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Regret region hypothesis testing—Application to the
left-retracted exponential distribution

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Summary.

In this paper we introduce a new optimal hypothesis testing with the regret region derived from inverting the minimum random interval based on the unbiased statistics for the parameter. Let T be the test statistic. The name of the regret region D comes from the fact that we regret not rejecting the null hypothesis H_0 if $T \in D$ and satisfy to reject H_0 if $T \notin D$. We call above test regret region hypothesis testing. As a reference concerning this topics we may refer to Tate *et. al.*(1959).

As an example we consider hypothesis testing for two parameters of the left-retracted exponential distribution. The author first considered a family of retracted distributions in Nogami(1982, 1985).

Keywords: hypothesis testing, minimum random interval, regret region, regret-probability function, retracted distributions, unbiased statistics, uniformly least regret test.

1. Introduction.

Let X_1, \dots, X_n be a random sample of size n taken from the density $f(x|\theta)$ with a real parameter θ . Let $T(X_1, \dots, X_n)$ be a statistic for θ . Let $:=$ be the defining property defining the right hand side of the equality by the left hand side of equality. We call $T(X_1, \dots, X_n)$ ($=:T$) an unbiased statistic for θ when $E(T)=\theta$. In this paper we first, in Sections 2, 3 and 4, consider two-edged test for one parameter to test, for example, the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta \neq \theta_0$ (with known real θ_0). In Section 5 we consider a two-edged test for two parameters. In Sections 6 and 7 we consider one-edged tests.

Let α be a real number such that $0 < \alpha < 1$. We call (U_1, U_2) a α random interval for the parameter γ if $P_\gamma[U_1 < \gamma < U_2] = \alpha$. To get the two-edged test of size $1-\alpha$ we find the minimum α random interval (U_1, U_2) for θ to minimize the

length $U_2 - U_1$ of the interval and construct the regret region derived from inverting this random interval for θ_0 . When we are doing so, we found that the minimum random intervals based on unbiased statistics for θ worked well (see Nogami(2001a)). So we name this test as the regret region hypothesis testing. We define the regret-probability function $\psi(\theta)$ by $\psi(\theta) := P_\theta(D)$, $\forall \theta$. Our test is unbiased of size $1 - \alpha$ when $\psi(\theta)$ is a concave function from below and maximizes $\psi(\theta)$ at θ_0 with $\psi(\theta_0) = \alpha$. (i.e. $\psi(\theta) \leq \psi(\theta_0) = \alpha$, $\forall \theta$) Furthermore, if the test with the regret region D_* has the property $P_\theta(D_*) \leq P_\theta(D)$, $\forall D$ and $\forall \theta$, then we call the test D_* to be the uniformly least regret test. To distinguish between this test and other tests like the likelihood-ratio test we needed new notations as above.

As the example we consider the left-retracted exponential distribution with the density

$$f(x|\theta, b) = b^{-1} e^{-(x-\theta)/b}, \quad \text{for } \theta < x \quad (1)$$

where $-\infty < \theta < \infty$ and $b > 0$. Author has been previously worked for a family of retracted distributions (see Nogami(1982, 1985)) under certain estimation problems.

Part of the properties of such family is also studied in Nogami(2001c). In Sections 2 and 6 we treat the tests for θ with known b and in Sections 3 and 7 we deal with the tests for θ with unknown b . In Section 4 we consider two-edged test for b . In Section 5 we introduce the two-edged test for (θ, b) .

Based on i.i.d. observations X_1, \dots, X_n from (1) we consider, in Sections 2 and 3 to test the null hypothesis $H_0: \theta = \theta_0$ versus the alternative hypothesis $H_1: \theta \neq \theta_0$. In Section 2, to obtain the optimal two-edged test we find the minimum random interval for θ , using the unbiased statistic $Y := \bar{X} - b$ ($= n^{-1} \sum_{i=1}^n X_i - b$) for θ and construct the regret region derived from inverting this random interval for θ_0 . In Section 3 we deal with the two-edged test for θ with unknown b . In Section 4 we consider the two-edged test for b . Let b_0 be a positive constant. In Section 5 we introduce the two-edged test for testing the hypotheses $H_0: \theta = \theta_0, b = b_0$ versus H_1 : At least one equality in H_0 fails. In Section 6 we derive the optimal one-edged test for testing the hypothesis $H_0': \theta \geq \theta_0$ versus the alternative hypothesis $H_1': \theta < \theta_0$, with known b . In Section 7 we deal with two one-edged tests for testing the hypotheses H_0' versus H_1' and testing the hypotheses $H_0'': \theta \leq \theta_0$ versus $H_1'': \theta > \theta_0$ when b is unknown.

2. The Two-edged test for θ with known b .

Let X_1, \dots, X_n be a random sample of size n taken from (1) with known b .

We consider the problem of testing the hypotheses $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$. We first derive necessary distributions to find the minimum random interval for θ .

As the statistic for θ we take $Y = \bar{X} - b$. We can easily check $E(Y) = \theta$. Let $X_{(1)}$ be the i -th smallest observation such that $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$. We find the joint density of variables $W = X_{(1)} + \dots + X_{(n)} (=X_1 + \dots + X_n)$, $V = X_{(1)}$, $Z_2 = X_{(2)}$, \dots , $Z_{n-1} = X_{(n-1)}$ as follows:

$$g(w, v, z_2, z_3, \dots, z_{n-1} | \theta) = \begin{cases} n! b^{-n} e^{-(w-n\theta)/b}, & \\ \quad \text{for } \theta \leq v \leq z_2 \leq z_3 \leq \dots \leq z_{n-1} \leq w - v - \sum_{i=2}^{n-1} z_i, & \\ 0, & \text{otherwise.} \end{cases}$$

Integrating out z_2 through z_{n-1} from the above density we get the joint density of (W, V) as follows:

$$g(w, v | \theta) = \int_v^w \int_{z_{n-3}}^{(w-v-\sum_{i=2}^{n-3} z_i)/3} \int_{z_{n-2}}^{(w-v-\sum_{i=2}^{n-2} z_i)/2} \dots \int g(w, v, z_2, \dots, z_{n-1} | \theta) dz_{n-1} dz_{n-2} \dots dz_2$$

$$= \begin{cases} (n/\Gamma(n-1)) b^{-n} e^{-(w-n\theta)/b} (w-nv)^{n-2}, & \text{for } \theta \leq v \leq w/n (< \infty) \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Taking the marginal density of W and furthermore, letting $t := 2(w-n\theta)/b$ we have the density of T so that

$$h_n(t) = (1/\Gamma(n)) e^{-t/2} t^{n-1} 2^{-n}, \quad \text{for } 0 \leq t; = 0, \text{ for } t < 0. \quad (3)$$

which is the chi-square density with $2n$ degrees of freedom.

Let r_1 and r_2 be real numbers such that $r_1 < r_2$. To find the minimum α random interval for θ we want to minimize $r_2 - r_1$ subject to

$$P_\theta[r_1 < Y - \theta < r_2] = \alpha. \quad (4)$$

But, by the transformation $t=2n(y+b-\theta)/b$ (4) is equivalent to

$$P[t_1 < T < t_2] = \alpha \quad (5)$$

where $t_i := 2n(r_i + b)/b$ for $i=1, 2$. Hence, we want to minimize $t_2 - t_1$ subject to the condition (5). Let λ be a Lagrange's multiplier and define

$$L = t_2 - t_1 - \lambda \left\{ \int_{t_1}^{t_2} h_n(t) dt - \alpha \right\}. \quad (6)$$

Then, $\partial L / \partial t_1 = 0 = \partial L / \partial t_2$ leads to

$$h_n(t_1) = h_n(t_2) \quad (= \lambda^{-1}). \quad (7)$$

Taking t_1 and t_2 which satisfy (7) and $\partial L / \partial \lambda = 0$, noticing that $t_1 < T := 2n(Y+b-\theta)/b < t_2$ and letting $t_3 := bt_1/(2n)$ and $t_4 := bt_2/(2n)$ we obtain the minimum α random interval for θ as follows:

$$(Y+b-t_4, Y+b-t_3). \quad (8)$$

Hence, by inverting (8) for θ_0 our regret region based on (W, V) becomes

$$\{(W, V) : \theta_0 + t_3 - b < Y < \theta_0 + t_4 - b \text{ and } V \geq \theta_0\}$$

where $Y = n^{-1}W - b$. Here, we emphasize the necessity of having the set $\{V \geq \theta_0\}$ in the regret region.

To check the unbiasedness of this test we obtain the regret-probability function of the test as follows:

$$\phi(\theta) = P_\theta[\theta_0 + t_3 - b < Y < \theta_0 + t_4 - b \text{ and } V \geq \theta_0]$$

$$= \begin{cases} \alpha \exp\{-n(\theta_0 - \theta)/b\}, & \text{for } \theta < \theta_0 \\ P[t_1 - 2n(\theta - \theta_0)/b < T < t_2 - 2n(\theta - \theta_0)/b], & \text{for } \theta_0 \leq \theta < \theta_0 + t_3 \\ P[0 < T \leq t_2 - 2n(\theta - \theta_0)/b], & \text{for } \theta_0 + t_3 \leq \theta < \theta_0 + t_4 \\ 0, & \text{for } \theta_0 + t_4 \leq \theta. \end{cases}$$

Hence, $d\phi(\theta)/d\theta > 0$ for $\theta < \theta_0$; $d\phi(\theta)/d\theta = 2nb^{-1} \{ h_n(t_1 - 2n(\theta - \theta_0)/b) - h_n(t_2 - 2n(\theta - \theta_0)/b) \}$

< 0 for $\theta_0 \leq \theta < \theta_0 + t_3$ because of (7) and (3), and $d\psi(\theta)/d\theta < 0$ for $\theta_0 + t_3 \leq \theta < \theta_0 + t_4$. Since $\psi'(\theta_0) = 0$ by (7) and $\psi(\theta_0) = \alpha$, we have that $\psi(\theta) \leq \alpha$ for real θ . Thus, This test is unbiased of size $1 - \alpha$.

In the next section we consider the two-edged test with unknown b .

3. The two-edged tests for θ with unknown b .

In this section we let b be unknown. We test the hypotheses $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$. Let W and V be as defined in Section 2. Since $(W - nV)/(n-1)$ is an unbiased statistic for b , we use the statistic

$$Z := (n-1)(n^{-1}W - \theta)/(W - nV)$$

and furthermore the test statistic

$$S := \{\log_e(n/(n-1)) + \log_e Z\}^{1/2} \quad (9)$$

whose density has one peak. Since from (2) and a little calculation, the p.d.f. of Z is obtained by

$$g_Z(z) = (n-1)^n / \{n^{n-1} z^n\}, \quad \text{for } (n-1)/n \leq z; = 0, \text{ otherwise,} \quad (10)$$

from the transformation (9) applied to (10) we obtain the density of S as follows:

$$g_S(s) = 2(n-1)\{\exp\{-(n-1)s^2\}\}s, \quad \text{for } 0 \leq s; = 0, \text{ for } s < 0. \quad (11)$$

Let s_1 and s_2 be positive numbers such that $s_1 < s_2$,

$$\int_{s_1}^{s_2} g_S(s) ds = \alpha$$

and

$$g_S(s_1) = g_S(s_2).$$

For such s_1 and s_2 we have that $s_1 < S < s_2$ or $\exp\{s_1^2\} < (W - n\theta)/(W - nV) < \exp\{s_2^2\}$.

Hence, the minimum α random interval for θ is

$$(n^{-1}(W-(W-nV)\exp\{s_2^2\}), n^{-1}(W-(W-nV)\exp\{s_1^2\})). \quad (12)$$

Thus, our regret region based on (W, V) for testing H_0 versus H_1 is as follows:

$$\{(W, V): n^{-1}(W-(W-nV)\exp\{s_2^2\}) < \theta_0 < n^{-1}(W-(W-nV)\exp\{s_1^2\}) \text{ and } \theta_0 \leq V\}.$$

In the next section we show two-edged tests for b .

4. The two-edged tests for b .

In this section we consider to test the hypotheses $H_0: b=b_0$ versus $H_1: b \neq b_0$.

We first assume that θ is known. Let T , W and V be as defined in Section 2. Namely, we use the test statistic $T:=2(W-n\theta)/b$. From Section 2 the density of T is given by (3). Taking positive numbers t_1 and t_2 which satisfy (5), (6), (7) and $t_1 \leq T \leq t_2$ we get the minimum α random interval for b as follows:

$$(2(W-n\theta)/t_2, 2(W-n\theta)/t_1) \quad (13).$$

By inverting (13) for b_0 our regret region based on W of size $1-\alpha$ becomes

$$n\theta + (b_0 t_1)/2 < W < n\theta + (b_0 t_2)/2.$$

Secondly, we assume that θ is unknown. We consider another test based on the statistic $Q:=2(W-nV)/b$. From (2) and a little computation we get the density $h_{n-1}(q)$ of Q where $h_n(q)$ is given by (3). Hereafter, we proceed the same way as above. Taking positive numbers t_1 and t_2 which satisfy (5), (6) and (7) with T and n there replaced by Q and $n-1$, respectively we obtain the minimum random interval for b as follows:

$$(2(W-nV)/t_2, 2(W-nV)/t_1). \quad (14)$$

Thus, by inverting (14) for b_0 the regret region of our test is as follows:

$$\{(W, V): 2(W-nV)/t_2 < b_0 < 2(W-nV)/t_1\}.$$

In the next section we introduce the two-edged test for (θ, b) .

5. The two-edged test for (θ, b) .

In this section we consider to test the hypothesis $H_0: \theta = \theta_0, b = b_0$ versus the alternative hypothesis H_1 : At least one equality in H_0 fails. Let W and V be as defined in Section 2. We let $S := (W-nV)^{1/2}$. To get the two-edged test for (θ, b) we use the statistic

$$Z := 2\sqrt{n-1}(n^{-1}W - \theta) / (\sqrt{b}S).$$

We first find the density of Z , get the minimum random interval for (θ, b) and from this random interval obtain the regret region based on (W, V) for the two-edged test with respect to (θ, b) .

We first find the density of Z and show that this density of Z has one peak. Let $U := n(V-\theta)/S$. To get the density of Z we first find the density of (S, V) from (2) of (W, V) , then the density of (S, U) from the density of (S, V) and finally the density of (S, Z) with the transformation $Z = \{2\sqrt{n-1}/(n\sqrt{b})\}(S+U)$ from the density of (S, U) . Obtained density of (S, Z) is as follows:

$$h_{S, Z}(s, z) = \begin{cases} (n/\{b^{n-1/2} \Gamma(n-1)\sqrt{n-1}\}) s^{2n-2} \exp\{-nzs/(2\sqrt{(n-1)b})\}, & (15) \\ 0, & \text{for } 0 \leq \{2\sqrt{n-1}s/(n\sqrt{b})\} \leq z \\ & \text{otherwise.} \end{cases}$$

Let

$$H_{2n-1}(z) := \int_{-\infty}^z h_{2n-1}(t) dt, \quad \forall z.$$

where $h_{2n-1}(z)$ is of form (3) with n there replaced by $2n-1$.

Integrating out s from (15) we get the density of z as follows:

$$h_z(z) = cz^{-(2n-1)} H_{2n-1}(n^2 z^2 / (2(n-1))), \quad \forall z \geq 0; = 0, \quad \forall z < 0,$$

where

$$c := (n-1)^{n-1} 2^{2n-1} \Gamma(2n-1) / \{n^{2n-2} \Gamma(n-1)\}.$$

Since $h_z(0) = 0 = h_z(-\infty)$, we show that there exists unique $z (> 0)$ such that $h_z'(z) = 0$. Now,

$$h_z'(z) = c \{ -(2n-1) z^{-2n} H_{2n-1}(n^2 z^2 / (2(n-1))) + (n^2 / (n-1)) z^{-(2n-2)} h_{2n-1}(n^2 z^2 / (2(n-1))) \}.$$

Let $\xi(z) := (n^2 z^2 / \{(n-1)(2n-1)\}) h_{2n-1}(n^2 z^2 / (2(n-1))) = c_1 z^2 h_{2n-1}(n^2 z^2 / (2(n-1)))$. We would like to show that there exists unique z such that $\xi(z) = H_{2n-1}(n^2 z^2 / (2(n-1)))$. Since

$$\xi'(z) = c_1 z \{ 4n - 2 - n^2 z^2 / (2(n-1)) \} h_{2n-1}(n^2 z^2 / (2(n-1)))$$

and

$$dH_{2n-1}(n^2 z^2 / (2(n-1))) / dz = (n^2 z / (n-1)) h_{2n-1}(n^2 z^2 / (2(n-1))),$$

we have that $\xi'(z) > dH_{2n-1}(n^2 z^2 / (n-1)) / dz (> 0)$ for $0 < z < \sqrt{2(2n-1)(n-1)} / n$. We also have that $\xi'(z) > 0$ for $0 < z < 2\sqrt{(2n-1)(n-1)} / n (= c_2)$; < 0 for $c_2 < z$ and $\xi(0) = H_{2n-1}(0) = 0$. Thus, there exists unique $z (> 0)$ such that $h_z'(z) = 0$. Therefore, $h_z(z)$ is the density with one peak.

To find the minimum α random interval for (θ, b) we let r_1 and r_2 be positive numbers such that $r_1 < r_2$ and want to minimize $r_2 - r_1$ subject to

$$P_\theta [r_1 < Z < r_2] = \alpha. \tag{16}$$

In the same fashion as (5), (6) and (7) we obtain

$$h_z(r_1) = h_z(r_2). \tag{17}$$

Hence, the minimum α random interval for (θ, b) is as follows:

$$\{(\theta, b): r_1 < 2\sqrt{n-1}(n^{-1}W-\theta)/\sqrt{b(W-nV)} < r_2\},$$

where r_1 and r_2 satisfy (16) and (17). Thus, our regret region based on (W, V) becomes as follows:

$$\{(W, V): r_1 < 2\sqrt{n-1}(n^{-1}W-\theta_0)/\sqrt{b_0(W-nV)} < r_2 \text{ and } \theta_0 \leq V\}.$$

In the next two sections we deal with the one-edged tests.

6. The one-edged test for θ with known b .

In this section we consider to test the hypotheses $H_0': \theta \geq \theta_0$ versus $H_1': \theta < \theta_0$ based on a random sample X_1, \dots, X_n from (1) with known b . Let Y, W, V and T be as defined in Section 2. Let $t_6 := bt_5/(2n)$ where t_5 is given by

$$\int_{t_5}^{\infty} h_n(t) dt = \alpha.$$

Here, $h_n(t)$ is given by (3). From Section 2 we can easily see that our test has the following regret region based on (W, V) :

$$\{(W, V): \theta_0 + t_6 - b < Y \text{ and } \theta_0 \leq V\}$$

where $Y := n^{-1}W - b$. Let $g_{Y, V}(y, v|\theta)$ be the joint density of (Y, V) . Then, from (2) and the relation $g_{Y, V}(y, v|\theta) = g(n(y+b), v|\theta)n$ we can easily get the regret-probability function of above test as follows:

$$\psi(\theta) = P_{\theta}[\theta_0 + t_6 - b < Y \text{ and } \theta_0 \leq V]$$

$$\psi(\theta) = \begin{cases} \alpha \exp\{-n(\theta_0 - \theta)/b\}, & \text{for } \theta < \theta_0 \\ P[t_5 - 2n(\theta - \theta_0)/b \leq T], & \text{for } \theta_0 \leq \theta < \theta_0 + t_6 \\ 1, & \text{for } \theta_0 + t_6 \leq \theta. \end{cases}$$

Since $d\phi(\theta)/d\theta > 0$ for $\theta < \theta_0 + t_s$ and hence $\phi(\theta) \leq \alpha = \phi(\theta_0)$ for $\theta < \theta_0$, our test is unbiased of size $1-\alpha$. Also, from Lehmann(1986, Problem 3(ii) in p. 112) this test is uniformly least regret test.

In the next section we consider one-edged tests for θ with unknown b .

7. The one-edged tests for θ with unknown b .

We consider to test the hypotheses $H_0' : \theta \geq \theta_0$ versus $H_1' : \theta < \theta_0$ based on a random sample X_1, \dots, X_n from (1) with unknown b . Let W and V be as in Section 2. Let S be as defined in Section 3.

Let r_1 be a positive number such that

$$\int_{r_1}^{\infty} g_s(s) ds = \alpha$$

where $g_s(s)$ is given by (11). From (12) in Section 3 we can easily see that the minimum α random interval for θ is obtained as follows:

$$(-\infty, n^{-1}(W - \exp\{r_1^2\}(W - nV))).$$

Hence, our regret region based on (W, V) of size $1-\alpha$ is as follows:

$$\{(W, V) : \theta_0 < n^{-1}(W - \exp\{r_1^2\}(W - nV)) \text{ and } \theta_0 \leq V\}.$$

We next consider to test the null hypothesis $H_0'' : \theta \leq \theta_0$ versus the alternative hypothesis $H_1'' : \theta > \theta_0$. Let r_2 be a positive number such that

$$\int_0^{r_2} g_s(s) ds = \alpha.$$

From (13) in Section 3 the minimum α random interval for θ is as follows:

$$(n^{-1}(W - \exp\{r_2^2\}(W - nV)), \infty).$$

Hence, our regret region based on (W, V) of size $1-\alpha$ is as follows:

$$\{(W, V) : n^{-1}(W - \exp\{r_2^2\}(W - nV)) < \theta_0\}.$$

8. Remark.

This paper is correction and refinement of Nogami(2003a, 2003c) and part of Nogami(2002a). Besides this paper we already presented the topics concerning regret region hypothesis testing in Nogami(2001b, 2002b, 2003b, 2003c).

In this paper we considered $\sup_{\theta \in H_0} (1 - \phi(\theta)) = 1 - \alpha$ as the size of the test.

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