{R}_{p++} \mid \boldsymbol{\lambda} \geqq \boldsymbol{e},\ \boldsymbol{\lambda}\mathbf{1} = M \,\}. \tag{2.4}$$

Proof: See Appendix. □

By Theorem 2.1, we have only to choose $\boldsymbol{\lambda} \in \Lambda$ and solve $(SC(\boldsymbol{\lambda}))$ to obtain an initial point in $D_E \cap D_V$. It was shown in Shigeno-Takahashi-Yamamoto [27] that n^2 suffices for M of (2.4), when we consider (mmF). The following theorem is also well known.

Theorem 2.2. *The set D_E is a connected union of the several faces of D.*

See Steuer [28] and Naccache [21] for the detail.

3. D.C. Optimization

A set S is said to be a *D.C. set* (difference of two convex sets) if $S = Q \backslash R$ for two convex sets Q and R. Similarly, a function f is said to be a *D.C. function* if $f = q - r$ for two convex functions q and r. The optimization problem described in terms of D.C. sets and/or D.C. functions is called *D.C. optimization problem*. D.C. optimization problem covers many of nonlinear programming problems such as location planning problem, engineering design problem, multilevel programming problem, and optimization problem over the efficient set.

Let $D = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \leqq 0 \,\}$, where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function. We assume that D is bounded for simplicity. Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function and assume

$$\mathrm{int}\,\{\, \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leqq 0 \,\} = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) < 0 \,\}. \tag{3.1}$$

In this section, we focus on a *canonical form D.C. problem*:

$$(CDC) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D,\ h(\boldsymbol{x}) \geqq 0, \end{array} \right.$$

where $\boldsymbol{p} \in \mathbb{R}_n$. Letting $H = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leqq 0 \,\}$, (CDC) can be written as

$$(CDC) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D \backslash \operatorname{int} H. \end{array} \right.$$

In the case where D is a polyhedron given by $D = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid B\boldsymbol{x} \leqq \boldsymbol{z},\ \boldsymbol{x} \geqq \boldsymbol{0} \,\}$ with $B \in \mathbb{R}^{m \times n}$ and $\boldsymbol{z} \in \mathbb{R}^m$, the above problem is called a *linear reverse convex problem* (LRCP). Figure 3.1 shows two-dimensional examples of the problems (CDC) and (LRCP), respectively. We explain two methods: the cut-and-split method and the outer approximation method. For other algorithms on D.C. optimization, the reader should refer to Tuy [31,32] and Horst-Tuy [15].

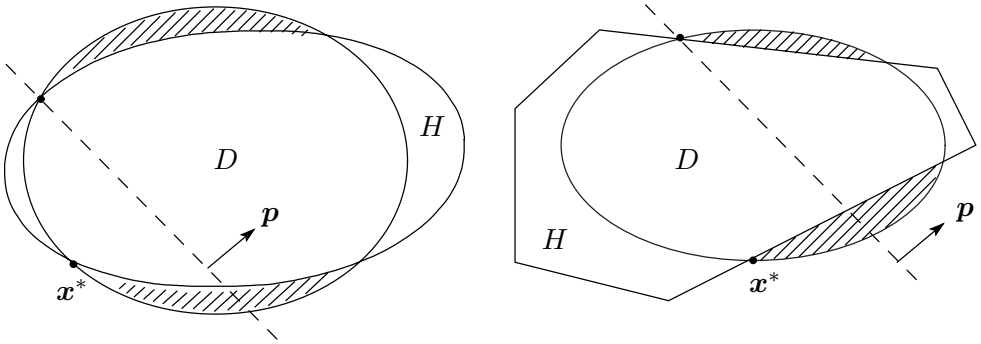


FIGURE 3.1. Two-dimensional examples of (CDC) and $(LRCP)$

3.1. **Cut and Split Method.**

To explain the cut-and-split (CS for short) method for $(LRCP)$ we assume that

$$D \subseteqq \mathbb{R}^n_+ \text{ and } \boldsymbol{0} \in D_V \cap \operatorname{int} H. \tag{3.2}$$

In the first step of the CS method for $(LRCP)$, we find an initial feasible solution $\bar{\boldsymbol{x}}$ and set up the family $\mathcal{S}$ of polyhedral cones, initially $\mathcal{S} := \{\mathbb{R}^n_+\}$. In each iteration, we calculate a lower bound with respect to K for each cone $K \in \mathcal{S}$, then we split the cone whose lower bound is minimum until the optimality condition is met. To calculate a lower bound we define a *concavity cut for* $K \backslash H$ as follows. Given a polyhedral cone K with a vertex at $\boldsymbol{0}$ and exactly n edges, let $\boldsymbol{u}^i$ denote the intersection point of ∂H and the ith edge of K for $i = 1, \ldots, n$. The *concavity cut*

$l_K(\boldsymbol{x}) \geqq 0$ *for* $K \backslash H$ is given by the linear function $l_K : \mathbb{R}^n \to \mathbb{R}$ such that

$$l_K(\boldsymbol{x}) = \boldsymbol{e}U^{-1}\boldsymbol{x} - 1, \tag{3.3}$$

where $U = [\boldsymbol{u}^1, \cdots, \boldsymbol{u}^n]$. Note that $l_K(\boldsymbol{0}) = -1$ and $l_K(\boldsymbol{u}^i) = 0$ for $i = 1, \ldots, n$. Figure 3.2 shows an example of the concavity cut. See Horst-Tuy [15] for the detail.

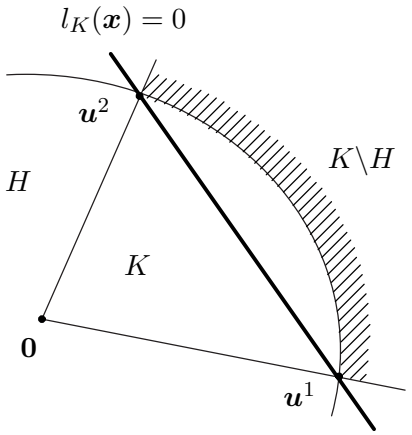


FIGURE 3.2. The concavity cut $l_K(\boldsymbol{x}) \geqq 0$ for $K \backslash H$

For each $K \in \mathcal{S}$ we solve the linear programming problem

$$(LP(K)) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D \cap K,\ l_K(\boldsymbol{x}) \geqq 0, \end{array} \right.$$

to obtain a solution $\boldsymbol{\omega}^K$ with the optimal value $\boldsymbol{p}\boldsymbol{\omega}^K$, which is a lower bound with respect to K. If $h(\boldsymbol{\omega}^K) \geqq 0$, i.e., $\boldsymbol{\omega}^K \in D_E$, and $\boldsymbol{p}\boldsymbol{\omega}^K < \boldsymbol{p}\bar{\boldsymbol{x}}$ for some K then we update the incumbent $\bar{\boldsymbol{x}}$ to $\boldsymbol{\omega}^K$. When there remain cones after discarding the cones K with $\boldsymbol{p}\boldsymbol{\omega}^K \geqq \boldsymbol{p}\bar{\boldsymbol{x}}$ if any, then we choose one of them and perform either the $\boldsymbol{\omega}$-*subdivision* or the *bisection*, which are defined below, and go to the next iteration.

For a cone K generated by n extreme rays with directions $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^n$, let $\boldsymbol{\omega} \in K$ be a point such that $\boldsymbol{\omega} = \sum_{j \in J} \theta_j \boldsymbol{r}^j$ for some $\theta_j > 0$ for $j \in J$, where $J \subseteqq \{1, \ldots, n\}$ is the index set of at least two elements, i.e., $|J| \geqq 2$. Let K_j denote the cone generated by $\{\boldsymbol{r}^1, \ldots, \boldsymbol{r}^n\} \backslash \{\boldsymbol{r}^j\} \cup \{\boldsymbol{\omega}\}$ for each $j \in J$. The cone K is then split into $|J|$ cones K_j. This splitting is called the $\boldsymbol{\omega}$-*subdivision* (See Figure 3.3). When $\boldsymbol{\omega} = (\boldsymbol{r} + \boldsymbol{r}')/2$ for $\boldsymbol{r}, \boldsymbol{r}' \in \{\boldsymbol{r}^1, \ldots, \boldsymbol{r}^n\}$ such that $\|\boldsymbol{r} - \boldsymbol{r}'\|$ is maximum, the subdivision is called the *bisection*.

The CS method is described as follows.

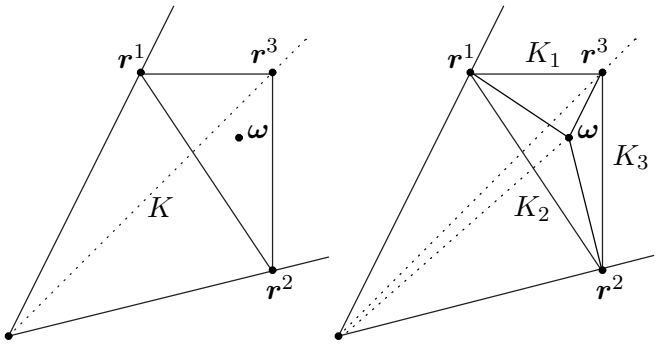


FIGURE 3.3. The $\boldsymbol{\omega}$-subdivision

/** CS method **/

$\langle 0 \rangle$ (initialization) Find an initial feasible solution $\bar{\boldsymbol{x}}$ of $(LRCP)$. Set $K_0 := \mathbb{R}^n_+$, $\mathcal{S} := \{K_0\}$, $\mathcal{R} := \mathcal{S}$, and $k := 0$.

$\langle k \rangle$ (iteration k) For each $K \in \mathcal{S}$, solve $\beta_K := \min\{\, \boldsymbol{p}\boldsymbol{x} \mid \boldsymbol{x} \in D \cap K,\ l_K(\boldsymbol{x}) \geqq 0 \,\}$ to obtain a solution $\boldsymbol{\omega}^K$, where $l_K(\boldsymbol{x}) \geqq 0$ is the concavity cut for $K \backslash H$.

$\langle k1 \rangle$ (update) Solve $\beta^* := \min\{\, \beta_K \mid K \in \mathcal{S},\ h(\boldsymbol{\omega}^K) \geqq 0 \,\}$ to obtain the cone K^*. If $\beta^* < \boldsymbol{p}\bar{\boldsymbol{x}}$ then set $\bar{\boldsymbol{x}} := \boldsymbol{\omega}^{K^*}$.

$\langle k2 \rangle$ (termination) Let $\mathcal{R}' := \{\, K \in \mathcal{R} \mid \beta_K < \boldsymbol{p}\bar{\boldsymbol{x}} \,\}$. If $\mathcal{R}' = \emptyset$ then stop, since $\bar{\boldsymbol{x}}$ solves $(LRCP)$.

$\langle k3 \rangle$ (subdivision) Solve $\min\{\, \beta_K \mid K \in \mathcal{R}' \,\}$ to obtain the cone K^{**}. Perform on K^{**}: the $\boldsymbol{\omega}$-subdivision for some $\boldsymbol{\omega} \in K^{**}$ or the bisection. Let $\mathcal{S}^{**}$ be the partition of K^{**}. Set $\mathcal{S} := \mathcal{S}^{**}$, $\mathcal{R} := \mathcal{S}^{**} \cap (\mathcal{R}' \backslash \{K^{**}\})$, $k := k+1$ and go to $\langle k \rangle$.

The convergence of the CS method critically depend on the subdivision rule of K^{**}. Let $\boldsymbol{\omega}^k = \boldsymbol{\omega}^{K^{**}}$ for iteration k. If the subdivision is *exhaustive*, namely any nested sequence of cones generated in the algorithm will shrink to a ray, there is at least one accumulation point $\boldsymbol{\omega}^*$ of $\{\boldsymbol{\omega}^k\}$ contained in ∂H, and hence $\boldsymbol{\omega}^*$ is feasible. Since $\boldsymbol{p}\boldsymbol{\omega}^k$ is a lower bound of $(LRCP)$ for all k, $\boldsymbol{\omega}^*$ is an optimal solution of $(LRCP)$.

3.2. **Outer Approximation Method.**

We explain the outer approximation method (OA for short) for (CDC), for which we assume that

$$\mathbf{0} \in D \cap \operatorname{int} H, \text{ and } \min\{\, \boldsymbol{p}\boldsymbol{x} \mid \boldsymbol{x} \in D \,\} = 0, \tag{3.4}$$

and the problem (CDC) is *regular*, i.e.,

$$D\backslash \operatorname{int} H = \operatorname{cl}(D\backslash H). \tag{3.5}$$

Figure 3.4 shows an example of (CDC) that is not regular. We see in the figure $\boldsymbol{x}^* \in D\backslash \operatorname{int} H$, while $\boldsymbol{x}^* \notin \operatorname{cl}(D\backslash H)$. The regularity assumption yields the optimality

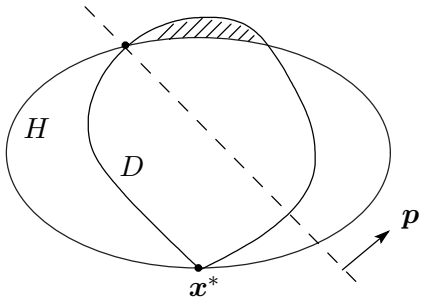


FIGURE 3.4. The case where (CDC) is not regular

condition stated in the following theorem. In the followings we denote

$$D(\bar{\boldsymbol{x}}) = \{\, \boldsymbol{x} \in D \mid \boldsymbol{p}\boldsymbol{x} \leqq \boldsymbol{p}\bar{\boldsymbol{x}} \,\}, \tag{3.6}$$

for $\bar{\boldsymbol{x}} \in \mathbb{R}^n$.

Theorem 3.1. (e.g. Horst-Tuy [15]) *Let $\boldsymbol{x}^*$ be a feasible solution of (CDC). Then $\boldsymbol{x}^*$ is an optimal solution if it satisfies*

$$D(\boldsymbol{x}^*) \subseteqq H. \tag{3.7}$$

Proof: See Appendix. □

Let $\boldsymbol{x}^*$ be an optimal solution of (CDC) and $\bar{\boldsymbol{x}}$ be the incumbent at iteration k. In the OA method, we construct infinitely many polytopes $P^0, P^1, \cdots, P^k, \cdots$ such that $P^0 \supseteqq P^1 \supseteqq \cdots \supseteqq P^k \supseteqq \cdots \supseteqq D(\boldsymbol{x}^*)$ until $P^k \subseteqq H$ for some k. We check the optimality condition $P^k \subseteqq H$ by evaluating $h(\boldsymbol{v})$ at each vertex $\boldsymbol{v}$ of P^k. If there is a vertex $\boldsymbol{v}$ of P^k such that $h(\boldsymbol{v}) > 0$, we cut off $\boldsymbol{v}$ from the current polytope P^k, i.e., we construct $P^{k+1} = P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid l(\boldsymbol{x}) \leqq 0 \,\}$ by a function $l : \mathbb{R}^n \to \mathbb{R}$ satisfying $l(\boldsymbol{v}) > 0$ and $l(\boldsymbol{x}) \leqq 0$ for all $\boldsymbol{x} \in D(\bar{\boldsymbol{x}})$.

Here we describe the OA method.

/** OA method **/

⟨0⟩ (initialization) Find an initial feasible solution $\bar{\boldsymbol{x}}$ of (CDC) and construct an initial polytope $P^0 \supseteqq D(\bar{\boldsymbol{x}})$. Let P_V^0 be the vertex set of P^0. Set $k := 0$.

⟨k⟩ (iteration k) Solve $\max\{\, h(\boldsymbol{v}) \mid \boldsymbol{v} \in P_V^k \,\}$ to obtain a vertex $\boldsymbol{v}^k$.

$\langle k1\rangle$ (termination) If $\boldsymbol{p}\bar{\boldsymbol{x}} = 0$ or $h(\boldsymbol{v}^k) \leqq 0$, i.e., $D(\bar{\boldsymbol{x}}) \subseteqq H$ then stop. We see that $\bar{\boldsymbol{x}}$ solves (CDC). Otherwise, obtain the point $\boldsymbol{x}^k \in [\boldsymbol{0}, \boldsymbol{v}^k] \cap \partial H$.

$\langle k2\rangle$ (cutting the polytope) If $\boldsymbol{x}^k \notin D$, set $P^{k+1} := P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid l(\boldsymbol{x}) \leqq 0 \,\}$, where $l(\boldsymbol{v}^k) > 0$ and $l(\boldsymbol{x}) \leqq 0$ for all $\boldsymbol{x} \in D(\bar{\boldsymbol{x}})$. Otherwise, set $\bar{\boldsymbol{x}} := \boldsymbol{x}^k$ and $P^{k+1} := P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}\boldsymbol{x} \leqq \boldsymbol{p}\boldsymbol{x}^k \,\}$.

$\langle k3\rangle$ Compute the vertex set P_V^{k+1} of P^{k+1}. Set $k := k+1$ and go to $\langle k\rangle$.

Remark 3.2. Defining the *set of all subgradient directions of f at a point $\boldsymbol{y}$*, denoted by $\partial f(\boldsymbol{y})$, as $\partial f(\boldsymbol{y}) = \{\, \boldsymbol{q} \in \mathbb{R}_n \mid f(\boldsymbol{x}) - f(\boldsymbol{y}) \geqq \boldsymbol{q}(\boldsymbol{x} - \boldsymbol{y}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^n \,\}$, the function $l : \mathbb{R}^n \to \mathbb{R}$ in Step $k2$ is given by $l(\boldsymbol{x}) = \boldsymbol{q}^k(\boldsymbol{x} - \boldsymbol{y}^k) + f(\boldsymbol{y}^k)$, where $\boldsymbol{y}^k \in [\boldsymbol{0}, \boldsymbol{v}^k] \cap \partial D$ and $\boldsymbol{q}^k \in \partial f(\boldsymbol{y}^k)$. When we consider $(LRCP)$, we choose an index $i \in \{1, \ldots, m\}$ such that $\boldsymbol{b}^i\boldsymbol{v}^k > z_i$, where $\boldsymbol{b}^i$ is the ith row of B, and set $l(\boldsymbol{x}) = \boldsymbol{b}^i\boldsymbol{x} - z_i$.

Remark 3.3. The subroutine computing the vertex set P_V^{k+1} is provided in e.g. Horst-Vries-Thoai [14], in which P_V^{k+1} is computed from the knowledge of P_V^k.

4. D.C. Optimization Methods for (mmF)

In this section we apply the cut-and-split method and the outer approximation method to (mmF). The problem (mmF) is written as

$$(mmF) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in X_M, \end{array} \right.$$

where $X = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}$ and $X_M = \{\, \boldsymbol{x} \in X \mid \nexists \boldsymbol{y} \in X \ :\ \boldsymbol{y} \geqq \boldsymbol{x},\ \boldsymbol{y} \neq \boldsymbol{x} \,\}$. Using the gap function

$$g(\boldsymbol{x}) = \max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x} \,\} - \boldsymbol{e}\boldsymbol{x}, \tag{4.1}$$

and the set $G = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid -g(\boldsymbol{x}) \leqq 0 \,\}$, (mmF) is also written as

$$(mmF) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in X \backslash \operatorname{int} G. \end{array} \right.$$

We exclude the trivial case where $X = \{\boldsymbol{0}\}$. This implies that $\boldsymbol{0} \notin X_M$ or $-g(\boldsymbol{0}) < 0$. We also assume that there is no t-s-path, which means $\boldsymbol{d}\boldsymbol{x} \geqq 0$ for all $\boldsymbol{x} \in X$. Note that all vertices of X are integer vectors by the total unimodularity of A and the integrality of $\boldsymbol{c}$. If $\boldsymbol{x} \in X$ is an integer vector, then $g(\boldsymbol{x})$ takes an

integer. As stated in Subsection 2.1, $g(\boldsymbol{x}) \geqq 0$ for all $\boldsymbol{x} \in X$. Additionally we have the following lemma.

Lemma 4.1. *$g(\boldsymbol{x}) > 0$ for all points $\boldsymbol{x}$ in the relative interior of X.*

Proof: Let $\boldsymbol{x}$ be a point in the relative interior of X, i.e., $A\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{0} < \boldsymbol{x} < \boldsymbol{c}$. Letting $\boldsymbol{x}' = (1+\varepsilon)\boldsymbol{x}$ for a sufficiently small $\varepsilon > 0$, we see that $A\boldsymbol{x}' = \boldsymbol{0}$ and $\boldsymbol{0} < \boldsymbol{x}' < \boldsymbol{c}$, i.e., $\boldsymbol{x}' \in X$. Therefore $g(\boldsymbol{x}) \geqq \boldsymbol{e}(\boldsymbol{x}' - \boldsymbol{x}) = \varepsilon \boldsymbol{e}\boldsymbol{x} > 0$. □

The following corollary is a direct consequence of Lemma 4.1.

Corollary 4.2. *$X_M \subsetneqq \partial X$.*

4.1. **Extension of Gap Function.**

The *domain of* g, denoted by $\operatorname{dom} g$, is given by $\operatorname{dom} g = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) > -\infty \,\}$. When we apply the CS method or the OA method to (mmF), we need to evaluate $-g(\boldsymbol{v})$ for $\boldsymbol{v}$ not contained in X. Since $g(\boldsymbol{v}) = -\infty$ if there is no point $\boldsymbol{y} \in X$ such that $\boldsymbol{y} \geqq \boldsymbol{v}$, no information about how far the point $\boldsymbol{v}$ is from the domain of g is available in this case. In this subsection we extend the gap function g to $\mathbb{R}^n$. The gap function $g(\boldsymbol{x})$ of (4.1) is given by the optimal value of the problem

$$(P_G(\boldsymbol{x})) \qquad \left| \begin{array}{ll} \max\limits_{\boldsymbol{y}} & \boldsymbol{e}\boldsymbol{y} - \boldsymbol{e}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{y} \leqq \boldsymbol{c}, \\ & \boldsymbol{y} \geqq \boldsymbol{x}, \end{array} \right.$$

whose dual problem is

$$(D_G(\boldsymbol{x})) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}} & \boldsymbol{\alpha}\boldsymbol{c} - \boldsymbol{\beta}\boldsymbol{x} - \boldsymbol{e}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e}, \\ & \boldsymbol{\alpha}, \boldsymbol{\beta} \geqq \boldsymbol{0}. \end{array} \right.$$

Note that $(D_G(\boldsymbol{x}))$ is always feasible, e.g. take $\boldsymbol{\pi} = \boldsymbol{\beta} = \boldsymbol{0}$ and $\boldsymbol{\alpha} \geqq \boldsymbol{e}$. Therefore, $(P_G(\boldsymbol{x}))$ is infeasible if and only if $(D_G(\boldsymbol{x}))$ is unbounded. Adding the upper bound constraints $\boldsymbol{\beta} \leqq \bar{\boldsymbol{\beta}}$ to $(D_G(\boldsymbol{x}))$ yields the following problem

$$(\overline{D_G(\boldsymbol{x})}) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}} & \boldsymbol{\alpha}\boldsymbol{c} - \boldsymbol{\beta}\boldsymbol{x} - \boldsymbol{e}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e}, \\ & \boldsymbol{\alpha} \geqq \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{\beta} \leqq \bar{\boldsymbol{\beta}}, \end{array} \right.$$

where $\bar{\boldsymbol{\beta}} \geqq \boldsymbol{0}$ will be specified in the following theorem. The dual problem of $(\overline{D_G(\boldsymbol{x})})$ is

$$(\overline{P_G(\boldsymbol{x})}) \quad \left| \begin{array}{ll} \max\limits_{\boldsymbol{y},\boldsymbol{t}} & \boldsymbol{e}\boldsymbol{y} - \boldsymbol{e}\boldsymbol{x} - \bar{\boldsymbol{\beta}}\boldsymbol{t} \\ \text{s.t.} & A\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{y} \leqq \boldsymbol{c}, \\ & \boldsymbol{y} + \boldsymbol{t} \geqq \boldsymbol{x},\ \boldsymbol{t} \geqq \boldsymbol{0}. \end{array} \right.$$

Then we define the *extended gap function* $\bar{g} : \mathbb{R}^n \to \mathbb{R}$ as

$$\bar{g}(\boldsymbol{x}) = \max\{\, \boldsymbol{e}\boldsymbol{y} - \bar{\boldsymbol{\beta}}\boldsymbol{t} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} + \boldsymbol{t} \geqq \boldsymbol{x},\ \boldsymbol{t} \geqq \boldsymbol{0} \,\} - \boldsymbol{e}\boldsymbol{x}. \tag{4.2}$$

Theorem 4.3.

(i) The domain of $\bar{g}$ is $\mathbb{R}^n$.

(ii) If $\bar{\boldsymbol{\beta}} \geqq n\boldsymbol{e}$ then $\bar{g} = g$ on the domain of g.

Proof: (i) For any $\boldsymbol{x} \in \mathbb{R}^n$, $(\overline{D_G(\boldsymbol{x})})$ has a feasible solution and the objective function is bounded. By the duality theorem of linear programming there is an optimal value of $(\overline{P_G(\boldsymbol{x})})$, and hence $\bar{g}(\boldsymbol{x}) > -\infty$ for any $\boldsymbol{x} \in \mathbb{R}^n$.
(ii) Let Ω and $\bar{\Omega}$ denote the feasible sets of $(D_G(\boldsymbol{x}))$ and $(\overline{D_G(\boldsymbol{x})})$, respectively, i.e.,

$$\begin{aligned} \Omega &= \{\, (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}_{m+2n} \mid \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e},\ \boldsymbol{\alpha}, \boldsymbol{\beta} \geqq \boldsymbol{0} \,\}, \text{ and} \\ \bar{\Omega} &= \{\, (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}_{m+2n} \mid \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e},\ \boldsymbol{\alpha} \geqq \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{\beta} \leqq \bar{\boldsymbol{\beta}} \,\}. \end{aligned}$$

By the theory of linear programming, if every vertex $\boldsymbol{v}$ of Ω satisfies $\boldsymbol{v} \leqq \bar{\boldsymbol{\beta}}$ then we have $\bar{g}(\boldsymbol{x}) = g(\boldsymbol{x})$ for all $\boldsymbol{x}$ in the domain of g. Replacing $\boldsymbol{\pi}$ by $\boldsymbol{\pi}^1 - \boldsymbol{\pi}^2$ with $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2 \geqq \boldsymbol{0}$ and introducing a slack variable vector $\boldsymbol{\gamma} \geqq \boldsymbol{0}$, Ω is rewritten as

$$\Omega = \left\{ \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} \middle| \begin{pmatrix} A^\top & -A^\top & I & -I & -I \end{pmatrix} \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} = \boldsymbol{1}, \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} \geqq \boldsymbol{0} \right\}.$$

Let $\boldsymbol{v}$ be a vertex of Ω. Then it is a basic solution of the system defining Ω, i.e., $\boldsymbol{v} = (\boldsymbol{w}^B, \boldsymbol{w}^N) = (B^{-1}\boldsymbol{1}, \boldsymbol{0})$ for some nonsingular $n \times n$ submatrix B of $(A^\top\ -A^\top\ I\ -I\ -I)$. Since the incidence matrix A is totally unimodular, i.e., each subdeterminant of A is -1, 0, or $+1$, so is $(A^\top\ -A^\top\ I\ -I\ -I)$. Therefore the matrix B^{-1} is composed of -1, 0 and $+1$, and hence $B^{-1}\boldsymbol{1} \leqq n\boldsymbol{1}$. This completes the proof. □

Theorem 4.3 yields an equivalent form of (mmF). Namely, fixing $\bar{\boldsymbol{\beta}} = n\boldsymbol{e}$ we can reformulate the problem (mmF) as

$$(mmF) \quad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in X \backslash \operatorname{int} \bar{G}, \end{array} \right.$$

where $\bar{G} = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid -\bar{g}(\boldsymbol{x}) \leqq 0 \,\}$.

4.2. **Local Search.**

For $\boldsymbol{v} \in X_M \cap X_V$, we define the set of all efficient vertices linked to $\boldsymbol{v}$ by

$$
\begin{aligned}
(4.3) \qquad N_M(\boldsymbol{v}) &= \{\, \boldsymbol{v}' \in X_M \cap X_V \mid [\boldsymbol{v}, \boldsymbol{v}'] \text{ is an edge of } X \,\} \\
&= \{\, \boldsymbol{v}' \in X_V \mid [\boldsymbol{v}, \boldsymbol{v}'] \text{ is an edge of } X \text{ and } -\bar{g}(\boldsymbol{v}') \geqq 0 \,\}.
\end{aligned}
$$

When we find a feasible solution $\boldsymbol{w} \in X_M$, we apply the Local Search procedure starting with $\boldsymbol{w}$ (LS($\boldsymbol{w}$) for short) for further improvement.

The procedure is described as follows.

/** LS($\boldsymbol{w}$) procedure **/

$\langle 0 \rangle$ (initialization) If $\boldsymbol{w} \notin X_V$ then solve $\min\{\, \boldsymbol{d}\boldsymbol{x} \mid \boldsymbol{x} \in F \,\}$, where F is the face of X containing $\boldsymbol{w}$ in its relative interior, to obtain a vertex $\boldsymbol{v}^0 \in X_M \cap X_V$, otherwise set $\boldsymbol{v}^0 := \boldsymbol{w}$. Set $k := 0$.

$\langle k \rangle$ (iteration k) Solve $\min\{\, \boldsymbol{d}\boldsymbol{v} \mid \boldsymbol{v} \in N_M(\boldsymbol{v}^k) \,\}$ to obtain a solution $\boldsymbol{v}^*$. If $\boldsymbol{d}\boldsymbol{v}^* \geqq \boldsymbol{d}\boldsymbol{v}^k$ then stop, $\boldsymbol{v}^k$ is the local optimal vertex of (mmF). Otherwise set $\boldsymbol{v}^{k+1} := \boldsymbol{v}^*$, $k := k + 1$ and go to $\langle k \rangle$.

Remark 4.4. If $\boldsymbol{w} \in X_M$, the face F of X containing $\boldsymbol{w}$ in its relative interior is contained in X_M by Theorem 2.2.

4.3. **Cut and Split Method for** (mmF)**.**

We can directly apply the CS method to (mmF) since it satisfies the assumptions of (3.2), i.e.,

$$
(4.4) \qquad X \subsetneqq \mathbb{R}^n_+ \text{ and } \mathbf{0} \in X_V \cap \operatorname{int} \bar{G}.
$$

To make the algorithm more efficient we combine the LS($\boldsymbol{w}$) procedure with the CS method, namely we apply the LS($\boldsymbol{w}$) procedure to obtain tighter upper bound every time we find a feasible solution $\boldsymbol{w} \in X_M$.

The CS method for (mmF) is described as follows.

/** CS method for (mmF) **/

$\langle 0 \rangle$ (initialization) Find an initial feasible vertex $\boldsymbol{w}^0 \in X_M \cap X_V$ of (mmF). If $N_M(\boldsymbol{w}^0) = \emptyset$ then stop. We see that $\boldsymbol{w}^0$ is the unique optimal solution of (mmF). Otherwise, apply the LS($\boldsymbol{w}^0$) procedure to obtain a local optimal vertex $\bar{\boldsymbol{x}} \in X_M \cap X_V$. Set $K_0 := \mathbb{R}^n_+$, $\mathcal{S} := \{K_0\}$, $\mathcal{R} := \mathcal{S}$, $\gamma := 0$ and $k := 0$.

$\langle k \rangle$ (iteration k) For each $K \in \mathcal{S}$, solve $\beta_K := \min \{ \boldsymbol{d}\boldsymbol{x} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0 \}$ to obtain a solution $\boldsymbol{\omega}^K$, where $l_K(\boldsymbol{x}) \geqq 0$ is the concavity cut for $K \backslash \bar{G}$.

$\langle k1 \rangle$ (update) Set $\mathcal{L} := \{ K \in \mathcal{S} \mid \boldsymbol{\omega}^K \in X_M \}$. If $\mathcal{L} \neq \emptyset$ then apply the LS($\boldsymbol{\omega}^K$) procedure to obtain a local optimal vertex $\boldsymbol{v}^K$ for each $K \in \mathcal{L}$. Solve $\min \{ \boldsymbol{d}\boldsymbol{v}^K \mid K \in \mathcal{L} \}$ to obtain the cone K^*. If $\boldsymbol{d}\boldsymbol{v}^{K^*} < \boldsymbol{d}\bar{\boldsymbol{x}}$, set $\bar{\boldsymbol{x}} := \boldsymbol{v}^{K^*}$.

$\langle k2 \rangle$ (termination) Set $\mathcal{R}' := \{ K \in \mathcal{R} \mid \beta_K < \boldsymbol{d}\bar{\boldsymbol{x}} \}$. If $\mathcal{R}' = \emptyset$ or $\boldsymbol{d}\bar{\boldsymbol{x}} - \gamma < 1$ then stop, since $\bar{\boldsymbol{x}}$ solves (mmF).

$\langle k3 \rangle$ (subdivision) Solve $\min \{ \beta_K \mid K \in \mathcal{R}' \}$ to obtain the cone K^{**}. If $\beta_{K^{**}} > \gamma$ then set $\gamma := \beta_{K^{**}}$. Let $\mathcal{S}^{**}$ be the partition of K^{**} obtained by the $\boldsymbol{\omega}$-subdivision on K^{**} for some direction $\boldsymbol{\omega} \in K^{**}$. Set $\mathcal{S} := \mathcal{S}^{**}$, $\mathcal{R} := \mathcal{S}^{**} \cap (\mathcal{R}' \backslash \{K^{**}\})$, $k := k+1$ and go to $\langle k \rangle$.

Remark 4.5. In Step $k3$ the direction $\boldsymbol{\omega} \in K^{**}$ is obtained as follows. Suppose that K^{**} is generated by the directions $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ such that $\boldsymbol{u}^i \in \partial \bar{G}$, initially $\boldsymbol{u}^i = \alpha^* \boldsymbol{e}^i$ with $\alpha^* = \max \{ \alpha \mid -\bar{g}(\alpha \boldsymbol{e}^i) \leqq 0 \}$ for each $i = 1, \ldots, n$. The function $l_{K^{**}} : \mathbb{R}^n \to \mathbb{R}$ defining the concavity cut $l_{K^{**}}(\boldsymbol{x}) \geqq 0$ for $K^{**} \backslash \bar{G}$ is given by $l_{K^{**}}(\boldsymbol{x}) = \boldsymbol{e}U^{-1}\boldsymbol{x} - 1$, where $U = [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n]$. Here we solve

$$\eta = \max \{ l_{K^{**}}(\boldsymbol{y}) \mid \boldsymbol{y} \in X \cap K^{**} \}, \tag{4.5}$$

to obtain a solution $\boldsymbol{y}^*$. Since K^{**} is in $\mathcal{R}'$, i.e., $\beta_{K^{**}} < \boldsymbol{d}\bar{\boldsymbol{x}}$, we see that $\eta \geqq 0$. If $\eta > 0$ then we perform $\boldsymbol{\omega}$-subdivision on K^{**} with $\boldsymbol{\omega} = \boldsymbol{y}^*$. If $\eta = 0$ then we discard K^{**} from $\mathcal{R}'$ and go back to Step $k2$. In this case there is no point $\boldsymbol{v} \in X_V \cap K^{**}$ such that $\boldsymbol{v} \neq \boldsymbol{u}^i$ for each $i = 1, \ldots, n$. Then we can discard K^{**} from further consideration, because at least one vertex of X solves (mmF).

Every time we obtain an optimal solution $\boldsymbol{y}^*$ of (4.5) with $\eta > 0$, we can perform $\boldsymbol{y}^*$-subdivision on K^{**}. This assertion follows from the following theorem.

Theorem 4.6. *Let K^{**} be a cone generated by the directions $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ such that $\boldsymbol{u}^i \in \partial \bar{G}$ for each $i = 1, \ldots, n$, and $\boldsymbol{y}^*$ be an optimal solution of* (4.5) *with $\eta > 0$. Then $\boldsymbol{y}^* \neq \alpha \boldsymbol{u}^i$ for any $i = 1, \ldots, n$ and for any $\alpha > 0$.*

Proof: Assume that $\boldsymbol{y}^*$ lies on an extreme ray of K^{**}, i.e., $\boldsymbol{y}^* = \alpha \boldsymbol{u}^j$ for some $\alpha > 0$ and $\boldsymbol{u}^j$. Since $0 < \eta = l_{K^{**}}(\boldsymbol{y}^*) = \boldsymbol{e}U^{-1}(\alpha \boldsymbol{u}^j) - 1 = \alpha - 1$, we have $\alpha > 1$. By the choice of $\boldsymbol{u}^j$, we have $\bar{g}((1+\varepsilon)\boldsymbol{u}^j) < 0$ for any $\varepsilon > 0$. Therefore we have $\bar{g}(\boldsymbol{y}^*) < 0$. On the other hand, $\boldsymbol{y}^* \in X \cap K^{**} \subsetneqq K$, which implies $\bar{g}(\boldsymbol{y}^*) \geqq 0$. This is a contradiction. □

Furthermore the following assertion is also available in this subdivision rule.

Theorem 4.7. *Let K be the cone generated by the directions $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n$ and $U = [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^n]$. If $\boldsymbol{u}^i \in X_M$ for each $i = 1, \ldots, n$ then there is an optimal solution $\boldsymbol{\omega}^K$ of $\min\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0 \,\}$ in X_M.*

Proof: Let $\boldsymbol{\omega}$ be an optimal solution of $\min\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0 \,\}$. Since $\boldsymbol{\omega} \in K$, there are nonnegative numbers $\mu_1, \ldots, \mu_n$ such that $\boldsymbol{\omega} = \sum_{i=1}^n \mu_i \boldsymbol{u}^i$. Also we see that $0 \leqq l_K(\boldsymbol{\omega}) = \boldsymbol{e}U^{-1}(\sum_{i=1}^n \mu_i \boldsymbol{u}^i) - 1 = \sum_{i=1}^n \mu_i - 1$, and hence $\sum_{i=1}^n \mu_i \geqq 1$. Note that $\boldsymbol{dx} \geqq 0$ for all $\boldsymbol{x} \in X$ by the assumption that a given network has no t-s-path. Let $\boldsymbol{u}^j$ attain $\min\{\, \boldsymbol{du}^i \mid i = 1, \ldots, n \,\}$. Then $\boldsymbol{du}^j \leqq \sum_{i=1}^n \mu_i \boldsymbol{du}^i = \boldsymbol{d\omega}$, in other words, $\boldsymbol{u}^j \in X_M$ solves $\min\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in X \cap K,\ l_K(\boldsymbol{x}) \geqq 0 \,\}$. □

We see that the set $\{\, \boldsymbol{x} \in K^{**} \mid l_{K^{**}}(\boldsymbol{x}) > 0 \,\}$ does not contain a vertex of X when η of (4.5) is zero. Suppose that an oracle is available that provides a vertex of X in $\{\, \boldsymbol{x} \in K^{**} \mid l_{K^{**}}(\boldsymbol{x}) > 0 \,\}$ whenever there are some, and take the vertex as the direction $\boldsymbol{\omega}$ in Step $k3$. Then, owing to the finiteness of X_V, the $\boldsymbol{\omega}$-subdivision is repeated at most $|X_V|$ times, and hence the CS method terminates after finitely many iterations. However, the oracle is costly and the authors estimate it NP-complete to check if $X_V \cap \{\, \boldsymbol{x} \in K^{**} \mid l_{K^{**}}(\boldsymbol{x}) > 0 \,\}$ is not empty. See Freund-Orlin [11].

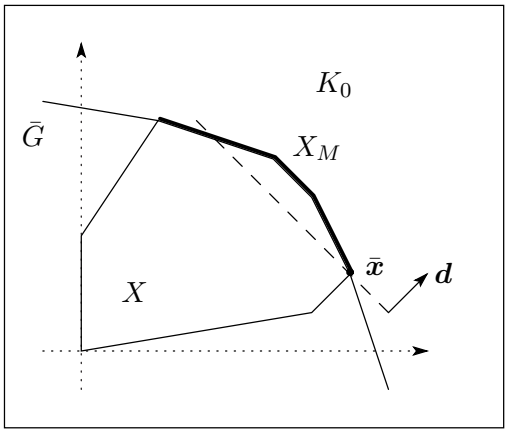
We illustrate the CS method for (mmF) in Figure 4.1. We first obtain a local optimal vertex $\bar{\boldsymbol{x}} \in X_M \cap X_V$ and set $K_0 := \mathbb{R}^n_+$ (See (a)). We determine the concavity cut $l_{K_0}(\boldsymbol{x}) \geqq 0$ for $K_0 \backslash \bar{G}$ and obtain a point $\boldsymbol{\omega}^{K_0}$ (See (b)). The value $\boldsymbol{d\omega}^{K_0}$ is a lower bound with respect to K_0. Since $\boldsymbol{\omega}^{K_0} \notin X_M$ and $\boldsymbol{d\omega}^{K_0} < \boldsymbol{d\bar{x}}$, we split the cone K_0 into K_1 and K_2 (See (c)). In the next iteration, points $\boldsymbol{\omega}^{K_1}$ and $\boldsymbol{\omega}^{K_2}$ are obtained (See (d)). We see that $\boldsymbol{\omega}^{K_2} \in X_M$, and hence apply the LS($\boldsymbol{\omega}^{K_2}$) procedure to obtain a better point $\boldsymbol{v}^{K_2}$ and update the incumbent $\bar{\boldsymbol{x}}$ to $\boldsymbol{v}^{K_2}$. The cone K_2 is discarded, since $\boldsymbol{d\omega}^{K_2} \geqq \boldsymbol{d\bar{x}}$. Meanwhile K_1 is split into K_3 and K_4 since $\boldsymbol{\omega}^{K_1} \notin X_M$ and $\boldsymbol{d\omega}^{K_1} < \boldsymbol{d\bar{x}}$ (See (e)). We obtain points $\boldsymbol{\omega}^{K_3}$ and $\boldsymbol{\omega}^{K_4}$ in the next iteration (See (f)) and continue the algorithm.

4.4. **Outer Approximation Method for** (mmF)**.**

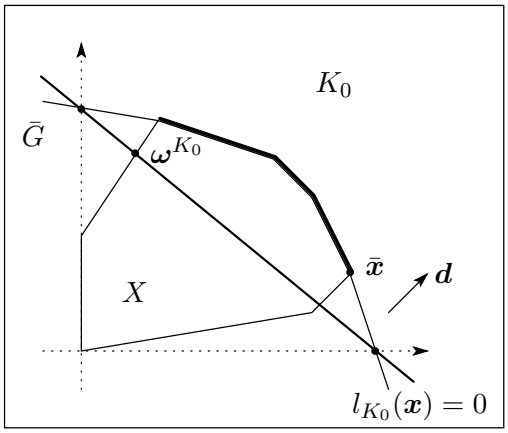
Unfortunately, the problem (mmF) is not regular, hence we introduce a positive tolerance ε, define $\bar{G}_\varepsilon = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid -\bar{g}(\boldsymbol{x}) + \varepsilon \leqq 0 \,\}$ and consider

$$(mmF_\varepsilon) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{dx} \\ \text{s.t.} & \boldsymbol{x} \in X \backslash \operatorname{int} \bar{G}_\varepsilon, \end{array} \right.$$

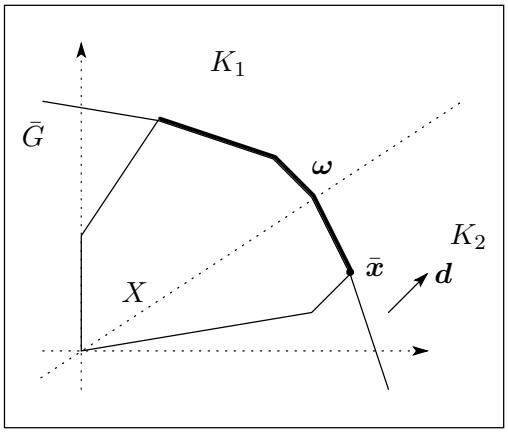
We call an optimal solution of (mmF_ε) an *ε-optimal solution of* (mmF). Before describing the OA method for (mmF), we discuss the range of ε together with



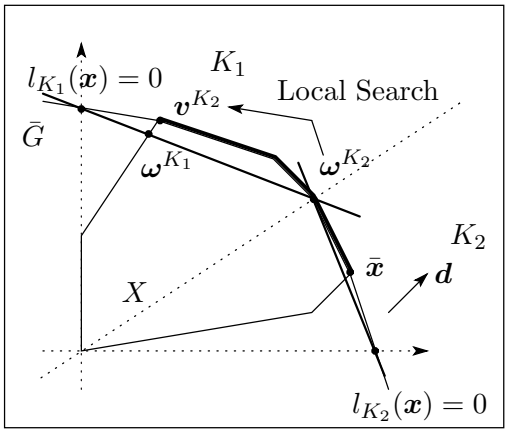
(a) initialization



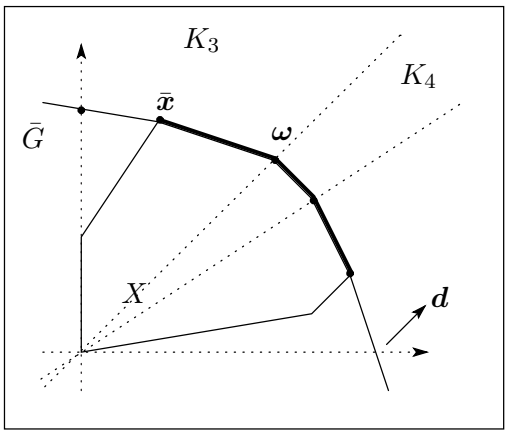
(b) obtaining $\boldsymbol{\omega}^{K_0}$



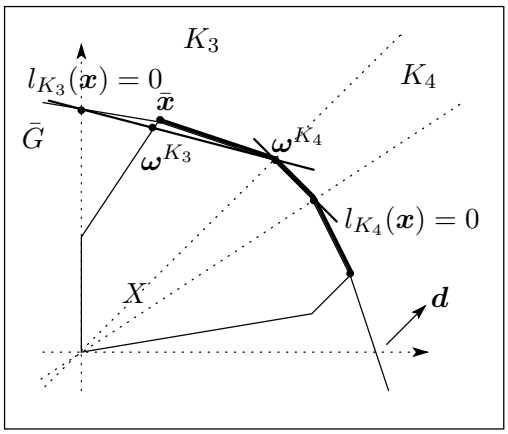
(c) splitting K_0 into K_1 and K_2



(d) obtaining $\boldsymbol{\omega}^{K_1}$ and $\boldsymbol{\omega}^{K_2}$



(e) splitting K_1 into K_3 and K_4



(f) obtaining $\boldsymbol{\omega}^{K_3}$ and $\boldsymbol{\omega}^{K_4}$

FIGURE 4.1. An example of the CS method for (mmF)

relation between (mmF) and (mmF_ε). First we assure the regularity of (mmF_ε) by the following theorem.

Theorem 4.8. *The problem (mmF_ε) is regular for any $\varepsilon > 0$.*

Proof: We show that for any $\varepsilon > 0$,

$$X\backslash \operatorname{int} \bar{G}_\varepsilon = \operatorname{cl}(X\backslash \bar{G}_\varepsilon). \tag{4.6}$$

($\supseteqq$) Since $X\backslash \operatorname{int} \bar{G}_\varepsilon$ is closed and $X\backslash \operatorname{int} \bar{G}_\varepsilon \supseteqq X\backslash \bar{G}_\varepsilon$, we have

$$X\backslash \operatorname{int} \bar{G}_\varepsilon = \operatorname{cl}(X\backslash \operatorname{int} \bar{G}_\varepsilon) \supseteqq \operatorname{cl}(X\backslash \bar{G}_\varepsilon).$$

($\subseteqq$) Let $\boldsymbol{x}$ be an arbitrary point of $X\backslash \operatorname{int} \bar{G}_\varepsilon$ and let $N_\delta(\boldsymbol{x})$ denote its δ-neighborhood, i.e., $N_\delta(\boldsymbol{x}) = \{\, \boldsymbol{x}' \in \mathbb{R}^n \mid \|\boldsymbol{x}' - \boldsymbol{x}\| < \delta \,\}$. We show that there is always a point, say $\boldsymbol{x}'$, in $N_\delta(\boldsymbol{x}) \cap (X\backslash \bar{G}_\varepsilon)$. If $-\bar{g}(\boldsymbol{x}) + \varepsilon < 0$ then by the continuity of $\bar{g}$, $-g(\boldsymbol{x}') + \varepsilon < 0$ for any point $\boldsymbol{x}'$ of some neighbourhood $N_\gamma(\boldsymbol{x})$ of $\boldsymbol{x}$. This implies $N_\gamma(\boldsymbol{x}) \subseteqq \bar{G}_\varepsilon$, and hence $\boldsymbol{x} \in \operatorname{int} \bar{G}_\varepsilon$. Therefore the assumption $\boldsymbol{x} \in X\backslash \operatorname{int} \bar{G}_\varepsilon$ implies that $\boldsymbol{x} \in X$ and $-\bar{g}(\boldsymbol{x}) + \varepsilon \geqq 0$. By Theorem 4.3, we have $\bar{g}(\boldsymbol{x}) = g(\boldsymbol{x})$. When $-\bar{g}(\boldsymbol{x}) + \varepsilon > 0$, take $\boldsymbol{x}$ as $\boldsymbol{x}'$. Clearly $\boldsymbol{x}' = \boldsymbol{x} \notin \bar{G}_\varepsilon$ and $\boldsymbol{x}' = \boldsymbol{x} \in N_\delta(\boldsymbol{x})$, and we have done. When $-\bar{g}(\boldsymbol{x}) + \varepsilon = 0$, i.e., $g(\boldsymbol{x}) = \bar{g}(\boldsymbol{x}) = \varepsilon$, there is an optimal solution $\boldsymbol{y}^*$ of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x} \,\}$ such that $\boldsymbol{e}(\boldsymbol{y}^* - \boldsymbol{x}) = \varepsilon$, and hence $\boldsymbol{y}^* \neq \boldsymbol{x}$. Take λ such that $0 < \lambda < \min\{\, 1,\, \delta/\|\boldsymbol{y}^* - \boldsymbol{x}\| \,\}$ and let $\boldsymbol{x}' = \lambda \boldsymbol{y}^* + (1 - \lambda)\boldsymbol{x}$. Since $\|\boldsymbol{x}' - \boldsymbol{x}\| = \lambda \|\boldsymbol{y}^* - \boldsymbol{x}\| < \delta$, we see $\boldsymbol{x}' \in N_\delta(\boldsymbol{x})$. Also we see that $\boldsymbol{x}' \in X$ by the convexity of X, and hence $g(\boldsymbol{x}') = \bar{g}(\boldsymbol{x}')$ by applying Theorem 4.3 again. Since $\boldsymbol{x}' \geqq \boldsymbol{x}$ and $\boldsymbol{x}' \neq \boldsymbol{x}$, we have

$$\begin{aligned}
\bar{g}(\boldsymbol{x}') &= g(\boldsymbol{x}') \\
&= \max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}' \,\} - \boldsymbol{e}\boldsymbol{x}' \\
&< \max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x} \,\} - \boldsymbol{e}\boldsymbol{x} \\
&= \boldsymbol{e}(\boldsymbol{y}^* - \boldsymbol{x}) = \varepsilon.
\end{aligned}$$

Therefore we see that $\boldsymbol{x}' \notin \bar{G}_\varepsilon$. This completes the proof. □

In the second place, we discuss an upper bound of ε, which will be useful for the convergence of the algorithms.

Lemma 4.9. *If $\varepsilon \in (0, 1)$ then $(\boldsymbol{0}, \boldsymbol{v}) \cap \partial \bar{G}_\varepsilon \neq \emptyset$ for any point $\boldsymbol{v}$ such that $-\bar{g}(\boldsymbol{v}) \geqq 0$.*

Proof: We have $-\bar{g}(\boldsymbol{0}) < 0$ by the assumption that $\boldsymbol{0} \notin X_M$. Note that $-\bar{g}(\boldsymbol{0})$ is integer by the integrality property of X, and hence $-\bar{g}(\boldsymbol{0}) + 1 \leqq 0$. Then we have $-\bar{g}(\boldsymbol{0}) + \varepsilon < 0$, i.e., $\boldsymbol{0} \in \operatorname{int} \bar{G}_\varepsilon$ for any $\varepsilon \in (0, 1)$. The continuity of $\bar{g}$ assures the

assertion. □

For the following lemma, we use δ_s to denote the number of arcs going out of source node s, i.e.,

$$\delta_s = |\{\, i \mid d_i = +1 \,\}|. \tag{4.7}$$

Lemma 4.10. *Let $\boldsymbol{x}^*$ and $\boldsymbol{x}^*_\varepsilon$ be an optimal solution and an ε-optimal solution of (mmF). Then $|\boldsymbol{d}\boldsymbol{x}^* - \boldsymbol{d}\boldsymbol{x}^*_\varepsilon| \leqq \varepsilon\delta_s$.*

Proof: Since $\boldsymbol{x}^* \in X$ and $-\bar{g}(\boldsymbol{x}^*) \geqq 0$, $\boldsymbol{x}^*$ is a feasible solution of (mmF_ε), and hence $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\boldsymbol{x}^*$. Let $\boldsymbol{y}^*_\varepsilon$ be an optimal solution of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^*_\varepsilon \,\}$. Clearly $\boldsymbol{y}^*_\varepsilon \in X_M$, i.e., $\boldsymbol{y}^*_\varepsilon$ is a feasible solution of (mmF), and hence $\boldsymbol{d}\boldsymbol{x}^* \leqq \boldsymbol{d}\boldsymbol{y}^*_\varepsilon$. We see that $y^*_{\varepsilon i} - x^*_{\varepsilon i} \leqq \varepsilon$ for each $i = 1, \ldots, n$, since $\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon \geqq \boldsymbol{0}$ and $\boldsymbol{e}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon) \leqq \varepsilon$. That implies $\boldsymbol{d}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon) \leqq \varepsilon|\{\, i \mid d_i = +1 \,\}| = \varepsilon\delta_s$, hence $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\boldsymbol{x}^* \leqq \boldsymbol{d}\boldsymbol{y}^*_\varepsilon \leqq \boldsymbol{d}\boldsymbol{x}^*_\varepsilon + \varepsilon\delta_s$.

□

Corollary 4.11. *If $\varepsilon \in (0,\, 1/\delta_s)$ then $\lceil \boldsymbol{d}\boldsymbol{x}^*_\varepsilon \rceil$ coincides with the optimal value of (mmF).*

In the sequel we choose ε from the open interval $(0,\, 1/\delta_s)$. Note that $\bar{g}(\boldsymbol{x}^*_\varepsilon) \leqq \varepsilon$ holds for an ε-optimal solution of (mmF). Therefore $\bar{g}(\boldsymbol{x}) \leqq 0$ for any accumulation point $\boldsymbol{x}$ of $\{\boldsymbol{x}^*_\varepsilon\}_{\varepsilon>0}$. This observation leads to the following corollary.

Corollary 4.12. *Let $\{\boldsymbol{x}^*_\varepsilon\}$ be a sequence of ε-optimal solutions of (mmF) for ε converging to 0. Then the accumulation point of $\{\boldsymbol{x}^*_\varepsilon\}$ is an optimal solution of (mmF).*

As seen in Subsection 3.2, the termination condition of the OA method is $S(\bar{\boldsymbol{x}}) \subseteqq \bar{G}_\varepsilon$, where

$$S(\bar{\boldsymbol{x}}) = \{\, \boldsymbol{x} \in X \mid \boldsymbol{p}\boldsymbol{x} \leqq \boldsymbol{p}\bar{\boldsymbol{x}} \,\}. \tag{4.8}$$

We relax this condition for (mmF) as shown in the following theorem.

Theorem 4.13. *Let $\bar{\boldsymbol{x}} \in X \backslash int\, \bar{G}_\varepsilon$ for some $\varepsilon \in (0,\, 1/\delta_s)$. If $S(\bar{\boldsymbol{x}}) \subseteqq \bar{G}_{\varepsilon'}$ for some $\varepsilon' \in (0,\, \varepsilon]$ then $\lceil \boldsymbol{d}\bar{\boldsymbol{x}} \rceil$ coincides with the optimal value of (mmF).*

Proof: Let $\boldsymbol{x}^*$ and $\boldsymbol{x}^*_\varepsilon$ be an optimal solution and an ε-optimal solution of (mmF), respectively. Since $\bar{\boldsymbol{x}}$ is a feasible solution of (mmF_ε), we have $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\bar{\boldsymbol{x}}$. It is also clear that $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\boldsymbol{x}^*$. If $\boldsymbol{d}\boldsymbol{x}^* < \boldsymbol{d}\bar{\boldsymbol{x}}$ then we have $\boldsymbol{x}^* \in S(\bar{\boldsymbol{x}}) \subseteqq \bar{G}_{\varepsilon'}$, and hence $\bar{g}(\boldsymbol{x}^*) \geqq \varepsilon' > 0$. This contradicts that $\bar{g}(\boldsymbol{x}^*) = 0$. Then we have $\boldsymbol{d}\bar{\boldsymbol{x}} \leqq \boldsymbol{d}\boldsymbol{x}^*$.

Lemma 4.10 and the fact that $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\bar{\boldsymbol{x}} \leqq \boldsymbol{d}\boldsymbol{x}^* \leqq \boldsymbol{d}\boldsymbol{x}^*_\varepsilon + \varepsilon\delta_s < \boldsymbol{d}\boldsymbol{x}^*_\varepsilon + 1$ complete the proof. □

Let P be a polytope such that $S(\bar{\boldsymbol{x}}) \subseteqq P$ for some $\bar{\boldsymbol{x}} \in X \backslash \operatorname{int} \bar{G}_\varepsilon$ and $\boldsymbol{v}^*$ be an optimal solution of $\max\{ -\bar{g}(\boldsymbol{v}) \mid \boldsymbol{v} \in P_V \}$. Since $\bar{\boldsymbol{x}} \in P$ and $\bar{\boldsymbol{x}} \notin \operatorname{int} \bar{G}_\varepsilon$, we see that $\bar{g}(\boldsymbol{v}^*) \leqq \bar{g}(\bar{\boldsymbol{x}}) \leqq \varepsilon$. Then the value $\varepsilon' = \bar{g}(\boldsymbol{v}^*)$ is smaller than or equal to ε. It is also clear that $P \subseteqq \bar{G}_{\varepsilon'}$ because for all $\boldsymbol{x} \in P$ we have $-\bar{g}(\boldsymbol{x}) \leqq -\bar{g}(\boldsymbol{v}^*)$, i.e., $-\bar{g}(\boldsymbol{x}) + \bar{g}(\boldsymbol{v}^*) = -\bar{g}(\boldsymbol{x}) + \varepsilon' \leqq 0$. Namely we have $\varepsilon' \leqq \varepsilon$ and $S(\bar{\boldsymbol{x}}) \subseteqq \bar{G}_{\varepsilon'}$. Therefore if $\varepsilon' > 0$ then the optimal value of (mmF) is obtained by Theorem 4.13.

We describe the OA method for (mmF) as follows.

/** OA method for (mmF) **/

$\langle 0 \rangle$ (initialization) Find an initial feasible vertex $\boldsymbol{w}^0 \in X_M \cap X_V$ of (mmF). If $N_M(\boldsymbol{w}^0) = \emptyset$ then stop. We see that $\boldsymbol{w}^0$ is the unique optimal solution of (mmF). Otherwise, apply the LS($\boldsymbol{w}^0$) procedure to obtain a local optimal vertex $\bar{\boldsymbol{x}} \in X_M \cap X_V$. Solve $c_{max} := \max\{ \boldsymbol{e}\boldsymbol{x} \mid \boldsymbol{x} \in X \}$ and construct an initial polytope $P^0 \supseteqq S(\bar{\boldsymbol{x}})$ by setting $P^0 := \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{e}\boldsymbol{x} \leqq c_{max},\ \boldsymbol{d}\boldsymbol{x} \leqq \boldsymbol{d}\bar{\boldsymbol{x}},\ \boldsymbol{x} \geqq \boldsymbol{0} \}$. Let P^0_V be the vertex set of P^0. Set $k := 0$.

$\langle k \rangle$ (iteration k) Solve $\max\{ -\bar{g}(\boldsymbol{v}) \mid \boldsymbol{v} \in P^k_V \}$ to obtain a vertex $\boldsymbol{v}^k$.

$\langle k1 \rangle$ (termination) If either $\boldsymbol{d}\bar{\boldsymbol{x}} = 0$ or $\bar{g}(\boldsymbol{v}^k) > 0$ then stop. The optimal value of (mmF) is $\lceil \boldsymbol{d}\bar{\boldsymbol{x}} \rceil$. Otherwise, obtain the point $\boldsymbol{x}^k \in (\boldsymbol{0}, \boldsymbol{v}^k) \cap \partial \bar{G}_\varepsilon$.

$\langle k2 \rangle$ (update) If $\boldsymbol{x}^k \in X$ then solve $\lambda^* := \max\{ \lambda \mid (1-\lambda)\boldsymbol{x}^k + \lambda \boldsymbol{v}^k \in \partial X,\ \lambda \in [0,1] \}$ and set $\boldsymbol{y}^k := (1-\lambda^*)\boldsymbol{x}^k + \lambda^* \boldsymbol{v}^k$.

$\langle k2.1 \rangle$ If $\boldsymbol{y}^k \in X_M$ then apply the LS($\boldsymbol{y}^k$) procedure to obtain the local optimal vertex $\boldsymbol{z}^k$. We obtain the point $\bar{\boldsymbol{y}}^k \in (\boldsymbol{0}, \boldsymbol{z}^k) \cap \partial \bar{G}_\varepsilon$.

$\langle k2.1.1 \rangle$ If $\boldsymbol{d}\bar{\boldsymbol{y}}^k \leqq \boldsymbol{d}\boldsymbol{x}^k$ then set $\bar{\boldsymbol{x}} := \bar{\boldsymbol{y}}^k$,
$P^{k+1} := P^k \cap \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{d}\boldsymbol{x} \leqq \boldsymbol{d}\bar{\boldsymbol{x}} \}$.

$\langle k2.1.2 \rangle$ Otherwise set $\bar{\boldsymbol{x}} := \boldsymbol{x}^k$ and
$P^{k+1} := P^k \cap \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{d}\boldsymbol{x} \leqq \boldsymbol{d}\bar{\boldsymbol{x}} \}$.

$\langle k2.2 \rangle$ If $\boldsymbol{y}^k \notin X_M$ then set $\bar{\boldsymbol{x}} := \boldsymbol{x}^k$ and
$P^{k+1} := P^k \cap \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{d}\boldsymbol{x} \leqq \boldsymbol{d}\bar{\boldsymbol{x}},\ l(\boldsymbol{x}) \leqq 0 \}$.

$\langle k3 \rangle$ If $\boldsymbol{x}^k \notin X$ then set $P^{k+1} := P^k \cap \{ \boldsymbol{x} \in \mathbb{R}^n \mid l(\boldsymbol{x}) \leqq 0 \}$.

$\langle k4 \rangle$ Compute the vertex set P^{k+1}_V of P^{k+1}. Set $k := k+1$ and go to $\langle k \rangle$.

Remark 4.14. The function $l : \mathbb{R}^n \to \mathbb{R}$ in Step k2.2 and Step k3 is given by one of the inequalities $\pm A\boldsymbol{x} \leqq 0$ and $\boldsymbol{x} \leqq \boldsymbol{c}$ not satisfied by the point $\boldsymbol{v}^k$, i.e.,

(i) $l(\boldsymbol{x}) = \boldsymbol{e}^j\boldsymbol{x} - c_j$ for some $j \in \{1, \ldots, n\}$ such that $v_j^k > c_j$, or
(ii) $l(\boldsymbol{x}) = \boldsymbol{a}^i\boldsymbol{x}$ or $-\boldsymbol{a}^i\boldsymbol{x}$ for some $i \in \{1, \ldots, m\}$ such that $\boldsymbol{a}^i\boldsymbol{v}^k > 0$ or $\boldsymbol{a}^i\boldsymbol{v}^k < 0$, respectively, where $\boldsymbol{a}^i$ is the ith row of A.

Lemma 4.15. *Let $\boldsymbol{z}^k$ be the local optimal vertex obtained by applying the LS($\boldsymbol{y}^k$) procedure in Step k2.1 at iteration k, and suppose $\boldsymbol{dz}^k > 0$. Then $\boldsymbol{dz}^{k'} < \boldsymbol{dz}^k$ for iteration k' such that $k' > k$.*

Proof: Let $\bar{\boldsymbol{x}}^k$ be the incumbent and $\bar{\boldsymbol{y}}^k$ be the point contained in $(\boldsymbol{0}, \boldsymbol{z}^k) \cap \partial\bar{G}_\varepsilon$ at iteration k. In the OA method for (mmF), we have $D(\bar{\boldsymbol{x}}^k) \subseteqq P^k \subseteqq \{\, \boldsymbol{x} \mid \boldsymbol{dx} \leqq \boldsymbol{d}\bar{\boldsymbol{x}}^k \,\}$ for each iteration k. Since $\boldsymbol{y}^k \in D(\bar{\boldsymbol{x}}^k)$ and $\boldsymbol{dz}^k > 0$, we see that $0 < \boldsymbol{d}\bar{\boldsymbol{y}}^k < \boldsymbol{dz}^k \leqq \boldsymbol{dy}^k \leqq \boldsymbol{d}\bar{\boldsymbol{x}}^k$. Then we have $\boldsymbol{d}\bar{\boldsymbol{x}}^{k+1} = \min\{\, \boldsymbol{dx}^k, \boldsymbol{d}\bar{\boldsymbol{y}}^k \,\} < \boldsymbol{dz}^k \leqq \boldsymbol{d}\bar{\boldsymbol{x}}^k$ in Step k2.1. This implies that $\boldsymbol{dz}^{k'} < \boldsymbol{dz}^k$ for iteration k' with $k' > k$. □

Theorem 4.16. *The OA method for (mmF) works correctly and terminates after finitely many iterations.*

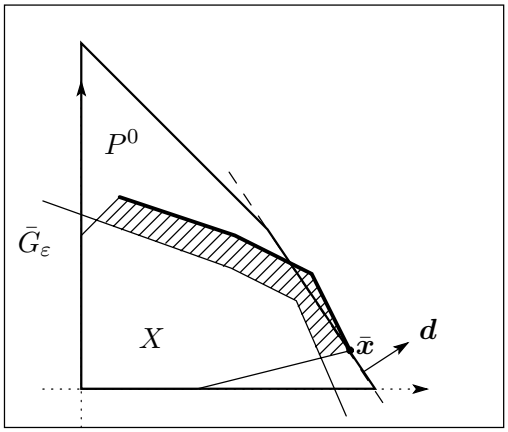
Proof: (correctness) If $N_M(\boldsymbol{w}^0) = \emptyset$ at iteration 0 then $\boldsymbol{w}^0$ solves (mmF) by the connectedness of X_M. When the algorithm terminates in Step k1, the optimal value of (mmF) is equal either to zero by the assumption that $\boldsymbol{dx} \geqq 0$ for all $\boldsymbol{x} \in X$, or to $\lceil \boldsymbol{d}\bar{\boldsymbol{x}} \rceil$ by Theorem 4.13. So the optimal value is obtained when the algorithm terminates either in Step 0 or in Step k1.

We suppose that the algorithm has not yet terminated and show that each step of the algorithm can be done. Let $\bar{\boldsymbol{x}}^k$ be the incumbent and $\boldsymbol{v}^k$ be an optimal solution of $\max\{\, -\bar{g}(\boldsymbol{v}) \mid \boldsymbol{v} \in P_V^k \,\}$ at iteration k, and suppose that $\boldsymbol{d}\bar{\boldsymbol{x}}^k > 0$ and $\bar{g}(\boldsymbol{v}^k) \leqq 0$. By Lemma 4.9, there is a point $\boldsymbol{x}^k \in (\boldsymbol{0}, \boldsymbol{v}^k) \cap \partial\bar{G}_\varepsilon$. Similarly, for all $\boldsymbol{z}^k \in X_M \cap X_V$ there is a point $\bar{\boldsymbol{y}}^k \in (\boldsymbol{0}, \boldsymbol{z}^k) \cap \partial\bar{G}_\varepsilon$ because $\bar{g}(\boldsymbol{z}^k) = 0$. Since $\boldsymbol{v}^k$ does not lie in the relative interior of X by Lemma 4.1 and the assumption that $\bar{g}(\boldsymbol{v}^k) \leqq 0$, we have $[\boldsymbol{x}^k, \boldsymbol{v}^k] \cap \partial X \neq \emptyset$ when $\boldsymbol{x}^k \in X$. Therefore λ^* in Step k1 is calculated and $\boldsymbol{y}^k$ is defined. When $\boldsymbol{x}^k \notin X$, clearly $\boldsymbol{v}^k \notin X$, and hence the function $l : \mathbb{R}^n \to \mathbb{R}$ can be defined in Step k3. To show that the function $l : \mathbb{R}^n \to \mathbb{R}$ is defined in Step k2.2 we have only to show that $\boldsymbol{v}^k \notin X$. Suppose the contrary, i.e., $\boldsymbol{v}^k \in X$. Then $\bar{g}(\boldsymbol{v}^k) = 0$ by the assumption that $\bar{g}(\boldsymbol{v}^k) \leqq 0$ and the fact that $\bar{g}(\boldsymbol{x}) \geqq 0$ for all $\boldsymbol{x} \in X$. This means $\boldsymbol{v}^k \in X_M$, which is a subset of ∂X by Corollary 4.2, and hence $\boldsymbol{y}^k = \boldsymbol{v}^k$ by the choice of $\boldsymbol{y}^k$. Then $\boldsymbol{y}^k \in X_M$, which contradicts that we are at iteration k2.2. Therefore we have seen that $\boldsymbol{v}^k \notin X$ in Step k2.2.
(finiteness) Suppose that the polytope P^ν at iteration ν meets the condition

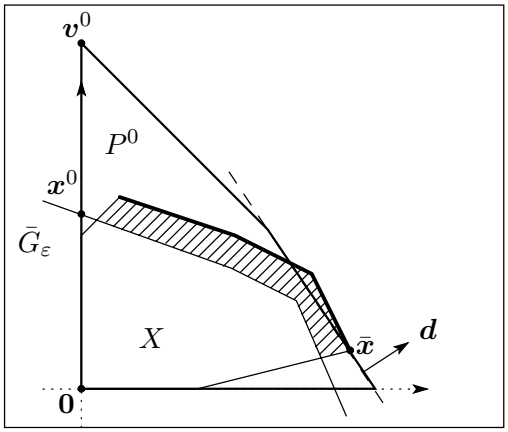
$$P^\nu \subseteqq X \text{ and } P^\nu \cap X_M = \emptyset, \tag{4.9}$$

after updated either in Step $k2$ or in Step $k3$, and consider the next iteration. Since $\boldsymbol{v}^\nu$ is chosen from P^ν, we have $\boldsymbol{v}^\nu \in X \backslash X_M$ and consequently $\bar{g}(\boldsymbol{v}^\nu) > 0$. Then the algorithm stops at Step $k1$. Therefore we have only to prove that (4.9) holds within a finite number of iterations. Note first that both Step $k2.2$ and Step $k3$ are done at most a finite number of times. Indeed, the polytope, say $P^{k'}$, with $2m+n$ cuts $l(\boldsymbol{x}) \leqq 0$ added to the initial polytope P^0 is contained in X. Therefore $\boldsymbol{v}^{k'}$ as well as $\boldsymbol{x}^{k'}$ lies in X, and hence we obtain that $\boldsymbol{y}^{k'} = \boldsymbol{v}^{k'} \in X_M$ in the same way as in the former part of this proof. Therefore we come to neither Step $k2.2$ nor Step $k3$ after iteration k'. Namely, Step $k2.1$ followed by Step $k4$ repeats itself after iteration k'. For iteration k with $k \geqq k'+1$, we have $\boldsymbol{y}^k \in X_M$. We then locate $\boldsymbol{z}^k \in X_M \cap X_V$ by applying the LS($\boldsymbol{y}^k$) procedure and obtain a point $\bar{\boldsymbol{y}}^k \in (\boldsymbol{0}, \boldsymbol{z}^k) \cap \partial \bar{G}_\varepsilon$. If $\boldsymbol{dz}^k = 0$ for some k then we set $\bar{\boldsymbol{x}}^{k+1} := \bar{\boldsymbol{y}}^k$ since $\boldsymbol{d}\bar{\boldsymbol{y}}^k = \boldsymbol{d}(\lambda \boldsymbol{0} + (1-\lambda)\boldsymbol{z}^k) = \lambda \boldsymbol{d0} + (1-\lambda)\boldsymbol{dz}^k = 0 \leqq \boldsymbol{dx}^k$. The incumbent value $\boldsymbol{d}\bar{\boldsymbol{x}}^{k+1}$ becomes zero, and hence the algorithm stops in Step $k1$ at the next iteration. If $\boldsymbol{dz}^k > 0$ for all k with $k \geqq k'+1$. We see that $\boldsymbol{dz}^{k+1} < \boldsymbol{dz}^k$ for all k by Lemma 4.15. Since $|X_M \cap X_V|$ is finite, we eventually obtain a point $\boldsymbol{z}^{\nu-1} \in X_M \cap X_V$ such that $\boldsymbol{dz}^{\nu-1} \leqq \boldsymbol{dz}$ for all $\boldsymbol{z} \in X_M \cap X_V$. Also we have $\boldsymbol{d}\bar{\boldsymbol{y}}^{\nu-1} < \boldsymbol{dz}^{\nu-1}$ by the choice of $\bar{\boldsymbol{y}}^{\nu-1}$. The polytope P^ν is then defined as $P^\nu := P^{\nu-1} \cap \{\, \boldsymbol{x} \mid \boldsymbol{dx} \leqq \boldsymbol{d}\bar{\boldsymbol{x}}^\nu \,\}$, where $\bar{\boldsymbol{x}}^\nu$ satisfies that $\boldsymbol{d}\bar{\boldsymbol{x}}^\nu = \min\{\, \boldsymbol{dx}^{\nu-1}, \boldsymbol{d}\bar{\boldsymbol{y}}^{\nu-1} \,\} < \boldsymbol{dz}^{\nu-1}$. This means that $P^\nu \cap (X_M \cap X_V) = \emptyset$. By Theorem 2.2, we see that $\boldsymbol{dz}^{\nu-1} \leqq \boldsymbol{dx}$ for all $\boldsymbol{x} \in X_M$. Therefore we conclude that $P^\nu \cap X_M = \emptyset$. □

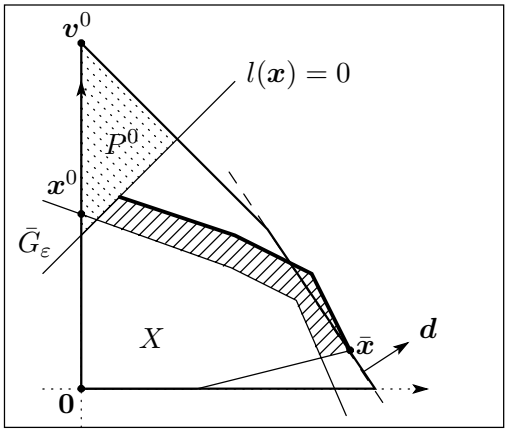
We illustrate the OA method for (mmF) in Figure 4.2. We obtain a local optimal vertex $\bar{\boldsymbol{x}} \in X_M \cap X_V$ and set up an initial polytope P^0 (See (a)). It is easy to enumerate all vertices of P^0 because this polytope is simply given by $P^0 := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{ex} \leqq c_{max},\ \boldsymbol{dx} \leqq \boldsymbol{d}\bar{\boldsymbol{x}},\ \boldsymbol{x} \geqq \boldsymbol{0} \,\}$. We obtain a point $\boldsymbol{v}^0$ which solves $\max\{\, -\bar{g}(\boldsymbol{v}) \mid \boldsymbol{v} \in P_V^0 \,\}$, and a point $\boldsymbol{x}^0 \in (\boldsymbol{0}, \boldsymbol{v}^0) \cap \partial \bar{G}_\varepsilon$ (See (b)). We see that $\boldsymbol{x}^0 \notin X$, and hence cut off $\boldsymbol{v}^0$ from P^0 (See (c)). Using P_V^0, we compute P_V^1. In the next iteration, we obtain points $\boldsymbol{v}^1, \boldsymbol{x}^1$ and $\boldsymbol{y}^1$. Since $\boldsymbol{y}^1 \in X_M$, we apply the LS($\boldsymbol{y}^1$) procedure to obtain a point $\boldsymbol{z}^1$, and obtain a point $\bar{\boldsymbol{y}}^1 \in (\boldsymbol{0}, \boldsymbol{z}^1) \cap \partial \bar{G}_\varepsilon$ (See (d)). We have $\bar{\boldsymbol{y}}^1 \in X \backslash \bar{G}_\varepsilon$ such that $\boldsymbol{d}\bar{\boldsymbol{y}}^1 < \min\{\, \boldsymbol{dx}^1, \boldsymbol{d}\bar{\boldsymbol{x}} \,\}$. We then update the incumbent $\bar{\boldsymbol{x}}$ to $\bar{\boldsymbol{y}}^1$ and construct P^2 by adding the cut $\boldsymbol{dx} \leqq \boldsymbol{d}\bar{\boldsymbol{x}}$ to P^1 (See (e)). We terminate at the next iteration because $\bar{g}(\boldsymbol{v}) > 0$ for all vertices $\boldsymbol{v}$ of P^2 (See (f)). The optimal value $\lceil \boldsymbol{d}\bar{\boldsymbol{x}} \rceil$ is obtained



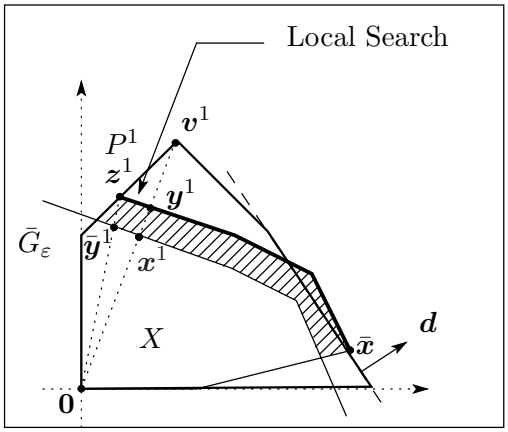
(a) polytope P^0



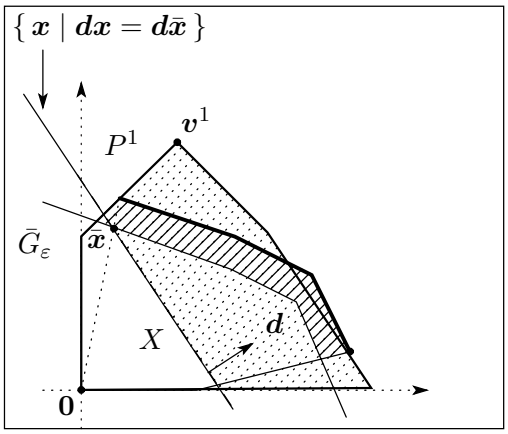
(b) obtaining $\boldsymbol{v}^0$ and $\boldsymbol{x}^0$



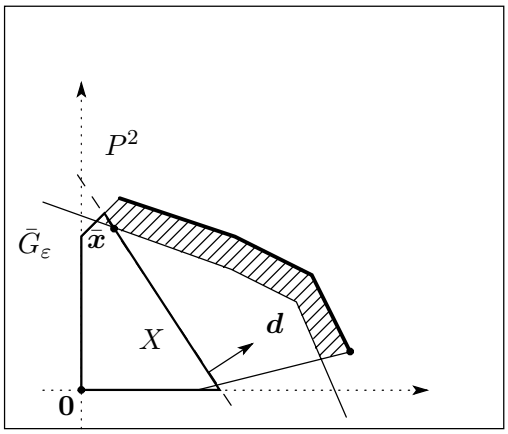
(c) cutting off $\boldsymbol{v}^0$ from P^0



(d) obtaining $\boldsymbol{v}^1$, $\boldsymbol{x}^1$, $\boldsymbol{y}^1$ and $\bar{\boldsymbol{y}}^1$



(e) cutting off $\boldsymbol{v}^1$ from P^1



(f) termination

FIGURE 4.2. An example of the OA method for (mmF)

5. Conclusion and Further Works

Computational experiment should be carried out to verify the efficiency of the algorithms we proposed in this paper. In many problems formulated as (P_E) the criterion matrix C has quite a small number p of rows. Some sophisticated algorithms for (P_E) take advantage of this property. The number p is however equal to the number of arcs, i.e., $p = n$ in (mmF). Therefore it is likely that a primitive method surpasses some sophisticated algorithms. The comparative studies of several algorithms for (mmF) are significant and required for further works.

The OA method for (mmF) provides the optimal value but not an optimal solution. The knowledge of the optimal value is not helpful to find an optimal solution, however, the following lemma is a clue to the way of finding an optimal solution.

Lemma 5.1. *Let $\boldsymbol{x}^*_\varepsilon$ be an ε-optimal solution of (mmF), and*

$$\Delta_\varepsilon = \{\, \boldsymbol{\xi} \in \mathbb{R}^n \mid A\boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{\xi} \geqq \boldsymbol{0},\ \boldsymbol{e}\boldsymbol{\xi} \leqq \varepsilon \,\}. \tag{5.1}$$

For $\varepsilon \in (0,1)$, if there is an integer vector $\boldsymbol{x}^*$ of dimension n such that $\boldsymbol{x}^* = \boldsymbol{x}^*_\varepsilon + \bar{\boldsymbol{\xi}}$ for some $\bar{\boldsymbol{\xi}} \in \Delta_\varepsilon$ then $\boldsymbol{x}^*$ is an optimal solution of (mmF).

Proof: (feasibility) We show that $\boldsymbol{x}^* = \boldsymbol{x}^*_\varepsilon + \bar{\boldsymbol{\xi}}$ is a feasible solution of (mmF), i.e., $\boldsymbol{x}^* \in X_M$. Let $\boldsymbol{y}^*$ be an optimal solution of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^* \,\}$. If the following assertion

$$\text{both } \boldsymbol{e}\boldsymbol{x}^* \text{ and } \boldsymbol{e}\boldsymbol{y}^* \text{ are integer, and } \ \boldsymbol{e}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{e}\boldsymbol{x}^* \leqq \boldsymbol{e}\boldsymbol{y}^* < \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + 1 \tag{5.2}$$

is true then we have $\boldsymbol{e}\boldsymbol{x}^* = \boldsymbol{e}\boldsymbol{y}^*$, and hence $\bar{g}(\boldsymbol{x}^*) = g(\boldsymbol{x}^*) = 0$, i.e., $\boldsymbol{x}^* \in X_M$. Now we show the assertion (5.2). Since $\boldsymbol{x}^*$ is an integer vector, clearly $\boldsymbol{e}\boldsymbol{x}^*$ is integer. By the integrality property of X, $\boldsymbol{e}\boldsymbol{y}^*$ is also integer. It is also clear that $\boldsymbol{e}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{e}\boldsymbol{x}^* \leqq \boldsymbol{e}\boldsymbol{y}^*$. Then the condition $\boldsymbol{e}\boldsymbol{y}^* < \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + 1$ remains to prove. Let $\boldsymbol{y}^*_\varepsilon$ be an optimal solution of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^*_\varepsilon \,\}$, and let $\boldsymbol{\xi}^* = \boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon$. We see that $A\boldsymbol{\xi}^* = A\boldsymbol{y}^*_\varepsilon - A\boldsymbol{x}^*_\varepsilon = \boldsymbol{0}$, $\boldsymbol{\xi}^* \geqq \boldsymbol{0}$ and $\boldsymbol{e}\boldsymbol{\xi}^* = \boldsymbol{e}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon) = \bar{g}(\boldsymbol{x}^*_\varepsilon) \leqq \varepsilon$, and hence $\boldsymbol{\xi}^* \in \Delta_\varepsilon$. Then $\boldsymbol{e}\boldsymbol{y}^*_\varepsilon = \boldsymbol{e}(\boldsymbol{x}^*_\varepsilon + \boldsymbol{\xi}^*) \leqq \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + \varepsilon < \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + 1$. The point $\boldsymbol{y}^*$ is a feasible solution of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^*_\varepsilon \,\}$, since $\boldsymbol{y}^* \in X$ and $\boldsymbol{y}^* \geqq \boldsymbol{x}^* = \boldsymbol{x}^*_\varepsilon + \bar{\boldsymbol{\xi}} \geqq \boldsymbol{x}^*_\varepsilon$. Then we see that $\boldsymbol{e}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{y}^*) \geqq 0$, and hence $\boldsymbol{e}\boldsymbol{y}^* \leqq \boldsymbol{e}\boldsymbol{y}^*_\varepsilon < \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + 1$.
(optimality) We show that $\boldsymbol{x}^*$ solves (mmF). Clearly, $\boldsymbol{d}\bar{\boldsymbol{\xi}} \leqq \boldsymbol{e}\bar{\boldsymbol{\xi}}$ since $\boldsymbol{d} \leqq \boldsymbol{e}$ and $\bar{\boldsymbol{\xi}} \geqq \boldsymbol{0}$. For any $\boldsymbol{v} \in X_M \cap X_V$, we see that $\bar{g}(\boldsymbol{v}) \leqq \varepsilon$, and $\boldsymbol{v}$ is an integer vector by the integrality property of X. Since $\boldsymbol{x}^*_\varepsilon = \boldsymbol{x}^* - \bar{\boldsymbol{\xi}}$ is an optimal solution of (mmF_ε), we have $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\boldsymbol{x}$ for all $\boldsymbol{x} \in X$ such that $\bar{g}(\boldsymbol{x}) \leqq \varepsilon$, and hence $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\boldsymbol{v}$ for all $\boldsymbol{v} \in X_M \cap X_V$. Then we see that $\boldsymbol{d}\boldsymbol{x}^* = \boldsymbol{d}\boldsymbol{x}^*_\varepsilon + \boldsymbol{d}\bar{\boldsymbol{\xi}} \leqq \boldsymbol{d}\boldsymbol{v} + \boldsymbol{e}\bar{\boldsymbol{\xi}} < \boldsymbol{d}\boldsymbol{v} + 1$. Since both $\boldsymbol{x}^*$ and $\boldsymbol{v}$ are integer vectors, we have $\boldsymbol{d}\boldsymbol{x}^* \leqq \boldsymbol{d}\boldsymbol{v}$ for all $\boldsymbol{v} \in X_M \cap X_V$. □

Appendix

The Proof of Theorem 2.1

Proof: ($\Leftarrow$) Assume that $\bar{\boldsymbol{x}} \in D$ is not an efficient point for (MC). There exists $\boldsymbol{y} \in D$ such that $C\boldsymbol{y} \geqq C\boldsymbol{x}$ and $C\boldsymbol{y} \neq C\boldsymbol{x}$. Then, $\bar{\boldsymbol{x}}$ is not an optimal solution of $(SC(\boldsymbol{\lambda}))$ for any $\boldsymbol{\lambda} \in \mathbb{R}_{p++}$. ($\Rightarrow$) Suppose $\bar{\boldsymbol{x}} \in D$ is an efficient point for (MC). Let $L_{\bar{\boldsymbol{x}}} = \mathrm{diag}\{l_1, \ldots, l_n\}$, where

$$l_i = \begin{cases} 1 & \text{if } \bar{x}_i = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, \ldots n). \tag{5.3}$$

If there exists a vector $\boldsymbol{u} \in \mathbb{R}^n$ satisfies the system

$$C\boldsymbol{u} \geqq \boldsymbol{0},\ C\boldsymbol{u} \neq \boldsymbol{0},\ L_{\bar{\boldsymbol{x}}}\boldsymbol{u} \geqq \boldsymbol{0},\ B\boldsymbol{u} = \boldsymbol{0}, \tag{5.4}$$

setting $\boldsymbol{x} = \bar{\boldsymbol{x}} + \theta\boldsymbol{u}$ for a sufficiently small $\theta > 0$, we see $\boldsymbol{x} \in D$ satisfies $C\boldsymbol{x} \geqq C\bar{\boldsymbol{x}}$ and $C\boldsymbol{x} \neq C\bar{\boldsymbol{x}}$. This contradicts that $\bar{\boldsymbol{x}}$ is an efficient point for (MC). Then, there is no vector $\boldsymbol{u} \in \mathbb{R}^n$ satisfies the system (5.4). Applying the Tucker's alternative theorem (See Mangasarian [20]), there are vectors $\boldsymbol{\lambda} \in \mathbb{R}_p$, $\boldsymbol{\mu} \in \mathbb{R}_n$ and $\boldsymbol{\nu} \in \mathbb{R}_m$ such that

$$\boldsymbol{\lambda}C + \boldsymbol{\mu}L_{\bar{\boldsymbol{x}}} + \boldsymbol{\nu}B = \boldsymbol{0},\ \boldsymbol{\lambda} > \boldsymbol{0},\ \boldsymbol{\mu} \geqq \boldsymbol{0}. \tag{5.5}$$

For any $\boldsymbol{x} \in \mathbb{R}^n$, we see that

$$\boldsymbol{\lambda}C(\bar{\boldsymbol{x}} - \boldsymbol{x}) + \boldsymbol{\mu}L_{\bar{\boldsymbol{x}}}(\bar{\boldsymbol{x}} - \boldsymbol{x}) + \boldsymbol{\nu}B(\bar{\boldsymbol{x}} - \boldsymbol{x}) = 0. \tag{5.6}$$

Therefore for any $\boldsymbol{x} \in D$, we have $\boldsymbol{\mu}L_{\bar{\boldsymbol{x}}}(\bar{\boldsymbol{x}} - \boldsymbol{x}) \leqq 0$ and $\boldsymbol{\nu}B(\bar{\boldsymbol{x}} - \boldsymbol{x}) = 0$, and hence $\boldsymbol{\lambda}C\bar{\boldsymbol{x}} \geqq \boldsymbol{\lambda}C\boldsymbol{x}$. It remains to proof the assertion that there exists $M > 0$ such that we can replace $\mathbb{R}_{p++}$ with Λ of (2.4). By the theory of the parametric linear program, D_E is the union of finitely many faces $F_1, \cdots, F_K$ of D such that F_k is the optimal face of $(SC(\boldsymbol{\lambda}^k))$ for some $\boldsymbol{\lambda}^k \in \mathbb{R}_{p++}$. Let $\alpha_k = 1/\min\{\, \lambda_i^k \mid i = 1, \ldots, p \,\}$ for $k = 1, \ldots, K$ and $M = \max\{\, \alpha_k(\boldsymbol{\lambda}^k\boldsymbol{1}) \mid k = 1, \ldots, K \,\}$. Then $\bar{\boldsymbol{\lambda}}^k = (M/\boldsymbol{\lambda}^k\boldsymbol{1})\boldsymbol{\lambda}^k \in \Lambda$ and F^k remains the optimal face of $(SC(\bar{\boldsymbol{\lambda}}^k))$ for each $k = 1, \ldots, K$. □

Proof of Theorem 3.1

Proof: Suppose that $\boldsymbol{x}^* \in D \backslash \mathrm{int}H$ is not an optimal solution of (CDC), i.e., there exists $\boldsymbol{y} \in D \backslash \mathrm{int}H$ such that $\boldsymbol{p}\boldsymbol{y} < \boldsymbol{p}\boldsymbol{x}^*$. Clearly, $\boldsymbol{y} \in D(\boldsymbol{x}^*)$ and $h(\boldsymbol{y}) \geqq 0$. If $h(\boldsymbol{y}) > 0$ then $\boldsymbol{y}$ is not contained in H, and hence $\boldsymbol{y} \in D(\boldsymbol{x}^*) \backslash H$. By the regularity assumption, if $h(\boldsymbol{y}) = 0$, i.e., $\boldsymbol{y} \in \partial H$ then we can take $\boldsymbol{y}' \in N_\delta(\boldsymbol{y}) \cap D$ such that $\boldsymbol{p}\boldsymbol{y}' < \boldsymbol{p}\boldsymbol{x}^*$ and $h(\boldsymbol{y}') > 0$ for a sufficiently small $\delta > 0$, where $N_\delta(\boldsymbol{y}) = \{\, \boldsymbol{y}' \in \mathbb{R}^n \mid \|\boldsymbol{y}' - \boldsymbol{y}\| < \delta \,\}$, and hence we see that $\boldsymbol{y}' \in D(\boldsymbol{x}^*) \backslash H$. □

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