Interior Point Trajectories and a Homogeneous Model for Nonlinear Complementarity Problems over Symmetric Cones

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Abstract

We study the continuous trajectories for solving monotone nonlinear mixed complementarity problems over symmetric cones. While the analysis in [5] depends on the optimization theory of convex log-barrier functions, our approach is based on the paper of Monteiro and Pang [17], where a vast set of conclusions concerning continuous trajectories is shown for monotone complementarity problems over the cone of symmetric positive semidefinite matrices. As an application of the results, we propose a homogeneous model for standard monotone nonlinear complementarity problems over symmetric cones and discuss its theoretical aspects.

Key words. Complementarity problem, symmetric cone, homogeneous algorithm, existence of trajectory, interior point method

1 Introduction

Let V be an n-dimensional real vector space and $(V, \circ)$ be a Euclidian Jordan algebra with an identity element e. We denote by K a symmetric cone of V which is a self-dual closed convex cone such that there exists an invertible map $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$ for every $x \in \text{int}K$ and $y \in \text{int}K$. It is known that a cone in V is symmetric if and only if it is the cone of squares of V given by $K = \{x \circ x : \ x \in V\}$.

Faybusovich [5] studied the linear monotone complementarity problem over symmetric cones of the form

$$\begin{aligned} \text{(LCP) Find} \quad & (x, y) \in K \times K \\ \text{s.t.} \quad & (x, y) \in (a, b) + L, \ x \circ y = 0 \end{aligned} \tag{1}$$

where $(a, b) \in V \times V$ and $L \subseteq K \times K$ is a linear subspace with $\text{dim}L = \text{dim}V$ having the monotonicity, i.e.,

$$\langle x, y \rangle \geq 0 \quad \text{if} \quad (x, y) \in L.$$

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The author showed the existence of the central path of the form

$$\{(x, y) \in \text{int}K \times \text{int}K : \; x \circ y = \mu e, \; \mu > 0\}$$

whenever the (LCP) has an interior feasible solution $(x, y) \in ((a, b) + L) \cap (\text{int}K \times \text{int}K)$, based on primal-dual interior point methods for linear programs ([11, 15], etc).

Note that the first extension of primal-dual methods to a more general setting than linear programs was achieved by Nesterov and Todd [19, 20] who developed the powerful theoretical concept of self-scaled barrier functions. It is known that the self-scaled cones associated with the self-scaled barriers are closely related to the symmetric cones ([1, 9, 10, 24]). See also [6, 25, 22, 23] for other extensions of primal-dual methods to the symmetric cones or the self-scaled cones.

In most of the papers cited above, the analyses depend on the optimization theory of convex barrier functions. In this paper, apart from the theories of barrier functions, we will discuss the trajectory of an interior point map in view of homeomorphisms of continuous maps. It should be noted that there have been studied various types of central paths using the theory of homeomorphisms for some special cases of symmetric cones, i.e., the non-negative orthant and the cone of symmetric positive semidefinite matrices (see, [7, 12, 14, 18], etc). We will extend these results and provide a set of properties concerning the existence of central paths for the monotone nonlinear complementarity problems over symmetric cones. As an application of the results, we will give a homogeneous model for solving the problems.

We consider the following nonlinear and mixed complementarity problem:

$$\begin{array}{ll} \text{(CP) Find} & (x, y, z) \in K \times K \times \Re^m \\ \text{s.t.} & F(x, y, z) = 0, \; x \circ y = 0 \end{array} \tag{2}$$

where $F : K \times K \times \Re^m \to V \times \Re^m$ is a continuous and differentiable map. We say that

- (CP) is *asymptotically feasible* if and only if there exists a bounded sequence $\{x^{(k)}, y^{(k)}, z^{(k)}\} \subset \text{int}K \times \text{int}K \times \Re^m$ such that

$$\lim_{k \to \infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0,$$

- (CP) is *asymptotically solvable* if and only if there exists a bounded sequence $\{x^{(k)}, y^{(k)}, z^{(k)}\} \subset \text{int}K \times \text{int}K \times \Re^m$ such that

$$\lim_{k \to \infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0 \text{ and } \lim_{k \to \infty} x^{(k)} \circ y^{(k)} = 0. \tag{3}$$

As long as we discuss the asymptotical properties, the map F is required to be defined on the set $\text{int}K \times \text{int}K \times \Re^m$ rather than on $K \times K \times \Re^m$. By this reason, together with our aim to design a homogeneous model for (CP), we first consider that the map F is only defined on the set $\text{int}K \times \text{int}K \times \Re^m$.

We will impose the following assumption on F:

Assumption 1.1 **(i)** *F is (x, y)-equilevel-monotone on its domain, i.e., for every (x, y, z) and (x', y', z') in the domain of F satisfying $F(x, y, z) = F(x', y', z')$, $\langle x - x', y - y' \rangle \geq 0$ holds.*

(ii) *F is z-bounded on its domain, i.e., for every $\{(x^{(k)}, y^{(k)}, z^{(k)})\}$ in the domain of F, if $\{(x^{(k)}, y^{(k)})\}$ and $\{F(x^{(k)}, y^{(k)}, z^{(k)})\}$ are bounded then the sequence $\{z^{(k)}\}$ is also bounded.*

(iii) *$F(x, y, z)$ is z-injective on its domain, i.e., if (x, y, z) and (x, y, z') lie in the domain of F and satisfy $F(x, y, z) = F(x, y, z')$ then $z = z'$.*

The above assumption is the same as the one imposed by Monteiro and Pang [17] for the case of the cone of symmetric matrices. The authors provided some optimization problems whose optimality condition can be formulated into a (CP) satisfying the above assumption. Note that, in contrast to the paper [17], the domain of the map F is not given explicitly in the assumption.

The paper is organized as follows.

In Section 2, we will summarize well-known results for symmetric cones and give some lemmas which are crucial in the succeeding discussions.

Section 3 is devoted to deriving a homeomorphism of the map $H : \text{int}K \times \text{int}K \times \Re^m \to V \times \Re^m$ given by

$$H := \begin{pmatrix} x \circ y \\ F(x, y, z) \end{pmatrix}. \tag{4}$$

The main result, Theorem 3.10, ensures that if Assumption 1.1 is satisfied with the domain $\text{int}K \times \text{int}K \times \Re^m$ then the system

$$H(x, y, z) = h$$

has a solution $(x, y, z) \in \mathcal{U} \times \Re^m$ for every $h \in \text{int}K \times H(\mathcal{U} \times \Re^m)$, where the set $\mathcal{U}$ is a subset of $\text{int}K \times \text{int}K$ defined by (7).

Suppose that there exists a sequence $\{h^{(k)}\} \subset \text{int}K \times H(\mathcal{U} \times \Re^m)$ satisfying $h^{(k)} \to 0$. Then the result implies that there exists a sequence $\{(x^{(k)}, y^{(k)}, z^{(k)})\} \subset \mathcal{U} \times \Re^m$ which is the set of solutions of the system

$$H(x^{(k)}, y^{(k)}, z^{(k)}) = h^{(k)}$$

for every k. It is easy to see that the sequence $\{(x^{(k)}, y^{(k)}, z^{(k)})\} \subset \mathcal{U} \times \Re^m$ satisfies (3). Thus, if $\{(x^{(k)}, y^{(k)}, z^{(k)})\}$ is bounded then (CP) is asymptotically solvable. In Section 4, we will discuss the asymptotical solvability of (CP) under Assumption 1.1 with the domain $K \times K \times \Re^m$. The obtained results are direct extensions of the ones in [17].

In Section 5, as an application of the results in Section 3, we will provide a homogeneous model for a special class of (CP)s. The first homogeneous model for nonlinear complementarity problems over the n-dimensional positive orthant was proposed by Andersen and Ye [3]. A remarkable feature of their model is that the associated trajectory gives certifications on the (strong) feasibility or the (strong) infeasibility of the original problem. We will show that the model has the same property in the case of *simple* Euclidean Jordan algebras. To our knowledge, this is a first homogeneous model having the property for (CP)s over the cone of symmetric matrices and/or the ones over the second order cone, which are the symmetric cones in the simple Euclidean Jordan algebras.

Some concluding remarks will be given in Section 6.

2 Some key lemmas for the symmetric cone

In this section, we give a summary of the theory of Euclidean Jordan algebra, which are used in the paper. Most of the results in this section can be found in the book of Faraut and Korányi[4]

Let V be n-dimensional real vector space and let $(V, \circ)$ be a Jordan algebra with the identity element e. Here $u \circ v$ is a bilinear map satisfying

(i) $u \circ v = v \circ u$,

(ii) $u \circ (v \circ u^2) = (u \circ v) \circ u^2$ where $u^2 = u \circ u$,

(iii) $u \circ e = e \circ u = u$

for every u and v in V. For $u \in V$, $L(u)$ denotes the multiplication by u linear operator satisfying

$$L(u)v = u \circ v$$

for every $v \in V$. We introduce the scalar product of the form

$$\langle u, v \rangle = \mathrm{tr}(u \circ v).$$

Then, for every $u \in V$, the linear operator $L(u)$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle v, L(u)w \rangle = \langle L(u)v, w \rangle$$

holds for $v, w \in V$. The set

$$K := \{x^2 : \ x \in V\}.$$

is the symmetric cone of V which has the following properties.

Proposition 2.1 **(i)** *The set K is a closed, pointed, convex and self-dual cone, and the set $\mathrm{int}K$ is an open, convex and self-dual cone.*

(ii) $\mathrm{int}K = \{u \in V : \ L(u) \succ 0\}$ *where $L(u) \succ 0$ means that $L(u)$ is positive definite.*

(iii) $\mathrm{int}K$ *is the connected component of e in the set*

$$\mathrm{Inv} := \{x \in V : \ x \text{ is invertible } \} = \{x \in V : \ \det(x) \neq 0\}.$$

(iv) $\mathrm{int}K = \{x^2 : \ x \in \mathrm{Inv}\} = \{P(x)e : x \in \mathrm{Inv}\}.$

Proof: See Theorem III.2.1 of [4] for (i) and (ii), Propositions III.2.2 and II.2.4 of [4] for (iii) and the proof of Theorem III.2.1 of [4] for (iv). ∎

The following proposition is crucial in our analysis.

Proposition 2.2 *For every $y \in \mathrm{int}K$ and $\eta > 0$, the set*

$$\{x \in K : \langle x, y \rangle \leq \eta\}$$

is compact.

Proof: See Corollary I.1.6 of [4]. ∎

Next we define the rank of a Jordan algebra and introduce the concept of Jordan frames.

For $u \in V$, the *degree* of u is the smallest integer q such that the set $\{e, u, u^2, \ldots, u^q\}$ is linearly independent. The *rank* r of V is the maximum of the degree of u over all $u \in V$.

An *idempotent* c is an element of V such that $c^2 = c$. An idempotent is *primitive* if it is nonzero and not given by the sum of two nonzero idempotents. A complete system of orthogonal idempotents is a set $\{c_1, c_2, \ldots, c_p\}$, where

$$c_j \circ c_j = c_j, \quad c_j \circ c_k = 0 \ (j \neq k), \quad \sum_{j=1}^{r} c_j = e.$$

A complete system of orthogonal primitive idempotents is called a *Jordan frame*. The following theorem shows that every Jordan frame has exactly r primitive idempotents.

Theorem 2.3 **(i)** *For every $u \in V$, there exist real numbers $\lambda_1, \ldots, \lambda_r$ and a Jordan frame $c_1, \ldots, c_r$ such that*

and

$$u = \sum_{j=1}^{r} \lambda_j c_j.$$

Here the numbers λ_j (with their multiplicities) are uniquely determined by u and λ_j's are called the eigenvalues *(multiplicities included) of u.*

(ii) *For the eigenvalues λ_j $(j = 1, 2, \ldots, r)$ of u, we have*

$$\det(u) = \Pi_{j=1}^{r} \lambda_j, \quad \mathrm{tr}(u) = \sum_{j=1}^{r} \lambda_j.$$

(iii) *Let c be an idempotent in a Jordan algebra, $c^2 = c$. The only possible eigenvalues of $L(c)$ are 0, $\frac{1}{2}$ and 1.*

Proof: See Theorem III.1.2 of [4] for (i) and (ii) and Proposition III.1.3 of [4] for (iii) . ∎

For $u \in V$, let $u = \sum_{j=1}^{r} \lambda_j c_j$ be a decomposition described in the above theorem. Then we can easily see that

$$u^2 = \sum_{j=1}^{r} \lambda_j^2 c_j$$

which implies that λ_j^2 $(j = 1, 2, \ldots, r)$ are the eiganvalues of u^2. Thus (ii) of the above theorem implies that

$$\langle u, u \rangle = \mathrm{tr}(u \circ u) = \sum_{j=1}^{r} \lambda_j^2.$$

Throughout the paper, we employ the following Frobenius norm as a norm on V:

$$\|u\| := \sqrt{\langle u, u \rangle} = \sqrt{\sum_{j=1}^{r} \lambda_j^2}. \tag{5}$$

Note that every eigenvalue of e is 1 and $\|e\| = \sqrt{r}$.

Let us observe the eigenvalues of $u \in K$. For every $u \in K$, there exists $v \in V$ such that $u = v^2$. Let $v = \sum_{j=1}^{r} \lambda_j c_j$ be a decomposition of u given by Theorem 2.3. Then we have

$$u = v^2 = \sum_{j=1}^{r} \lambda_j^2 c_j^2 = \sum_{j=1}^{r} \lambda_j^2 c_j$$

which implies that all the eigenvalues of $u \in K$ are nonnegative. In addition, by (iii) of Proposition 2.1 and by (ii) of Theorem 2.3, we can see that $\lambda_j^2 \neq 0$ for every j whenever $u \in \mathrm{int}K$. Consequently, we obtain the following corollary.

Corollary 2.4 *Let $u \in V$ and let $\sum_{j=1}^{r} \lambda_j c_j$ be a decomposition of u given by Theorem 2.3.*

(i) *If $u \in K$ then $\lambda_j \geq 0$ $(j = 1, 2, \dots, r)$.*

(ii) *If $u \in \mathrm{int}K$ then $\lambda_j > 0$ $(j = 1, 2, \dots, r)$.*

Here we introduce the notion of the *quadratic representation* of $(V, \circ)$. Given $x \in V$, we define

$$P(x) := 2L(x)^2 - L(x^2).$$

and

$$\begin{aligned} P(x, y) &:= \frac{1}{2} D_y P(x) \\ &= \frac{1}{2}(P(x + y) - P(x) - P(y)) \\ &= L(x)L(y) + L(y)L(x) - L(x \circ y). \end{aligned} \tag{6}$$

Proposition 2.5 *For every $x, y \in V$,*

(i) $P(x)e = x^2$,

(ii) *$P(x)$ is invertible if and only if x is invertible,*

(iii) $P(P(y)x) = P(y)P(x)P(y)$.

If $x, y \in V$ are invertible then

(iv) $P(x)x^{-1} = x$,

(v) $P(x)^{-1} = P(x^{-1})$,

(vi) *$P(x)y$ is invertible,*

(vii) $(P(x)y)^{-1} = P(x^{-1})y^{-1}$,

(viii) $P(x)\mathrm{int}K = \mathrm{int}K$.

For every $x \in \mathrm{int}K$,

(ix) $P(x)^{-1/2} = P(x^{-1/2})$.

Proof: (i) follows from the definition of $P(x)$.

See Proposition II.3.1 of [4] for the proofs of (ii), (iv) and (v), Proposition II.3.3 of [4] for (iii), (vi) and (vii), and Proposition III.2.2 of [4] for (viii).

Using the results (i), (iii) and (v), we obtain (ix) as follows:

$$\begin{aligned}[P(x)^{-1/2}]^2 &= P(x)^{-1} \\ &= P(x^{-1}) & \text{(by (v))} \\ &= P(P(x^{-1/2}e)) & \text{(by (i))} \\ &= P(x^{-1/2})P(e)P(x^{-1/2}) & \text{(by (iii))} \\ &= [P(x^{-1/2})]^2.\end{aligned}$$

∎

The following is a collection of technical facts which will be often used in the succeeding sections. Before we proceed, we give a definition of the *star-shaped* set in a vector space.

Definition 2.6 *A subset C of a vector space is said to be* star-shaped *if there exists $c^0 \in C$ such that the line segment connecting to c^0 to any other point in C is contained entirely in C.*

Lemma 2.7 **(i)** *$u \in K$ if and only if $\langle u, v\rangle \geq 0$ for every $v \in K$.*

(ii) *For every $u \in K$ and $v \in K$, $\langle u, v\rangle = 0$ if and only if $u \circ v = 0$.*

(iii) *For every $u \in \text{int}K$ and $v \in K$, $\langle u, v\rangle = 0$ if and only if $v = 0$.*

(iv) *Define*

$$\mathcal{U} := \{(x, y) \in \text{int}K \times \text{int}K : \ x \circ y \in \text{int}K\}. \tag{7}$$

Then

$$\mathcal{U} = \{(x, y) \in K \times K : \ x \circ y \in \text{int}K\}.$$

(v) *For every $(x, y) \in \mathcal{U}$, $L(x)L(y) + L(y)L(x) \succ 0$.*

(vi) *For every $(x, y) \in \mathcal{U}$ and $(\Delta x, \Delta y) \in V \times V$, if*

$$\langle \Delta x, \Delta y\rangle \geq 0, \tag{8}$$

$$x \circ \Delta y + y \circ \Delta x = 0 \tag{9}$$

hold then $\Delta x = \Delta y = 0$.

(vii) *$\mathcal{U}$ is a nonempty and open subset of $\text{int}K \times \text{int}K$ which is star-shaped.*

(viii) *For every $x \in \text{int}K$, $(x, x) \in \mathcal{U}$.*

(ix)

$$\begin{aligned}&\text{int}K \times \{\alpha e : \ \alpha \in \Re_{++}\} \subset \mathcal{U}, \quad \{\alpha e : \ \alpha \in \Re_{++}\} \times \text{int}K \subset \mathcal{U}, \\ &K \times \{\alpha e : \ \alpha \in \Re_{+}\} \subset \text{cl}(\mathcal{U}), \quad \{\alpha e : \ \alpha \in \Re_{+}\} \times K \subset \text{cl}(\mathcal{U})\end{aligned}$$

where

$$\Re_+ := \{\alpha \in \Re : \ \alpha \geq 0\} \quad \textit{and} \quad \Re_{++} := \{\alpha \in \Re : \ \alpha > 0\}.$$

Proof: (i): Since the set K is self-dual, we obtain (i).
(ii): By the definition of the scalar product, it is clear that $\langle u, v\rangle = 0$ if $u \circ v = 0$. Suppose that $\langle u, v\rangle = 0$. Since $u, v \in K \subset V$, by (i) of Theorem 2.3 and (i) of Corollary 2.4, there exist Jordan frames e_i $(i = 1, \ldots, r)$ and f_j $(j = 1, \ldots, s)$ and nonnegative real numbers $\lambda_i \geq 0$ $(i = 1, \ldots, r)$ and $\mu_j \geq 0$ $(j = 1, \ldots, s)$ such that

$$u = \sum_{i=1}^{r} \lambda_i e_i, \quad v = \sum_{j=1}^{s} \mu_j f_j.$$

Then $\langle u, v\rangle = 0$ implies that

$$0 = \langle u, v\rangle = \sum_{i,j} \lambda_i \mu_j \langle e_i, f_j\rangle$$

and we have

$$\lambda_i \mu_j = 0 \ \text{ or } \ \langle e_i, f_j\rangle = 0 \tag{10}$$

for every (i, j), since $\lambda_i \mu_j \geq 0$ and $\langle e_i, f_j\rangle \geq 0$. Note that

$$\langle e_i, f_j\rangle = \langle e_i, f_j^2\rangle = \langle e_i \circ f_j, f_j\rangle = \langle f_j, f_j \circ e_i\rangle = \langle f_j, L(e_i) f_j\rangle \geq 0$$

holds for every (i, j). Since $L(e_i)$ is self-adjoint and positive semidefnite (see (ii) of Proposition 2.1), if $\langle e_i, f_j\rangle = 0$ then

$$0 = \langle f_j, L(e_i) f_j\rangle \ \text{ and } \ L(e_i) f_j = 0.$$

Therefore, (10) turns out to be

$$\lambda_i \mu_j = 0 \ \text{ or } \ e_i \circ f_j = 0$$

which ensures that

$$u \circ v = \sum_{i,j} \lambda_i \mu_j (e_i \circ f_j) = 0.$$

(iii): It is clear that $\langle u, v\rangle = 0$ if $u \in \text{int}K$ and $v = 0$. Suppose that $\langle u, v\rangle = 0$ for some $u \in \text{int}K$ and $v \in K$. Then by a similar discussion in (ii) above, we have

$$0 = \langle u, v\rangle = \sum_{j=1}^{s} \mu_j \langle u, f_j\rangle = \sum_{j=1}^{s} \mu_j \langle f_j, u \circ f_j\rangle = \sum_{j=1}^{s} \mu_j \langle f_j, L(u) f_j\rangle \tag{11}$$

for a decomposition $v = \sum_{j=1}^{s} \mu_j f_j$. Since $L(u)$ is positive definite by (ii) of Proposition 2.1 and since $f_j \neq 0$, we see that $\langle f_j, L(u) f_j\rangle > 0$ for every $j = 1, \ldots, s$ Thus $\mu_j = 0$ $(j = 1, \ldots, s)$ follows from $\mu_j \geq 0$ $(j = 1, \ldots, s)$ and (11); which implies $v = 0$.
(iv): Let

$$\bar{\mathcal{U}} := \{(x, y) \in K \times K : \ x \circ y \in \text{int}K\}.$$

It is obvious that $\mathcal{U} \supseteq \bar{\mathcal{U}}$. Suppose that $(x, y) \in \bar{\mathcal{U}}$ and $x \in K \setminus \text{int}K$. Let $x = \sum_{i=1}^{r} \lambda_i e_i$ and $y = \sum_{j=1}^{s} \mu_j f_j$ be decompositions of x and y given by (i) of Theorem 2.3. Since $x \in K \setminus \text{int}K$, by (i) and (ii) of Corollary 2.4, there exists an index $\bar{i}$ such that $\lambda_{\bar{i}} = 0$. Define $z := \sum_{k=1}^{s} \nu_k f_k$ where

$$\nu_k = \begin{cases} 1 \text{ if } \lambda_k = 0, \\ 0 \text{ otherwise.} \end{cases}$$

Then $z \neq 0$, $z^2 = z \in K$, and

$$\begin{aligned}
\langle x \circ y, z \rangle &= \langle \sum_{i,j} \lambda_i \mu_j (e_i \circ f_j), z \rangle \\
&= \sum_{i,j,k} \lambda_i \mu_j \nu_k \langle e_i \circ f_j, f_k \rangle \\
&= \sum_{i,j,k} \lambda_i \mu_j \nu_k \langle L(f_j) e_i, f_k \rangle \\
&= \sum_{i,j,k} \lambda_i \mu_j \nu_k \langle e_i, L(f_j) f_k \rangle \\
&= \sum_{i,j,k} \lambda_i \mu_j \nu_k \langle e_i, f_j \circ f_k \rangle \\
&= \sum_{i,j} \lambda_i \mu_j \nu_j \langle e_i, f_j^2 \rangle \\
&= \sum_{i,j} \lambda_i \mu_j \nu_j \langle e_i, f_j \rangle \\
&= 0
\end{aligned}$$

where the last equality follows from the definition of ν_k $(k = 1, \ldots, s)$. This leads to a contradiction. In fact,

$$\begin{aligned}
0 = \langle x \circ y, z \rangle &= \langle x \circ y, z^2 \rangle \\
&= \langle z, (x \circ y) \circ z \rangle \\
&= \langle z, L(x \circ y) z \rangle,
\end{aligned}$$

but $z \neq 0$ and $L(x \circ y)$ is positive definite since $x \circ y \in \text{int}K$. Thus, $x \in \text{int}K$. Similarly, we can see that $y \in \text{int}K$.

(v): We first show that $P(x, y)$ defined by (6) is positive definite for every $x, y \in \text{int}K$. Suppose that $x, y \in \text{int}K$. Using the results of Proposition 2.5, we see that

$$\begin{aligned}
P(x)^{-1/2} P(x+y) P(x)^{-1/2} &= P(x^{-1/2}) P(x+y) P(x^{-1/2}) && \text{(by (ix))} \\
&= P(P(x^{-1/2})(x+y)) && \text{(by (iii))} \\
&= P(P(x^{-1/2})x + P(x^{-1/2})y) \\
&= P(P(x^{-1/2}) P(x^{1/2}) e + P(x^{-1/2}) y) && \text{(by (i))} \\
&= P(e + P(x^{-1/2}) y). && \text{(by (ix))}
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(x)^{-1/2} P(y) P(x)^{-1/2} &= P(x^{-1/2}) P(y) P(x^{-1/2}) && \text{(by (ix))} \\
&= P(P(x^{-1/2})(y)). && \text{(by (iii))}
\end{aligned}$$

Therefore, we have

$$P(x+y) - P(x) - P(y) = P(x)^{1/2} [P(e + P(x^{-1/2}) y) - P(e) - P(P(x^{-1/2}) y)] P(x)^{1/2}.$$

Let $z := P(x^{-1/2})y$. It follows from the equation (6) that

$$\begin{aligned} P(e+z) - P(e) - P(z) &= 2[L(e)L(z) + L(z)L(e) - L(e \circ z)] \\ &= 2[L(z) + L(z) - L(z)] \\ &= 2L(z). \end{aligned}$$

Note that $z = P(x^{1/2})y \in \text{int}K$ since $x^{1/2}$ is invertible and $y \in \text{int}K$ (see (viii) of Proposition 2.5). Thus we observe that

$$P(x+y) - P(x) - P(y) = 2L(z) \succ 0.$$

Since the fact $x \circ y \in \text{int}K$ implies that $L(x \circ y)$ is positive definite, using (6) again, we can conclude that

$$\begin{aligned} L(x)L(y) + L(y)L(x) &= P(x,y) + L(x \circ y) \\ &= [P(x+y) - P(x) - P(y)] + L(x \circ y) \succ 0. \end{aligned}$$

(vi): Let $(x,y) \in \mathcal{U}$. Since $x \in \text{int}K$, by (ii) of Proposition 2.1, $L(x)$ is invertible. Suppose that $(\Delta x, \Delta y)$ satisfy (8) and (9). The equation (9) implies

$$\Delta y + L(x)^{-1}L(y)\Delta x = 0$$

and

$$\langle \Delta x, \Delta y \rangle + \langle \Delta x, L(x)^{-1}L(y)\Delta x \rangle = 0.$$

By (8), we have

$$\langle \Delta x, L(x)^{-1}L(y)\Delta x \rangle \leq 0.$$

Define $\Delta\tilde{x} = L(x)^{-1}\Delta x$. Then

$$\begin{aligned} 0 &\geq \langle \Delta x,\ L(x)^{-1}L(y)\Delta x \rangle \\ &= \langle L(x)\Delta\tilde{x},\ L(x)^{-1}L(y)L(x)\Delta\tilde{x} \rangle \\ &= \langle \Delta\tilde{x},\ L(y)L(x)\Delta\tilde{x} \rangle \\ &= \langle \Delta\tilde{x},\ (L(x)L(y) + L(y)L(x))\Delta\tilde{x} \rangle / 2 \end{aligned}$$

which implies that $\Delta\tilde{x} = 0$ by the facts $(x,y) \in \mathcal{U}$ and (v) above. Finally, we see that

$$\Delta x = L(x)\Delta\tilde{x} = 0 \text{ and } \Delta y = -L(x)^{-1}L(y)\Delta x = 0.$$

(vii): By the fact $(e,e) \in \mathcal{U}$ and by the continuity of the operators $x \circ y$ and $L(x)L(y)+L(y)L(x)$, the set $\mathcal{U}$ is a nonempty open subset of $\text{int}K \times \text{int}K$.

Let $(x,y) \in \mathcal{U}$. For $\theta \in [0,1]$, define

$$(x(\theta), y(\theta)) := (\theta e + (1-\theta)x, \theta e + (1-\theta)y) = \theta(e,e) - (1-\theta)(x,y).$$

To see that the set $\mathcal{U}$ is star-shaped, it suffices to show that $(x(\theta), y(\theta)) \in \mathcal{U}$ for every $\theta \in [0,1]$. Since the set $\text{int}K \times \text{int}K$ is convex, for every $\theta \in [0,1]$, we have $(x(\theta), y(\theta)) \in \text{int}K \times \text{int}K$.

In addition, $x(\theta) \circ y(\theta)$ turns out to be

$$\begin{aligned} x(\theta) \circ y(\theta) &= (\theta e + (1-\theta)x) \circ (\theta e + (1-\theta)y) \\ &= \theta^2 e + \theta(1-\theta)(x+y) + (1-\theta)^2 x \circ y \end{aligned}$$

where $e \in \text{int}K$, $x+y \in \text{int}K$ and $x \circ y \in \text{int}K$. By the convexity of the cone $\text{int}K$, we see that $x(\theta) \circ y(\theta) \in \text{int}K$ and hence $(x(\theta), y(\theta)) \in \mathcal{U}$ for every $\theta \in [0,1]$.
(viii): For every $x \in \text{int}K$, (iv) of Proposition 2.1 implies that $x \circ x \in \text{int}K$ and hence $(x,x) \in \mathcal{U}$.
(ix): Since $\text{int}K$ is a convex cone, for every $x \in \text{int}K$ and $\alpha \in \Re_{++}$, it must hold that $\alpha e \in \text{int}K$ and

$$x \circ (\alpha e) = \alpha x \in \text{int}K.$$

Thus we see that

$$\text{int}K \times \{\alpha e : \ \alpha \in \Re_{++}\} \subset \mathcal{U}$$

and hence

$$K \times \{\alpha e : \ \alpha \in \Re_{+}\} \subset \text{cl}(\mathcal{U}).$$

By the symmetricity of the product $x \circ y = y \circ x$, we obtain the assertion. ∎

Next we consider a special class of Jordan algebras. A Jordan algebra is called *simple* if it cannot be represented as the sum of two Jordan algebras.

Proposition 2.8 *Let V be a simple Euclidean Jordan algebra.*

(i) *For every $u \in V$,*

$$\text{Tr}(L(u)) = \frac{n}{r}\text{tr}(u).$$

(ii) *For every nonzero idempotent c of V,*

$$\sqrt{\frac{r}{2n}} \le \|c\| \le \sqrt{r}$$

where the norm $\|u\|$ is defined by (5).

Proof: See Proposition III.4.2 of [4] for (i). Since every nonzero idempotent c is an element of K, by (i) of Corollary 2.4, all eigenvalues of c are nonnegative and $\text{tr}(c)$ is positive. The assertion (ii) follows from (i) of the proposition, (iii) of Theorem 2.3 and

$$0 < \|c\| = \sqrt{\langle c, c\rangle} = \sqrt{\text{tr}(c^2)} = \sqrt{\text{tr}(c)} = \sqrt{\frac{r}{n}\text{Tr}(L(u))}.$$

∎

It is known that Simple Euclidean Jordan algebras can be classified into the following five cases, which gives a classification for symmetric cones.

Theorem 2.9 *Let V be a simple Euclidean Jordan algebra. Then V is isomorphic to one of the following algebras, where the operator is defined by $X \circ Y = \frac{1}{2}(XY + YX)$ for the matrix algebras.*

(i) *The algebra $(\mathcal{E}_{n+1}, \circ)$, the algebra of quadratic forms in $\Re^{n+1}$ under the operation $u \circ v = (u^T v, u_0\bar{v} + v_0\bar{u})$ where $u = (u_0, \bar{u}), v = (v_0, \bar{v}) \in \Re^{n+1}$.*

(ii) *The algebra $(\mathcal{S}_n, \circ)$ of $n \times n$ symmetric matrices.*

(iii) *The algebra $(\mathcal{H}_n, \circ)$ of $n \times n$ complex Hermitian matrices.*

(iv) *The algebra $(\mathcal{Q}_n, \circ)$ of $n \times n$ quaternion Hermitian matrices.*

(v) *The exceptional Albert algebra, i.e., the algebra $(\mathcal{O}_3, \circ)$ of 3×3 octonian Hermitian matrices.*

Proof: See Chapter V in [4]. ∎

3 Homeomorphism of an interior point map

In this section, we will extend the results in [17] to the case of symmetric cones and show the homeomorphism of an interior point map using the results in Section 2.

The arguments used in the section are quite analogous to the ones in [17]. However, we restrict the domain of the map F to $\text{int}K \times \text{int}K \times \Re^m$ and it affects subtle details of the proofs. Therefore, we will give the proofs unless they completely coincide with the ones in [17].

Monteiro and Pang's approach in [17] is based on a theory of local homeomorphic maps (Section 2 and the Appendix of [16], Chapter 5 of Ortega and Rheinboldt [21] and Chapter 3 of Ambrosetti and Prodi [2]). The following quite general results are key propositions in their analysis.

Here we introduce some notations and definitions. If M and N are two metric space, we denote the set of continuous functions from M to N by $C(M, N)$ and the set of homeomorphisms from M onto N by $\text{Hom}(M, N)$. For given $G \in C(M, N)$, $D \subseteq M$ and $E \subseteq N$, we define

$$G(D) := \{G(u) : \ u \in D\},$$
$$G^{-1}(E) := \{u \in M : \ G(u) \in E\}.$$

We will also denote "G restricted to the pair (D, E)" by $G|_{(D,E)}$.

Definition 3.1 (Section 2 of [16], Section 2.2 of [17]) (i) *A metric space M is* connected *if there exists no partition (V_1, V_2) of M for which V_1 and V_2 are nonempty and open.*

(ii) *A metric space M is* path-connected *if for any two points $u_0, u_1 \in M$, there exists a path, i.e., a continuous function $p : [0, 1] \to M$ such that $p(0) = u_0$ and $p(1) = u_1$.*

(iii) *A metric space m is* simply-connected *if it is path-connected and for any path $p : [0, 1] \to M$ with $p(0) = p(1) = u$, there exists a continuous map $\alpha : [0, 1] \times [0, 1] \to M$ such that $\alpha(s, 0) = p(s)$and $\alpha(s, 1) = u$ for all $s \in [0, 1]$ and $\alpha(0, t) = \alpha(1, t) = u$ for all $t \in [0, 1]$.*

(vi) *The map $G \in C(M, N)$ is said to be* proper *with respect to the set $E \subseteq N$ if the set $G^{-1}(K) \subseteq M$ is compact for every compact set $K \subseteq E$. If G is proper with respect to N, we will simply say that G is proper.*

It is easy to see that every star-shaped set (see Definition 2.6) in a normed vector space is simply-connected.

Proposition 3.2 (Theorem 1 of [16], Proposition 1 of [17]) *Let M and N be metric spaces such that M is path-connected and N is simply-connected. Suppose that $G : M \to \Re^n$ is a local homeomorphism. Then G is proper if and only if $G \in \mathrm{Hom}(M, N)$.*

Proposition 3.3 (Corollary 1 of [16], Proposition 2 of [17]) *Let $G \in C(M, N)$, $M_0 \subseteq M$ and $N_0 \subseteq N$.*

(i) *Suppose that G, $M_0 \subseteq M$ and $N_0 \subseteq N$ satisfy the following conditions:*

- **(a)** *$G|_{(M_0,N)}$ is a local homeomorphism,*
- **(b)** *$G(M_0) \cap N_0 \neq \emptyset$,*
- **(c)** *$G(M \setminus M_0) \cap N_0 = \emptyset$, and*
- **(d)** *G is proper with respect to a subset E such that $N_0 \subseteq E \subseteq N$.*

Then $G|_{(M_0 \cap G^{-1}(N_0), N_0)}$ is a proper local homeomorphism.

(ii) *Suppose that G, $M_0 \subseteq M$ and $N_0 \subseteq N$ satisfy the conditions (a) – (d) in (ii) above and the additional condition below:*

- **(e)** *N_0 is connected.*

Then $G(M_0) \supseteq N_0$ and $G(\mathrm{cl}(M_0)) \supseteq E \cap \mathrm{cl}(N_0)$.

Proposition 3.4 (Corollary 3 of [16], Proposition 3 of [17]) *Let M be a path-connected metric space and let V be an n-dimensional real vector space. Suppose that $G : M \to V$ is a local homeomorphism and that $G^{-1}([y_0, y_1])$ is compact for any pair of points $y_0, y_1 \in G(M)$. Then, $G|_{(M,G(M))} \in \mathrm{Hom}(M, G(M))$ and $G(M)$ is convex.*

The following result is analogous to Lemma 2 of [17], but the restriction on the domain F to $\mathrm{int}K \times \mathrm{int}K \times \Re^m$ has an effect on the set for which the map H is shown to be proper.

Lemma 3.5 (cf. Lemma 2 of [17]) *Let $F : \mathrm{int}K \times \mathrm{int}K \times \Re^m \to V \times \Re^m$ be a continuous map which satisfies Assumption 1.1. Let H be the map defined by (4). If the map H restricted to $\mathcal{U} \times \Re^m$ is a local homeomorphism, then the map H is proper with respect to $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$.*

Proof: Let C be a compact subset of $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$. We first show that $H^{-1}(C)$ is closed. Suppose that there exists a sequence $\{(x^{(k)}, y^{(k)}, z^{(k)})\} \subset H^{-1}(C)$ such that $(x^{(k)}, y^{(k)}, z^{(k)}) \to (\bar{x}, \bar{y}, \bar{z})$. For each k, we see that

$$(x^{(k)}, y^{(k)}, z^{(k)}) \in \mathrm{int}K \times \mathrm{int}K \times \Re^m, \quad H(x^{(k)}, y^{(k)}, z^{(k)}) \in C.$$

The continuity of H and the closedness of C ensure that

$$(\bar{x}, \bar{y}, \bar{z}) \in K \times K \times \Re^m, \quad H(\bar{x}, \bar{y}, \bar{z}) \in C.$$

Since C is a subset of $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$, the fact $H(\bar{x}, \bar{y}, \bar{z}) \in C$ implies that

$$\bar{x} \circ \bar{y} \in \mathrm{int}K.$$

Thus, by (vi) of Lemma 2.7, we have $(\bar{x}, \bar{y}) \in \mathrm{int}K \times \mathrm{int}K$ and hence $(\bar{x}, \bar{y}, \bar{z}) \in H^{-1}(C)$.

Next, let us show that $H^{-1}(C)$ is bounded. Suppose that $H^{-1}(C)$ is unbounded. Then there exists a sequence $\{(x^{(k)}, y^{(k)}, z^{(k)})\} \subseteq H^{-1}(C)$ for which

$$\lim_{k\to\infty} \sqrt{\|x^{(k)}\|^2 + \|y^{(k)}\|^2 + \|z^{(k)}\|^2} = \infty$$

holds. Since C is compact, there exists an $F^\infty \in F(\mathcal{U} \times \Re^m)$ such that

$$F^\infty = \lim_{k\to\infty} F(x^{(k)}, y^{(k)}, z^{(k)}). \tag{12}$$

Here $F^\infty \in F(\mathcal{U} \times \Re^m)$ ensures the existence of an $(x^\infty, y^\infty, z^\infty) \in \mathcal{U} \times \Re^m$ satisfying

$$F(x^\infty, y^\infty, z^\infty) = F^\infty. \tag{13}$$

Note that, since $(x^\infty, y^\infty) \in \mathrm{int}K \times \mathrm{int}K$, there exists an $\eta > 0$ such that

$$\mathcal{N}_\infty := \{(x, y, z) \in \mathcal{U} \times \Re^m : \ (x - \eta^{-1}e, y - \eta^{-1}e) \in \mathrm{int}K \times \mathrm{int}K, \ \|x\| < \eta, \ \|y\| < \eta\}$$

contains (x^∞, y^∞). Since $\mathcal{N}_\infty$ is an open set and every local homeomorphism maps open sets onto open sets, the set $H(\mathcal{N}_\infty)$ and hence the set $F(\mathcal{N}_\infty)$ are open. Thus, by (12), it can be seen that for every sufficient large $k \geq k_0$, we obtain $F(x^{(k)}, y^{(k)}, z^{(k)}) \in F(\mathcal{N}_\infty)$, which implies that there exists $(\tilde{x}^{(k)}, \tilde{y}^{(k)}, \tilde{z}^{(k)}) \in \mathcal{N}_\infty$ satisfying $F(x^{(k)}, y^{(k)}, z^{(k)}) = F(\tilde{x}^{(k)}, \tilde{y}^{(k)}, \tilde{z}^{(k)})$ for every $k \geq k_0$. Here, (i) of Assumption 1.1 implies that

$$\langle \tilde{x}^{(k)} - x^{(k)}, \tilde{y}^{(k)} - y^{(k)} \rangle \geq 0$$

and hence

$$\langle \tilde{x}^{(k)}, y^{(k)} \rangle + \langle x^{(k)}, \tilde{y}^{(k)} \rangle \leq \langle \tilde{x}^{(k)}, \tilde{y}^{(k)} \rangle + \langle x^{(k)}, y^{(k)} \rangle. \tag{14}$$

We see that $\{x^{(k)} \circ y^{(k)}\}$ is bounded since $\{H(x^{(k)}, y^{(k)}, z^{(k)})\}$ lies in the compact set C. Therefore there exists a $\gamma > 0$ such that

$$\mathrm{tr}(x^{(k)} \circ y^{(k)}) = \langle x^{(k)}, y^{(k)} \rangle \leq \gamma \tag{15}$$

for every k. In addition, since $(\tilde{x}^{(k)}, \tilde{y}^{(k)}) \in \mathcal{N}_\infty$, it follows that

$$\langle \tilde{x}^{(k)} - \eta^{-1}e, y^{(k)} \rangle \geq 0, \ \langle x^{(k)}, \tilde{y}^{(k)} - \eta^{-1}e \rangle \geq 0, \tag{16}$$

$$\|\tilde{x}^{(k)}\| \leq \eta, \ \|\tilde{y}^{(k)}\| \leq \eta. \tag{17}$$

Consequently, we have

$$\begin{aligned} \langle \eta^{-1}e, y^{(k)} \rangle + \langle x^{(k)}, \eta^{-1}e \rangle &\leq \langle \tilde{x}^{(k)}, y^{(k)} \rangle + \langle x^{(k)}, \tilde{y}^{(k)} \rangle \quad \text{(by (16))} \\ &\leq \langle \tilde{x}^{(k)}, \tilde{y}^{(k)} \rangle + \langle x^{(k)}, y^{(k)} \rangle \quad \text{(by (14))} \\ &\leq \eta^2 + \gamma \quad \text{(by (17) and (15))} \end{aligned} \tag{18}$$

for every $k \geq k_0$. Since $\eta^{-1}e \in \mathrm{int}K$, by Proposition 2.2, the inequality (18) ensures that the set $\{(x^{(k)}, y^{(k)}) : k \geq k_0\} \subset \mathrm{int}K \times \mathrm{int}K$ is bounded. This implies that $\{z^{(k)}\}$ is also bounded by (ii) of Assumption 1.1. Thus we obtain a contradiction. ∎

In the following two lemmas and Theorem 3.8, we consider that the map F is affine and defined on $V \times V \times \Re^m$. Theorem 3.8 leads us to a technical result, Lemma 3.9, which is a crucial lemma to establish the result for nonlinear maps F, Theorem 3.8.

Lemma 3.6 (cf. Lemma 3 of [17]) *Let $F : V \times V \times \Re^m \to V \times \Re^m$ be an affine map and let F^0 be the linear part of F.*

(i) *F is (x, y)-equilevel-monotone if and only if for every $(\Delta x, \Delta y, \Delta z) \in V \times V \times \Re^m$, $F^0(\Delta x, \Delta y, \Delta z) = 0$ implies that $\langle \Delta x, \Delta y\rangle \geq 0$.*

(ii) *F is z-injective if and only if for every $\Delta z \in \Re^m$, $F^0(0, 0, \Delta z) = 0$ implies that $\Delta z = 0$.*

(iii) *F is z-injective if and only if F is z-bounded.*

Proof: The proof is completely the same as the one of Lemma 3 in [17]. ∎

Lemma 3.7 (cf. Lemma 4 of [17]) *Let $F : V \times V \times \Re^m$ be an affine map which is equilevel-monotone and z-injective. Then H restricted to $\mathcal{U} \times \Re^m$ is a local homeomorphism.*

Proof: Since $\mathcal{U} \times \Re^m$ is an open set by (vii) of Lemma 2.7, it suffices to show that the derivative map $H'(x, y, z) : V \times V \times \Re^m \to V \times V \times \Re^m$ is an isomorphism for every $(x, y, z) \in \mathcal{U} \times \Re^m$. Let $(x, y, z) \in \mathcal{U} \times \Re^m$ and suppose that $H'(x.y, z)(\Delta x, \Delta y, \Delta) = 0$. Then

$$H'(x.y, z)(\Delta x, \Delta y, \Delta) = \begin{pmatrix} \Delta x \circ y + x \circ \Delta y \\ F^0(\Delta x, \Delta y, \Delta z) \end{pmatrix}$$

and we see that

$$\Delta x \circ y + x \circ \Delta y = 0, \tag{19}$$
$$F^0(\Delta x, \Delta y, \Delta z) = 0 \tag{20}$$

where $F^0 : V \times V \times \Re^m$ is the linear part of F. Since F is (x, y)-equilevel-monotone, by (i) of Lemma 3.6, the equality (20) implies $\langle \Delta x, \Delta y\rangle \geq 0$. Combining this with (19), by (vi) of Lemma 2.7, we have $\Delta x = \Delta y = 0$. Thus, by the z-injectivity of F, we obtain $\Delta z = 0$ from (ii) of Lemma 3.6. By the linearity of $H'(x, y, z)$, the fact that $(\Delta x, \Delta y, \delta z) = (0, 0, 0)$ whenever $H'(x.y, z)(\Delta x, \Delta y, \Delta) = 0$ implies the isomorphism of $H'(x.y, z)(\Delta x, \Delta y, \Delta)$ for every $(x, y, z) \in \mathcal{U} \times \Re^m$. ∎

Theorem 3.8 (cf. Theorem 1 of [17]) *Let $F : V \times V \times \Re^m \to V \times \Re^m$ be an affine map which is (x, y)-equilevel-monotone, z-injective and z-bounded on $V \times V \times \Re^m$. Then the map H defined by (4) satisfies that*

(i) *H is proper with respect to* $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$*,*

(ii) *H maps $\mathcal{U} \times \Re^m$ homeomorphically onto* $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$*.*

Proof: Define

$$\begin{aligned} &M := \mathrm{int}K \times \mathrm{int}K \times \Re^m,\ N := V \times V \times \Re^m,\ E := \mathrm{int}K \times F(\mathcal{U} \times \Re^m), \\ &M_0 := \mathcal{U} \times \Re^m,\ N_0 := \mathrm{int}K \times F(\mathcal{U} \times \Re^m),\ G := H|_{(M,N)}. \end{aligned} \tag{21}$$

We can easily see that

$$N_0 \subseteq E \subseteq N, \quad M_0 \subseteq H^{-1}(N_0). \tag{22}$$

(i): Since F is equilevel-bounded and z-injective, Lemma 3.7 and (iii) of Lemma 3.6 ensure that $H|_{(M_0,N)} = G|_{(M_0,N)}$ is a local homeomorphism and z-bounded. Thus the map F satisfies Assumption 1.1 and by Lemma 3.5, $H|_{(M,E)}$ is proper with respect to $E = \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$.
(ii): Note that (i) above ensures that the requirement (d) of Proposition 3.3 is satisfied. We show that another requirements (a)–(c) are also satisfied.

Lemma 3.7 implies that $G|_{(M_0,N)} = H_{(M_0,N)}$ is a local homeomorphism, i.e., the requirement (a) holds.

Let us show that $H(M_0) \cap N_0 \neq \emptyset$ and $H(M \setminus M_0) \cap N_0 = \emptyset$. The former follows from the fact that $(e, e, 0) \in \mathcal{U} \times \Re^m$ and $H(e, e, 0) \in H(M_0) \cap N_0$. To show the latter, suppose that $H(M \setminus M_0) \cap N_0 \neq \emptyset$. Then there exists a triplet $(x, y, z) \in \mathrm{int}K \times \mathrm{int}K \times \Re^m$ such that $(x, y) \notin \mathcal{U}$ and $H(x, y, z) \in N_0$. Then, by the definitions (4) and (21) of H and N_0, the fact $H(x, y, z) \in N_0$ implies that $F(x, y, z) \in F(\mathcal{U} \times \Re^m)$ and hence $(x, y) \in \mathcal{U}$, which is a contradiction.

Consequently, we have shown that (a)–(d) of Proposition 3.3 are satisfied. Since (22) ensures the relation $M_0 \subseteq M_0 \cap H^{-1}(N_0) = M_0 \cap G^{-1}(N_0)$, we obtain that the map H restricted to

$$(M_0, N_0) = (\mathcal{U} \times \Re^m, \mathrm{int}K \times F(\mathcal{U} \times \Re^m))$$

is a proper local homeomorphism.

Next, we show that $H(\mathcal{U} \times \Re^m) = \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$ by using (ii) of Proposition 3.3. It is clear that

$$G(M_0) = H(\mathcal{U} \times \Re^m) \subseteq \mathrm{int}K \times F(\mathcal{U} \times \Re^m) = N_0.$$

To obtain the inverse inclusion, we should mention that the set N_0 is connected. In fact, by (vii) of Lemma 2.7, $\mathcal{U} \times \Re^m$ is star-shaped and hence path-connected. Since F is continuous, the sets $F(\mathcal{U} \times \Re^m)$ and $N_0 = \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$ are also path-connected, and hence connected. Thus, we can apply (ii) of Proposition 3.3 and obtain that

$$H(\mathcal{U} \times \Re^m) = G(M_0) \supseteq N_0 = \mathrm{int}K \times F(\mathcal{U} \times \Re^m).$$

Finally, let us show that $G \in H(M_0, N_0)$. In (vii) of Lemma 2.7, we have seen that the set $\mathcal{U}$ is star-shaped. Since F is affine, both of the set M_0 and N_0 are star-shaped and hence simply-connected. By the local homeomorphism of G, the assertion $G \in H(M_0, N_0)$ follows from Proposition 3.2. ∎

We proceed to give an important lemma concerning the set $\mathcal{U}$ using the above theorem. Note that the set $\mathcal{U}$ is defined regardless of the map F and the following lemma is applicable for the case where the map F is nonlinear and not necessarily affine.

Lemma 3.9 (cf. Lemma 5 of [17]) *For every* $(x_0, y_0), (x_1, y_1) \in \mathcal{U}$*, if* $\langle x_0 - x_1, y_0 - y_1 \rangle \geq 0$ *and* $x_0 \circ y_0 = x_1 \circ y_1$ *then* $(x_0, y_0) = (x_1, y_1)$*.*

Proof: Let $\Delta x = x_0 - x_1$ and $\Delta y = y_0 - y_1$. Suppose that $\Delta x \neq 0$ or $\Delta y \neq 0$. Without any loss of generality, we may assume that $\Delta x \neq 0$.

First, we will show that there exists a linear map $M : V \to V$ such that $M(\Delta x) = \Delta y$ and $\langle w, M(w) \rangle \geq 0$ for every $w \in V$. If $\Delta y = 0$ then $M = O$ satisfies the requirements. Otherwise, we can consider the following two cases:

Case 1 ($\langle \Delta x, \Delta y \rangle > 0$): Let $L = \{v : \langle v, \Delta y \rangle = 0\}$. Then $\Delta x \notin L$ and the subspace generated by Δx and the subspace L span V. Thus, every $w \in V$ is given by $w = \alpha \Delta x + v$ for some $v \in L$ and $\alpha \in \Re$. Since $\Delta x \notin L$, there exists a unique linear map M satisfying

$$M(\Delta x) = \Delta y \text{ and } M(v) = 0 \text{ for every } v \in L.$$

and for every $w \in V$, we have

$$\begin{aligned} \langle w, M(w) \rangle &= \langle \alpha \Delta x + v, M(\alpha \Delta x + v) \rangle \\ &= \langle \alpha \Delta x + v, \alpha \Delta y \rangle \\ &= \alpha^2 \langle \Delta x, \Delta y \rangle \geq 0. \end{aligned}$$

Case 2 ($\langle \Delta x, \Delta y \rangle = 0$): Let $L_1 = \{v : \langle v, \Delta y \rangle = 0 \text{ and } \langle v, \Delta y \rangle = 0\}$. Then $\Delta x, \Delta y \notin L_1$ and the subspace generated by $\{\Delta x, \Delta y\}$ and L_1 span V. Thus, every $w \in V$ is given by $w = \alpha \Delta x + \beta \Delta y + v$ for some $v \in L_1$ and $\alpha, \beta \in \Re$. Since $\Delta x, \Delta y \notin L$, there exists a unique linear map M satisfying

$$M(\Delta x) = \Delta y, \; M(\Delta y) = -\frac{\|\Delta y\|^2}{\|\Delta x\|^2} \Delta x \text{ and } M(v) = 0 \text{ for every } v \in L_1$$

and for every $w \in V$, we have

$$\begin{aligned} \langle w, M(w) \rangle &= \langle \alpha \Delta x + \beta \Delta y + v, M(\alpha \Delta x + \beta \Delta y + v) \rangle \\ &= \langle \alpha \Delta x + \beta \Delta y + v, \alpha \Delta y - \beta \frac{\|\Delta y\|^2}{\|\Delta x\|^2} \Delta x \rangle \\ &= \alpha \beta \|\Delta y\|^2 - \alpha \beta \frac{\|\Delta y\|^2}{\|\Delta x\|^2} \|\Delta x\|^2 = 0. \end{aligned}$$

Define $F(x, y) = y - M(x)$ for every $(x, y) \in V \times V$. While the map F will be affected by the choice of $(\Delta x, \Delta y) = (x_0 - x_1, y_0 - y_1)$, it always satisfies the assumptions of Theorem 3.8 with $m = 0$ and that $F(x_0, y_0) = F(x_1, y_1)$. Thus, by (ii) of Theorem 3.8, the associated map H restricted to $\mathcal{U}$ is one-to-one. Since (x_0, y_0) and (x_1, y_1) satisfy $x_0 \circ y_0 = x_1 \circ y_1$ and $F(x_0, y_0) = F(x_1, y_1)$, we obtain that $H(x_0, y_0) = H(x_1, y_1)$ and hence the desired result $(x_0, y_0) = (x_1, y_1)$. ∎

The following is our main result.

Theorem 3.10 (cf. Theorem 2 of [17]) *Suppose that a continuous map* $F : \text{int} K \times \text{int} K \times \Re^m \to V \times \Re^m$ *satisfies Assumption 1.1. Then the map* H *defend by (4) satisfies that*

(i) *H is proper with respect to* $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$*,*

(ii) *H maps $\mathcal{U} \times \Re^m$ homeomorphically onto* $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$*.*

Proof: The proof is similar to the one of Theorem 3.8, while we should note that the domain of the map H is restricted on the set $\mathrm{int}K \times \mathrm{int}K \times \Re^m$.

Define the sets M, N, E, M_0 and N_0, and the map G as in (21). Then (22) holds even in this case.

(i): To show the local homeomorphism of $H|_{(M_0,N)} = G|_{(M_0,N)}$, we use Lemma 3.9 instead of Lemma 3.7. Since $H|_{(M_0,N)}$ is a continuous map from an open subset of $V \times V \times \Re^m$ into the same space, by the domain invariance theorem, it suffices to show that $H|_{(M_0,N)}$ is one-to-one.

Suppose that $(\hat{x}, \hat{y}, \hat{z}), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathcal{U} \times \Re^m$ satisfy $H(\hat{x}, \hat{y}, \hat{z}) = H(\tilde{x}, \tilde{y}, \tilde{z})$, i.e.,

$$F(\hat{x}, \hat{y}, \hat{z}) = F(\tilde{x}, \tilde{y}, \tilde{z}), \quad \hat{x} \circ \hat{y} = \tilde{x} \circ \tilde{y}.$$

Since F is (x, y)-equilevel-monotone, we see that

$$\langle \hat{x} - \tilde{x}, \hat{y} - \tilde{y} \rangle \geq 0, \quad \hat{x} \circ \hat{y} = \tilde{x} \circ \tilde{y}.$$

By Lemma 3.9, the above relations imply that

$$(\hat{x}, \hat{y}) = (\tilde{x}, \tilde{y})$$

and by the z-injectivity of F, we have

$$(\hat{x}, \hat{y}, \hat{z}) = (\tilde{x}, \tilde{y}, \tilde{z}).$$

Thus, $H|_{(M_0,N)}$ is one-to-one and maps M_0 homeomorphically onto $H(M_0)$. By the local homeomorphism of $H|_{(M_0,N)}$ together with the equilevel-monotonicity and the z-boundedness of the map F, Lemma 3.5 ensures that H is proper with respect to the set $E = \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$.

(ii): In the proof of (i) above, we have already seen that (a) and (d) of Proposition 3.3 hold. By the same discussions as in the proof of (ii) of Theorem 3.8, we can see that the assumptions (b) and (c) of Proposition 3.3 are also satisfied. Since F is continuous, $F(\mathcal{U} \times \Re^m)$ and $N_0 = \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$ are also path-connected. Thus, as in the proof of (ii) of Theorem 3.8 again, we obtain that

$$H(\mathcal{U} \times \Re^m) = G(M_0) = N_0 = \mathrm{int}K \times F(\mathcal{U} \times \Re^m).$$

∎

In what follows, we discuss the convexity of the set $F(\mathcal{U} \times \Re^m)$, which is a key property for finding a trajectory and observing its limiting behavior in our homogeneous model described in Section 5. To do this, we impose the additional assumption below on the map F, which has been introduced in [17].

Assumption 3.11 *F is (x, y)-everywhere-monotone on the domain with respect to the set $V \times \Re^m$, i.e., there exist continuous functions ϕ from the domain of F to the set $V \times \Re^m$ and $c : (V \times \Re^m) \times (V \times \Re^m) \to \Re$ such that*

$$c(r, r) = 0$$

for every $r \in V \times \Re^m$ *and*

$$\langle x - x', y - y' \rangle \geq \langle r - r', \phi(x,y,z) - \phi(x',y',z') \rangle_{V \times \Re^m} + c(r, r')$$

holds for every (x,y,z) *and* (x',y',z') *in the domain of* F *satisfying* $F(x,y,z) = r$ *and* $F(x',y',z') = r'$.

Here we define

$$\langle (a,b), (a',b') \rangle_{V \times \Re^m} = \langle a, a' \rangle + b^T b'$$

for every $(a,b), (a',b') \in V \times \Re^m$.

It can be easily seen that F is (x,y)-equilevel-monotone whenever F is (x,y)-everywhere-monotone.

Theorem 3.12 (cf. Theorem 3 of [17]) *Suppose that a continuous map* $F : \mathrm{int}K \times \mathrm{int}K \times \Re^m \to V \times \Re^m$ *satisfies Assumptions 1.1 and 3.11. Then the set* $F(\mathcal{U} \times \Re^m)$ *is an open convex set.*

Proof: It suffices to show that $H(\mathcal{U} \times \Re^m)$ is open and convex since $H(\mathcal{U} \times \Re^m) = \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$ holds by (ii) of Theorem 3.10. Since $\mathcal{U} \times \Re^m$ is open (see (vii) of Lemma 2.7), (ii) of Theorem 3.10 implies that the set $H(\mathcal{U} \times \Re^m)$ is also open. In what follows, we will show that the set $H(\mathcal{U} \times \Re^m)$ is convex. Define

$$M := \mathcal{U} \times \Re^m, \ N := V \times V \times \Re^m, \ G := H|_{(M,N)}.$$

Then the set M is path-connected by (vii) of Lemma 2.7 and G is a local homeomorphism by (ii) of Theorem 3.10. Thus, to obtain the assertion from Proposition 3.4, we have only to show that for any $w_0, w_1 \in G(M)$, $G^{-1}([w_0.w_1])$ is compact.

Let $w_0, w_1 \in G(M)$. Then there exist $(\tilde{x}, \tilde{y}, \tilde{z}), (\hat{x}, \hat{y}, \hat{z}) \in \mathcal{U} \times \Re^m$ such that

$$G(\tilde{x}, \tilde{y}, \tilde{z}) = w_0 \text{ and } G(\hat{x}, \hat{y}, \hat{z}) = w_1.$$

Define $E = [G(\tilde{x}, \tilde{y}, \tilde{z}), G(\hat{x}, \hat{y}, \hat{z})]$. We will show that every sequence $\{(x^{(k)}, y^{(k)}, z^{(k)})\} \subset G^{-1}(E)$ has an accumulation point in $G^{-1}(E)$. Let $\{(x^{(k)}, y^{(k)}, z^{(k)})\} \subset G^{-1}(E)$ be given. Then there exists a sequence $\{\tau_k\} \subset [0,1]$ for which

$$x^{(k)} \circ y^{(k)} = \tau_k \tilde{x} \circ \tilde{y} + (1 - \tau_k) \hat{x} \circ \hat{y}, \tag{23}$$
$$F^{(k)} = \tau_k \tilde{F} + (1 - \tau_k) \hat{F} \tag{24}$$

hold for every $k \geq 0$, where we define $F^{(k)} := F(x^{(k)}, y^{(k)}, z^{(k)})$, $\tilde{F} := F(\tilde{x}, \tilde{y}, \tilde{z})$ and $\hat{F} := F(\hat{x}, \hat{y}, \hat{z})$. Since we assume that F is (x,y)-everywhere-monotone, by (24), we have

$$\begin{aligned} \langle \tilde{x} - x^{(k)}, \tilde{y} - y^{(k)} \rangle &\geq \langle \tilde{F} - F^{(k)}, \tilde{\phi} - \phi^{(k)} \rangle + c(\tilde{F}, F^{(k)}) \\ &= (1 - \tau_k) \langle \tilde{F} - \hat{F}, \tilde{\phi} - \phi^{(k)} \rangle + c(\tilde{F}, F^{(k)}) \end{aligned}$$

where $\tilde{\phi} := \phi(\tilde{x}, \tilde{y}, \tilde{z})$ and $\phi^{(k)} := \phi(x^{(k)}, y^{(k)}, z^{(k)})$. It follows that

$$\begin{aligned} \langle x^{(k)}, \tilde{y} \rangle + \langle y^{(k)}, \tilde{x} \rangle \leq \langle \tilde{x}, \tilde{y} \rangle &+ \langle x^{(k)}, y^{(k)} \rangle - c(\tilde{F}, F^{(k)}) \\ &- (1 - \tau_k) \langle \tilde{F} - \hat{F}, \tilde{\phi} - \phi^{(k)} \rangle. \end{aligned} \tag{25}$$

Similarly, we have

$$\langle x^{(k)}, \hat{y}\rangle + \langle y^{(k)}, \hat{x}\rangle \leq \langle \hat{x}, \hat{y}\rangle + \langle x^{(k)}, y^{(k)}\rangle - c(\hat{F}, F^{(k)}) \\ -\tau_k\langle \hat{F} - \tilde{F}, \hat{\phi} - \phi^{(k)}\rangle. \tag{26}$$

Multiplying (25) by τ_k, (26) by $(1-\tau_k)$ and adding them, we see that

$$\begin{aligned}&\langle x^{(k)}, \tau_k\tilde{y} + (1-\tau_k)\hat{y}\rangle + \langle y^{(k)}, \tau_k\tilde{x} + (1-\tau_k)\hat{x}\rangle \\ &\leq \tau_k\langle \tilde{x}, \tilde{y}\rangle + (1-\tau_k)\langle \hat{x}, \hat{y}\rangle + \langle x^{(k)}, y^{(k)}\rangle \\ &\quad +\tau_k(1-\tau_k)\langle \hat{F} - \tilde{F}, \tilde{\phi} - \hat{\phi}\rangle - \tau_k c(\tilde{F}, F^{(k)}) - (1-\tau_k)c(\hat{F}, F^{(k)})\end{aligned} \tag{27}$$

holds for every $k \geq 0$. Note that the convexity of $\mathrm{int}K$ guarantees that $\tau_k\tilde{y} + (1-\tau_k)\hat{y} \in \mathrm{int}K$ and $\tau_k\tilde{x} + (1-\tau_k)\hat{x} \in \mathrm{int}K$ for every $\tau_k \in [0,1]$. Thus, there exists an $\eta > 0$ such that

$$([\tau_k\tilde{y} + (1-\tau_k)\hat{y}] - \eta e, [\tau_k\tilde{x} + (1-\tau_k)\hat{x}] - \eta e) \in \mathrm{int}K \times \mathrm{int}K$$

for every $\tau_k \in [0,1]$. This implies that

$$\langle x^{(k)}, [\tau_k\tilde{y} + (1-\tau_k)\hat{y}] - \eta e\rangle + \langle y^{(k)}, [\tau_k\tilde{x} + (1-\tau_k)\hat{x}] - \eta e\rangle \geq 0$$

and hence

$$\langle x^{(k)}, \eta e\rangle + \langle y^{(k)}, \eta e\rangle \leq \langle x^{(k)}, \tau_k\tilde{y} + (1-\tau_k)\hat{y}\rangle + \langle y^{(k)}, \tau_k\tilde{x} + (1-\tau_k)\hat{x}\rangle \tag{28}$$

for every $k \geq 0$. Note that the continuity of c, the boundedness of $\{\tau_k\}$, the boundedness of $\{F^{(k)}\}$, and (27) imply that the right hand side of (28) is bounded. By Proposition 2.2, we can conclude that $\{(x^{(k)}, y^{(k)})\} \subset \mathrm{int}K \times \mathrm{int}K$ is bounded. The z-boundedness of F ensures that $\{(x^{(k)}, y^{(k)}, z^{(k)})\} \subset \mathcal{U} \times \Re^m$ is also bounded. Thus, $\{(x^{(k)}, y^{(k)}, z^{(k)})\}$ has an accumulation point $(\bar{x}, \bar{y}, \bar{z}) \in \mathrm{cl}(\mathcal{U} \times \Re^m) = \mathrm{cl}(\mathcal{U}) \times \Re^m$.

Let us show that $(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{U} \times \Re^m$. The equation (23), $\tilde{x} \circ \tilde{y} \in \mathrm{int}K$, $\hat{x} \circ \hat{y} \in \mathrm{int}K$ and the convexity of $\mathrm{int}K$ guarantee that $\bar{x} \circ \bar{y} \in \mathrm{int}K$ and hence $(\bar{x}, \bar{y}) \in \mathcal{U} \subseteq \mathrm{int}K \times \mathrm{int}K$ by (iv) of Lemma 2.7. Thus, $(\bar{x}, \bar{y}, \bar{z})$ lies in the domain of G and $G(\bar{x}, \bar{y}, \bar{z})$ is well defined. Since G is continuous and the set E is closed, we finally see that $G(\bar{x}, \bar{y}, \bar{z}) \in E$ and $(\bar{x}, \bar{y}, \bar{z}) \in G^{-1}(E)$. ∎

4 Solvability of (CP)

In this section, we discuss the solvability of (CP) assuming that the map F is defined and continuous on the set $K \times K \times \Re^m$ instead of $\mathrm{int}K \times \mathrm{int}K \times \Re^m$. The following results are direct extensions of the ones in [17] to the case of symmetric cones and obtained by quite similar discussions in the previous section. We will only give the differences arising from the expansion of the domain of the map F in the proofs below.

Lemma 4.1 (cf. Lemma 2 of [17] and Lemma 3.5) *Let $F : K \times K \times \Re^m \to V \times \Re^m$ be a continuous map which is (x,y)-equilevel-monotone and z-bounded on $K \times K \times \Re^m$. Let H be the map defined by (4). If the map H restricted to $\mathcal{U} \times \Re^m$ is a local homeomorphism, then the map H is proper with respect to $V \times F(\mathcal{U} \times \Re^m)$.*

Proof: Let C be a compact subset of $V \times F(\mathcal{U} \times \Re^m)$. Since the domain $K \times K \times \Re^m$ of F is closed, by the continuity of H, the set $H^{-1}(C)$ is always closed. The argument to obtain the boundedness of $H^{-1}(C)$ is the same as in the proof of Lemma 3.5. ∎

Theorem 4.2 (cf. Theorem 2 of [17] and Theorem 3.10) *Suppose that a continuous map $F : K \times K \times \Re^m \to V \times \Re^m$ satisfies Assumption 1.1. Then the map H defend by (4) satisfies that*

(i) *H is proper with respect to $V \times F(\mathcal{U} \times \Re^m)$,*

(ii) *H maps $\mathcal{U} \times \Re^m$ homeomorphically onto* $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$*, and*

(iii) *$H(K \times K \times \Re^m) \supseteq K \times F(\mathcal{U} \times \Re^m)$.*

Proof: Define the sets M, N, M_0 and N_0, and the map G as in (21), and the set E by

$$E := V \times F(\mathcal{U} \times \Re^m).$$

(i): The assumption imposed here is stricter than the one in Theorem 3.10. Thus the local homeomorphism of $H|_{(M_0,N)} = G|_{(M_0,N)}$ is similarly obtained from Lemma 3.9. Using Lemma 4.1 instead of Lemma 3.5, we can see that H is proper with respect to $E = V \times F(\mathcal{U} \times \Re^m)$, i.e., the assertion (i) holds.
(ii): In the above discussion, we have shown that (d) of Proposition 3.3 holds. Since the sets M, N and N_0 are the same as in Theorem 3.10, we can see that the assumptions (a)–(c) of Proposition 3.3 are also satisfied and that

$$H(\mathcal{U} \times \Re^m) = G(M_0) = N_0 = \mathrm{int}K \times F(\mathcal{U} \times \Re^m).$$

holds.
(iii): Since $\mathcal{U}$ is star-shaped (see (vii) of Lemma 2.7), $\mathcal{U} \times \Re^m$ is connected and hence N_0 is also connected by the continuity of F. Combining this with the facts

$$K \times K \times \Re^m = \mathrm{cl}(M) \supseteq \mathrm{cl}(M_0) \;\; \text{and} \;\; E \cap \mathrm{cl}(N_0) = K \times F(\mathcal{U} \times \Re^m),$$

the assertion (iii) follows from (ii) of Proposition 3.3. ∎

The following corollary can be obtained as a direct consequence of Theorem 3.12.

Corollary 4.3 (cf. Theorem 3 of [17] and Theorem 3.12) *Suppose that a continuous map $F : K \times K \times \Re^m \to V \times \Re^m$ satisfies Assumptions 1.1 and 3.11. Then the set $F(\mathcal{U} \times \Re^m)$ is an open convex set.*

The solvability of (CP) follows from the above results.

Corollary 4.4 (cf. Corollary 1 of [17]) *Suppose that the map $F : K \times K \times \Re^m \to V \times \Re^m$ is continuous and that $0 \in F(\mathcal{U} \times \Re^m)$, which implies that (CP) has an interior feasible point $(\bar{x}, \bar{y}, \bar{z}) \in \mathrm{int}K \times \mathrm{int}K \times \Re^m$ such that $\bar{x} \circ \bar{y} \in \mathrm{int}K$ and $F(\bar{x}, \bar{y}, \bar{z}) = 0$. Let $p : [0, 1] \to K \times F(\mathcal{U} \cap \Re^m)$ be a path for which*

$$p(0) = 0 \quad \text{and} \quad p(t) \in \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$$

hold and suppose that the set $\{p(t) : \; t \in [0, 1]\}$ is bounded.

(i) *If the map F satisfies Assumption 1.1 then there exists a unique path $(x, y, z) : (0, 1] \to \mathrm{int}K \times \mathrm{int}K \times \Re^m$ such that*

$$H(x(t), y(t), z(t)) = p(t) \text{ for every } t \in (0, 1]$$

and $\{(x(t), y(t), z(t)) : t \in (0, 1]\}$ is bounded. Thus (CP) is asymptotically solvable. In addition, every accumulation point of $\{(x(t), y(t), z(t)) : t \in (0, 1]\}$ is a solution of (CP).

(ii) *If F satisfies Assumptions 1.1 and 3.11 then for each $h \in \mathrm{int}K \times F(\mathcal{U} \cap \Re^m)$, we can take*

$$p(t) = th, \text{ for every } t \in [0, 1]$$

as a path $p : [0, 1] \to K \times F(\mathcal{U} \cap \Re^m)$ satisfying the requirements above.

Proof: (i): The assertion follows from Theorem 4.2: (ii) of the theorem implies the unique existence of $(x(t), y(t), z(t))$ for every $t \in (0, 1]$, (i) implies the boundedness of $(x(t), y(t), z(t))$, and (iii) implies that every accumulation point of $\{(x(t), y(t), z(t))\}$ is a solution of (CP).
(ii): Since $(0, 0) \in K \times F(\mathcal{U} \times \Re^m)$, Corollary 4.3 ensures that

$$\{p(t) : p(t) = th,\ t \in [0, 1]\} \subset K \times F(\mathcal{U} \times \Re^m)$$

for every $h \in \mathrm{int}K \times F(\mathcal{U} \cap \Re^m)$. It is obvious that $\{p(t) : t \in [0, 1]\}$ is bounded, $p(0) = 0$ and $p(t) \in \mathrm{int}K \times F(\mathcal{U} \times \Re^m)$ for every $t \in (0, 1]$. ∎

5 A homogeneous model for (CP)

In this section, as an application of the results in Section 3, we give a homogenous model for solving a special class of (CP)s where the map $F : K \times K \to V$ is of the form

$$F(x, y) = y - \psi(x) \tag{29}$$

for a continuous map $\psi : K \to V$.

Our model is a natural extension of the homogeneous model proposed by Andersen and Ye [3]. One of the remarkable features of their model is that the associated trajectory gives certifications on the strong feasibility or the strong infeasibility of the original problem. Our results, Theorems 5.4 and 5.5, show that our homogeneous model inherits the property even for the case of symmetric cones.

Throughout this section, we impose the following assumption on ψ.

Assumption 5.1 *The map $\psi : K \to V$ in (29) is monotone on the set K, i.e.,*

$$\langle \psi(x) - \psi(x'), x - x' \rangle \geq 0$$

for every $x, x' \in K$.

Proposition 5.2 *Suppose that $S \subseteq K$ and $\psi : S \to V$ is monotone on the set S. Then the map $F : S \times K \to V$ given by (29) is (x, y)-everywhere-monotone on the set $S \times K$ with $m = 0$.*

Proof: Define $\phi : S \times K \to V$ and $c : V \times V \to \Re$ by

$$\phi(x,y) := x \quad \text{and} \quad c := 0.$$

Let $r := F(x,y)$ and $r' := F(x',y')$ where $(x,y),(x',y') \in S \times K$. Then we see that

$$\psi(x) - \psi(x') = (y - y') - (r - r'),$$

and the monotonicity of ψ implies that

$$\begin{aligned}
0 &\leq \langle \psi(x) - \psi(x'), x - x' \rangle \\
&= \langle (y - y') - (r - r'), x - x' \rangle \\
&= \langle y - y', x - x' \rangle - \langle r - r', x - x' \rangle \\
&= \langle y - y', x - x' \rangle - \langle r - r', \phi(x,y) - \phi(x',y') \rangle + c(r,r').
\end{aligned}$$

Thus, by the definition of (x,y)-everywhere-monotonicity in Assumption 3.11, the map F is (x,y)-everywhere-monotone on the set $S \times K$ with $m = 0$. ∎

Define the sets

$$\Re_+ := \{\tau \in \Re : \ \tau \geq 0\} \quad \text{and} \quad \Re_{++} := \{\tau \in \Re : \ \tau > 0\}.$$

For a given (CP) with a map F of the form (29), we consider the following homogeneous model (HCP):

$$\begin{aligned}
\text{(HCP) Find} \quad & (x,\tau,y,\kappa) \in (K \times \Re_{++}) \times (K \times \Re_+) \\
\text{s.t.} \quad & F_{\mathrm{H}}(x,\tau,y,\kappa) = 0, \ (x,\tau) \circ_{\mathrm{H}} (y,\kappa) = 0
\end{aligned} \tag{30}$$

where $F_{\mathrm{H}} : (K \times \Re_{++}) \times (K \times \Re_+) \to (V \times \Re)$ and $(x,\tau) \circ_{\mathrm{H}} (y,\kappa)$ are given by

$$F_{\mathrm{H}}(x,\tau,y,\kappa) := \begin{pmatrix} y - \tau\psi(x/\tau, z/\tau) \\ \kappa + \langle \psi(x/\tau), x \rangle \end{pmatrix} \tag{31}$$

and

$$(x,\tau) \circ_{\mathrm{H}} (y,\kappa) := \begin{pmatrix} x \circ y \\ \tau\kappa \end{pmatrix}. \tag{32}$$

For ease of notation, we use the following symbols

$$V_{\mathrm{H}} := V \times \Re, \quad K_{\mathrm{H}} := K \times \Re_+, \quad x_{\mathrm{H}} := (x,\tau) \in V_{\mathrm{H}}, \quad y_{\mathrm{H}} := (y,\kappa) \in V_{\mathrm{H}}. \tag{33}$$

It should be noted that the set K_{H} is a Cartesian product of two symmetric cones K and $\Re_+$ and given by

$$K_{\mathrm{H}} = \left\{ x_{\mathrm{H}}^2 = \begin{pmatrix} x^2 \\ \tau^2 \end{pmatrix} : \ x_{\mathrm{H}} \in V_{\mathrm{H}} \right\}.$$

Thus the closed convex cone $K_{\rm H}$ is the symmetric cone of $V_{\rm H}$. It can be easily seen that $\text{int}K_{\rm H} = \text{int}K \times \Re_{++}$.

We also introduce the scalar product $\langle (x,\tau),(y,\kappa)\rangle_{\rm H}$ associated to the bilinear product $\circ_{\rm H}$ as

$$\langle (x,\tau),(y,\kappa)\rangle_{\rm H} := \langle x,y\rangle + \tau\kappa. \tag{34}$$

For every $x_{\rm H} = (x,\tau) \in V_{\rm H}$, the linear operator

$$L_{\rm H}(x_{\rm H}) := \begin{pmatrix} L(x) & 0 \\ 0^T & \tau \end{pmatrix}$$

is self-adjoint with respect to $\langle \cdot,\cdot\rangle$, i.e.,

$$\langle v_{\rm H}, L_{\rm H}(x_{\rm H})w_{\rm H}\rangle = \langle L_{\rm H}(x_{\rm H})v_{\rm H}, w_{\rm H}\rangle$$

for every $v_{\rm H}, w_{\rm H} \in V_{\rm H}$.

Let us define the map $\psi_{\rm H}$ by

$$\psi_{\rm H}(x_{\rm H}) = \psi_{\rm H}(x,\tau) := \begin{pmatrix} \tau\psi(x/\tau) \\ -\langle \psi(x/\tau), x\rangle \end{pmatrix} \tag{35}$$

for every $x_{\rm H} = (x,\tau) \in K \times \Re_{++}$. Then the map $F_{\rm H}$ is given by

$$F_{\rm H}(x_{\rm H}, y_{\rm H}) = y_{\rm H} - \psi_{\rm H}(x_{\rm H}). \tag{36}$$

We also define the set

$$\mathcal{U}_{\rm H} := \{(x_{\rm H}, y_{\rm H}) \in \text{int}K_{\rm H} \times \text{int}K_{\rm H} : \ x_{\rm H} \circ_{\rm H} y_{\rm H} \text{int}K_{\rm H}\}. \tag{37}$$

It is clear that the set $\mathcal{U}_{\rm H}$ has the properties described in Lemma 2.7 with $\mathcal{U} = \mathcal{U}_{\rm H}$.

The following proposition shows that a monotonicity of the map $F_{\rm H}$ on the set $\text{int}K_{\rm H} \times \text{int}K_{\rm H}$ can be obtained if the map ψ is monotone on the set K.

Proposition 5.3 *Suppose that $\psi : K \to V$ satisfies Assumption 5.1. Then we have*

(i) *the map $\psi_{\rm H}$ is monotone on $\text{int}K_{\rm H}$, and*

(ii) *the map $F_{\rm H}$ is $(x_{\rm H}, y_{\rm H})$-everywhere-monotone on $\text{int}K_{\rm H} \times \text{int}K_{\rm H}$.*

Thus, the map $F_{\rm H}$ with the domain $\text{int}K_{\rm H} \times \text{int}K_{\rm H}$ satisfies Assumptions 1.1 and 3.11 with $m = 0$ whenever the map ψ is monotone on K.

Proof: (i): For every $x_{\rm H}, x'_{\rm H} \in \text{int}K_{\rm H}$, it follows from the definition (35) that

$$\begin{aligned}
&\langle \psi_{\rm H}(x_{\rm H}) - \psi_{\rm H}(x'_{\rm H}), x_{\rm H} - x'_{\rm H}\rangle_{\rm H} \\
&= \langle \tau\psi(x/\tau) - \tau'\psi(x'/\tau'), x - x'\rangle - (\tau - \tau')[\langle \psi(x/\tau), x\rangle - \langle \psi(x'/\tau'), x'\rangle] \\
&= \langle \tau\psi(x/\tau), x - x'\rangle - \langle \tau'\psi(x'/\tau'), x - x'\rangle - (\tau - \tau')\langle \psi(x/\tau), x\rangle + (\tau - \tau')\langle \psi(x'/\tau'), x'\rangle \\
&= -\tau\langle \psi(x/\tau), x'\rangle - \tau'\langle \psi(x'/\tau'), x\rangle + \tau'\langle \psi(x/\tau), x\rangle + \tau\langle \psi(x'/\tau'), x'\rangle \\
&= -\tau\tau'\langle \psi(x/\tau), x'/\tau'\rangle - \tau\tau'\langle \psi(x'/\tau'), x/\tau\rangle + \tau\tau'\langle \psi(x/\tau), x/\tau\rangle + \tau\tau'\langle \psi(x'/\tau'), x'/\tau'\rangle \\
&= \tau\tau'\langle \psi(x/\tau), (x/\tau) - (x'/\tau')\rangle - \tau\tau'\langle \psi(x'/\tau'), (x/\tau) - (x'/\tau')\rangle \\
&= \tau\tau'\langle \psi(x/\tau) - \psi(x'/\tau'), (x/\tau) - (x'/\tau')\rangle \\
&\geq 0
\end{aligned}$$

where the last inequality follows from the monotonicity of ψ. Thus the map ψ_{H} is monotone on the set $\mathrm{int}K_{\mathrm{H}}$.
(ii): The assertion follows from (i) above and Proposition 5.2 with $S = \mathrm{int}K_{\mathrm{H}}$. ∎

In the theorem below, the assertions (i)–(iii) follow only from the construction (35) of the map ψ_{H} while we assume that the map ψ is monotone. Note that we assume that $(V, \circ)$ is a *simple* Euclidean Jordan algebra in the assertion (v).

Theorem 5.4 (cf. Theorem 1 of [3]) *Suppose that $\psi : K \to V$ satisfies Assumption 5.1.*

(i) *For every $x_{\mathrm{H}} \in \mathrm{int}K_{\mathrm{H}}$,*

$$\langle x_{\mathrm{H}}, \psi_{\mathrm{H}}(x_{\mathrm{H}})\rangle_{\mathrm{H}} = 0.$$

(ii) *Every asymptotically feasible solution $(\hat{x}_{\mathrm{H}}, \hat{y}_{\mathrm{H}})$ of (HCP) is an asymptotically complementary solution.*

(iii) *(HCP) is asymptotically feasible.*

(iv) *(CP) has a solution if and only if (HCP) has an (asymptotical) solution $(x^*_{\mathrm{H}}, y^*_{\mathrm{H}}) = (x^*, \tau^*, y^*, \kappa^*)$ with $\tau^* > 0$. In this case, $(x^*/\tau^*, y^*/\tau^*)$ is a solution of (CP).*

(v) *Suppose that $(V, \circ)$ is a simple Euclidean Jordan algebra. Then (CP) is strongly infeasible if and only if (HCP) has an asymptotical solution $(x^*, \tau^*, y^*, \kappa^*)$ with $\kappa^* > 0$.*

Proof: (i): By a simple calculation, we have

$$\begin{aligned}
&\langle x_{\mathrm{H}}, \psi_{\mathrm{H}}(x_{\mathrm{H}})\rangle_{\mathrm{H}} \\
&= \langle x, \tau\psi(x/\tau)\rangle - \tau\langle \psi(x/\tau), x\rangle \\
&= 0.
\end{aligned}$$

(ii): Suppose that $(\hat{x}_{\mathrm{H}}, \hat{y}_{\mathrm{H}})$ is an asymptotically feasible solution. Then there exists a bounded sequence $(x^{(k)}_{\mathrm{H}}, y^{(k)}_{\mathrm{H}}) \in \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}}$ such that

$$\lim_{k\to\infty} F_{\mathrm{H}}(x^{(k)}_{\mathrm{H}}, y^{(k)}_{\mathrm{H}}) \lim_{k\to\infty} (y^{(k)}_{\mathrm{H}} - \psi_{\mathrm{H}}(x^{(k)}_{\mathrm{H}})) = 0.$$

Here (i) above implies that

$$\begin{aligned}
\langle x^{(k)}_{\mathrm{H}}, y^{(k)}_{\mathrm{H}}\rangle_{\mathrm{H}} &= \langle x^{(k)}_{\mathrm{H}}, y^{(k)}_{\mathrm{H}}\rangle_{\mathrm{H}} - \langle x^{(k)}_{\mathrm{H}}, \psi_{\mathrm{H}}(x^{(k)}_{\mathrm{H}})\rangle_{\mathrm{H}} \\
&= \langle x^{(k)}_{\mathrm{H}}, y^{(k)}_{\mathrm{H}} - \psi_{\mathrm{H}}(x^{(k)}_{\mathrm{H}})\rangle_{\mathrm{H}}
\end{aligned}$$

holds for every $k \geq 0$. Thus, we see that

$$\lim_{k\to\infty} \langle x^{(k)}_{\mathrm{H}}, y^{(k)}_{\mathrm{H}}\rangle_{\mathrm{H}} = 0.$$

By the definition (34) of $\langle\cdot,\cdot\rangle_{\mathrm{H}}$ and (ii) of Lemma 2.7, we obtain that $(\hat{x}_{\mathrm{H}}, \hat{y}_{\mathrm{H}})$ is an asymptotically complementary solution.

(iii): For every $k \geq 0$, define

$$x^{(k)} := (1/2)^k e \in \text{int}K,\ \tau_k := (1/2)^k \in \Re_{++},\ y^{(k)} := (1/2)^k e \in \text{int}K,\ \kappa_k := (1/2)^k \in \Re_{++}.$$

Then the sequence $\{(x_{\rm H}^{(k)}, y_{\rm H}^{(k)})\} = \{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k)\}$ is bounded and

$$\lim_{k\to\infty}\Big(y^{(k)} - \tau_k\psi(x^{(k)}/\tau_k)\Big) = \lim_{k\to\infty}\Big((1/2)^k e - (1/2)^k\psi(e,0)\Big) = 0,$$
$$\lim_{k\to\infty}\Big(\kappa_k + \langle\psi(x^{(k)}/\tau_k), x^{(k)}\rangle\Big) = \lim_{k\to\infty}\Big((1/2)^k + (1/2)^k\langle\psi(e,0), e\rangle\Big) = 0.$$

Thus, the bounded sequence $\{(x_{\rm H}^{(k)}, y_{\rm H}^{(k)})\} \subset \text{int}K_{\rm H} \times \text{int}K_{\rm H}$ satisfies

$$\lim_{k\to\infty}(y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)}))\lim_{k\to\infty} F_{\rm H}(x^{(k)}, y_{\rm H}^{(k)}) = 0$$

which implies that (HCP) is asymptotically feasible.
(iv): It is easy to see that if $(x^*, \tau^*, y^*, \kappa^*) \in (K \times \Re_+) \times (K \times \Re_+)$ is a solution of (HCP) with $\tau^* > 0$ then

$$y^*/\tau^* - \psi(x^*/\tau^*) = 0 \ \text{ and } \ x^* \circ y^* = 0$$

which shows that $(x^*/\tau^*, y^*/\tau^*) \in K \times K$ is a solution of (CP).

Conversely, let $(\hat{x}, \hat{y}) \in K \times K$ be a solution of (CP). Then

$$\hat{y} - 1 \cdot \psi(\hat{x}/1) = 0 \ \text{ and } \ (\hat{x}, 1) \circ_{\rm H} (\hat{y}, 0) = 0$$

and $(\hat{x}, 1, \hat{y}, 0) \in (K \times \Re_{++}) \times (K \times \Re_+)$ is a solution of (HCP).
(v): By Proposition 5.2, the monotonicity of the map ψ on the set K implies that the map F defined by (29) satisfies Assumptions 1.1 and 3.11 with the domain $K \times K$ and $m = 0$. Thus the set $F(\mathcal{U})$ is open and convex by Theorem 3.12.

If (CP) is strongly infeasible, then we must have $0 \notin \text{cl}(F(\mathcal{U}))$. Since the set $\text{cl}(F(\mathcal{U}))$ is a closed convex set, by the separating hyperplane theorem, there exist $a \in V$ with $\|a\| = 1$ and $\xi \in \Re$ such that

$$\langle a, b\rangle \geq \xi > 0 \ \text{ for every } b \in \text{cl}(F(\mathcal{U})). \tag{38}$$

Since F is continuous on the set $\text{cl}(\mathcal{U}) \subseteq K \times K$, we can see that $F(\text{cl}(\mathcal{U})) \subset \text{cl}(F(\mathcal{U}))$. In fact, if $b \in F(\text{cl}(\mathcal{U}))$ then there exists a sequence such that

$$(x^{(k)}, y^{(k)}) \in \mathcal{U}, \quad \lim_{k\to\infty}(x^{(k)}, y^{(k)}) = (\bar{x}, \bar{y}) \in \text{cl}(\mathcal{U}), \quad F(\bar{x}, \bar{y}) = b,$$

and the continuity of F on the set $\text{cl}(\mathcal{U})$ implies that

$$\lim_{k\to\infty} F(x^{(k)}, y^{(k)}) = F(\bar{x}, \bar{y}) = b.$$

Therefore (38) implies that

$$\begin{aligned}\langle a, F(x,y)\rangle &= \langle a, y - \psi(x)\rangle \\ &= \langle a, y\rangle - \langle a, \psi(x)\rangle \\ &\geq \xi > 0 \ \text{ for every } (x,y) \in \text{cl}(\mathcal{U}).\end{aligned} \tag{39}$$

Note that (ix) of Lemma 2.7 ensures that the above relation (39) holds at $(x, y) = (0, \alpha\bar{y})$ for every fixed $\bar{y} \in K$ and every $\alpha > 0$. Thus, it must be true that

$$\langle a, \bar{y} \rangle \geq 0 \text{ for every } \bar{y} \in K$$

which implies that $a \in K$. Similarly, since $(x, 0) \in \mathrm{cl}(\mathcal{U})$ for every $x \in K$, it follows from (39) that

$$-\langle a, \psi(x) \rangle \geq \xi > 0 \tag{40}$$

for every $x \in K$. Thus, combining with the fact that $a \in K$, we see that

$$-\langle a, \psi(\beta a) \rangle \geq \xi > 0 \text{ for every } \beta \geq 0. \tag{41}$$

From the monotonicity of the map ψ on the set $K \times K$, we also see that for every $x \in K$ and $\beta \geq 0$,

$$\begin{aligned} 0 &\leq \langle \beta x - x, \psi(\beta x) - \psi(x) \rangle \\ &= (\beta - 1)\langle x, \psi(\beta x) - \psi(x) \rangle. \end{aligned}$$

Thus, for every $\beta \geq 1$,

$$\langle x, \psi(\beta x) - \psi(x) \rangle \geq 0 \tag{42}$$

which implies that

$$\lim_{\beta \to \infty} \langle x, \psi(\beta x) \rangle / \beta \geq 0. \tag{43}$$

For each $x \in K$, denote

$$\psi^{\infty}(x) := \lim_{\beta \to \infty} \frac{\psi(\beta x)}{\beta} \tag{44}$$

where $\psi^{\infty}(x)$ represents the limit of any subsequence and its value may have ∞ or $-\infty$.

We first claim that $\psi^{\infty}(a) \in K$. Let $\{\beta_k\}$ be a subsequence such that $\beta_k \to +\infty$ and for each k, let

$$\frac{\psi(\beta_k a)}{\beta_k} = \sum_{i=1}^{r} \lambda_i^{(k)} c_i^{(k)} \tag{45}$$

be a decomposition given by (i) of Theorem 2.3. We also define

$$\lambda_k := \min\{\lambda_i^{(k)} \ (i = 1, 2, \ldots, r)\}, \quad j_k \in \arg\min\{\lambda_i^{(k)} \ (i = 1, 2, \ldots, r)\} \text{ and } c^{(k)} := c_{j_k}^{(k)}. \tag{46}$$

Note that $\{c^{(k)}\}$ is a sequence of a primitive (i.e., nonzero) idempotent of a simple Euclidean Jordan algebra $(V, \circ)$ and hence it satisfies

$$\sqrt{\frac{r}{2n}} \leq \|c^{(k)}\| \leq \sqrt{r} \text{ for each } k \tag{47}$$

by (ii) of Proposition 2.8.

Suppose that $\psi^{\infty}(a) \notin K$ along the subsequence $\{\beta_k\}$. Then there exist a $\delta > 0$ for which

$$\lambda_k \leq -\delta$$

for sufficiently large k's. Define

$$x^{(k)} := a + \epsilon c^{(k)}$$

for $\epsilon > 0$. Then we can see that

$$\begin{aligned}
\langle x^{(k)}, \psi(\beta_k x^{(k)})\rangle/\beta_k &= \langle a + \epsilon c^{(k)}, \psi(\beta_k x^{(k)})\rangle/\beta_k \\
&= \langle a, \psi(\beta_k x^{(k)})\rangle/\beta_k + \epsilon\langle c^{(k)}, \psi(\beta_k x^{(k)})\rangle/\beta_k \\
&< \epsilon\langle c^{(k)}, \psi(\beta_k x^{(k)})\rangle/\beta_k \qquad \text{(by (40))} \\
&= \epsilon\left(\langle c^{(k)}, \psi(\beta_k x^{(k)}) - \psi(\beta_k a)\rangle/\beta_k + \langle c^{(k)}, \psi(\beta_k a)\rangle/\beta_k\right). \qquad (48)
\end{aligned}$$

Here the definitions (45) and (46) and the boundedness (47) of $\{c^{(k)}\}$ ensure that

$$\langle c^{(k)}, \psi(\beta_k a)\rangle/\beta_k = \lambda_k\langle c^{(k)}, c^{(k)}\rangle \leq -\delta\frac{r}{2n} < 0 \tag{49}$$

for sufficiently large k's. In addition, since we set $x^{(k)} = a + \epsilon c^{(k)}$, by the continuity of ψ and the boundedness of $\{c^{(k)}\}$, we have

$$\langle c^{(k)}, \psi(\beta_k x^{(k)}) - \psi(\beta_k a)\rangle/\beta_k = \mathcal{O}(\epsilon) \tag{50}$$

for sufficiently small ϵ's. Thus, by (49) and (50),

$$\langle c^{(k)}, \psi(\beta_k x^{(k)}) - \psi(\beta_k a)\rangle/\beta_k + \langle c^{(k)}, \psi(\beta_k a)\rangle/\beta_k \leq -\delta\frac{r}{4n} < 0$$

holds for sufficiently large k's and sufficiently small ϵ's. Therefore, by (48), we obtain that

$$\langle x^{(k)}, \psi(\beta_k x^{(k)})\rangle/\beta_k \leq -\epsilon\delta\frac{r}{4n} < 0$$

for all such k's and ϵ's. Since $x^{(k)} = a + \epsilon c^{(k)} \in K$, by fixing a suitably small $\epsilon > 0$, the above inequality contradicts to (43) and we must have $\psi^\infty(a) \in K$.

Next we claim that $\psi^\infty(a)$ is bounded. Let $\{\beta_k\}$ be a subsequence along which $\psi(\beta_k a)/\beta_k$ tends to $\psi^\infty(a)$. By the facts $\beta_k a \in K$ for every k, $e \in K$ and ψ is monotone on K, we see that

$$\begin{aligned}
0 &\leq \langle \beta_k a - e, \psi(\beta_k a) - \psi(e)\rangle/\beta_k \\
&= \langle a, \psi(\beta_k a)\rangle - \langle e, \psi(\beta_k a)/\beta_k\rangle - \langle a, \psi(e)\rangle + \langle e, \psi(e)/\beta_k\rangle \\
&< -\langle e, \psi(\beta_k a)/\beta_k\rangle - \langle a, \psi(e)\rangle + \langle e, \psi(e)/\beta_k\rangle
\end{aligned}$$

where the last inequality follows from (40). Taking a limit as $k \to \infty$ from both sides, we have

$$\langle e, \psi^\infty(a)\rangle \leq -\langle a, \psi(e)\rangle$$

which implies that $\psi^\infty(a) \in K$ is bounded (see Proposition 2.2). Note that $\langle a, \psi(\beta_k a)\rangle \leq -\xi$ from (41) and $\langle a, \psi(\beta_k a)\rangle \geq \langle a, \psi(a)\rangle$ from (42). Thus $\{\langle a, \psi(\beta_k a)\rangle\}$ is bounded. To summarize, by setting

$$\begin{aligned}
&x^* := a \in K, \quad \tau^* := \lim_{\beta_k \to \infty} \frac{1}{\beta_k} = 0, \\
&y^* := \psi^\infty(a) = \lim_{k\to\infty} \psi(\beta_k a)/\beta_k \in K, \quad \kappa^* := \lim_{k\to\infty} -\langle a, \psi(\beta_k a)\rangle \geq \xi > 0
\end{aligned}$$

(HCP) has an asymptotical solution $(x^*, \tau^*, y^*, \kappa^*) \in (K \times \Re_+) \times (K \times \Re_+)$ with $\kappa^* > 0$.

Conversely, suppose that there exists a bounded sequence $(x^{(k)}, \tau_k, y^{(k)}, \kappa_k) \in (\mathrm{int}K \times \Re_{++}) \times (\mathrm{int}K \times \Re_{++})$ such that

$$\lim_{k\to\infty} y^{(k)} = \lim_{k\to\infty} \tau_k \psi(x^{(k)}/\tau_k) \in K, \quad \lim_{k\to\infty} \kappa_k = \lim_{k\to\infty} -\langle x^{(k)}, \psi(x^{(k)}/\tau_k)\rangle \geq \xi > 0.$$

Let us show that there is no feasible point $(x, y) \in K \times K$ satisfying $y - \psi(x) = 0$. Suppose that $(x, y) \in K \times K$ and $y - \psi(x) = 0$. Since ψ_{H} is monotone on $(K \times \Re_{++}) \times (K \times \Re_+)$, by the definition (35), we have

$$\begin{aligned}
0 &\leq \langle (x^{(k)}, \tau_k) - (x, 1), \psi_{\mathrm{H}}(x^{(k)}, \tau_k) - \psi_{\mathrm{H}}(x, 1)\rangle \\
&= \langle x^{(k)} - x, \tau_k \psi(x^{(k)}/\tau_k) - \psi(x)\rangle + (\tau_k - 1)\left(\langle x, \psi(x)\rangle - \langle x^{(k)}, \psi(x^{(k)}/\tau_k)\rangle\right) \\
&= \langle x^{(k)}, \tau_k \psi(x^{(k)}/\tau_k)\rangle + \langle x, \psi(x)\rangle \\
&\qquad + (\tau_k - 1)\langle x, \psi(x)\rangle - (\tau_k - 1)\langle x^{(k)}, \psi(x^{(k)}/\tau_k)\rangle \\
&\qquad - \langle x^{(k)}, \psi(x)\rangle - \langle x, \tau_k \psi(x^{(k)}/\tau_k)\rangle \\
&= \tau_k \langle x, \psi(x)\rangle + \langle x^{(k)}, \psi(x^{(k)}/\tau_k)\rangle - \langle x^{(k)}, \psi(x)\rangle - \langle x, \tau_k \psi(x^{(k)}/\tau_k)\rangle
\end{aligned}$$

and hence

$$\begin{aligned}
&\langle x^{(k)}, \psi(x^{(k)}/\tau_k)\rangle \\
&\geq \langle x^{(k)}, \psi(x)\rangle + \langle x, \tau_k \psi(x^{(k)}/\tau_k)\rangle - \tau_k \langle x, \psi(x)\rangle \\
&= \langle x^{(k)}, y\rangle + \langle x, \tau_k \psi(x^{(k)}/\tau_k)\rangle - \tau_k \langle x, y\rangle. \qquad (51)
\end{aligned}$$

Here $\lim_{k\to\infty} \tau_k = 0$ since $\lim_{k\to\infty} \kappa_k \geq \xi > 0$. In addition, it follows from the assumption that $\langle x^{(k)}, y\rangle \geq 0$ and that

$$\lim_{k\to\infty} \langle x, y^{(k)}\rangle = \langle x, \lim_{k\to\infty} \tau_k \psi(x^{(k)}/\tau_k)\rangle \geq 0.$$

Thus the relation (51) ensures that

$$\lim_{k\to\infty} \langle x^{(k)}, \psi(x^{(k)}/\tau_k)\rangle \geq 0$$

which contradicts to

$$\kappa_k := -\langle x^{(k)}, \psi(x^{(k)}/\tau_k)\rangle \geq \xi > 0.$$

Also, any limit of $x^{(k)}$ gives a separating hyperplane, i.e., a certificate proving infeasibility.

∎

We are going to show that a central path of the homogeneous model (HCP) is well defined. As we will see in (iii) of Theorem 5.5, any limit point of the path lets us know if (HCP) has an asymptotically complementarity solution $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*) = (x^*, \tau^*, y^*, \kappa^*)$ with $\tau^* > 0$ or if it has such a solution with $\kappa^* > 0$. Therefore, in view of (iv) and (v) of Theorem 5.4, if we find a limit of the path then we can determine whether (CP) is strongly feasible, strongly infeasible or other possible cases.

Let us consider the map

$$H_{\mathrm{H}} := \begin{pmatrix} x_{\mathrm{H}} \circ_{\mathrm{H}} y_{\mathrm{H}} \\ F_{\mathrm{H}}(x_{\mathrm{H}}, y_{\mathrm{H}}) \end{pmatrix} \qquad (52)$$

and choose an initial point $(x_{\rm H}^{(0)}, y_{\rm H}^{(0)})$ such that

$$(x_{\rm H}^{(0)}, y_{\rm H}^{(0)}) \in {\rm int}K_{\rm H} \times {\rm int}K_{\rm H} \text{ and } x_{\rm H} \circ_{\rm H} y_{\rm H} \in {\rm int}K_{\rm H}.$$

For simplicity, we set

$$(x_{\rm H}^{(0)}, y_{\rm H}^{(0)}) = (x^{(0)}, \tau_0, y^{(0)}, \kappa_0) = (e, 1, e, 1) \in {\rm int}K_{\rm H} \times {\rm int}K_{\rm H}.$$

Define

$$h_{\rm H}^{(0)} := \begin{pmatrix} p_{\rm H}^{(0)} \\ f_{\rm H}^{(0)} \end{pmatrix} := \begin{pmatrix} x_{\rm H}^{(0)} \circ_{\rm H} y_{\rm H}^{(0)} \\ F_{\rm H}(x_{\rm H}^{(0)}, y_{\rm H}^{(0)}) \end{pmatrix} = \begin{pmatrix} e_{\rm H} \\ y_{\rm H}^{(0)} - \psi_{\rm H}(x_{\rm H}^{(0)}) \end{pmatrix} \tag{53}$$

where $e_{\rm H} = (e, 1)$ is the identity element in $V_{\rm H}$ satisfying

$$\text{tr}(e_{\rm H}) = \text{rank}(V_{\rm H}) = r + 1. \tag{54}$$

We consider the system

$$H_{\rm H}(x_{\rm H}, y_{\rm H}) = t h_{\rm H}^{(0)}$$

for each $t \in (0, 1]$.

Theorem 5.5 (cf. Theorem 2 of [3]) *Suppose that $\psi : K \to V$ satisfies Assumption 5.1. Define $h_{\rm H}^{(0)}$ by (53).*

(i) *For any $t \in (0, 1]$, there exists a point $(x_{\rm H}(t), y_{\rm H}(t)) \in {\rm int}K_{\rm H} \times {\rm int}K_{\rm H}$ such that*

$$H_{\rm H}(x_{\rm H}(t), y_{\rm H}(t)) = t h_{\rm H}^{(0)}.$$

(ii) *The set*

$$P := \{(x_{\rm H}, y_{\rm H}) : \ H_{\rm H}(x_{\rm H}(t), y_{\rm H}(t)) = t h_{\rm H}^{(0)}, \ t \in (0, 1]\}$$

forms a bounded path in ${\rm int}K_{\rm H} \times {\rm int}K_{\rm H}$. Any accumulation point $(x_{\rm H}(0), y_{\rm H}(0))$ is an asymptotically complementary solution of (HCP).

(iii) *If (HCP) has an asymptotically complementarity solution $(x_{\rm H}^*, y_{\rm H}^*) = (x^*, \tau^*, y^*, \kappa^*)$ with $\tau^* > 0$ ($\kappa^* > 0$, respectively), then any accumulation point*

$$(x_{\rm H}(0), y_{\rm H}(0)) = (x(0), \tau(0), y(0), \kappa(0))$$

of the bounded path P satisfies $\tau(0) > 0$ ($\kappa(0) > 0$, respectively).

Proof: (i): It follows from Proposition 5.3 that the map $F_{\rm H}$ defined by (31) satisfies Assumptions 1.1 and 3.11. Thus, by Theorem 3.12, the set $H_{\rm H}(\mathcal{U}_{\rm H})$ with

$$\mathcal{U}_{\rm H} := \{(x_{\rm H}, y_{\rm H}) \in {\rm int}K_{\rm H} \times {\rm int}K_{\rm H} : \ x_{\rm H} \circ_{\rm H} y_{\rm H} \in {\rm int}K_{\rm H}\}$$

is an open convex subset of ${\rm int}K_{\rm H} \times {\rm int}K_{\rm H}$. Here we have already seen that

$$0 \in {\rm cl}(H_{\rm H}(\mathcal{U}_{\rm H}))$$

in (ii) and (iii) of Theorem 5.4. Since the set $H_{\rm H}(\mathcal{U}_{\rm H})$ is convex, the fact above implies that

$$th_{\rm H}^{(0)} \in H_{\rm H}(\mathcal{U}_{\rm H})$$

for every $t \in (0,1]$. Combining this with the homeomorphism of the map $H_{\rm H}$ in Theorem 3.10, we obtain the assertion (i).
(ii): The homeomorphism of the map $H_{\rm H}$ also ensures that the set P forms a path in $\mathrm{int}K_{\rm H} \times \mathrm{int}K_{\rm H}$. It suffices to show that the path P is bounded.

Let $(x_{\rm H}, y_{\rm H}) = (x, \tau, y, \kappa) \in P$. Then there exists a $t \in (0,1]$ for which

$$x_{\rm H} \circ_{\rm H} y_{\rm H} = te_{\rm H} \quad \text{and} \quad y_{\rm H} - \psi_{\rm H}(x_{\rm H}) = tf_{\rm H}^{(0)} \tag{55}$$

hold and

$$\begin{aligned}
&\langle x_{\rm H}, f_{\rm H}^{(0)}\rangle_{\rm H} \\
&= \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} - \langle x_{\rm H}, \psi_{\rm H}(x_{\rm H}^{(0)})\rangle_{\rm H} \\
&\qquad\qquad \text{(by (53))} \\
&= \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - \langle x_{\rm H}, \psi_{\rm H}(x_{\rm H}^{(0)})\rangle_{\rm H} \\
&= \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - \langle x_{\rm H}^{(0)}, tf_{\rm H}^{(0)} + \psi_{\rm H}(x_{\rm H})\rangle_{\rm H} - \langle x_{\rm H}, \psi_{\rm H}(x_{\rm H}^{(0)})\rangle_{\rm H} \\
&\qquad\qquad \text{(by (55))} \\
&= \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - t\langle x_{\rm H}^{(0)}, f_{\rm H}^{(0)}\rangle_{\rm H} - \langle x_{\rm H}^{(0)}, \psi_{\rm H}(x_{\rm H})\rangle_{\rm H} - \langle x_{\rm H}, \psi_{\rm H}(x_{\rm H}^{(0)})\rangle_{\rm H} \\
&\geq \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - t\langle x_{\rm H}^{(0)}, f_{\rm H}^{(0)}\rangle_{\rm H} - \langle x_{\rm H}, \psi_{\rm H}(x_{\rm H})\rangle_{\rm H} - \langle x_{\rm H}^{(0)}, \psi_{\rm H}(x_{\rm H}^{(0)})\rangle_{\rm H} \\
&\qquad\qquad \text{(by the monotonicity of } \psi_{\rm H}) \\
&= \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - t\langle x_{\rm H}^{(0)}, f_{\rm H}^{(0)}\rangle_{\rm H} \\
&\qquad\qquad \text{(by (i) of Theorem 5.4)} \\
&= \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - t\langle x_{\rm H}^{(0)}, y_{\rm H}^{(0)} - \psi_{\rm H}(x_{\rm H}^{(0)})\rangle_{\rm H} \\
&\qquad\qquad \text{(by (53))} \\
&= \langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} - t\langle x_{\rm H}^{(0)}, y_{\rm H}^{(0)}\rangle_{\rm H} \\
&\qquad\qquad \text{(by (i) of Theorem 5.4).}
\end{aligned}$$

In addition, for the same $t \in (0,1]$, we have

$$\begin{aligned}
t\langle x_{\rm H}, f_{\rm H}^{(0)}\rangle_{\rm H} &= \langle x_{\rm H}, tf_{\rm H}^{(0)}\rangle_{\rm H} \\
&= \langle x_{\rm H}, y_{\rm H} - \psi_{\rm H}(x_{\rm H})\rangle_{\rm H} \qquad \text{(by (55))} \\
&= \langle x_{\rm H}, y_{\rm H}\rangle_{\rm H} \qquad \text{(by (i) of Theorem 5.4)} \\
&= \mathrm{tr}(te_{\rm H}) \qquad \text{(by (55))} \\
&= t(r+1) \qquad \text{(by (54))} \\
&= t\langle x_{\rm H}^{(0)}, y_{\rm H}^{(0)}\rangle_{\rm H}.
\end{aligned}$$

Therefore, we obtain that

$$\langle x_{\rm H}, y_{\rm H}^{(0)}\rangle_{\rm H} + \langle y_{\rm H}, x_{\rm H}^{(0)}\rangle_{\rm H} \leq \langle x_{\rm H}, f_{\rm H}^{(0)}\rangle_{\rm H} + t\langle x_{\rm H}^{(0)}, y_{\rm H}^{(0)}\rangle_{\rm H}$$

$$
\begin{aligned}
&= \langle x_{\rm H}^{(0)}, y_{\rm H}^{(0)} \rangle_{\rm H} + t \langle x_{\rm H}^{(0)}, y_{\rm H}^{(0)} \rangle_{\rm H} \\
&= (1+t) \langle x_{\rm H}^{(0)}, y_{\rm H}^{(0)} \rangle_{\rm H} \\
&= (1+t)(r+1) \leq 2(r+1).
\end{aligned}
$$

Thus, by Proposition 2.2, the set P is bounded.
(iii): Let $(x_{\rm H}^*, y_{\rm H}^*) = (x^*, \tau^*, y^*, \kappa^*)$ be an asymptotical solution for (HCP). Then there exists a bounded sequence

$$
\{(x_{\rm H}^{(k)}, y_{\rm H}^{(k)})\} = \{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k)\} \subset {\rm int} K_{\rm H} \times {\rm int} K_{\rm H}
$$

such that

$$
\lim_{k \to \infty} (x_{\rm H}^{(k)}, y_{\rm H}^{(k)}) = (x_{\rm H}^*, y_{\rm H}^*), \ \lim_{k \to \infty} y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)}) = 0 \ \text{ and } \ \lim_{k \to \infty} x_{\rm H}^{(k)} \circ_{\rm H} y_{\rm H}^{(k)} = 0.
$$

Let $(x_{\rm H}(t), y_{\rm H}(t)) = (x(t), \tau(t), y(t), \kappa(t))$ be any point on the path P. Then,

$$
x_{\rm H}(t) \circ_{\rm H} y_{\rm H}(t) = t e_{\rm H} \ \text{ and } \ y_{\rm H}(t) - \psi_{\rm H}(x_{\rm H}(t)) = t f_{\rm H}^{(0)}. \tag{56}
$$

By the boundedness of the set P as we have seen in (ii) above, there exists an $\epsilon \in (0, 1]$ such that

$$
\|x_{\rm H}(t)\| \leq 1/\epsilon \ \text{ and } \ \|y_{\rm H}(t)\| \leq 1/\epsilon \tag{57}
$$

holds for every $t \in (0, 1]$. In addition, for each $t \in (0, 1]$, there exists an index $k(t)$ such that for every $k \geq k(t)$, we have

$$
\|x_{\rm H}^{(k)} - x_{\rm H}^*\| \leq \epsilon, \ \ \|y_{\rm H}^{(k)} - y_{\rm H}^*\| \leq \epsilon \text{ and } \|y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)})\| \leq t\epsilon. \tag{58}
$$

Here, by the monotonicity of $\psi_{\rm H}$,

$$
\begin{aligned}
&\langle x_{\rm H}(t) - x_{\rm H}^{(k)}, y_{\rm H}(t) - y_{\rm H}^{(k)} \rangle_{\rm H} \\
&= \langle x_{\rm H}(t) - x_{\rm H}^{(k)}, \psi_{\rm H}(x_{\rm H}) - \psi_{\rm H}(x_{\rm H}^{(k)}) \rangle_{\rm H} \\
&\qquad + \langle x_{\rm H}(t) - x_{\rm H}^{(k)}, y_{\rm H}(t) - \psi_{\rm H}(x_{\rm H}(t)) \rangle_{\rm H} - \langle x_{\rm H}(t) - x_{\rm H}^{(k)}, y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)}) \rangle_{\rm H} \\
&\geq \langle x_{\rm H}(t) - x_{\rm H}^{(k)}, y_{\rm H}(t) - \psi_{\rm H}(x_{\rm H}(t)) \rangle_{\rm H} - \langle x_{\rm H}(t) - x_{\rm H}^{(k)}, y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)}) \rangle_{\rm H}
\end{aligned}
$$

and hence, for every $t \in (0, 1]$ and every $k \geq k(t)$,

$$
\begin{aligned}
&\langle x_{\rm H}(t), y_{\rm H}^{(k)} \rangle_{\rm H} + \langle y_{\rm H}(t), x_{\rm H}^{(k)} \rangle_{\rm H} \\
&\leq \langle x_{\rm H}(t), y_{\rm H}(t) \rangle_{\rm H} + \langle x_{\rm H}^{(k)}, y_{\rm H}^{(k)} \rangle_{\rm H} \\
&\qquad - \langle x_{\rm H}(t) - x_{\rm H}^{(k)}, y_{\rm H}(t) - \psi_{\rm H}(x_{\rm H}(t)) \rangle_{\rm H} \\
&\qquad + \langle x_{\rm H}(t) - x_{\rm H}^{(k)}, y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)}) \rangle_{\rm H} \\
&= \langle x_{\rm H}(t), y_{\rm H}(t) \rangle_{\rm H} + \langle x_{\rm H}^{(k)}, y_{\rm H}^{(k)} \rangle_{\rm H} \\
&\qquad - \langle x_{\rm H}(t), y_{\rm H}(t) - \psi_{\rm H}(x_{\rm H}(t)) \rangle_{\rm H} - \langle x_{\rm H}^{(k)}, y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)}) \rangle_{\rm H} \\
&\qquad + \langle x_{\rm H}^{(k)}, y_{\rm H}(t) - \psi_{\rm H}(x_{\rm H}(t)) \rangle_{\rm H} + \langle x_{\rm H}(t), y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)}) \rangle_{\rm H}
\end{aligned}
$$

$$
\begin{aligned}
&= \langle x_{\rm H}^{(k)}, y_{\rm H}(t) - \psi_{\rm H}(x_{\rm H}(t))\rangle_{\rm H} + \langle x_{\rm H}(t), y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)})\rangle_{\rm H} \\
&\qquad\qquad\qquad \text{(by (i) of Theorem 5.4)} \\
&= \langle x_{\rm H}^{(k)}, t f_{\rm H}^{(0)}\rangle_{\rm H} + \langle x_{\rm H}(t), y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)})\rangle_{\rm H} \\
&\qquad\qquad\qquad \text{(by (56))} \\
&\le t\|x_{\rm H}^{(k)}\|\|f_{\rm H}^{(0)}\| + \|x_{\rm H}(t)\|\|y_{\rm H}^{(k)} - \psi_{\rm H}(x_{\rm H}^{(k)})\| \\
&\le t(\|x_{\rm H}^{*}\| + \epsilon)\|h_{\rm H}^{(0)}\| + t \\
&\qquad\qquad\qquad \text{(by (57) and (58))} \\
&\le t\delta
\end{aligned}
$$

where $\delta := 1 + (\|x_{\rm H}^{*}\| + 1)\|h_{\rm H}^{(0)}\| > 0$. Note that (56) implies

$$x_{\rm H}(t) = t y_{\rm H}(t)^{-1} \text{ and } y_{\rm H}(t) = t x_{\rm H}(t)^{-1}.$$

Combining the above relations, it must hold that for every $t \in (0, 1]$ and $k \ge k(t)$

$$
\begin{aligned}
t\delta &\ge \langle x_{\rm H}(t), y_{\rm H}^{(k)}\rangle_{\rm H} + \langle y_{\rm H}(t), x_{\rm H}^{(k)}\rangle_{\rm H} \\
&= \langle t y_{\rm H}(t)^{-1}, y_{\rm H}^{(k)}\rangle_{\rm H} + \langle t x_{\rm H}(t)^{-1}, x_{\rm H}^{(k)}\rangle_{\rm H} \\
&= t\left\{ \langle y(t)^{-1}, y^{(k)}\rangle + \frac{\kappa_k}{\kappa(t)} + \langle x(t)^{-1}, x^{(k)}\rangle + \frac{\tau_k}{\tau(t)} \right\}.
\end{aligned}
$$

Since $\langle y(t)^{-1}, y^{(k)}\rangle > 0$ and $\langle x(t)^{-1}, x^{(k)}\rangle > 0$, we finally obtain that

$$\frac{\kappa_k}{\kappa(t)} < \delta \quad \text{and} \quad \frac{\tau_k}{\tau(t)} < \delta$$

for every $t \in (0, 1]$ and $k \ge k(t)$. Thus, the assertion (iii) follows from the facts $\kappa_k \to \kappa^*$, $\tau_k \to \tau^*$ and $\delta > 0$. ∎

6 Concluding remarks

In this paper, we studied the homeomorphism of the interior point map defined by (4) for monotone complementarity problems over symmetric cones associated with Euclidean Jordan algebras. As an application of our results, we provided a homogeneous model for the problems and showed the existence of a trajectory. We also showed that if the algebra is simple then any limit point of the trajectory gives us a certification of the strong feasibility or the strong infeasibility.

While there remain many issues to be investigated, we would like to mention three of them below:

- development of numerical algorithms and their complexity analyses,

- extension of the results for more general complementarity problems, e.g., the mixed complementarity problems for which the map F is given by

$$F(x, y, z) = \begin{pmatrix} y - \psi_1(x, z) \\ \psi_2(x, z) \end{pmatrix}$$

for some continuous mapping $\psi := (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$,

- providing optimization problems whose local optimality condition can be stated as a complementarity problem over symmetric cones.

Recently, the concept of P-properties for linear and/or nonlinear transformations on Euclidean Jordan algebra was introduced by Gowda, Sznajder and Tao [8] and by Tao and Gowda [26], aiming to provide non-monotone properties on the algebra. For the case of the n-dimensional positive orthant, a homogeneous model for P_0 complementarity problems has been proposed in [27]. In contrast to the results in this paper, however, the lack of the monotonicity of the map F prevents us to show that the image of the map F is convex, which corresponds to our result, Theorem 3.12. As we have seen in the theorem, the convexity of the image of F is a key property to obtain a certificate proving strong infeasibility of the original problem. It may be another interesting issue to determine whether the property can be obtained even if the Jordan algebra is not simple, rather than for non-monotone cases.

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