CUSTOMERS SELECTION PROBLEM WITH IDLING PROFIT

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Abstract

This paper deals with a decision problem on whether or not to accept orders from customers sequentially arriving at a custom production company where an idling profit is yielded by manufacturing products with standard specifications when no backorder exists. We discuss the admission control problem and pricing control problem in an identical framework. Properties of the optimal decision rule maximizing the total expected present discounted net profit gained over an infinite planning horizon are examined and clarified. It is shown that when the idling profit is great, the optimal policies may not be monotone in the number of orders in the system.

1 Introduction

This paper deals with the problem of selecting which profitable orders to accept out of customers sequentially arriving in a custom production company, such as a shipbuilding company, advertising agency, consulting company, design office, construction firm, and so on. In discussions on the problem, the two kinds of opportunity loss described below should be taken into consideration:

1. *Opportunity loss I* . Suppose that orders from all arriving customers are accepted irrespective of their profitabilities. In this case, the production process soon becomes full; with the result that orders from customers arriving thereafter can not be accepted however high their profitabilities may be. This leads to an opportunity loss that if adequate allowance were kept in the production lines by having rejected less profitable orders in advance, the company could have enjoyed upcoming profitable orders. We shall refer to this loss as Opportunity loss I.
2. *Opportunity loss II* . Excessively refraining from accepting orders due to the apprehension that Opportunity loss I could occur causes a reduced number of backorders. This time, the production process soon becomes idle, implying the opportunity loss that if more orders had been accepted in advance, the profit could have been gained from them. We shall refer to this loss as Opportunity loss II.

Both Opportunity losses cause a diminishment in the long run profit. The objective therefore is to find an optimal customers selection rule so as to maximize an expected long run profit through keeping an appropriate level of backorders by controlling the number of orders to accept in advance with the aim to avoid Opportunity losses I and II. This problem is usually called the customers selection problem.

This class of problems has been studied as the admission control problem and the pricing control problem. In the former, a customer offers the price for his order, and judging from this, the company decides whether or not to accept. In the latter, by contrast, the system offers a price for an order, and judging from this, the customer decides whether or not to place an order with the company.

Optimal policies in the admission control problem were originally considered by Heyman [2]. This was later applied to the queueing system with a finite customer class and finite-capacity by Miller [10] and continued by Lippman and Ross [7] for a single-server with uncountable customer classes. In [3] a model for a discrete-time process was formulated by Ikuta. Optimal pricing policies in pricing control were discussed by Low [8], more recently by You [15] and Feng and Xiao [1] for yield management, and by Fu et al. [9] for queueing staffing problem. These two problems, i.e., the admission control and the pricing control problems are separately formulated and analyzed in Yoon and Lewis's [14], which is the latest report. In our paper we show that both problems can be treated in an identical framework(see [5, Ikuta] for a general discussion on the integration of admission control problem and pricing contol problem) and prove that the optimal policies are not always monotone in the number of backorders through theoretical analysis (see Figure 8.2).

We introduce the *idling profit*, which is not taken into account in any other papers. The idling profit is yielded when there is no backorder in the system. For example, consider a custom production company manufacturing products with general specifications and with special ones; let us refer to the production of products with general specifications as standard production, and that of products with special specifications as custom production. If all the products with special specifications that have been accepted so far have been completed and the production process has become idle, they shift custom production to standard production, which yields a idling profit. In this paper we examine the problem mainly focusing on the relationship of the optimal policies with the idling profit. Furthermore, we introduce a search cost. The search cost is paid to search for a customer where without paying a search cost at a point in time, no customer arrives at the next point in time. The introduction of the search cost inevitably yields the option of *skipping* the search or not.

The objective here is to find the optimal decision rule so as to maximize the total expected present discounted net profit gained over an infinite planning horizon, the total expected present discounted value of prices of orders accepted whether in the admission control problem or in the pricing control problem *plus* the idling profits *minus* the search costs *minus* the penalty costs.

Section 2 provides a strict definition of the model of the problem treated in the paper. Section 3 defines some functions and examines their properties, and this will be used in the analysis of the subsequent sections. In Section 4 the optimal equation of the model is derived, and in Section 5 it is transformed for convenience of discussion in the subsequent sections. In Section 6 the properties of the optimal decision rule are examined, and these are summarized in Section 7. Section 8 discusses some important aspects of the problem through numerical experiments, and Section 9 considers the practical implications of the results obtained in the above sections and summarizes the conclusions derived, while in Section 10 we suggest some subjects of study to be tackled in the future.

2 Model

The model examined in the paper is defined on the eight assumptions below:

A1. The model is defined as a discrete-time sequential stochastic decision process with an infinite planning horizon. Let points in time be equally spaced on the axis of the planning horizon, and let the time interval between successive points in time be called the period.

A2. It is only when a search is enacted by paying a *search cost* $c \geq 0$ at a point in time that a customer arrives at the next point in time with a probability λ ($0 < \lambda \leq 1$).

A3. Let the prices offered by subsequently appearing customers, $w, w', \cdots$, in the admission control problem and the maximum permissible ordering prices of subsequently appearing customers, $w, w', \cdots$, in the pricing control problem be both independent and identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ. Then, in the pricing control problem, if the system offers a price z to an appearing customer, the probability of the customer placing the order with the system is given by

$$p(z) = \Pr\{z \leq w\}. \tag{2.1}$$

In both the admission control and pricing control problems, for certain given numbers a and b $(0 < a < b < \infty)$ let us define the probability density function as follows;

$$f(w) = 0, \quad w < a, \qquad f(w) > 0, \quad a \le w \le b, \qquad f(w) = 0, \quad b < w \tag{2.2}$$

where clearly $a < \mu < b$. Throughout the paper, let us denote the expectation of a given function $g(w)$ as to w by $\mathbf{E}[g(w)]$.

A4. With a probability q $(0 < q < 1)$ an order in the system at a certain point in time is completed and goes out of the system at the next point in time.

A5. When there exists no backorder in the system, an *idling profit* $s \ge 0$ is yielded by engaging in other economic activities using the idle production line.

A6. Let the discount factor be denoted by $\beta < 1$.

A7. By $n > 1$ let us denote the maximum permissible number of orders which can be held in the system at any instance (A model with $n = 1$ is examined in [13, Son]).

A8. Let us assume the process to start with no backorder.

Here, note that the decision on the problem is based on the following three rules:

1) The rule whether or not to accept an order from arriving customers in the admission control problem.

2) The rule as to the ordering price to offer in the pricing control problem.

3) The rule whether to continue or to skip the search in both problems.

The objective is to find the optimal decision rule so as to maximize the total expected present discounted net profit gained over an infinite planning horizon, the total expected present discounted value of prices of orders accepted or placed *plus* the idling profits *minus* the total expected present discounted value of search costs.

3 Notations and Definitions

For any real number x let us define the following three functions, which will be used to describe the optimal equation. The properties of its optimal decision rule are investigated using their properties, which will be stated in Section 6.1.

$$T(x) = \begin{cases} \mathbf{E}[\max\{w - x, 0\}] & \text{for the admission control problem,} \\ \max_z p(z)(z - x) & \text{for the pricing control problem,} \end{cases} \tag{3.1}$$

$$L(x) = \lambda\beta T(x) - c, \tag{3.2}$$

$$M(x) = L(x) + \beta x,$$

called, respectively, the T-, L-, and M-functions [*]. In pricing control, by $z(x)$ let us designate the z attaining the maximum of $p(z)(z - x)$ on $z \in (-\infty, \infty)$ for a given x if it exists; i.e., $T\big(z(x)\big) = p\big(z(x)\big)\big(z(x) - x\big)$. Note $T(0) > 0$ (See [5]). Further, define

$$\alpha = \lambda\beta T(0) - c, \tag{3.3}$$

$$\gamma = (1 - \beta(1 - q))^{-1} > 1. \tag{3.4}$$

Here, it can be easily shown that

$$1 - \gamma q\beta = \gamma(1 - \beta) > 0. \tag{3.5}$$

For expressional simplicity, we define the following notations seen in Table 3.1.

[*] An example of T-function is shown in App. B, which is used for numerical experiments conducted in Section 8

Table 3.1: Definition of notations

Notation	Definition	Notation	Definition	Notation	Definition
C	continuing the search	$\langle\mathtt{C}\rangle$	Each corresponding decision is optimal	$\langle\mathtt{O}(z)\rangle$	It is optimal to offer the price z for an order in pricing control
K	skipping the search	$\langle\mathtt{K}\rangle$			
A	accepting an order	$\langle\mathtt{A}\rangle$		$\langle\mathtt{A}(w)\rangle$	It is optimal to accept an appearing order w in admission control
R	rejecting an order	$\langle\mathtt{R}\rangle$		$\langle\mathtt{R}(w)\rangle$	It is optimal to reject an appearing order w in admission control

*We do not use S as a notation representing "skipping the search" because it is often used for representing "stopping the search"

4 Optimal Equations

Either if the search was skipped at the previous point in time or if no customer has appeared with probability $1-\lambda$ regardless of having conducted the search at the previous point in time, it follows that no customer appears at the present point in time. For convenience, we shall refer to such a situation as "the system has a *fictitious order* ϕ".

- In both admission control and pricing control, by $u(\phi, i)$ we shall denote the maximum total expected present discounted net profits starting from a state of having the fictitious order ϕ and i ($0 \le i \le n$) orders in the system; let us refer to such a situation as the state (ϕ, i).
- In the admission control problem, by $u(w, i)$ let us denote the maximum total expected present discounted net profit starting with i ($0 \le i < n$) orders in the system and an arriving customer who offers a price w.
- In the pricing control problem, by $u(1, i)$ let us denote the maximum total expected present discounted net profit starting with i ($0 \le i < n$) orders in the system and an arriving customer to whom the system offers a price z.

Here, note that when in state (ϕ, n), even if a customer appears, it can not be accepted due to the assumption of $i \le n$; accordingly, the present state (ϕ, n) remains unchanged at the next point in time if no order in the system is completed with probability $1-q$.

Since the expectation of immediate reward at any point in time is clearly finite, using the conventional way outlined in the discussion of the Markovian decision process [11, Ross](p29-30), we can easily show that $|u(\phi, i)| \le M/(1-\beta)$ for a sufficiently large $M > 0$, i.e., $u(\phi, i)$ is finite. Furthermore, $u(w, i)$ and $u(1, i)$ are also finite. Now, for convenience in the later discussions, let us define

$$h_i = u(\phi, i) - u(\phi, i+1), \quad 0 \le i < n. \tag{4.1}$$

Then the optimal equations for both cases can be described as follows.

1. **Admission control problem**:

$$u(\phi,0) \;=\; \max\begin{cases} \texttt{C}: \beta\big(\lambda\,\mathbf{E}[u(\xi,0)]+(1-\lambda)u(\phi,0)\big)-c+s, \\ \texttt{K}: \beta u(\phi,0)+s, \end{cases} \tag{4.2}$$

$$u(\phi,i) \;=\; \max\begin{cases} \texttt{C}: (1-q)\beta\big(\lambda\,\mathbf{E}[u(\xi,i)]+(1-\lambda)u(\phi,i)\big) \\ \qquad\qquad +q\beta\big(\lambda\,\mathbf{E}[u(\xi,i-1)]+(1-\lambda)u(\phi,i-1)\big)-c, \\ \texttt{K}: (1-q)\beta u(\phi,i)+q\beta u(\phi,i-1), \end{cases} \quad 1\le i<n, \tag{4.3}$$

$$u(\phi,n) \;=\; \max\begin{cases} \texttt{C}: (1-q)\beta u(\phi,n)+q\beta\big(\lambda\,\mathbf{E}[u(\xi,n-1)]+(1-\lambda)u(\phi,n-1)\big)-c, \\ \texttt{K}: (1-q)\beta u(\phi,n)+q\beta u(\phi,n-1), \end{cases} \tag{4.4}$$

$$u(w,i) \;=\; \max\begin{cases} \texttt{A}: w+u(\phi,i+1) \\ \texttt{R}: u(\phi,i) \end{cases} \tag{4.5}$$

$$\;=\; \max\{w-h_i,0\}+u(\phi,i), \quad 0\le i<n. \qquad \square \tag{4.6}$$

2. **Pricing control problem**:

$$u(\phi,0) \;=\; \max\begin{cases} \texttt{C}: \beta\big(\lambda u(1,0)+(1-\lambda)u(\phi,0)\big)-c+s, \\ \texttt{K}: \beta u(\phi,0)+s, \end{cases} \tag{4.7}$$

$$u(\phi,i) \;=\; \max\begin{cases} \texttt{C}: (1-q)\beta(\lambda u(1,i)+(1-\lambda)u(\phi,i)\big) \\ \qquad\qquad +q\beta\big(\lambda u(1,i-1)+(1-\lambda)u(\phi,i-1)\big)-c, \\ \texttt{K}: (1-q)\beta u(\phi,i)+q\beta u(\phi,i-1), \end{cases} \quad 1\le i<n, \tag{4.8}$$

$$u(\phi,n) \;=\; \max\begin{cases} \texttt{C}: (1-q)\beta u(\phi,n)+q\beta\big(\lambda u(1,n-1)+(1-\lambda)u(\phi,n-1)\big)-c, \\ \texttt{K}: (1-q)\beta u(\phi,n)+q\beta u(\phi,n-1), \end{cases} \tag{4.9}$$

$$u(1,i) \;=\; \max_z\{p(z)\big(z+u(\phi,i+1)\big)+(1-p(z))u(\phi,i)\} \tag{4.10}$$

$$\;=\; \max_z p(z)(z-h_i)+u(\phi,i), \quad 0\le i<n. \qquad \square \tag{4.11}$$

See Lemma 6.3 for the unique existence of the solution of the above equations.

5 Transformation of Optimal Equations

Let us define

$$v(i) \;=\; \left\{\begin{array}{ll} \mathbf{E}[u(w,i)] & \text{for the admission control problem} \\ u(1,i) & \text{for the pricing control problem} \end{array}\right\}, \quad 0\le i<n. \tag{5.1}$$

Then since $u(w,i)\ge w$ and $u(1,i)\ge \max_z p(z)z$ from Eqs. (4.5) and (4.10), we obtain, respectively, $\mathbf{E}[u(w,i)]\ge\mu=T(0)$ and $u(1,i)\ge\max_z p(z)z=T(0)$, hence

$$v(i) \;\ge\; T(0), \quad 0\le i<n. \tag{5.2}$$

Then using Eq. (3.1), we can immediately rearrange both Eq. (4.2) to Eq. (4.5) and Eq. (4.7) to Eq. (4.10) into the identical expression below.

$$u(\phi,0) = \max\{\lambda\beta v(0) + (1-\lambda)\beta u(\phi,0) - c, \beta u(\phi,0)\} + s, \tag{5.3}$$

$$u(\phi,i) = \max\left\{\begin{array}{l} (1-q)\beta(\lambda v(i) + (1-\lambda)u(\phi,i)) \\ \qquad +q\beta\big(\lambda v(i-1) + (1-\lambda)u(\phi,i-1)\big) - c, \\ (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1) \end{array}\right\}, \quad 1 \le i < n, \tag{5.4}$$

$$u(\phi,n) = \max\left\{\begin{array}{l} (1-q)\beta u(\phi,n) + q\beta\big(\lambda v(n-1) + (1-\lambda)u(\phi,n-1)\big) - c, \\ (1-q)\beta u(\phi,n) + q\beta u(\phi,n-1), \end{array}\right\}, \tag{5.5}$$

$$v(i) = T(h_i) + u(\phi,i) \quad \text{or equivalently} \quad T(h_i) = v(i) - u(\phi,i), \quad 0 \le i < n. \tag{5.6}$$

Further, Eq. (5.3) to Eq. (5.5) can be rewritten, respectively,

$$u(\phi,0) = \beta u(\phi,0) + \max\{\lambda\beta\big(v(0) - u(\phi,0)\big) - c, 0\} + s, \tag{5.7}$$

$$u(\phi,i) = (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1)$$
$$+ \max\{\lambda(1-q)\beta(v(i) - u(\phi,i)) + \lambda q\beta\big(v(i-1) - u(\phi,i-1)\big) - c, 0\}, \quad 1 \le i < n, \tag{5.8}$$

$$u(\phi,n) = (1-q)\beta u(\phi,n) + q\beta u(\phi,n-1) + \max\{\lambda q\beta\big(v(n-1) - u(\phi,n-1)\big) - c, 0\}, \tag{5.9}$$

which can be immediately rearranged into

$$u(\phi,0) = \big(\max\{\lambda\beta\big(v(0) - u(\phi,0)\big) - c, 0\} + s\big)/(1-\beta), \tag{5.10}$$

$$u(\phi,i) = \gamma q\beta u(\phi,i-1)$$
$$+\gamma\max\{\lambda(1-q)\beta\big(v(i) - u(\phi,i)\big) + \lambda q\beta\big(v(i-1) - u(\phi,i-1)\big) - c, 0\}, \quad 1 \le i < n, \tag{5.11}$$

$$u(\phi,n) = \gamma q\beta u(\phi,n-1) + \gamma\max\{\lambda q\beta\big(v(n-1) - u(\phi,n-1)\big) - c, 0\} \tag{5.12}$$

where γ is defined by Eq. (3.4). Hence, using Eq. (5.6), we can rewrite Eq. (5.10) to and Eq. (5.12) as follows.

$$u(\phi,0) = (\max\{\lambda\beta T(h_0) - c, 0\} + s)/(1-\beta), \tag{5.13}$$

$$u(\phi,i) = \gamma q\beta u(\phi,i-1) + \gamma\max\{\lambda(1-q)\beta T(h_i) + \lambda q\beta T(h_{i-1}) - c, 0\}, \quad 1 \le i < n, \tag{5.14}$$

$$u(\phi,n) = \gamma q\beta u(\phi,n-1) + \gamma\max\{\lambda q\beta T(h_{n-1}) - c, 0\}. \tag{5.15}$$

Further, using *L- function* defined in Eq. (3.2), we can rewrite Eq. (5.13) to Eq. (5.15) as follows.

$$u(\phi,0) = (\max\{L(h_0), 0\} + s)/(1-\beta), \tag{5.16}$$

$$u(\phi,i) = \gamma q\beta u(\phi,i-1) + \gamma\max\{(1-q)L(h_i) + qL(h_{i-1}), 0\}, \quad 1 \le i < n, \tag{5.17}$$

$$u(\phi,n) = \gamma q\beta u(\phi,n-1) + \gamma\max\{qL(h_{n-1}) - (1-q)c, 0\}. \tag{5.18}$$

Below, for convenience, let

$$Q_0 = L(h_0), \tag{5.19}$$

$$Q_i = (1-q)L(h_i) + qL(h_{i-1}), \quad 1 \le i < n, \tag{5.20}$$

$$Q_n = qL(h_{n-1}) - (1-q)c. \tag{5.21}$$

Then Eq. (5.16) to Eq. (5.18) can be rewritten as follows.

$$u(\phi,0) \; = \; (\max\{Q_0,0\}+s)/(1-\beta), \tag{5.22}$$

$$u(\phi,i) \; = \; \gamma q\beta u(\phi,i-1)+\gamma\max\{Q_i,0\}, \quad 1\le i\le n. \tag{5.23}$$

Now, Noting Eqs. (5.23) and (3.5), we can rewrite Eq. (4.1) with $i=0$ as follows.

$$h_0 \; = \; u(\phi,0)-u(\phi,1)=\gamma(1-\beta)u(\phi,0)-\gamma\max\{Q_1,0\}. \tag{5.24}$$

Rearranging Eq. (5.24) by substituting Eq. (5.22) yields

$$h_0 \; = \; \gamma\max\{Q_0,0\}-\gamma\max\{Q_1,0\}+\gamma s. \tag{5.25}$$

Similarly, we obtain

$$h_i \; = \; \gamma q\beta h_{i-1}+\gamma\max\{Q_i,0\}-\gamma\max\{Q_{i+1},0\}, \quad 1\le i< n, \tag{5.26}$$

Regarding h_i as a function of s, let us represent h_i and Q_i by, respectively, $h_i(s)$ and $Q_i(s)$, i.e.,

$$Q_0(s) \; = \; L(h_0(s)), \tag{5.27}$$

$$Q_i(s) \; = \; (1-q)L(h_i(s))+qL(h_{i-1}(s)), \quad 1\le i<n, \tag{5.28}$$

$$Q_n(s) \; = \; qL(h_{n-1}(s))-(1-q)c. \tag{5.29}$$

Here, by s_i let us denote the smallest solution of $Q_i(s)=0$, if it exists, i.e.,

$$s_i \; = \; \min\{s \mid Q_i(s)=0\}. \tag{5.30}$$

From all the above it can be easily seen that the optimal decision rules for any given i can be prescribed as follows.

□ *Optimal Decision Rule 5.1*

1. Admission control problem:
   i. If $Q_i>0$, then $\langle \texttt{C}\rangle_i$ [†] , or else $\langle \texttt{K}\rangle_i$ for $0\le i\le n$.
   ii. If $w>h_i$, then $\langle \texttt{A(w)}\rangle_i$, or else $\langle \texttt{R(w)}\rangle_i$ for $0\le i<n$.
2. Pricing control problem:
   i. If $Q_i>0$, then $\langle \texttt{C}\rangle_i$, or else $\langle \texttt{K}\rangle_i$ for $0\le i\le n$.
   ii. $\langle \texttt{O}(z_i)\rangle$ with $z_i=z(h_i)$ for $0\le i<n$.

6 Analysis

6.1 Preliminaries

Lemma 6.1

(a) $a\le z(x)$ *for all* x.

(b) *If* $x<b$, *then* $x<z(x)<b$, *and if* $x\ge b$, *then* $z(x)=b$.

(c) $z(x)$ *is nondecreasing in* x.

(d) *There exists a finite* $x^\star<a$ *such that if* $x<\,(>)\,x^\star$, *then* $z(x)=\,(>)\,a$.

(e) $T(x)$ *is nonincreasing on* $(-\infty,\infty)$, *strictly decreasing on* $(-\infty,b)$, *and convex on* $(-\infty,\infty)$.

(f) $T(x)\ge 0$ *on* $(-\infty,\infty)$.

(g) $T(x)>0$ *on* $(-\infty,b)$, *and* $T(x)=0$ *on* $[b,\infty)$.

(h) $\lim_{x\to\infty}T(x)=0$ *and* $\lim_{x\to-\infty}T(x)=\infty$.

[†] The notation $\langle \texttt{C}\rangle_i$ implies that continuing the search is optimal in state (ϕ,i).

(i) $\nu T(x) + x$ *is nondecreasing in* x *if* $\nu \le 1$ *and strictly increasing in* x *if* $\nu < 1$.

Proof. See [5, Ikuta] [15, You][‡] . ∎

Note. It is not yet proven which of $z(x^\star) > a$ or $z(x^\star) = a$ is true in [5]. If $F(w)$ is a uniform distribution on $[a, b]$ with $0 < a < b$, then $x^\star = 2a - b$ (See App. B).

Lemma 6.2

(a) $L(x)$ *is nonincreasing on* $(-\infty, \infty)$*, strictly decreasing on* $(-\infty, b)$*, and convex on* $(-\infty, \infty)$.

(b) $L(x) \ge -c$ *on* $(-\infty, \infty)$.

(c) $L(x) \le \lambda\beta T(0) - c$ *on* $[0, \infty)$*,* $L(x) > -c$ *on* $(-\infty, b)$*, and* $L(x) = -c$ *on* $[b, \infty)$.

(d) $\lim_{x\to\infty} L(x) = -c$ *and* $\lim_{x\to-\infty} L(x) = \infty$.

(e) *If* $L(x) > 0$*, then* $x < b$.

(f) *If* $L(x) = L(y) > 0$*, then* $x = y$*, and if* $L(x) > L(y)$*, then* $x < y$.

(g) $M(x)$ *is nondecreasing in* x *and* $M(b) = \beta b - c$.

Proof. (a-d) Immediate from Lemma 6.1(e-h).

(e) $L(x) = \lambda\beta T(x) - c > 0$, from which $T(x) > c/\lambda\beta \ge 0$, hence from Lemma 6.1(g) $x < b$.

(f) Clear from (a).

(g) $M(x) = L(x) + \beta x = \beta(\lambda T(x) + x) - c$, which is nondecreasing in x due to Lemma 6.1(i). Further, for $x \ge b$ we get $M(x) = \beta x - c$ due to (c), hence $M(b) = \beta b - c$. ∎

Lemma 6.3 *The system of equations* Eq. (4.2) *to* Eq. (4.5) *and* Eq. (4.7) *to* Eq. (4.10) *has a unique solution.*

Proof. See App. A.2. ∎

Lemma 6.4

(a) $u(\phi, i)$ *and* $v(i)$ *are nonincreasing in* i *where* $u(\phi, i) \ge 0$ *for* $0 \le i \le n$.

(b) $h_i \ge 0$ *for* $0 \le i < n$.

Proof. See App. A.3. ∎

6.2 Case of $\alpha \le 0$

Lemma 6.5 $Q_i \le 0$ *for* $0 \le i \le n$.

Proof. Assume $\alpha \le 0$. Then from Lemmas 6.4(b) and 6.2(c) we have $0 \ge \alpha = \lambda\beta T(0) - c \ge L(h_i)$ for $0 \le i < n$. Hence (1) $0 \ge L(h_0) = Q_0$, (2) $0 \ge \lambda\beta T(0) - c = (1-q)(\lambda\beta T(0) - c) + q(\lambda\beta T(0) - c) \ge (1-q)L(h_i) + qL(h_{i-1}) = Q_i$ for $1 \le i < n$, and (3) $0 \ge \lambda\beta T(0) - c > \lambda q\beta T(0) - c = q(\lambda\beta T(0) - c) - (1-q)c \ge qL(h_{n-1}) - (1-q)c = Q_n$. ∎

[‡] As [5] includes the contents of [15], readers should refer to [5]

6.3 Case of $\alpha > 0$

Lemma 6.6

(a) $u(\phi, i) > 0$ *for* $0 \le i \le n$.

(b) *If* $Q_i \le 0$ *for a given* i *such as* $1 \le i < n$, *then* $h_{i-1} > h_i$, *hence* $h_{i-1} \ge h_i$.

Proof. (a) First, note $u(\phi, i) \ge 0$ for all i from Lemma 6.4(a). Hence, from Eqs. (5.3) and (5.2) we have $u(\phi, 0) \ge \beta(\lambda v(0) + (1-\lambda)u(\phi, 0)) - c + s \ge \lambda\beta T(0) - c + s = \alpha + s > 0$. Suppose $u(\phi, i-1) > 0$. Then from Eqs. (5.4) and (5.5) we can get $u(\phi, i) \ge (1-q)\beta u(\phi, i) + q\beta u(\phi, i-1) > 0$ for $1 \le i \le n$.

(b) Let $Q_i \le 0$ for a given i such as $1 \le i < n$. Then from Eq. (5.23) we have $u(\phi, i) = \gamma q\beta u(\phi, i-1)$, hence $u(\phi, i+1) = \gamma q\beta u(\phi, i) + \gamma \max\{Q_{i+1}, 0\} = (\gamma q\beta)^2 u(\phi, i-1) + \gamma \max\{Q_{i+1}, 0\}$. Accordingly, we get

$$\begin{aligned} h_i - h_{i-1} &= 2u(\phi, i) - u(\phi, i-1) - u(\phi, i+1) \\ &= 2\gamma q\beta u(\phi, i-1) - u(\phi, i-1) - (\gamma q\beta)^2 u(\phi, i-1) - \gamma \max\{Q_{i+1}, 0\} \\ &= -(1-\gamma q\beta)^2 u(\phi, i-1) - \gamma \max\{Q_{i+1}, 0\} < 0 \end{aligned}$$

due to $u(\phi, i-1) > 0$ from (a) and $1 > \gamma q\beta$ from Eq. (3.5); accordingly, $h_{i-1} > h_i$, hence $h_{i-1} \ge h_i$. ∎

Lemma 6.7 *Let* $h_{i-1} < h_i$ *for a given* i *such as* $1 \le i < n$. *Then* $h_{i-1} < h_i < \cdots < h_{n-1} < b$ *and* $Q_j > 0$ *for* j *with* $i \le j < n$.

Proof. Let $h_{i-1} < h_i$ for a given i such as $1 \le i < n$. Then $Q_i > 0$ from the contrapositions of Lemma 6.6(b); accordingly, from Lemma 6.2(a) we get

$$0 < Q_i = (1-q)L(h_i) + qL(h_{i-1}) \le (1-q)L(h_{i-1}) + qL(h_{i-1}) = L(h_{i-1}),$$

implying $h_{i-1} < b$ due to Lemma 6.2(e). Further, from Eq. (5.26) we have

$$\begin{aligned} h_i &= \gamma q\beta h_{i-1} + \gamma Q_i - \gamma \max\{Q_{i+1}, 0\} \\ &= \gamma q\beta h_{i-1} + \gamma(1-q)L(h_i) + \gamma qL(h_{i-1}) - \gamma \max\{Q_{i+1}, 0\} \\ &\le \gamma q\big(\beta h_{i-1} + L(h_{i-1})\big) + \gamma(1-q)L(h_i) \\ &= \gamma qM(h_{i-1}) + \gamma(1-q)L(h_i). \end{aligned} \tag{6.1}$$

Assume $h_i \ge b$. Then $L(h_i) = -c \le 0$ from Lemma 6.2(c), hence $h_i \le \gamma qM(h_{i-1})$. Since $h_{i-1} < b$, from Lemma 6.2(g) we get $h_i \le \gamma qM(b) = \gamma q(\beta b - c) \le \gamma q\beta b < b$ due to Eq. (3.5), which is a contradiction. Hence, it must be $h_{i-1} < h_i < b$. Here, let us assume $Q_{i+1} \le 0$. Then $h_{i+1} < h_i < b$ from Lemma 6.6(b) and the above result. Further, from Lemma 6.2(a) we have

$$0 \ge Q_{i+1} = (1-q)L(h_{i+1}) + qL(h_i) > (1-q)L(h_i) + qL(h_i) = L(h_i) \quad \cdots (1^*).$$

From Eq. (6.1) and Lemma 6.2(g) we have

$$\begin{aligned} h_i &\le \gamma qM(h_i) + \gamma(1-q)L(h_i) \\ &= \gamma q\big(\beta h_i + L(h_i)\big) + \gamma(1-q)L(h_i) \\ &= \gamma q\beta h_i + \gamma L(h_i), \end{aligned}$$

from which we have $(1-\gamma q\beta)h_i \le \gamma L(h_i)$. Using Eq. (3.5), we can rewrite the above inequality $\gamma(1-\beta)h_i \le \gamma L(h_i)$, i.e., $(1-\beta)h_i \le L(h_i)$. Since $h_i \ge 0$ from Lemma 6.4(b), we have $L(h_i) > 0$, which contradicts (1*), hence it must be $Q_{i+1} > 0$. From this and $Q_i > 0$ we can rewrite Eq. (5.26) as follows.

$$h_i \;=\; \gamma q\beta h_{i-1} + \gamma(Q_i - Q_{i+1}).$$

Since $\gamma(Q_i - Q_{i+1}) = \gamma q L(h_{i-1}) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1})$, we obtain

$$\begin{aligned} h_i \;&=\; \gamma q\big(\beta h_{i-1} + L(h_{i-1})\big) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}) \\ &=\; \gamma q M(h_{i-1}) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}). \end{aligned}$$

Noting the assumption of $h_{i-1} < h_i$, from Lemma 6.2(g) we get

$$\begin{aligned} h_i \;&\le\; \gamma q M(h_i) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}) \\ &=\; \gamma q(\beta h_i + L(h_i)) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}) \\ &=\; \gamma q\beta h_i + \gamma(1-q)\big(L(h_i) - L(h_{i+1})\big), \end{aligned}$$

from which we have

$$(1-\gamma q\beta)h_i \le \gamma(1-q)\big(L(h_i) - L(h_{i+1})\big).$$

Here, since $h_{i-1} \ge 0$ from Lemma 6.4(b), we have $h_i > 0$ due to the assumption of $h_{i-1} < h_i$. From this result and $1 > \gamma q\beta$ due to Eq. (3.5) we obtain $(1-\gamma q\beta)h_i > 0$, so that $L(h_{i+1}) < L(h_i)$, implying $h_i < h_{i+1}$. Repeating the same procedure as the above leads to the completion of the proof. ∎

Lemma 6.8 *Let $h_{i-1} \le h_i$ for a given i such as $1 \le i < n$. Then $h_{i-1} \le h_i \le \cdots \le h_{n-1} < b$ and $Q_j > 0$ for j with $i \le j < n$.*

Proof. Almost the same as in the proof of Lemma 6.7. ∎

Lemma 6.9 *If $Q_i > 0$ for a given i such as $0 \le i < n$, then $Q_j > 0$ for $i \le j < n$.*

Proof. Let $Q_i > 0$ for a given i such as $1 \le i < n$. First, let $h_{i-1} < h_i$. Then $Q_{i+1} > 0$ from Lemma 6.7. Next, let $h_{i-1} \ge h_i$. Then since $L(h_{i-1}) \le L(h_i)$ due to Lemma 6.2(a), we get $0 < Q_i = (1-q)L(h_i) + qL(h_{i-1}) \le L(h_i)$. Here, assume $Q_{i+1} \le 0$, i.e., $(1-q)L(h_{i+1}) + qL(h_i) \le 0$. Then $h_i \ge h_{i+1}$ due to Lemma 6.6(b). Noting $L(x)$ is convex on $(-\infty, \infty)$ from Lemma 6.2(a), we have

$$L((1-q)h_{i+1} + qh_i) \le (1-q)L(h_{i+1}) + qL(h_i) \le 0 < L(h_i),$$

from which we have $(1-q)h_{i+1} + qh_i > h_i$, hence $h_i < h_{i+1}$ due to the assumption of $q < 1$, which is a contradiction. Hence, it must be $Q_{i+1} > 0$. Repeating the same procedure leads to the completion of the induction. Let $Q_0 > 0$. Then since $L(h_0) > 0$ from Eq. (5.19). Here, assuming $Q_1 \le 0$, we can also derive a contradiction in quite the same way as the above, hence it must be $Q_1 > 0$. ∎

Lemma 6.10 *$h_i(s)$ is nondecreasing in s for $i \ge 0$.*

Proof. See App. A.4. ∎

Lemma 6.11 *$\lim_{s\to\infty} h_i(s) = \infty$ and $\lim_{s\to-\infty} h_i(s) = -\infty$ for $i \ge 0$.*

Proof. Since $L(h_i) \le \lambda\beta T(0) - c$ for $0 \le i < n$ due to Lemmas 6.4(b) and 6.2(c), noting Eq. (5.25), we have

$$h_0(s) \ \geq\ -\gamma \max\{Q_1, 0\} + \gamma s = -\gamma \max\{(1-q)L(h_1) + qL(h_0), 0\} + \gamma s \geq -\gamma \max\{\lambda\beta T(0) - c, 0\} + \gamma s,$$

$$h_0(s) \ \leq\ \gamma \max\{Q_0, 0\} + \gamma s = \gamma \max\{L(h_0), 0\} + \gamma s \leq \gamma \max\{\lambda\beta T(0) - c, 0\} + \gamma s,$$

from which $\lim_{s\to\infty} h_0(s) = \infty$ and $\lim_{s\to-\infty} h_0(s) = -\infty$. Let $\lim_{s\to\infty} h_{i-1}(s) = \infty$ and $\lim_{s\to-\infty} h_{i-1}(s) = -\infty$. Then noting Eq. (5.26), in the same way as the above we obtain

$$h_i(s) \ \geq\ \gamma q\beta h_{i-1}(s) - \gamma \max\{\lambda\beta T(0) - c, 0\},$$

$$h_i(s) \ \leq\ \gamma q\beta h_{i-1}(s) - \gamma \max\{\lambda\beta T(0) - c, 0\}.$$

Therefore, $\lim_{s\to\infty} h_i(s) = \infty$ and $\lim_{s\to-\infty} h_i(s) = -\infty$. Hence, by induction the assertion holds for $0 \leq i \leq n-1$. From this result and Eq. (5.29) we can prove $\lim_{s\to\infty} h_{n-1}(s) = \infty$ and $\lim_{s\to-\infty} h_{n-1}(s) = -\infty$ in the same way as the above. ▮

Lemma 6.12 *$Q_i(s)$ is nonincreasing in s for all $i \geq 0$.*

Proof. Since $h_i(s)$ is nondecreasing in s for $0 \leq i < n$ due to Lemma 6.10, from Lemma 6.2(a) we can easily see that $Q_i(s)$ is nonincreasing in s. ▮

Lemma 6.13 *For $0 \leq i \leq n$ we have:*

(a) *There exists $s_i > 0$.*

(b) *If $s < (\geq)\ s_i$, then $Q_i(s) > (\leq)\ 0$.*

Proof. (a) From Lemmas 6.11 and 6.2(d) we clearly have $\lim_{s\to-\infty} Q_i(s) = \infty$. Further, for a sufficiently large s we have $h_i(s) \geq b$ due to Lemma 6.11, hence $\lim_{s\to\infty} Q_i(s) = -c \leq 0$. Thus, it follows that s_i exists.

(b) Immediate from the definition of s_i and Lemma 6.12. ▮

Lemma 6.14

(a) *Let $s = 0$.*

1 *$Q_0(s) > 0$.*

2 *If $h_0 = 0$, then $h_0 = h_1$.*

3 *If $h_0 > 0$, then $h_0 < h_1$.*

(b) *If $s_0 \leq s$, then $h_0 > h_1$.*

Proof. (a) Let $s = 0$.

(a1) Assume $\lambda\beta v(0) + (1-\lambda)\beta u(\phi, 0) - c \leq \beta u(\phi, 0)$ from Eq. (5.3). Then since $u(\phi, 0) = \beta u(\phi, 0)$, we have $\beta = 1$ due to $u(\phi, 0) > 0$ from Lemma 6.6(a), which contradicts the assumption of $\beta < 1$. Accordingly, we have $u(\phi, 0) = \lambda\beta v(0) + (1-\lambda)\beta u(\phi, 0) - c = \lambda\beta(v(0) - u(\phi, 0)) + \beta u(\phi, 0) - c = \lambda\beta T(h_0) + \beta u(\phi, 0) - c$ due to Eq. (5.6), from which

$$0 < u(\phi, 0) \ =\ (\lambda\beta T(h_0) - c)/(1-\beta) = L(h_0)/(1-\beta), \tag{6.2}$$

hence $L(h_0) > 0$, i.e., $Q_0(s) > 0$.

(a2) $Q_0 = L(h_0) > 0$ due to (a1), hence $Q_1 > 0$ due to Lemma 6.6(b). Accordingly, from Eq. (5.26) with $s = 0$ we have $h_0 = \gamma(Q_0 - Q_1)$, i.e.,

$$h_0 = \gamma\big(L(h_0) - (1-q)L(h_1) - qL(h_0)\big) = \gamma(1-q)\big(L(h_0) - L(h_1)\big), \tag{6.3}$$

from which we obtain that if $h_0 = 0$, then $L(h_1) = L(h_0) = L(0) = \lambda\beta T(0) - c = \alpha > 0$, hence, $h_1 = 0$ due to Lemma 6.2(f).

(a3) From Eq. (6.3), if $h_0 > 0$, then $L(h_0) > L(h_1)$, implying $h_0 < h_1$ due to Lemma 6.2(f).

(b) Let $s_0 \leq s$. Then from Lemma 6.13(b) we have $Q_0(s) = L(h_0(s)) \leq 0$. Assume $h_0 \leq h_1$. Then $Q_1 = (1-q)L(h_1)+qL(h_0) \leq (1-q)L(h_0)+qL(h_0) = L(h_0) \leq 0$, implying $h_0 > h_1$ due to Lemma 6.6(b). This is a contradiction, hence it must be $h_0 > h_1$. ∎

Let us define

$$s^* = \min\{s \mid h_0(s) > h_1(s)\}.$$

Lemma 6.15 *We have $s_0 \geq s^* > 0$ where if $s \geq (<)\ s^*$, then $h_0 > (\leq)\ h_1$.*

Proof. From Lemma 6.14 we have $h_0 \leq h_1$ for $s = 0$ and $h_0 > h_1$ for $s \geq s_0$, implying that there exists a positive $s^* \leq s_0$ such as $h_0(s) > h_1(s)$; accordingly, the latter half of the assertion is clearly true. ∎

Lemma 6.16 *If $s = 0$, then h_i is nondecreasing in i and $Q_i > 0$ for $0 \leq i < n$.*

Proof. The former half is immediate from Lemmas 6.14(a2) and 6.8 with $i = 1$. The latter half is evident from Lemmas 6.14(a1) and 6.8. ∎

Theorem 6.1

(a) *Let $\alpha \leq 0$. Then $\langle \mathtt{K} \rangle_{0 \leq i \leq n}$.*

(b) *Let $\alpha > 0$.*

- 1 *Let $s_0 \leq s$. Then $\langle \mathtt{K} \rangle_{0 \leq i < n}$ or there exists $i^*(\, 0 < i^* < n\,)$ such that $\langle \mathtt{K} \rangle_{0 \leq i < i^*}$ and $\langle \mathtt{C} \rangle_{i^* \leq i < n}$.*
- 2 *Let $s < s_0$.*
  - i $\langle \mathtt{C} \rangle_{1 \leq i < n}$.
  - ii *Let $s^* \leq s$. Then h_i is not always nondecreasing in i.*
  - iii *Let $s < s^*$.*
    - 1 $h_0 \leq h_1$.
    - 2 *If $h_0 = h_1$, then h_i is nondecreasing in i with $h_i < b$ for $0 \leq i < n$.*
    - 3 *If $h_0 < h_1$, then h_i is strictly increasing in i with $h_i < b$ for $0 \leq i < n$.*

Proof. (a) Evident from Lemma 6.5.

(b) Let $\alpha > 0$. Here note that $s^* \leq s_0$ from Lemma 6.15.

(b1) Let $s_0 \leq s$. Clearly $Q_0(s) \leq 0$ from Lemma 6.13(b with $i = 0$), hence $\langle \mathtt{K} \rangle_0$. From this result and the fact that once continuing the search is optimal for a certain i, i.e., $\langle \mathtt{C} \rangle_i$, then so also is for all i' with $i \leq i' < n$ due to Lemma 6.9. Accordingly, the assertion clearly holds.

(b2) Let $s < s_0$.

(b2i) Then $Q_0(s) > 0$ from Lemma 6.13(b with $i = 0$), hence $Q_i(s) > 0$ for $0 \leq i < n$ from Lemma 6.9, thus $\langle \mathtt{C} \rangle_{0 \leq i < n}$.

(b2ii) Let $s^* \leq s$. Then since $h_0 > h_1$ from Lemma 6.15, it follows that h_i is not always nondecreasing in i.

(b2iii) Let $s < s^*$.

(b2iii1-b2iii3) Immediate from, respectively, Lemmas 6.15, 6.8, and 6.7. ∎

7 Optimal Decision Rule

For explanatory convenience, let us define the two assertions below:

Assertion SP: Conducting only a standard production without searching for customers is always better than doing a custom production with searching for customers.

Assertion CP: Conducting custom production and shifting to standard production when the backorder is exhausted is always better than only doing a standard production without searching for customers.

We assumed A8 in Section 2, that is, the production starts with no backorder, i.e., $i = 0$. Then if skipping the search is optimal, i.e., $\langle \mathtt{K} \rangle_0$, and since no customer appears, it follows that the number of backorders remains forever zero, i.e., $i = 0$ over the entire planning horizon. Accordingly, it eventually follows that Assertion SP holds. Consequently, the optimal decision rule *5.1* can be restated as follows.

□ *Optimal Decision Rule 7.1*

(a) Let $\alpha \leq 0$ or "$\alpha > 0$ and $s_0 \leq s$". Then $\langle \mathtt{K} \rangle_0$ (Theorem 6.1(a,b1)), hence Assertion SP holds for the reason stated above.

(b) Let $\alpha > 0$ and $s < s_0$. Then since $\langle \mathtt{C} \rangle_{0 \leq i < n}$ (Lemma 6.1(b2i)), it is optimal to conduct the search by paying a search cost c, implying that Assertion CP holds for $0 \leq i < n$. If $i = n$, any of continuing the search and skipping the search may be optimal; more precisely, if $s < s_n$, then $\langle \mathtt{C} \rangle_n$, or else $\langle \mathtt{K} \rangle_n$ (Lemma 6.13(b) with $i = n$).

   1 Let $s^* \leq s$. Then h_i is not always nondecreasing in i (Theorem 6.1(b2ii)); in other words, as seen in Figure 8.2, there exists a $i^*(s) \geq 1$ such that h_i is decreasing in $i \leq i^*(s)$ and increasing in $i > i^*(s)$.

   2 Let $s < s^*$. Then h_i is nondecreasing in i with $h_0 \leq h_1$ where if $h_0 < h_1$, then h_i is strictly increasing in i (Theorem 6.1(b2iii2,b2iii3)).

In pricing control it should be noted that the monotonicity of h_i in i stated above is inherited to the optimal price z_i due to Lemma 6.1(c). Since $z_i = z(h_i)$, from Lemma 6.1(d) we see that $z_i = a$ if $h_i < x^\star$.

8 Numerical Examples

Here, let us show some numerical examples of the optimal decision rule summarized in Section 7.

8.1 Admission control problem

Let $F(w)$ be the uniform distribution on $[0.01, 1.01]$, i.e., $a = 0.01$ and $b = 1.01$, and let $\lambda = 0.95$, $q = 0.35$, $\beta = 0.99$ and $c = 0.01$. Then from Eq. (B.1) we have

$$T(x) = \begin{cases} 0.51 - x & \text{for} \quad x < 0.01, \\ 0.5(1.01 - x)^2 & \text{for} \quad 0.01 \leq x < 1.01, \\ 0 & \text{for} \quad 1.01 \leq x. \end{cases}$$

In this case, $T(0) = 0.51$, hence $\alpha = \lambda\beta T(0) - c = 0.47 > 0$.

I. **s^* and s_0**: Performing numerical calculations, we obtain $s^* = 0.1330293\cdots \simeq 0.133$ and $s_0 = 0.3259868\cdots \simeq 0.326$. Accordingly, if the idling profit $s \geq 0.326$, the assertion SP holds, and if $s < 0.326$, the assertion SP holds.

   Relationship of s^* and s_0 with related parameters λ, q, β, and c:

   1. Figure 8.1 illustrates the relationships of s^* and s_0 with the four related parameters λ, q, β, and c where the calculations are made by setting one of the four parameters as a variable with all the others being fixed. Here, it is to be noted that each of the coordinates planes of the four graphs

is divided into the three regions; $\mathcal{R}\langle\texttt{K}\rangle$ for $s_0 \le s$, $\breve{\mathcal{R}}\langle\texttt{C}\rangle$ for $s^* \le s < s_0$, and $\acute{\mathcal{R}}\langle\texttt{C}\rangle$ for $s < s^*$. In the region $\mathcal{R}\langle\texttt{K}\rangle$, not conducting the search, i.e., skipping the search is always optimal, and in both regions $\breve{\mathcal{R}}\langle\texttt{C}\rangle$ and $\acute{\mathcal{R}}\langle\texttt{C}\rangle$, conducting the search is always optimal where h_i is unimodal in i on $\breve{\mathcal{R}}\langle\texttt{C}\rangle$ and nondecreasing in i on $\acute{\mathcal{R}}\langle\texttt{C}\rangle$ (see Figure 8.2).

2. From Figure 8.1 it can be seen that (1) s^* is nonincreasing in c and nondecreasing in λ and β, (2) s_0 is nonincreasing in c and nondecreasing in λ and q, and (3) s^* and s_0 are unimodal in, respectively, q and β. That s^* is unimodal in q implies that for a certain given s there exists q' and q'' with $q' < q''$ such that if $q \le q'$, then $(q, s) \in \breve{\mathcal{R}}\langle\texttt{C}\rangle$, if $q' < q \le q''$, then $(q, s) \in \acute{\mathcal{R}}\langle\texttt{C}\rangle$, and if $q'' \le q$, then *again* $(q, s) \in \breve{\mathcal{R}}\langle\texttt{C}\rangle$; in other words, there exists two critical values of q such that the shape of h_i changes from "unimodal" to "nondecreasing" at $q = q'$ and from "nondecreasing" to "unimodal" at $q = q''$.

II. $\boldsymbol{h_i}$: Figure 8.2 depicts the relationships of h_i with the number of backorders i and the idling profit s. The figure tells us that (1) if $s < 0.133$, then h_i is strictly increasing in $i \ge 0$, (2) if $s = 0.133$, then $h_0 \simeq h_1 = 0.3731556\cdots$ and h_i is strictly increasing in $i \ge 1$, and (3) if $0.133 < s < 0.326$, then h_i is unimodal, i.e., there exists a $i^*(s) \ge 1$ such that h_i is strictly decreasing in $i \le i^*(s)$ and strictly increasing in $i \ge i^*(s)$. Further, the figure shows that h_i is nondecreasing in s for all i (Lemma 6.10) and that if i is sufficiently large, then h_i coincides with h_i with $s = 0.000$. The latter finding reflects the fact that as the number of backorders becomes larger, since the possibility of the backorder being exhausted gets smaller, the effect of s on h_i is gradually diminished.

8.2 Pricing control problem

Let $F(w)$ be the uniform distribution on $[2, 3]$, i.e., $a = 2$ and $b = 3$, and let $\lambda = 0.75$, $q = 0.55$, $\beta = 0.99$ and $c = 0.05$. Then from Eq. (B.3) we get

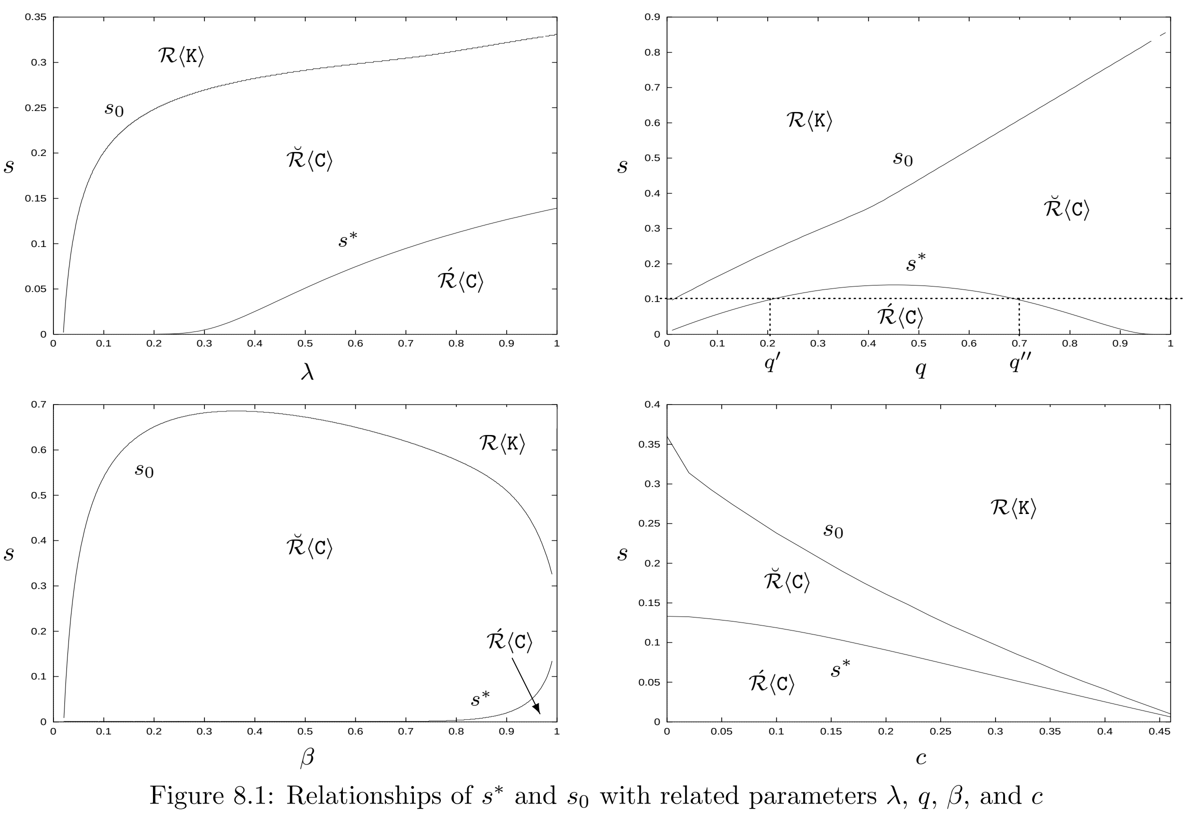

Figure 8.1: Relationships of s^* and s_0 with related parameters λ, q, β, and c

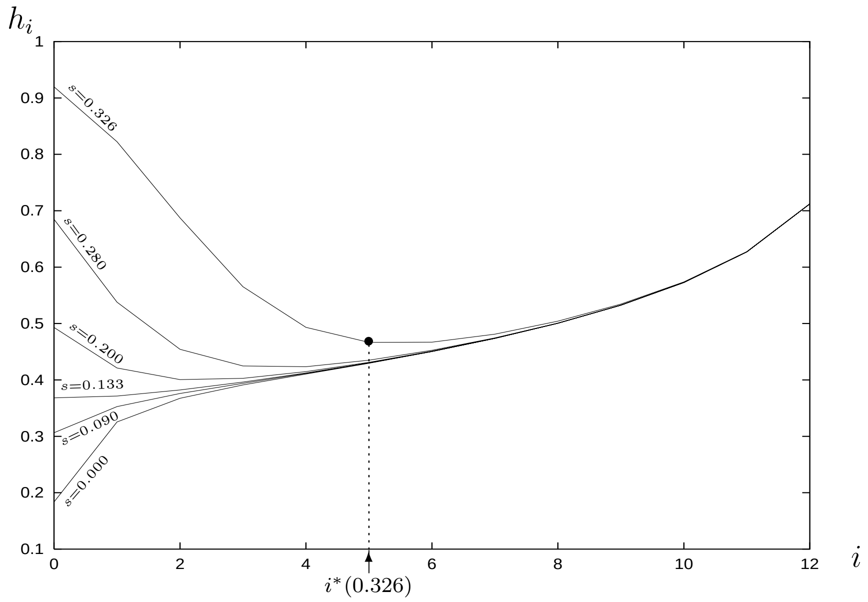

Figure 8.2: Graph of h_i where $s^* = 0.133$ and $s_0 = 0.326$. Here, note that if $s < 0.133$, then h_i is strictly increasing in i and if $0.133 \le s < 0.326$, then h_i is unimodal in i.

$$T(x) = \begin{cases} 2 - x & \text{for} \quad x < 1 \quad \rightarrow \quad z(x) = 2, \\ 0.25(3 - x)^2 & \text{for} \quad 1 \le x < 3 \quad \rightarrow \quad z(x) = (x+3)/2, \\ 0 & \text{for} \quad 3 \le x \quad \rightarrow \quad z(x) = 3. \end{cases}$$

In this case, $T(0) = 2$, hence $\alpha = \lambda\beta T(0) - c = 1.435 > 0$. Further, $x^\star = 2a - b = 1$.

I. **s^* and s_0**: Performing numerical calculations, we obtain $s^* = 0.4986829\cdots \simeq 0.499$ and $s_0 = 1.3828263\cdots \simeq 1.383$. Accordingly, if the idling profit $s \ge 1.383$, the assertion SP holds, and if $s < 1.383$, the assertion CP holds. In the case we obtain almost the same graphs as Figure 8.1.

II. **h_i and z_i**: Figure 8.3 depicts the relationships of h_i and $z_i (= z(h_i))$ with the number of backorders i and the idling profit s.

1. The graph on the left tells us that (1) if $s < 0.499$, then h_i is strictly increasing in $i \ge 0$, (2) if $s = 0.499$ then $h_0 \simeq h_1 = 0.8993454\cdots$ and h_i is strictly increasing in $i \ge 1$, (3) if $0.499 \le s < 1.383$, there exists a $i^*(s) \ge 1$ such that h_i is strictly decreasing in $i \le i^*(s)$ and strictly increasing in $i \ge i^*(s)$.
2. The graph on the right shows the optimal ordering price z_i. Now, note that there exists i such that $h_i < x^\star = 2a - b = 1$ in the graph of h_i. Since $z_i = z(h_i) = a$ for $h_i < x^\star = 1$ due to Lemma d, it follows that $z_i = z(h_i)$ for such i becomes equal to $a = 2$; in other words, $z_i = z(h_i)$ is truncated by a, the low bound of the distribution. Further, it should be noted that there exists $h_i < a$ such that its corresponding optimal ordering price z_i becomes greater than a, i.e., $z_i = z(h_i) > a$.

9 Conclusions and Considerations

Now, let us examine the practical implications of the optimal decision rule described in Section 7.

A. Let $\alpha \le 0$ or equivalently $\lambda\beta T(0) \le c$, implying that the search cost c is sufficiently large to be greater than or equal to $\lambda\beta T(0)$. Then not conducting the search; in other words, skipping the search always becomes optimal, i.e., $\langle \mathtt{K} \rangle_0$. In this case, the assertion SP holds.

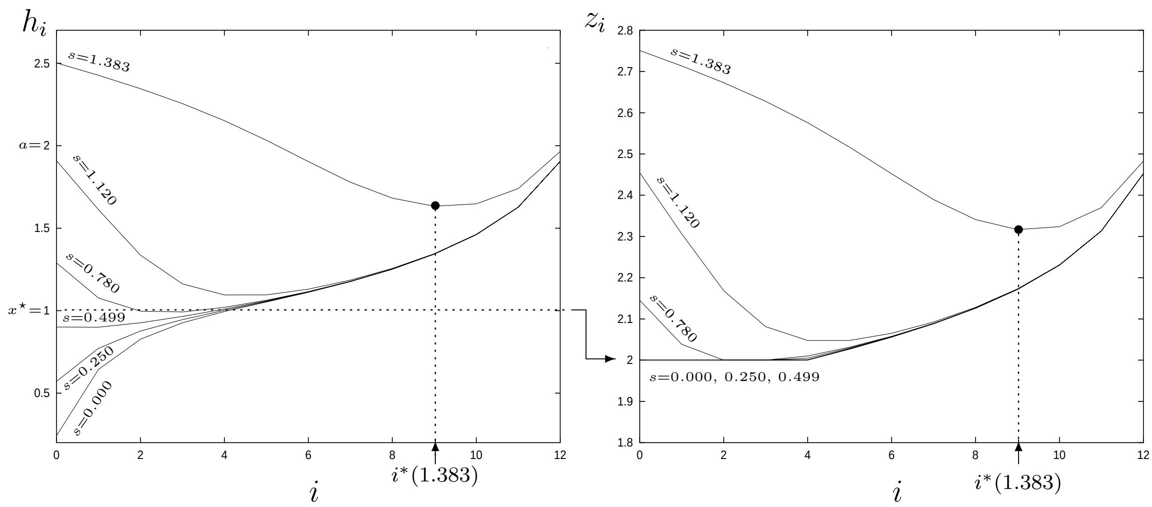

Figure 8.3: Graphs of h_i and z_i

B. Let $\alpha > 0$ or equivalently $\lambda\beta T(0) > c$, implying that the search cost c is sufficiently small to be smaller than $\lambda\beta T(0)$, including $c = 0$. Then it can be conjectured that conducting the search is always optimal, i.e., $\langle \texttt{C} \rangle_{0\le i<n}$; Is this always the case? Unfortunately the answer is negative for the reasons stated below.

1. Let $s_0 \le s$. In this case, even though the search cost is sufficiently small, if the idling profit is sufficiently large to be greater than or equal to s_0, it becomes optimal to skip the search in order to enjoy the idling profit, i.e., $\langle \texttt{K} \rangle_0$. Accordingly, the conjecture stated above is false. In this case, the number of backorders remains ever zero, hence it follows that the standard production is always conducted, i.e., the assertion SP holds.

2. Let $s < s^*$, that is, both the search cost c and idling profit s are sufficiently small. In the case, it is optimal to conduct the search, i.e., $\langle \texttt{C} \rangle_{0\le i<n}$, with the resultant conclusion that the assertion CP holds. Accordingly, the conjecture stated above is true. Further, in this case, let us not forget the fact that the optimal selection criterion h_i in admission control and the optimal ordering price z_i in pricing control both increase in the number of backorders i as seen in Figures 8.2 and 8.3 (Theorem 6.1(b2iii)). Below, let us consider the implication of the monotonicity of h_i and z_i in i.

   i. Let the number of backorders i be sufficiently small. Then in order to avoid Opportunity loss II, the system should accept any order however low in price it may be; of course, although there exists a low bound. This implies that the optimal selection criterion h_i in admission control and the optimal ordering price z_i in pricing control must be set to be low.

   ii. Let the number of backorders i be sufficiently large. Then in order to avoid Opportunity loss I, the system should reject orders with low price by setting the high selection criterion in admission control and the offering high price in pricing control; as the result that only orders with a high price are accepted in admission control and that a high price is offered in pricing control.

   iii. The above two considerations imply that the optimal selection criterion h_i in admission control and the optimal ordering price z_i in pricing control should be set to be increasing in the number of backorders i. The monotonicity of h_i and z_i brings about the following dynamic behavior for the movement of the number of backorders i. First, let us consider the admission control problem. When the number of backorders is small, since the selection criterion is low, the number of orders accepted becomes large; accordingly, the number of backorders increases.

Since the selecting criterion becomes high as the number of backorders increases, the number of orders accepted becomes small; therefore, the number of backorders becomes small, hence it follows that the number of backorders decreases. The above fact can be restated as follows. The smaller the number of backorders may become, the stronger the force of making itself large may become; on the contrary, the larger the number of backorders may become, the stronger the force of making itself small may become. Such a movement in the number of backorders looks just like a free oscillation of pendulum, always moving toward the vertical, the most stable position. The above consideration leads us to the implication that the number of backorders fluctuates while all the time being pulled toward the equilibrium point in the stochastic sense. Stabilization of the number of backorders is also what management desires. In the pricing control problem the same consideration as the above can be given.

3. Let $s^* \leq s < s_0$, i.e., the idling profit be neither sufficiently large nor sufficiently small. For example, if $0.133 \leq s < 0.326$ in Figure 8.2 and if $0.499 \leq s < 1.383$ in Figure 8.3, there exists a $i^*(s) \geq 1$ such that h_i is decreasing in $i \leq i^*(s)$ and h_i is increasing in $i \geq i^*(s)$; in other words, both of the optimal selection criterion h_i and the optimal ordering price z_i are not always increasing in the number of backorders i, i.e., h_i and z_i are both unimodal in i. This implies the following. Let the number of backorders i be sufficiently small. Then if rejecting orders by setting the high selection criterion in admission control, the probability of production process becoming idle is large; as a result, the system can enjoy the idling profit. Further, as the number of backorders increases until $i = i^*(s)$ and goes cross $i^*(s)$, since the influence of idling profits on the selection criterion and the ordering price get weaker, they become nondecreasing in i as in the case of $s = 0.000$. Now, that the h_i takes such a shape stated above in admission control first tells us the following. For an appearing customer with certain value w there exists such $i^{'} < i^{''}$ that if $i \leq i^{'}$, rejecting the order of customer is optimal, if $i^{'} < i \leq i^{''}$, accepting it is optimal, and if $i^{''} < i$, *again* rejecting it is optimal; that is, it follows that there exist *double critical values* in terms of i at both of which rejecting and accepting become indifferent. In the pricing control problem the same consideration as the above can be also given.

10 Suggested Future Study

In this paper we have proposed a basic model for a customer selection problem with idling profit. In order to make the model more practical, the following points should be necessarily investigated and tackled.

1. A custom production company may have multiple production lines.
2. An order held in the system may be canceled owing to customer unavoidable circumstances.
3. Orders may be processed through a series of processes; in this case, a scheduling problem, or a sequencing problem of orders to be processed arises.
4. It is rather natural and practical to think that any order has an appointed date of delivery, and this is not considered in our model. The period up to the time when an accepted order should be delivered after its completion is different among the orders accepted. In this case, the optimal customer selection criterion and the optimal ordering price may depend on the length of the period. Further, in this case, a model can be considered into which an assumption is introduced that delay of delivery is permitted, and this will inevitably be accompanied by a penalty.
5. Thus far we have implicitly assumed that a customer once turned away can not be solicited in the future. The future availability of a rejected customer, that is assumed in usual models of optimal stopping problems [4] [6] [12], should be also introduced in our model.

Appendix

A. Proofs

A.1 Recurrent Equations

Let us define the following recurrent relations corresponding to Eq. (5.3) to Eq. (5.6).

$$u_t(\phi,0) \;=\; \max\{\lambda\beta v_{t-1}(0)+(1-\lambda)\beta u_{t-1}(\phi,0)-c, \beta u_{t-1}(\phi,0)\}+s, \quad t\ge 1, \tag{A.1}$$

$$u_t(\phi,i) \;=\; \max\left\{\begin{array}{l}(1-q)\beta\big(\lambda v_{t-1}(i)+(1-\lambda)u_{t-1}(\phi,i)\big)\\ \quad +q\beta(\lambda v_{t-1}(i-1)+(1-\lambda)u_{t-1}(\phi,i-1))-c,\\ (1-q)\beta u_{t-1}(\phi,i)+q\beta u_{t-1}(\phi,i-1)\end{array}\right., \quad 1\le i<n, \quad t\ge 1, \tag{A.2}$$

$$u_t(\phi,n) \;=\; \max\left\{\begin{array}{l}(1-q)\beta u_{t-1}(\phi,n)\\ \qquad +q\beta(\lambda v_{t-1}(n-1)+(1-\lambda)u_{t-1}(\phi,n-1))-c,\\ (1-q)\beta u_{t-1}(\phi,n)+q\beta u_{t-1}(\phi,n-1)\end{array}\right\}, \quad t\ge 1 \tag{A.3}$$

where $u_0(\phi,0)=0$ for all i. Further, as ones corresponding to Eqs. (4.5) and (4.10) for $0\le i<n$ let us define, respectively,

$u_t(w,i)=\max\{w+u_t(\phi,i+1),u_t(\phi,i)\}$, for the admission control problem,

$u_t(1,i)=\max_z\{p(z)\big(z+u_t(\phi,i+1)\big)+(1-p(z))u_t(\phi,i)\}$, for the pricing control problem,

and then define, respectively, $v_t(i) = \mathbf{E}[u_t(w,i)]$ and $v_t(i) = u_t(1,i)$. Then $v_t(i)$ can eventually be rewritten as follows.

$$v_t(i) \;=\; \left\{\begin{array}{l}\mathbf{E}[\max\{w+u_t(\phi,i+1),u_t(\phi,i)\}],\\ \max\limits_z\{p(z)\big(z+u_t(\phi,i+1)\big)+(1-p(z))u_t(\phi,i)\}\end{array}\right\}, \quad 0\le i<n. \tag{A.4}$$

Accordingly, letting

$$h_{it}=u_t(\phi,i)-u_t(\phi,i+1), \quad 0\le i<n, \quad t\ge 0, \tag{A.5}$$

from Eqs. (A.4) and (3.1) we have

$$v_t(i)=T(h_{it})+u_t(\phi,i), \quad 0\le i<n, \quad t\ge 0. \tag{A.6}$$

A.2 Lemma 6.3

For any given vector $\boldsymbol{x}=(x_0,x_1,\cdots,x_n)'$ let us define the norm $\|\boldsymbol{x}\|=\max\{|x_0|,|x_1|,\cdots,|x_n|\}$ where clearly $\|\boldsymbol{x}\|\ge|x_i|$ for $0\le i\le n$. Further, by D_iu let us denote the right hand sides of Eq. (4.2) to Eq. (4.4) and Eq. (4.7) to Eq. (4.9), and let $\boldsymbol{D}u=(D_0u,D_1u,\cdots,D_nu)'$ and $\boldsymbol{u}=(u(\phi,0),u(\phi,1),\cdots,u(\phi,n))'$. Let $\hat{u}(w,i)$ and $\hat{u}(1,i)$ be the bounded functions of $i=0,1,\cdots,n$. Then using the definition of $v(i)$ and $\hat{v}(i)$, from Eq. (4.5) with $i=0$ we have

$$|v(0)-\hat{v}(0)|\le\mathbf{E}[\max\{|u(\phi,1)-\hat{u}(\phi,1)|,|u(\phi,0)-\hat{u}(\phi,0)|\}]=\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|, \tag{A.7}$$

and from Eq. (4.10) we get

$$|v(0)-\hat{v}(0)| \;\le\; \max_z\{p(z)|u(\phi,1)-\hat{u}(\phi,1)|+(1-p(z))|u(\phi,0)-\hat{u}(\phi,0)|\}\le\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|.$$

Accordingly, from (4.7) we obtain

$$|D_0u-D_0\hat{u}| \;\le\; \max\left\{\begin{array}{l}\lambda\beta|v(0)-\hat{v}(0)|+(1-\lambda)\beta|u(\phi,0)-\hat{u}(\phi,0)|,\\ \beta|u(\phi,0)-\hat{u}(\phi,0)|\end{array}\right\}\le\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|.$$

Similarly, we get $|D_iu-D_i\hat{u}|\le\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|$ for $1\le i\le n$. Thus by definition we have $\|\boldsymbol{D}u-\boldsymbol{D}\hat{u}\|\le\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|$, implying that $\boldsymbol{D}u$ is a contraction mapping. Hence, the assertion holds. ▮

A.3 Lemma 6.4

(a) Clearly, $u_0(\phi, i)$ in nonincreasing in i, hence from Eq. (A.4) $v_0(i)$ is nonincreasing in i. Assume $u_{t-1}(\phi, i)$ is nonincreasing in i, hence $v_{t-1}(i)$ is nonincreasing in i. Then we have

$$\begin{aligned} u_t(\phi,0) &\geq u_t(\phi,0) - s \\ &= \max\left\{ \begin{array}{l} (1-q)\beta(\lambda v_{t-1}(0) + (1-\lambda)u_{t-1}(\phi,0)) + q\beta(\lambda v_{t-1}(0) + (1-\lambda)u_{t-1}(\phi,0)) - c \\ (1-q)\beta u_{t-1}(\phi,0) + q\beta u_{t-1}(\phi,0) \end{array} \right\} \\ &\geq u_t(\phi,1). \end{aligned}$$

In almost the same way as this, for $2 \leq i \leq n-1$ we get $u_t(\phi, i-1) \geq u_t(\phi, i)$. Now, noting that $v_{t-1}(i) \geq u_{t-1}(\phi, i)$ from Eq. (A.6) due to Lemma 6.1(f) and that $u_{t-1}(\phi, i) \geq u_{t-1}(\phi, i+1)$ due to the induction hypothesis, from Eq. (A.2) with $i = n-1$ we obtain

$$u_t(\phi, n-1) \geq \max\left\{ \begin{array}{l} (1-q)\beta(\lambda u_{t-1}(\phi,n) + (1-\lambda)u_{t-1}(\phi,n)) \\ \qquad\qquad +q\beta(\lambda v_{t-1}(n-1) + (1-\lambda)u_{t-1}(\phi,n-1)) - c \\ (1-q)\beta u_{t-1}(\phi,n) + q\beta u_{t-1}(\phi,n-1) \end{array} \right\} = u_t(\phi,n).$$

Hence, since $u_t(\phi, i)$ is nonincreasing in $i \in [0, n]$, for $1 \leq i \leq n-1$ from Eq. (A.4) we immediately have $v_t(i-1) \geq v_t(i)$. Thus, $u(\phi, i)$ and $v(i)$ are nonincreasing in, respectively, $i \in [0, n]$ and $i \in [0, n-1]$. Now, since we can easily show that $u(\phi, 0) \geq s/(1-\beta) \geq 0$ from Eq. (5.13), we have $u(\phi, i) \geq \gamma q\beta u(\phi, i-1) \geq 0$ for $1 \leq i \leq n$ from Eq. (5.23), hence by induction $u(\phi, i) \geq 0$ for $i = 0, 1, \cdots, n$.

(b) Immediate from Eq. (4.1) and (a). ▮

A.4 Lemma 6.10

Let $u_0(\phi, i) = 0$ for all i. Then $h_{i0}(s) = 0$ for all i, which can be regarded as nondecreasing in $s \geq 0$. Assume that $h_{i,t-1}(s)$ is nondecreasing in $s \geq 0$ for all i.

1. *Proof for the monotonicity of $h_0(s)$ in s.* Let us define $\mathcal{S}_0^- = \{s \mid Q_0(s) \leq 0,\ s \geq 0\}$ and $\mathcal{S}_0^+ = \{s \mid Q_0(s) > 0,\ s \geq 0\}$. Then let us consider the following two cases of $s \in \mathcal{S}_0^-$ and $s \in \mathcal{S}_0^+$.
   i. Case of $s \in \mathcal{S}_0^-$. Then since the optimal decision in state $(\phi, 0)$ is to skip the search, from Eqs. (5.3) and (5.4) with $i = 1$ the optimal equations become

   $$\begin{aligned} u(\phi,0) &= \beta u(\phi,0) + s, \\ u(\phi,1) &= \max\left\{ \begin{array}{l} (1-q)\beta(\lambda v(1) + (1-\lambda)u(\phi,1)\big) + q\beta\big(\lambda v(1) + (1-\lambda)u(\phi,0)\big) - c, \\ (1-q)\beta u(\phi,1) + q\beta u(\phi,0). \end{array} \right. \end{aligned}$$

   Here, let us define the following recurrent relations corresponding to the above equations.

   $$\begin{aligned} u_t(\phi,0) &= \beta u_{t-1}(\phi,0) + s, \\ u_t(\phi,1) &= (1-q)\beta u_{t-1}(\phi,1) + q\beta u_{t-1}(\phi,0) \\ &\qquad + \max\{\lambda(1-q)\beta(v_{t-1}(1) - u_{t-1}(\phi,1)) + \lambda q\beta(v_{t-1}(0) - u_{t-1}(\phi,0)) - c, 0\}. \end{aligned}$$

   Accordingly, noting $T(h_{it}(s)) = v_t(i) - u_t(\phi, i)$ in Eq. (A.6), from Eq. (A.5) with $i = 0$ we have

   $$h_{0t}(s) = (1-q)\beta h_{0,t-1}(s) + s - \max\{(1-q)L(h_{1,t-1}(s)) + qL(h_{0,t-1}(s)), 0\}.$$

   Since $L(h_{i,t-1}(s))$ with $i = 0, 1$ are both nonincreasing in s from the induction hypothesis and Lemma 6.2(a), we immediately obtain that $h_{0t}(s)$ is nodecreasing in $s \in \mathcal{S}_0^-$.
   ii. Case of $s \in \mathcal{S}_0^+$. Then $Q_1 > 0$ due to Lemma 6.9. Accordingly, since the optimal decisions are to continue the search in both states $u(\phi, 0)$ and $(\phi, 0)$, the optimal equations Eqs. (5.3) and (5.4) with $i = 1$ become

$$
\begin{aligned}
u(\phi,0) &= \lambda\beta v(0) + (1-\lambda)\beta u(\phi,0) - c + s, \\
u(\phi,1) &= (1-q)\beta(\lambda v(1) + (1-\lambda)u(\phi,1)) + q\beta(\lambda v(0) + (1-\lambda)u(\phi,0)) - c.
\end{aligned}
$$

Here, let us define the following recurrent relations corresponding to the above equations.

$$
\begin{aligned}
u_t(\phi,0) &= \beta u_{t-1}(\phi,0) + \lambda\beta(v_{t-1}(0) - u_{t-1}(\phi,0)) - c + s, \\
u_t(\phi,1) &= (1-q)\beta u_{t-1}(\phi,1) + q\beta u_{t-1}(\phi,0) \\
&\quad + \lambda(1-q)\beta(v_{t-1}(1) - u_{t-1}(\phi,1)) + \lambda q\beta(v_{t-1}(0) - u_{t-1}(\phi,0)) - c.
\end{aligned}
$$

Accordingly, noting $T(h_{it}(s)) = v_t(i) - u_t(\phi,i)$ in Eq. (A.6), from Eq. (A.5) with $i=0$ we have

$$
\begin{aligned}
h_{0t}(s) &= (1-q)\beta h_{0,t-1}(s) - \lambda(1-q)\beta T(h_{1,t-1}(s)) + \lambda(1-q)\beta T(h_{0,t-1}(s)) + s \\
&= (1-q)M(h_{0,t-1}(s)) - (1-q)L(h_{1,t-1}(s)) + s,
\end{aligned}
$$

which is nondecreasing in $s \in \mathcal{S}_0^+$ from Lemmas 6.2(g,a).

Now, since $h_{0t}(s)$ is continuous in $s \in \mathcal{S}_0^- \cup \mathcal{S}_0^+$, it eventually follows that $h_{0t}(s)$ is nondecreasing in $s \geq 0$.

2. *Proof for the monotonicity of $h_i(s)$ in s for $1 \leq i \leq n-2$.* From Eqs. (A.2) and (A.5) for $1 \leq i \leq n-2$ we have

$$
\begin{aligned}
h_{it}(s) &= (1-q)\beta u_{t-1}(\phi,i) + q\beta u_{t-1}(\phi,i-1) \\
&\quad + \max\{\lambda(1-q)\beta(v_{t-1}(i) - u_{t-1}(\phi,i)) + \lambda(1-q)\beta(v_{t-1}(i-1) - u_{t-1}(\phi,i-1)) - c, 0\} \\
&\quad - (1-q)\beta u_{t-1}(\phi,i+1) - q\beta u_{t-1}(\phi,i) \\
&\quad - \max\{\lambda(1-q)\beta(v_{t-1}(i+1) - u_{t-1}(\phi,i+1)) + \lambda(1-q)\beta(v_{t-1}(i) - u_{t-1}(\phi,i)) - c, 0\} \\
&= (1-q)\beta h_{i,t-1}(s) + q\beta h_{i-1,t-1}(s) + \max\{(1-q)L(h_{i,t-1}(s)) + qL(h_{i-1,t-1}(s)), 0\} \\
&\quad - \max\{(1-q)L(h_{i+1,t-1}(s)) + qL(h_{i,t-1}(s)), 0\}. \qquad \text{(A.8)}
\end{aligned}
$$

Accordingly, for any $s < s'$, from Eq. (A.8) we obtain

$$
\begin{aligned}
h_{it}(s) - h_{it}(s') &\leq (1-q)\beta\big(h_{i,t-1}(s) - h_{i,t-1}(s')\big) + q\beta\big(h_{i-1,t-1}(s) - h_{i-1,t-1}(s')\big) \\
&\quad + \max\{(1-q)\big(L(h_{i,t-1}(s)) - L(h_{i,t-1}(s'))\big) + q\big(L(h_{i-1,t-1}(s)) - L(h_{i-1,t-1}(s'))\big), 0\} \\
&\quad + \max\{(1-q)\big(L(h_{i+1,t-1}(s')) - L(h_{i+1,t-1}(s))\big) + q\big(L(h_{i,t-1}(s')) - L(h_{i,t-1}(s))\big), 0\}
\end{aligned}
$$

Since $L(h_{i,t-1}(s')) \leq L(h_{i,t-1}(s))$ for all i due to the induction hypothesis and Lemma 6.2(a), we have

$$
\begin{aligned}
h_{it}(s) - h_{it}(s') &\leq (1-q)\Big(\big(\beta h_{i,t-1}(s) + L(h_{i,t-1}(s))\big) - \big(\beta h_{i,t-1}(s') + L(h_{i,t-1}(s'))\big)\Big) \\
&\quad + q\Big(\big(\beta h_{i-1,t-1}(s) + L(h_{i-1,t-1}(s))\big) - \big(\beta h_{i-1,t-1}(s') + L(h_{i-1,t-1}(s'))\big)\Big) \\
&= (1-q)\Big(M(h_{i,t-1}(s)) - M(h_{i,t-1}(s'))\Big) + q\Big(M(h_{i-1,t-1}(s)) - M(h_{i-1,t-1}(s'))\Big) \leq 0
\end{aligned}
$$

due to Lemma 6.2(g). Hence, $h_{it}(s)$ in nondecreasing in s for $1 \leq i \leq n-2$.

3. *Proof for the monotonicity of $h_{n-1}(s)$ in s.* From Eqs. (A.2) with $i = n-1$, (A.3), and (A.6) we obtain

$$
\begin{aligned}
h_{n-1,t}(s) &= (1-q)\beta h_{n-1,t-1}(s) + q\beta h_{n-2,t-1}(s) \\
&\quad + \max\{(1-q)L(h_{n-1,t-1}(s)) + qL(h_{n-2,t-1}(s)), 0\} - \max\{qL(h_{n-1,t-1}(s)) - (1-q)c, 0\}. \qquad \text{(A.9)}
\end{aligned}
$$

Accoridngly, in almost the same way as the proof of $1 \le i < n-1$, from Eq. (A.9) we can also prove $h_{n-1,t}(s) \le h_{n-1,t}(s')$, i.e., $h_{n-1,t}(s)$ is nondecreasing in s.

From all the above it eventually follows that $h_i(s)$ is nondecreasing in s for $0 \le i < n$. ▮

B. Example of T-function

Let us show an example of T-function, which is used for the numerical experiments conducted in Section 8. Let $F(w)$ be the uniform distribution on $[a, b]$ with $0 < a < b < \infty$.

1. *Admissio control problem*: Noting that $T(x)$ can be rewritten $T(x) = \int_x^b (w-x)dF(w)$, we have $T(0) = 0.5(b+a)$ and

$$T(x) = \begin{cases} 0.5(b+a) - x & \text{for} \quad x < a, \\ 0.5(b-x)^2/(b-a) & \text{for} \quad a \le x < b, \\ 0 & \text{for} \quad b \le x. \end{cases} \tag{B.1}$$

2. *Pricing contro problem*: It is evident from Eqs. (2.1) and (2.2) that $p(z) = 1$ for $z < a$, $p(z) = (b-z)/(b-a)$ for $a \le z < b$, and $p(z) = 0$ for $b \le z$. For convenience, let $g(z, x) = p(z)(z-x)$, hence $T(x) = \max_z g(z, x)$. Further, for any real numbers z and x let us define $y(z, x) = (b-z)(z-x)/(b-a)$, and by $z^*(x)$ let us denote the z attaining the maximum of $y(z, x)$ for any given $x \in (-\infty, \infty)$. Then $g(z, x)$ can be expressed as follows.

$$g(z,x) = \begin{cases} z - x & \text{for} \quad z < a, \\ y(z,x) & \text{for} \quad a \le z < b, \\ 0 & \text{for} \quad b \le z. \end{cases} \tag{B.2}$$

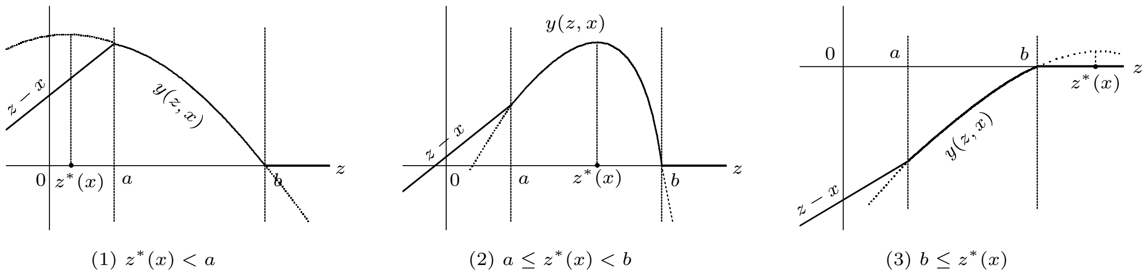

Figure 2.4: graphs of $g(z, x)$

Here, note that $g(z, x)$ is maximized at $z = z(x)$ by the definition where $a \le z(x) \le b$ from Lemma 6.1(a,b). Accordingly, since $\partial y(z, x)/\partial z = (-2z + x + b)/(b-a)$, we obtain $z^*(x) = (x+b)/2$. Then $g(z, x)$ can be depicted as any one of the three graphs in Figure 2.4, depending on a value that $z^*(x)$ takes on. From the figure it is immediately seen that (1) if $z^*(x) < a$, hence $x < 2a - b$, then $z(x) = a$, thus $T(x) = p(a)(a-x) = a - x$, (2) if $2a - b \le z^*(x) < b$, hence $2a \le x < b$, then $z(x) = (x+b)/2$, thus $T(x) = p\big((x+b)/2\big)\big((x+b)/2 - x\big) = 0.25(b-x)^2/(b-a)$, and (3) if $z^*(x) \ge b$, hence $x \ge b$, then $z(x) = b$, thus $T(x) = p(b)(b-x) = 0$, which can be summarized as follows.

$$T(x) = \begin{cases} a - x & \text{for} \quad x < 2a - b & \rightarrow \quad z(x) = a, \\ 0.25(b-x)^2/(b-a) & \text{for} \quad 2a - b \le x < b & \rightarrow \quad z(x) = (x+b)/2, \\ 0 & \text{for} \quad b \le x & \rightarrow \quad z(x) = b. \end{cases} \tag{B.3}$$

from which we see that $x^\star = 2a - b$. Accordingly, if $x^\star = 2a - b > 0$, then $T(0) = a$, or else $T(0) = 0.25\, b^2/(b-a)$.

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