CUSTOMERS SELECTION PROBLEM WITH IDLING PROFIT WHERE ONLY ONE CUSTOMER IS ALLOWED TO BE HELD

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Abstract

This paper deals with a decision problem on whether or not to accept orders from customers sequentially arriving at a custom production company where an idling profit is yielded by conducting production of products with standard specifications when there exists no backorder in the company and where a penalty is paid for not keeping an appointed date of delivery. We discuss the admission control problem and pricing control problem in an identical framework. Properties of the optimal decision rule maximizing the total expected present discounted net profit gained over an infinite planning horizon are examined and clarified.

1 Introduction

This paper deals with the problem of selecting which profitable orders to accept out of sequentially arriving customers in a custom production company, such as a shipbuilding company, advertising agency, consulting company, design office, construction firm, and so on. In the problem, we should note that the two kinds of opportunity loss described below closely relate to the problem.

1. *Opportunity loss I*. Suppose that orders from all arriving customers are accepted irrespective of their profitabilities. In this case, the production process soon becomes full; with the result that orders from customers arriving thereafter can not be accepted, however high their profitability potential may be, for the reason that if they are accepted, they can not be processed by the appointed date for delivery. This leads to an opportunity loss that if adequate allowance had been kept in the production lines by having rejected less profitable orders in advance, the company could have enjoyed upcoming profitable orders. We shall refer to this loss as Opportunity loss I.
2. *Opportunity loss II*. Excessively refraining from accepting orders due to apprehension that Opportunity loss I could occur causes a reduced number of backorders. This time, the production process soon becomes idle, implying an opportunity loss where if more orders had been accepted in advance, profit could have been gained from them. We shall refer to this loss as Opportunity loss II.

Both Opportunity losses cause a diminishment in the long run profit. The objective here therefore is to find an optimal customers selection rule so as to maximize an expected long run profit through keeping an appropriate level of backorders by controlling the number of orders to accept in advance with the aim to avoid Opportunity losses I and II. This problem is usually called the customers selection problem.

This class of problems has been studied as the admission control problem and the pricing control problem. In the former, a customer offers a price for his order, and judging from this, the company decides whether or not to accept. In the latter, by contrast, the system offers a price for an order, and judging from this, the customer decides whether or not to place an order with the company.

Optimal policies in the admission control problem were originally considered by Heyman [2]. This was later applied to the queueing system with a finite customer class and finite-capacity by Miller [10] and continued by Lippman and Ross [7] for a single-server with uncountable customer classes. In [3] a model for a discrete-time process was formulated by Ikuta. Optimal pricing policies in pricing control were discussed by Low [8], more recently by You [16] and Feng and Xiao [1] for yield management, and by Fu et al. [9] for queueing staffing problem. These two problems, i.e., the admission control and the pricing control problems are separately formulated and analyzed in Yoon and Lewis's [15], which is the latest report. In our paper we show that both problems can be treated in an identical framework; see [5, Ikuta] for a general discussion on the integration of admission control problem and pricing control problem.

In this paper a search cost is paid to search for a customer where without paying a search cost at a point in time, no customer arrives at the next point in time. The introduction of the search cost inevitably yields the option of *skipping* the search or not. Furthermore, we introduce *penalty cost* and *idling profit*, which are not taken into account in any other papers. A penalty cost penalizes the company (system) for not keeping an appointed date of delivery; if an order accepted can not be completed and delivered up to the appointed date, the company has to pay a penalty to the customer for the period delayed. The idling profit is yielded when there is no backorder in the system. For example, consider a custom production company manufacturing products with special and general specifications. When all the products with special specification that have been accepted to a point have been completed and the production process has become idle, the production is shifted to general specification products to yield profit again. We clarify the relationship of the optimal policies with the penalty cost, the search cost, and the idling profit.

The objective here is to find the optimal decision rule to maximize the total expected present discounted net profit gained over an infinite planning horizon, the total expected present discounted value of prices of orders accepted whether in the admission control problem or in the pricing control problem *plus* the idling profits *minus* the search costs *minus* the penalty costs.

Here by n let us denote the maximum number of orders that can be held in the system. The way of analysis for the case of $n = 1$ is very different from that for the case of $n \geq 2$. In this paper we only discuss the case of $n = 1$; see [14, Son] for the case of $n \geq 2$.

Section 2 provides a strict definition for the model. Section 3 describes the optimal equation of the model, and this is transformed in Section 4. Three functions are defined in Section 5. The properties of the optimal decision rule are clarified in Section 6, and based on this the optimal decision rule is prescribed in Section 7. Section 8 summarizes the conclusions obtained in the previous sections and Section 9 suggests some subjects of study to be tackled in the future.

2 Model

□ *Assumptions* The model examined in the paper is defined on the assumptions below:

A1. The model is defined as a discrete-time sequential stochastic decision process with an infinite planning horizon. Let points in time be equally spaced on the axis of the planning horizon, and let the time interval between successive points in time be called the period.

A2. It is only when a search is enacted by paying a *search cost* $c \geq 0$ at a point in time that a customer arrives at the next point in time with a probability λ ($0 < \lambda \leq 1$).

A3. Let the prices offered by subsequently appearing customers, $w, w', \cdots$, in the admission control problem and the maximum permissible ordering prices of subsequently appearing customers, $w, w', \cdots$, in the pricing control problem be both independent and identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ. Then, in the pricing control problem, if the system offers a price z to an appearing customer, the probability of the customer placing the order with the system is given by

$$p(z) = \Pr\{z \leq w\}. \tag{2.1}$$

In both the admission control and pricing control, for certain given numbers a and b ($0 < a < b < \infty$) let us define the probability density function as follows;

$$f(w) = 0, \quad w < a, \qquad f(w) > 0, \quad a \le w \le b, \qquad f(w) = 0, \quad b < w \tag{2.2}$$

where clearly $a < \mu < b$. Throughout the paper, let us denote the expectation of a given function $g(w)$ as to w by $\mathbf{E}[g(w)]$.

A4. With a probability q ($0 < q < 1$) an order in the system at a certain point in time is completed and goes out of the system at the next point in time.

A5. When there exists no backorder in the system, an *idling profit* $s \ge 0$ is yielded by engaging in other economic activities using the idle production line.

A6. A contract is assumed to be signed with all the clients that any order accepted is delivered within τ periods, and a clause is added that if the contract can not be honored, then a penalty $\theta \ge 0$ is paid for a period delayed. Accordingly, for an order accepted l periods ago if $\tau \le l$, then the penalty must be paid, or else it does not need to be paid; here note that $\tau = l$ implies that the order has not yet been completed at the latest date signed, hence at least one period is needed for its completion, and the penalty θ must be paid for the order.

A7. Let the discount factor be denoted by $\beta < 1$.

□ *Decision rules* The decision on the problem is based on the following three rules:

(1) The rule whether or not to accept an order from arriving customers in the admission control problem.

(2) The rule as to the ordering price to offer in the pricing control problem.

(3) The rule whether to continue or to skip the search in both problems.

□ *Relationship with optimal stopping problems* From the viewpoint of the optimal stopping problem, our model can be interpreted as follows. In the standard model of the optimal stopping problem [13, Sakaguchi], once an offer is accepted, the process is assumed to terminate at that time. If the assumption is changed into one where even if an offer (customer) is accepted, another one can be accepted at some period thereafter, deterministic or stochastic, then the variation can be basically reduced to our model. By introducing different concepts, as stated in Section 9, in various optimal stopping problems, we may develop different variations of our model.

□ *Notations* For convenience in the later discussions let us define

$$\eta \;=\; (1-q)\beta < 1, \tag{2.3}$$

$$\gamma \;=\; (1-\eta)^{-1} > 1, \tag{2.4}$$

$$\rho \;=\; \gamma\eta^{\tau}\theta \ge 0, \tag{2.5}$$

$$\chi \;=\; c\gamma(1-q)/q + \gamma s + \rho \tag{2.6}$$

where it can be easily seen that

$$1 - \gamma q\beta \;=\; \gamma(1-\beta) > 0. \tag{2.7}$$

For expressional simplicity, we define the notations such as in Table 2.1.

3 Optimal Equations

Either if the search was skipped at the previous point in time or if no customer has appeared with probability $1-\lambda$ regardless of having conducted the search at the previous point in time, it follows that

Table 2.1: Definition of notations

Notation	Definition	Notation	Definition	Notation	Definition
$\mathtt{C}$	continuing the search	$\langle\mathtt{C}\rangle$	Each corresponding decision is optimal	$\langle\mathtt{O}(z)\rangle$	It is optimal to offer the price z for an order in pricing control
$\mathtt{K}$	skipping the search	$\langle\mathtt{K}\rangle$			
$\mathtt{A}$	accepting an order	$\langle\mathtt{A}\rangle$		$\langle\mathtt{A}(w)\rangle$	It is optimal to accept an appearing order w in admission control
$\mathtt{R}$	rejecting an order	$\langle\mathtt{R}\rangle$		$\langle\mathtt{R}(w)\rangle$	It is optimal to reject an appearing order w in admission control

*We do not use $\mathtt{S}$ as a notation representing "skipping the search" because it is often used for representing "stopping the search"

no customer appears at the present point in time. For convenience, we shall refer to such a situation as "the system has a *fictitious customer* ϕ", called the state (ϕ).

In both admission control and pricing control, by $u(\phi)$ and $u(\phi,l)$ we shall denote the maximum total expected present discounted net profits starting from state (ϕ), provided, respectively, that there exists no order in the system and that there exists an order accepted l periods ago in the system. Since the expectation of immediate reward at any point in time is clearly finite, using the conventional way outlined in the discussion of the Markovian decision process [11, Ross](p29-30), we can easily show that $|u(\phi)| \leq M/(1-\beta)$ and $|u(\phi,l)| \leq M/(1-\beta)$, $l \geq 0$, for a sufficiently large $M > 0$, i.e., $u(\phi)$ is finite and $u(\phi,l)$ is bounded in $l \geq 0$. Suppose the system is a state (ϕ,l) at a certain point in time. Then if $\tau \leq l$, the penalty θ must be paid at the present point in time, or else it does not need to be paid. For convenience in the discussions that follow, let us define

$$h \;=\; u(\phi) - u(\phi,0). \tag{3.1}$$

1. *Admission control problem*: By $u(w)$ let us denote the maximum total expected present discounted net profit starting with an appearing customer w and with no order in the system. Then we get

$$u(\phi) \;=\; \max\left\{\begin{array}{l} \mathtt{C}: \beta\big(\lambda\,\mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)\big) - c + s, \\ \mathtt{K}: \beta u(\phi) + s, \end{array}\right. \tag{3.2}$$

$$u(\phi,l) \;=\; -\theta I(\tau \leq l) + \max\left\{\begin{array}{l} \mathtt{C}\text{:}\,(1-q)\beta u(\phi,l+1) \\ \qquad\qquad + q\beta\big(\lambda\,\mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)\big) - c, \\ \mathtt{K}: (1-q)\beta u(\phi,l+1) + q\beta u(\phi) \end{array}\right\}, \quad l \geq 0, \tag{3.3}$$

$$u(w) \;=\; \max\left\{\begin{array}{l} \mathtt{A}: w + u(\phi,0), \\ \mathtt{R}: u(\phi) \end{array}\right\} \geq w. \tag{3.4}$$

The last equation can be rearranged as

$$u(w) \;=\; \max\{w-h, 0\} + u(\phi). \qquad \square \tag{3.5}$$

2. *Pricing control problem*: By $u(1)$ let us denote the maximum total expected present discounted net profit starting with an appearing customer and with no order in the system. Then we have

$$u(\phi) = \max \begin{cases} \texttt{C}: \beta\big(\lambda u(1) + (1-\lambda)u(\phi)\big) - c + s, \\ \texttt{K}: \beta u(\phi) + s, \end{cases} \tag{3.6}$$

$$u(\phi, l) = -\theta I(\tau \le l) + \max \left\{ \begin{array}{l} \texttt{C}: (1-q)\beta u(\phi, l+1) \\ \qquad\qquad + q\beta\big(\lambda u(1) + (1-\lambda)u(\phi)\big) - c, \\ \texttt{K}: (1-q)\beta u(\phi, l+1) + q\beta u(\phi) \end{array} \right\}, \quad l \ge 0, \tag{3.7}$$

$$u(1) = \max_z \{p(z)\big(z + u(\phi, 0)\big) + (1 - p(z))u(\phi)\} \ge \max_z p(z)z. \tag{3.8}$$

Eq. (3.8) can be rearranged as

$$u(1) = \max_z p(z)(z-h) + u(\phi). \tag{3.9}$$

See Lemma 4.1 for the proof of the unique existence of the solutions for the above system of equations.

4 Transformation of the Optimal Equations

Let us define the following functions:

1. For any real number x let

$$T(x) = \begin{cases} \mathbf{E}[\max\{w - x, 0\}] & \text{for the admission control problem,} \\ \max_z p(z)(z-x) & \text{for the pricing control problem,} \end{cases} \tag{4.1}$$

called the *T-function* [5, Ikuta] [16, You]. Note $T(0) > 0$ (See [5]). In the pricing control problem, for a given x by $z(x)$ let us designate the z attaining the maximum of $p(z)(z-x)$ on $(-\infty, \infty)$ if it exists, i.e., $T(x) = p(z(x))(z(x) - x)$.

2. Let us define

$$L(h) = \lambda\beta T(h) - c, \qquad \dot{L}(h) = \lambda q\beta T(h) - c. \tag{4.2}$$

When regarding h as a function of s, i.e., $h = h(s)$, by s^* and $\dot{s}^*$ let us denote the solutions of $L(h(s)) = 0$ and $\dot{L}(h(s)) = 0$, respectively, if they exist, i.e.,

$$L(h(s^*)) = 0, \qquad \dot{L}(h(\dot{s}^*)) = 0. \tag{4.3}$$

If multiple solutions exist in each of the above two equations, let us define the *smallest* of them as s^* and $\dot{s}^*$, respectively.

Further, let us define

$$v(0) = \begin{cases} \mathbf{E}[u(w)] & \text{for the admission control problem,} \\ u(1) & \text{for the pricing control problem.} \end{cases} \tag{4.4}$$

Then since $\mathbf{E}[u(w)] \ge \mu = T(0)$ from Eq. (3.4) and $u(1) \ge \max_z p(z)z = T(0)$ from Eq. (3.8), we have

$$v(0) \ge T(0). \tag{4.5}$$

Now using the definition Eq. (4.1), we can express both of "Eqs. (3.2), (3.3), and (3.5)" and "Eqs. (3.6), (3.7), and (3.9)" by the identical equations as below.

$$u(\phi) = \max\{\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - c, \beta u(\phi)\} + s, \tag{4.6}$$

$$u(\phi, l) = -\theta I(\tau \le l) + \max \left\{ \begin{array}{l} (1-q)\beta u(\phi, l+1) + q\beta\big(\lambda v(0) + (1-\lambda)u(\phi)\big) - c \\ (1-q)\beta u(\phi, l+1) + q\beta u(\phi) \end{array} \right\}, \quad l \ge 0, \tag{4.7}$$

$$v(0) = T(h) + u(\phi). \tag{4.8}$$

Further, Eqs. (4.6) and (4.7) can be rearranged into, respectively,

$$u(\phi) \;=\; \beta u(\phi) + \max\{\lambda\beta\big(v(0) - u(\phi)\big) - c, 0\} + s, \tag{4.9}$$

$$u(\phi, l) \;=\; (1-q)\beta u(\phi, l+1) + q\beta u(\phi) + \max\{\lambda q\beta\big(v(0) - u(\phi)\big) - c, 0\} - \theta I(\tau \le l), \;\; l \ge 0. \tag{4.10}$$

Using Eq. (4.8), we can rearrange Eqs. (4.9) and (4.10) as follows, respectively,

$$u(\phi) \;=\; (\max\{L(h), 0\} + s)/(1-\beta) \ge 0, \tag{4.11}$$

$$u(\phi, l) \;=\; (1-q)\beta u(\phi, l+1) + q\beta u(\phi) + \max\{\dot{L}(h), 0\} - \theta I(\tau \le l), \quad l \ge 0. \tag{4.12}$$

Lemma 4.1 *The system of* Eqs. (4.6) *to* (4.8) *has a unique solution.*

Proof. See App. A. ▮

Lemma 4.2 Eq. (4.12) *with* $l = 0$ *can be rewritten*

$$u(\phi, 0) \;=\; \gamma q\beta u(\phi) + \gamma \max\{\dot{L}(h), 0\}) - \rho. \tag{4.13}$$

Proof. See App. B. ▮

Now, Eqs. (3.5) and (4.11) and Eqs. (3.9) and (4.12) tell us that the optimal decision rules can be prescribed as follows.

□ *Optimal Decision Rule 4.1*

1. Admission control problem:
   i. If $w > h$, then $\langle \mathtt{A}(w)\rangle$, or else $\langle \mathtt{R}(w)\rangle$.
   ii. If $L(h) > 0\,\big(\dot{L}(h) > 0\big)$, then $\langle \mathtt{C}\rangle_\phi\,(\langle \mathtt{C}\rangle_l)^*$, or else $\langle \mathtt{K}\rangle_\phi\,(\langle \mathtt{K}\rangle_l)$.
2. Pricing control problem:
   i. $\langle \mathtt{O}(z(h))\rangle$.
   ii. If $L(h) > 0\,\big(\dot{L}(h) > 0\big)$, then $\langle \mathtt{C}\rangle_\phi\,(\langle \mathtt{C}\rangle_l)$, or else $\langle \mathtt{K}\rangle_\phi\,(\langle \mathtt{K}\rangle_l)$.

Note. The optimal decision rule is independent of l.

5 Preliminaries

To begin with, for any real number x let us define the following three functions, which becomes inevitably necessary in the analyses of Section 6.

$$G(x) \;=\; \gamma\big(\max\{L(x), 0\} - \max\{\dot{L}(x), 0\}\big) - x + \gamma s + \rho, \tag{5.1}$$

$$B_1(x) \;=\; T(x) - (x - \gamma s - \rho)/\gamma\lambda(1-q)\beta, \tag{5.2}$$

$$B_2(x) \;=\; T(x) - (x + \gamma(c - s) - \rho)/\gamma\lambda\beta. \tag{5.3}$$

Further, by x^*, x_1^*, and x_2^* let us denote the solution of, respectively, $G(x) = 0$, $B_1(x) = 0$, and $B_2(x) = 0$, if they exist, i.e.,

$$G(x^*) = 0, \qquad B_1(x_1^*) = 0, \qquad B_2(x_2^*) = 0. \tag{5.4}$$

For convenience in the later discussions, let us denote x_1^* and x_2^* for $s = 0$ by $x_1^*(0)$ and $x_2^*(0)$, respectively. Now, noting Eqs. (4.13) and (2.7), we can rewrite Eq. (3.1)

$$h \;=\; u(\phi) - u(\phi, 0) = \gamma(1-\beta)u(\phi) - \gamma\max\{\dot{L}(h), 0\} + \rho. \tag{5.5}$$

Rearranging Eq. (5.5) by substituting Eq. (4.11) yields

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*The notations $\langle \mathtt{C}\rangle_\phi$ and $\langle \mathtt{K}\rangle_l$ imply that conducting the search and skipping the search, respectively, is optimal in state (ϕ) and (ϕ, l) for $l \ge 0$.

$$h \;=\; \gamma\big(\max\{L(h),0\} - \max\{\dot{L}(h),0\}\big) + \gamma s + \rho, \tag{5.6}$$

which can be eventually rewritten

$$G(h) = 0; \tag{5.7}$$

in other words, the h, defined by Eq. (3.1), is given by the solution of $G(x) = 0$.

Lemma 5.1

(a) *$z(x)$ exists with $z(x) \geq a$.*

(b) *$a \leq z(x) \leq b$ for all x where $z(x) = b$ if $x \geq b$.*

(c) *If $x < b$, then $x < z(x) < b$.*

(d) *$z(x)$ is nondecreasing in x.*

Proof. See [5]. ▮

Lemma 5.2 *In both admission control and pricing control we have*:

(a) *$T(x)$ is continuous and nonincreasing on $(-\infty,\infty)$ and strictly decreasing on $(-\infty,b]$.*

(b) *$T(x) \geq 0$ on $(-\infty,\infty)$.*

(c) *$T(x) > 0$ on $(-\infty,b)$, and $T(x) = 0$ on $[b,\infty)$.*

(d) *If $T(x) = \;(>)\, 0$, then $b \leq \;(>)\, x$.*

(e) *For any given $y > 0$ the equation $T(x) = y$ has a unique solution, less than b.*

(f) *$\lim_{x\to\infty} T(x) = 0$ and $\lim_{x\to-\infty} T(x) = \infty$.*

(g) *$\nu T(x) + x$ is nondecreasing in x and strictly increasing in x if $\nu < 1$.*

(h) *$\lim_{x\to\infty} \nu T(x) + x = \infty$, and if $\nu < 1$, then $\lim_{x\to-\infty} \nu T(x) + x = -\infty$.*

Proof. See [5]. ▮

Lemma 5.3

(a) *$G(x)$ is strictly decreasing in x.*

(b) *$G(x) < (>)\, 0$ for any sufficiently large $x > 0\,(\,x < 0)$.*

(c) *$G(x)$ is nonincreasing in c and strictly decreasing in q and τ for all x.*

(d) *$G(x)$ is nondecreasing in λ and strictly increasing in β and θ for all x.*

(e) *$G(x)$ is strictly increasing in s for all x.*

Proof. See App. C. ▮

Lemma 5.4

(a) *$B_1(x)$ and $B_2(x)$ are both strictly decreasing in x where $B_1(x) > (<)\; 0$ and $B_2(x) > (<)\, 0$ for any sufficiently small (large) x.*

(b) *x_1^* and x_2^* are both uniquely exist, which are positive if $\lambda\beta T(\rho) > c$.*

(c) *$\chi > (= (<))\, x \Leftrightarrow B_1(x) > (= (<))\, B_2(x)$ where $B_1(\chi) = B_2(\chi)$.*

(d) *$x_2^* > \chi \Leftrightarrow x_1^* > \chi$ and $x_2^* \leq \chi \Leftrightarrow x_1^* \leq \chi$.*

Proof. See App. D. ▮

6 Analysis

In this section, we shall prove many assertions in the relationship with s. For this reason let us regard h, $G(x)$, x_1^*, and x_2^* as a function of s, i.e., $h = h(s)$, $G(x, s)$, $x_1^* = x_1^*(s)$, and $x_2^* = x_2^*(s)$. However, for explanatory simplicity let us employ the notations "h, $G(x)$, x_1^*, and x_2^*" in the Lemmas 6.1 to 6.4 except Lemma 6.4(d) and "$h(s)$, $G(x, s)$, $x_1^*(s)$, and $x_2^*(s)$" in their proofs.

Lemma 6.1

(a) *x^* uniquely exists with $x^* = h$, $x^* \geq s$, and $x^* \geq \rho$.*

(b) *If $s < (b - \rho)/\gamma$ (, hence $b > \rho$ due to $s \geq 0$), then $h < b$, or else $h \geq b$.*

(c) *h is nonincreasing in c and strictly decreasing in q and τ.*

(d) *h is nondecreasing in λ and strictly increasing in β and θ.*

(e) *h is strictly increasing in s with $\lim_{s\to\infty} h = \infty$ and $\lim_{s\to-\infty} h = -\infty$.*

Proof. Below, note that $L(x) = \lambda\beta T(x) - c \geq \lambda q\beta T(x) - c = \dot{L}(x)$ for any x because $T(x) \geq 0$ due to Lemma 5.2(b).

(a) The unique existence of x^*, independent of s, is immediate from Lemma 5.3(a,b), hence $h = x^*$ from Eq. (5.7). Since $\gamma > 1$ due to Eq. (2.4), we have

$$G(s, s) = \gamma\big(\max\{L(s), 0\} - \max\{\dot{L}(s), 0\}\big) + (\gamma - 1)s + \rho \geq 0,$$

$$G(\rho, s) = \gamma\big(\max\{L(\rho), 0\} - \max\{\dot{L}(\rho), 0\}\big) + \gamma s \geq 0,$$

hence $x^* \geq s$ and $x^* \geq \rho$ due to Lemma 5.3(a).

(b) Since $L(b) = \dot{L}(b) = -c$ due to Eq. (4.2) and Lemma 5.2(c), we obtain $G(b, s) = -b + \gamma s + \rho$. If $s < (b - \rho)/\gamma$, then $G(b, s) < 0$, implying $h < b$, or else $G(b, s) \geq 0$, implying $h \geq b$.

(c,d) Evident from Lemma 5.3(c,d), respectively.

(e) The former half is immediate from Lemma 5.3(e). Suppose $h(s)$ converges to a finite $\bar{h}$ as $s \to \infty$. Then since $h(s) < \bar{h}$ for any s, we have $G(\bar{h}, s) < G\big(h(s), s\big) = 0$ due to Lemma 5.3(a); accordingly, $\lim_{s\to\infty} G(\bar{h}, s) \leq 0$ due to Lemma 5.3(e). However, $\lim_{s\to\infty} G(\bar{h}, s) = \infty$ from Eq. (5.1), which is a contradiction. Thus $h(s)$ must diverge as $s \to \infty$. Similarly also proven $\lim_{s\to-\infty} h(s) = -\infty$. ▮

Below, by s_b let us denote the solution of $h(s) = b$ if it exists, i.e., $h(s_b) = b$.

Lemma 6.2

(a) *s_b uniquely exists.*

(b) *Both $L(h)$ and $\dot{L}(h)$ are strictly decreasing in $s < s_b$ and nonincreasing in s on $(-\infty, \infty)$.*

(c) *$L(h) > 0$ and $\dot{L}(h) > 0$ for any sufficiently small s.*

(d) *If $c > 0$, then $L(h) < 0$ and $\dot{L}(h) < 0$ for any sufficiently large s.*

(e) *$L(h) > \dot{L}(h)$ for $s < s_b$ and $L(h) = \dot{L}(h) = -c$ for $s \geq s_b$.*

Proof. (a) Immediate from Lemma 6.1(e).

(b) First, for any $s < s' < s_b$ since $h(s) < h(s') < h(s_b) = b$ due to Lemma 6.1(e), we get $L(h(s)) > L(h(s'))$ and $\dot{L}(h(s)) > \dot{L}(h(s'))$ from Lemma 5.2(a), hence the former half of the assertion holds. From Lemmas 6.1(e) and 5.2(a) it is immediately seen that $L(h(s))$ and $\dot{L}(h(s))$ are nonincreasing in s.

(c) Let $c > 0$. Then from Lemmas 6.1(e) and Lemma 5.2(e) we easily see that there exists an $\bar{s}$ such that $T\big(h(\bar{s})\big) = c/\lambda\beta > 0$ and $h(\bar{s}) < b$. Now, for $s < \bar{s}$ we have $h(s) < h(\bar{s}) < b$ due to Lemma 6.1(e).

Accordingly, from Lemma 5.2(a) we get $T\big(h(s)\big) > T\big(h(\bar{s})\big) = c/\lambda\beta$, hence $0 < \lambda\beta T\big(h(s)\big) - c = L\big(h(s)\big)$. Let $c = 0$. Then for $s < s_{_b}$ we obtain $h(s) < h(s_{_b}) = b$, hence $L\big(h(s)\big) = \lambda\beta T\big(h(s)\big) > \lambda\beta T(b) = 0$ due to Lemma 5.2(b,c)). The proof of $\dot{L}\big(h(s)\big) > 0$ is also shown in quite the same way as the above.

(d) Let $c > 0$. Then for $s_{_b} \le s$ we have $b = h(s_{_b}) \le h(s)$ from Lemma 6.1(e), hence $L(h(s)) = \dot{L}(h(s)) = -c < 0$ due to Lemma 5.2(c).

(e) For any $s < s_{_b}$ we have $h(s) < h(s_{_b}) = b$ due to Lemma 6.1(e), hence $T(h(s)) > 0$ due to Lemma 5.2(c); accordingly, $L(h(s)) = \lambda\beta T(h(s)) - c > \lambda q\beta T(h(s)) - c = \dot{L}(h(s))$. For $s_{_b} \le s$ we get $b = h(s_{_b}) \le h(s)$, hence $L(h(s)) = \dot{L}(h(s)) = -c$ due to Lemma 5.2(c). ∎

From Lemma 6.2 we can depict Figure 6.1.

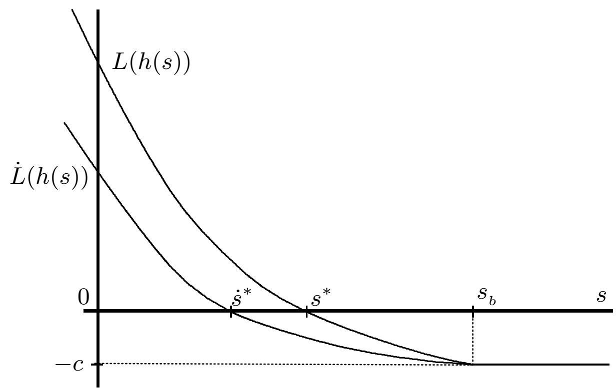

Figure 6.1: Graphs of $L(h(s))$ and $\dot{L}(h(s))$ with $c > 0$

Lemma 6.3

(a) *Let $c = 0$. Then $L(h) \ge 0$ and $\dot{L}(h) \ge 0$.*

(b) *Let $c > 0$.*

1 *Both s^* and $\dot{s}^*$ uniquely exist with $\dot{s}^* < s^* < s_{_b}$.*

2 *If $s^* \le s$, then $L(h) \le 0$ and $\dot{L}(h) \le 0$.*

3 *If $s < s^*$, then $L(h) > 0$.*

4 *If $\dot{s}^* \le s$, then $\dot{L}(h) \le 0$.*

5 *If $s < \dot{s}^*$, then $L(h) > 0$ and $\dot{L}(h) > 0$.*

Proof. (a) If $c = 0$, then $L(h) = \lambda\beta T(h) \ge 0$ and $\dot{L}(h) = \lambda q\beta T(h) \ge 0$ due to Lemma 5.2(b).

(b) Let $c > 0$ (see Figure 6.1).

(b1) The unique existence of s^* and $\dot{s}^*$ are immediate from Lemma 6.2(b,c,d). The latter half is evident from Lemma 6.2(e).

(b2-b5) Evident from Lemma 6.2(b,e). ∎

Below, let us regard s^* and $\dot{s}^*$ as functions of c, i.e., $s^*(c)$ and $\dot{s}^*(c)$, and let c^* and $\dot{c}^*$ be the solution of $s^*(c) = 0$ and $\dot{s}^*(c) = 0$, respectively, if they exist, i.e.,

$$s^*(c^*) \;=\; 0, \qquad \dot{s}^*(\dot{c}^*) = 0. \tag{6.1}$$

Lemma 6.4

(a) *Both $s^*(c)$ and $\dot{s}^*(c)$ are strictly decreasing in c.*

(b) *If* $c = 0$*, then* $s^* = \dot{s}^* = s_{_b} = (b - \rho)/\gamma$.

(c) $c^* = \lambda\beta T(\rho)$.

(d) $\dot{c}^* = q\big(x_1^*(0) - \rho\big)/\gamma(1-q)$.

(e) *If* $\rho < b$*, then* $\dot{c}^* \le \lambda q\beta T(\rho) < c^*$.

Proof. (a) Let $c < c'$. Then since $L\big(h(s^*(c))\big) = 0$ and $L\big(h(s^*(c'))\big) = 0$, we have $c = \lambda\beta T\big(h(s^*(c))\big)$ and $c' = \lambda\beta T\big(h(s^*(c'))\big)$; accordingly $\lambda\beta T\big(h(s^*(c))\big) = c < c' = \lambda\beta T\big(h(s^*(c'))\big)$. Hence $h\big(s^*(c)\big) > h\big(s^*(c')\big)$ because if not so, we have the contradiction of $T\big(h(s^*(c))\big) \ge T\big(h(s^*(c'))\big)$ due to Lemma 5.2(a). Accordingly, we obtain $s^*(c) > s^*(c')$ from Lemma 6.1(e). Similarly proven for $\dot{s}^*(c)$.

(b) Let $c = 0$. Then since $L(x) = \lambda\beta T(x) \ge 0$ and $\dot{L}(x) = \lambda q\beta T(x) \ge 0$ for all x due to Lemma 5.2(b), we have $G(x, s) = \gamma\lambda(1-q)\beta T(x) - x + \gamma s + \rho$. Hence letting $\bar{s} = (b-\rho)/\gamma$, we get $0 = G\big(h(\bar{s}), \bar{s}\big) = \gamma\lambda(1-q)\beta T(h(\bar{s})) - h(\bar{s}) + b$ or equivalently

$$\gamma\lambda(1-q)\beta T(h(\bar{s})) = h(\bar{s}) - b. \tag{6.2}$$

Since $T(h(\bar{s})) \ge 0$ due to Lemma 5.2(b), we obtain $h(\bar{s}) \ge b$ or equivalently $h(\bar{s}) \ge b = h(s_{_b})$ due to the definition of $s_{_b}$. Thus, $s_{_b} \le \bar{s}$ due to Lemma 6.1(e). Here, suppose $s_b < \bar{s}$. Then $h(\bar{s}) > h(s_b) = b$, hence $T(h(\bar{s})) > 0$ from Eq. (6.2), so that $h(\bar{s}) < b$ due to Lemma 5.2(d), which is a contradiction. Therefore it must be $\bar{s} = s_{_b}$. Now, if $s < s_{_b}$, then $h(s) < h(s_{_b})$ from Lemma 6.1(e), hence $L\big(h(s)\big) > L\big(h(s_{_b})\big) = 0$ due to Lemma 6.2(b,e with $c = 0$), and if $s_{_b} \le s$, then $L\big(h(s)\big) = 0$ due to Lemma 6.2(e with $c = 0$). Thus we have $s^* = s_{_b}$ due to the definition of s^*. The proof of $\dot{s}^* = s_{_b}$ is the same as the above.

(c) From Eqs. (4.2) and (4.3) we have, for any c,

$$c \;=\; \lambda\beta T\big(h(s^*)\big), \tag{6.3}$$

$$c \;=\; \lambda q\beta T\big(h(\dot{s}^*)\big). \tag{6.4}$$

Since $0 = L(h(s^*)) \ge \dot{L}(h(s^*))$ due to Lemma 5.2(b) and $h(s^*)$ is the solution of $G(x, s^*) = 0$, from Eq. (5.1) we get

$$0 \;=\; G\big(h(s^*), s^*\big) = -h(s^*) + \gamma s^* + \rho,$$

hence $h(s^*) = \gamma s^* + \rho$. Accordingly, if $c = c^*$, then $s^* = s^*(c^*) = 0$, hence since $h(s^*) = \rho$, from Eq. (6.3) we have $c^* = \lambda\beta T\big(h(s^*)\big) = \lambda\beta T(\rho)$.

(d) Rearranging Eq. (5.1) with $x = h\big(\dot{s}^*\big)$ by substituting (6.4) yields

$$\begin{aligned}
0 = G\big(h(\dot{s}^*), \dot{s}^*\big) \;&=\; \gamma\big(\max\{\lambda\beta T\big(h(\dot{s}^*)\big) - c, 0\} - \max\{\lambda q\beta T\big(h(\dot{s}^*)\big) - c, 0\}\big) - h(\dot{s}^*) + \gamma\dot{s}^* + \rho \\
&=\; \gamma\max\{\lambda\beta T\big(h(\dot{s}^*)\big) - \lambda q\beta T\big(h(\dot{s}^*)\big), 0\} - h(\dot{s}^*) + \gamma\dot{s}^* + \rho \\
&=\; \gamma\max\{\lambda(1-q)\beta T\big(h(\dot{s}^*)\big), 0\} - h(\dot{s}^*) + \gamma\dot{s}^* + \rho \\
&=\; \gamma\lambda(1-q)\beta T\big(h(\dot{s}^*)\big) - h(\dot{s}^*) + \gamma\dot{s}^* + \rho
\end{aligned}$$

due to Lemma 5.2(b), from which

$$T\big(h(\dot{s}^*)\big) \;=\; \big(h(\dot{s}^*) - \gamma\dot{s}^* - \rho\big)/\gamma\lambda(1-q)\beta,$$

implying

$$B_1\big(h(\dot{s}^*)\big) \;=\; 0, \tag{6.5}$$

hence $x_1^*(\dot{s}^*) = h(\dot{s}^*)$ from Lemma 5.4(b). If $c = \dot{c}^*$, then $\dot{s}^* = \dot{s}^*(\dot{c}^*) = 0$ due to Eq. (6.1), thus $x_1^*(0) = h(0)$. Accordingly,

$$T\big(x_1^*(0)\big) \;=\; (x_1^*(0)-\rho)/\gamma\lambda(1-q)\beta. \tag{6.6}$$

Now, from Eq. (6.4) we have $\dot{c}^* = \lambda q\beta T(h(\dot{s}^*)) = \lambda q\beta T(h(0)) = \lambda q\beta T\big(x_1^*(0)\big)$, from which $T\big(x_1^*(0)\big) = \dot{c}^*/\lambda q\beta$, hence from Eq. (6.6) $\dot{c}^*/\lambda q\beta = \big(x_1^*(0)-\rho\big)/\gamma\lambda(1-q)\beta$ or equivalently $\dot{c}^* = q\big(x_1^*(0)-\rho\big)/\gamma(1-q)$.

(e) Since $T(x_1^*(0)) \geq 0$ due to Lemma 5.2(b), we have $x_1^*(0) \geq \rho$ from Eq. (6.6), thus $T(\rho) \geq T(x_1^*(0))$ from Lemma 5.2(a). Accordingly, from Eqs. (6.6) and (d) we have

$$T(\rho) \geq \frac{x_1^*(0)-\rho}{\gamma\lambda(1-q)\beta} = \frac{q(x_1^*(0)-\rho)/\gamma(1-q)}{\lambda q\beta} = \frac{\dot{c}^*}{\lambda q\beta},$$

hence $\dot{c}^* \leq \lambda q\beta T(\rho)$. If $\rho < b$, then since $T(\rho) > 0$ from Lemma 5.2(c), we obtain $\dot{c}^* \leq \lambda q\beta T(\rho) < \lambda\beta T(\rho) = c^*$ due to the assumption of $q < 1$ and (c). ∎

Lemma 6.5

(a) *Let $c^* > c$.*

1 *$u(\phi) > 0$.*

2 *$s^* > 0$.*

3 *Let $s < s^*$.*

i *If $\dot{s}^* \leq s$, then $h = x_2^* \leq \chi$.*

ii *If $s < \dot{s}^*$, then $h = x_1^* > \chi$.*

(b) *Let $c^* \leq c$. Then $\dot{L}(h) \leq L(h) \leq 0$.*

Proof. (a) Let $c^* > c$.

(a1) From Eqs. (4.6), (4.5), and (4.11) we have $u(\phi) \geq \lambda\beta v(0) + (1-\lambda)\beta u(\phi) - c + s \geq \lambda\beta T(0) - c + s$. Further since $T(0) \geq T(\rho)$ from Eq. (2.5) and Lemma 5.2(a), from the above inequality we get $u(\phi) \geq \lambda\beta T(\rho) - c + s = c^* - c + s > 0$ due to Lemma 6.4(c) and the assumption of $s \geq 0$.

(a2) Let $s = 0$. Assume $\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - c \leq \beta u(\phi)$. Then $u(\phi) = \beta u(\phi)$ from Eq. (4.6), leading to $\beta = 1$ due to $u(\phi) > 0$ from (a1), which contradicts the assumption of $\beta < 1$. Accordingly, it must be $\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - c > \beta u(\phi)$, which can be rearranged into $0 < \lambda\beta(v(0) - u(\phi)) - c = \lambda\beta T(h) - c = L(h)$ from Eq. (4.8), implying $s^* > 0$ due to Lemma 6.2(b).

(a3) Let $s < s^*$. Then $L(h) > 0$ from Lemma 6.3(b3). Here note $\dot{s}^* < s^*$ from Lemma 6.3(b1).

(a3i) Let $\dot{s}^* \leq s$. Then $\dot{L}(h) \leq 0$ from Lemma 6.3(b4). Accordingly, from Eq. (5.6) we have $h = \gamma L(h) + \gamma s + \rho = \gamma\lambda\beta T(h) - \gamma(c-s) + \rho$, hence $T(h) - \big(h + \gamma(c-s) - \rho\big)/\gamma\lambda\beta = 0$, i.e., $B_2(h) = 0$ from Eq. (5.3), implying that h defined by Eq. (3.1) is given by x_2^*, which is the unique solution of $B_2(x) = 0$ due to Lemma 5.4(b), i.e., $h = x_2^*$, hence

$$T(h) = T(x_2^*) = \big(x_2^* + \gamma(c-s) - \rho\big)/\gamma\lambda\beta. \tag{6.7}$$

Now, from the assumption of $\dot{s}^* \leq s$ we obtain $0 \geq \dot{L}(h) = \lambda q\beta T(h) - c$ due to Lemma 6.3(b4). Rearranging the inequality by substituting Eq. (6.7) produces

$$0 \geq qx_2^*/\gamma - c(1-q) - qs - q\rho/\gamma,$$

hence $x_2^* \leq c\gamma(1-q)/q + \gamma s + \rho = \chi$ from Eq. (2.6).

(a3ii) If $s < \dot{s}^*$, then $\dot{L}(h) > 0$ from Lemma 6.3(b5). Accordingly, from Eq. (5.6) we get $h = \gamma(L(h) - \dot{L}(h)) + \gamma s + \rho = \gamma\lambda(1-q)\beta T(h) + \gamma s + \rho$, hence $T(h) - (h - \gamma s - \rho)/\gamma\lambda(1-q)\beta = 0$, i.e., $B_1(h) = 0$ from Eq. (5.2). This implies that h defined by Eq. (3.1) is also given by x_1^* which is the unique solution of $B_1(x) = 0$ due to Lemma 5.4(b), i.e., $h = x_1^*$. Hence

$$T(h) = T(x_1^*) = (x_1^* - \gamma s - \rho)/\gamma\lambda(1-q)\beta. \tag{6.8}$$

Now, from the assumption of $s < \dot{s}^*$ and Lemma 6.3(b5) we have $0 < \dot{L}(h) = \lambda q\beta T(h) - c$. Rearranging this inequality by substituting Eq. (6.8) yields $q(x_1^* - \gamma s - \rho)/\gamma(1-q) - c > 0$ or equivalently $x_1^* > c\gamma(1-q)/q + \gamma s + \rho = \chi$ from Eq. (2.6).

(b) Let $c \geq c^*$. Then $c \geq \lambda\beta T(\rho)$ due to Lemma 6.4(c). Now, since $h \geq \rho$ from Lemma 6.1(a), we have $T(h) \leq T(\rho)$ from Lemma 5.2(a). Hence, $0 \geq \lambda\beta T(\rho) - c \geq \lambda\beta T(h) - c \geq \lambda q\beta T(h) - c$ due to Lemma 5.2(b), i.e., $\dot{L}(h) \leq L(h) \leq 0$. ▮

7 Optimal Decision Rule

The following theorem prescribes the optimal decision rule.

Theorem 7.1

(a) *Let $\rho \geq b$. Then $\langle \mathtt{K} \rangle_\phi$ and $\langle \mathtt{K} \rangle_l$.*

(b) *Let $\rho < b$.*

- 1 *Let $c = 0$. Then $\langle \mathtt{C} \rangle_\phi$ and $\langle \mathtt{C} \rangle_l$.*
- 2 *Let $c > 0$.*
  - i *If $c^* \leq c$, then $\langle \mathtt{K} \rangle_\phi$ and $\langle \mathtt{K} \rangle_l$.*
  - ii *If $c < c^*$, then*
    - 1 *If $s^* \leq s$, then $\langle \mathtt{K} \rangle_\phi$ and $\langle \mathtt{K} \rangle_l$.*
    - 2 *If $s < s^*$, then*
      - i $\langle \mathtt{C} \rangle_\phi$.
      - ii *If $x_1^* \leq (>)\chi$, then $\langle \mathtt{K} \rangle_l (\langle \mathtt{C} \rangle_l)$, or if $x_2^* > (\leq)\chi$, then $\langle \mathtt{C} \rangle_l (\langle \mathtt{K} \rangle_l)$.*

Proof.

(a) Let $\rho \geq b$. Then $T(\rho) = 0$ due to Lemma 5.2(c); accordingly, $0 = \lambda\beta T(\rho) = c^* \leq c$ from Lemma 6.4(c) and the assumption of $c \geq 0$. Hence the assertion holds from Lemma 6.5(b).

(b1,b2i,b2ii1,b2ii2i) Immediate from Lemmas 6.3(a), 6.5(b), 6.3(b2), and 6.3(b3), respectively.

(b2ii2ii) We immediately obtain the following relationships from Lemmas 5.4(d), 6.3(b4,b5), and the contrapositions of Lemma 6.5(a3i,a3ii).

$$
\begin{array}{ccccccccc}
x_1^* > \chi & \overset{\text{Lemma 5.4(d)}}{\Longleftrightarrow} & x_2^* > \chi & \overset{\text{Lemma 6.5(a3i)}}{\Longrightarrow} & s < \dot{s}^* & \overset{\text{Lemma 6.3(b5)}}{\Longrightarrow} & \dot{L}(h) > 0 & \Longrightarrow & \langle \mathtt{C} \rangle_l \\
x_2^* \leq \chi & \underset{\text{Lemma 5.4(d)}}{\Longleftrightarrow} & x_1^* \leq \chi & \underset{\text{Lemma 6.5(a3ii)}}{\Longrightarrow} & \dot{s}^* \leq s & \underset{\text{Lemma 6.3(b4)}}{\Longrightarrow} & \dot{L}(h)) \leq 0 & \Longrightarrow & \langle \mathtt{K} \rangle_l
\end{array}
$$
▮

Theorem 7.2 *In the pricing control problem we have*

(a) *$z(h)$ is nondecreasing in λ, β, s, and θ and nonincreasing in q, c, and τ.*

(b) *If $s < (b-\rho)/\gamma$, then $h < z(h) < b$, or else $z(h) = b$.*

Proof. (a) Immediate from Lemmas 6.1(c,d,e), and 5.1(d).

(b) Evident from Lemmas 6.1(b) and 5.1(b,c). ▮

8 Conclusions and Considerations

A. *Optimal decision rules.* Theorem 7.1 provides the most important conclusions obtained in the paper, which can be summarized as in Table 8.2.

Table 8.2: Summary of optimal decision rules

<table>
<tr><th>ρ</th><th>c</th><th>s</th><th>χ</th><th>State (ϕ)</th><th>State (ϕ, l)</th></tr>
<tr><td rowspan="5">$\rho < b$</td><td>$c = 0$</td><td colspan="2"></td><td rowspan="2">$\langle \mathtt{C} \rangle_\phi$</td><td rowspan="2">$\langle \mathtt{C} \rangle_l$</td></tr>
<tr><td rowspan="3">$0 < c < c^*$</td><td rowspan="2">$s < s^*$</td><td>$\chi < x_1^*$</td></tr>
<tr><td>$x_1^* \leq \chi$</td><td>$\langle \mathtt{C} \rangle_\phi$</td><td>$\langle \mathtt{K} \rangle_l$</td></tr>
<tr><td>$s^* \leq s$</td><td></td><td rowspan="3">$\langle \mathtt{K} \rangle_\phi$</td><td rowspan="3">$\langle \mathtt{K} \rangle_l$</td></tr>
<tr><td>$c^* \leq c$</td><td colspan="2"></td></tr>
<tr><td>$\rho \geq b$</td><td colspan="3"></td></tr>
</table>

B. *Relationships of the optimal decision rules with parameters.* If τ is sufficiently small or θ is sufficiently large, then $b \leq \rho = \gamma\eta^\tau\theta$, implying that skipping the search is optimal in both states (ϕ) and (ϕ, l). On the contrary, if τ is sufficiently large or θ is sufficiently small, then $b > \rho = \gamma\eta^\tau\theta$, hence we can depict the optimal decision rule in Table 8.2 as in Figure 8.2 where both $s^*(c)$ and $\dot{s}^*(c)$ are strictly decreasing in c with $c^* = \lambda\beta T(\rho)$, $\dot{c}^* = q(x_1^*(0) - \rho)/\gamma(1-q)$ (Lemma 6.4(a,c,d)), $\dot{s}^*(c) < s^*(c)$ for $c > 0$ (Lemma 6.3(b1)), and $\dot{s}^*(0) = s^*(0) = (b - \rho)/\gamma > 0$ (Lemma 6.4(b)). The three regions $\mathit{\Omega}(\mathtt{K}, \mathtt{K})$, $\mathit{\Omega}(\mathtt{C}, \mathtt{K})$, and $\mathit{\Omega}(\mathtt{C}, \mathtt{C})$ in Figure 8.2 correspond to the optimal decisions of, respectively, "skipping in both states (ϕ) and (ϕ, l)", "continuing in state (ϕ) and skipping in state (ϕ, l)", and "continuing in both states (ϕ) and (ϕ, l)". Here note that the three regions are independent of l. The figure tells us the following two points:

1. When the search cost c or the idling profit s is sufficiently large, skipping the search becomes optimal in both states (ϕ) and (ϕ, l), implying, respectively, that it is profitable to avoid the search cost through skipping the search or that it becomes profitable to enjoy the idling profits while emptying the process by skipping the search.
2. When the search cost c and the idling profit s are sufficiently small, continuing the search becomes optimal in both states $(\phi, 0)$ and $(\phi, 1)$, implying, respectively, that it is reasonable to enjoy the profit from an order obtained through conducting the search and that it does not become profitable even when emptying the process by skipping the search.

C. *Properties of h.*

1. In the admission control problem the optimal selection criterion, on which the system decides whether to accept an appearing customer or not, is given by h, and in the pricing control problem the optimal price, on which an appearing customer decides whether to place his order in the system or not, is given by the function $z(h)$. Further, it is only in the regions $\mathit{\Omega}(\mathtt{C}, \mathtt{K}) \cup \mathit{\Omega}(\mathtt{C}, \mathtt{C})$ that the decisions stated above are to be made.
2. The h is given by the unique solution x^* of the equation $G(x) = 0$, i.e., $h = x^*$; refer to Lemma 6.1 for the properties of h.
3. Let $(s, c) \in \mathit{\Omega}(\mathtt{C}, \mathtt{C}) \cup \mathit{\Omega}(\mathtt{C}, \mathtt{K})$, hence $h < b$ (Lemma 6.1(b)). Then the optimal decisions can be prescribed as follows.
   1) In admission control, if $w > h$, an order with value w appearing after having conducted the search is accepted, or else rejected.
   2) In pricing control, the optimal price $z(h)$ is in the interval (h, b) (Theorem 7.2(b)).

D. *The monotonicities of h and $z(h)$ in the parameters* (Lemma 6.1(c,d,e) and Theorem 7.2(a)).

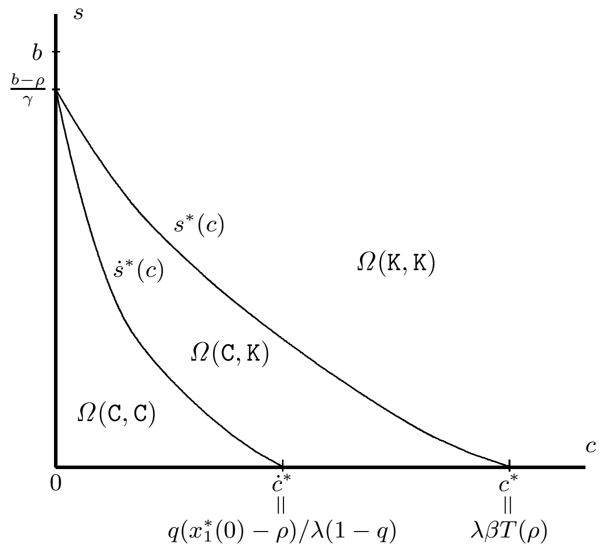

Figure 8.2: Three regions encircled by the functions $\dot{s}^*(c)$ and $s^*(c)$ and the axes c, s when $b > \rho$.

1. *Both h and $z(h)$ are nondecreasing in λ, β, s, and θ.* This implies that the larger the customer appearing probability λ, the discount factor β, the idling profit s, and the penalty cost θ may be, it is reasonable to accept orders with higher values in the admission control problem and to offer higher prices in the pricing control problem, and vice versa.
2. *Both h and $z(h)$ are nonincreasing in q, c, and τ.* This implies that the larger the service completion probability q, the search cost c, and the date of delivery τ may be, the inverse of the above can be said, i.e., it is reasonable to accept orders, even if their values are smaller, in admission control and to offer smaller prices in pricing control, and vice versa.

9 Future Studies

In order to make the model posed in this paper more realistic, some points must be investigated; especially, the following points could be challenged and should be examined.

1. We have assumed so far that $n = 1$, i.e., no more than one customer can be held at any instance. However, we should also examine the case of $n \geq 2$ from a practical viewpoint. Nevertheless, for the reason that the mathematical treatment for $n = 1$ has a specific way of analysis which is not apparent in the analysis of the case of $n \geq 2$, we excluded this case in the present paper.
2. An order under going processing may be canceled due to customer related unavoidable circumstances. According to business usage, if a customer cancels the contract, he should pay a penalty to the company. The introduction of cancellation is sure to have influence on the optimal decision rule; this will become an important subject of study to be tackled.
3. Thus far we have implicitly assumed that a customer once turned away can not be solicited in the future. The future availability of a rejected customer, that is assumed in usual models of optimal stopping problems [4] [6] [12], should be also introduced in our model, and this is also a subject of study to be examined from a practical viewpoint.

Appendix

A. Lemma 4.1

Consider any given vector $\boldsymbol{x} = (x_\phi, x_0, x_1, \cdots)'$ let us define the norm $\|\boldsymbol{x}\| = \max\{|x_\phi|, |x_0|, x_1|, \cdots\}$ where $\|\boldsymbol{x}\| \geq |x_i|$ for $i = \phi, 0, 1, \cdots$. Further, by $D_\phi u$ and $D_l u$ let us denote the right hand sides of

Eqs. (4.6) and (4.7), and let $\boldsymbol{D}u = (D_\phi u, D_0 u, D_1 u, \cdots)'$ and $\boldsymbol{u} = (u(\phi), u(\phi,0), u(\phi,1), \cdots)'$. Then noting Eq. (4.4), from Eq. (3.4) we have

$$\begin{aligned} |v(0) - \hat{v}(0)| &\leq |\mathbf{E}[\max\{w + u(\phi,0), u(\phi)\} - \max\{w + \hat{u}(\phi,0), \hat{u}(\phi)\}]| \\ &\leq \max\{|u(\phi) - \hat{u}(\phi)|, |u(\phi,0) - \hat{u}(\phi,0)|\} \leq \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|, \end{aligned}$$

and from Eq. (3.8) we get

$$\begin{aligned} |v(0) - \hat{v}(0)| &\leq \max_z\{p(z)|u(\phi,0) - \hat{u}(\phi,0)| + (1-p(z))|u(\phi) - \hat{u}(\phi)|\} \\ &\leq \max_z\{p(z)\|\boldsymbol{u} - \hat{\boldsymbol{u}}\| + (1-p(z))\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|\} = \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|. \end{aligned}$$

Accordingly, from Eq. (4.6) we obtain

$$|D_\phi u - D_\phi \hat{u}| \leq \max \left\{ \begin{array}{l} \lambda\beta|v(0) - \hat{v}(0)| + (1-\lambda)\beta|u(\phi) - \hat{u}(\phi)|, \\ \beta|u(\phi) - \hat{u}(\phi)| \end{array} \right\} \leq \beta\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|.$$

Similiarly, from Eq. (4.7) we get $|D_l u - D_l \hat{u}| \leq \beta\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|$ for $l \geq 0$. Thus $\|\boldsymbol{D}u - \boldsymbol{D}\hat{u}\| \leq \beta\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|$, implying that $\boldsymbol{D}u$ is a contraction mapping; accordingly, the assertion holds. ▮

B. Lemma 4.2

For simplicity, let $R = q\beta u(\phi) + \max\{\dot{L}(h), 0\}$. Then Eq. (4.12) can be rewritten

$$u(\phi,l) = \eta u(\phi,l+1) + R - \theta I(\tau \leq l), \quad l \geq 0, \tag{B.1}$$

from which we can develop $u(\phi,0)$ as follows.

$$\begin{aligned} u(\phi,0) &= \eta u(\phi,1) + R \\ &= \eta(\eta u(\phi,2) + R) + R = \eta^2 u(\phi,2) + (1+\eta)R \\ &\quad\vdots \\ &= \eta^\tau u(\phi,\tau) + (1 + \eta + \cdots + \eta^{\tau-1})R \\ &= \eta^\tau u(\phi,\tau) + R(1-\eta^\tau)/(1-\eta).. \end{aligned} \tag{B.2}$$

Further, from Eq. (B.1) with $l = \tau$ we get

$$\begin{aligned} u(\phi,\tau) &= \eta u(\phi,\tau+1) + R - \theta \\ &= \eta(\eta u(\phi,\tau+2) + R - \theta) + R - \theta = \eta^2 u(\phi,\tau+2) + (1+\eta)(R-\theta) \\ &\quad\vdots \\ &= \eta^j u(\phi,\tau+j) + (1 + \eta + \cdots + \eta^{j-1})(R-\theta) \\ &= \eta^j u(\phi,\tau+j) + (R-\theta)(1-\eta^j)/(1-\eta). \end{aligned}$$

Accordingly, since $\eta^j \to 0$ as $j \to \infty$ due to $\eta < 1$, we obtain $\lim_{j\to\infty} n^j u(\phi,\tau+j) = 0$, hence

$$u(\phi,\tau) = (R-\theta)/(1-\eta). \tag{B.3}$$

Rearranging Eq. (B.2) by substituting Eq. (B.3) produces

$$u(\phi,0) = \eta^\tau\big(R-\theta\big)/(1-\eta) + R(1-\eta^\tau)/(1-\eta) = \big(R - \eta^\tau\theta\big)/(1-\eta). \tag{B.4}$$

Noting Eqs. (2.4) and (2.5), we can arrange Eq. (B.4) as follows.

$$u(\phi,0) = \gamma R - \gamma\eta^\tau\theta = \gamma q\beta u(\phi) + \gamma\max\{\dot{L}(h), 0\}) - \rho. \quad ▮$$

C. Lemma 5.3

(a) Eq. (5.1) can be rearranged into

$$G(x) = \gamma \max\{\lambda\beta T(x) - c, 0\} - \max\{x + \gamma\lambda q\beta T(x) - \gamma c, x\} + \gamma s + \rho, \tag{C.1}$$

the first term of the right hand side of which is nonincreasing in x from Lemma 5.2(a). Since $1 > \gamma q\beta \geq \gamma\lambda q\beta$ from Eq. (2.7), it follows that $x + \gamma\lambda q\beta T(x) - \gamma c$ is strictly increasing in x from Lemma 5.2(g), hence the entire right hand side of Eq. (C.1) is strictly decreasing in x.

(b) Applying Lemma 5.2(f) to Eq. (5.1) leads to

$$\lim_{x\to\infty} G(x) \;=\; \gamma\big(\max\{-c, 0\} - \max\{-c, 0\}\big) - \lim_{x\to\infty} x + \gamma s + \rho = -\infty$$

Since $G(x) \geq -\max\{x + \gamma\lambda q\beta T(x) - \gamma c, x\} + \gamma s + \rho$ from Eq. (C.1), applying Lemma 5.2(h) to this inequality yields

$$\lim_{x\to-\infty} G(x) \;\geq\; -\max\{\lim_{x\to-\infty} \big(x + \gamma\lambda q\beta T(x)\big) - \gamma c,\ \lim_{x\to-\infty} x\} + \gamma s + \rho = \infty.$$

(c) For convenience, let us rewrite Eq. (5.1) as follows.

$$G(x) = \gamma\big(\lambda\beta \max\{T(x) - c/\lambda\beta, 0\} - \lambda q\beta \max\{T(x) - c/\lambda q\beta, 0\}\big) - x + \gamma s + \rho.$$

For any given $c > 0$ let $x_1(c)$ and $x_2(c)$ be the solution of $T(x) = c/\lambda q\beta$ and $T(x) = c/\lambda\beta$, respectively. Then since $c/\lambda q\beta > c/\lambda\beta > 0$, clearly both $x_1(c)$ and $x_2(c)$ uniquely exist from Lemma 5.2(e) where $x_1(c) < b$ and $x_2(c) < b$. In addition, since $T(x_1(c)) = c/\lambda q\beta > c/\lambda\beta = T(x_2(c))$, we have $x_1(c) < x_2(c)$ due to Lemma 5.2(a). It is evident that $x_1(c)$ and $x_2(c)$ are both strictly decreasing in c. Now, let $c' = c+\varepsilon$ for any infinitesimal $\varepsilon > 0$, hence $c' > c$. Then $x_1(c') < x_1(c) < x_2(c') < x_2(c)$ (see Figure 3.3). Below, describing $G(x)$ as $G(x, c)$, let us examine the relationship of $G(x, c)$ and c. First, Eq. (5.1) for each c and c' can be rewritten as follows, respectively.

$$G(x, c) \;=\; \begin{cases} \gamma\big(\lambda(1-q)\beta T(x) + s\big) - x + \rho, & \text{on} \quad \mathrm{I} \cup \mathrm{II} \quad \cdots \ (1), \\ \gamma(\lambda\beta T(x) - c + s) - x + \rho, & \text{on} \quad \mathrm{III} \cup \mathrm{IV} \quad \cdots \ (2), \\ -x + \gamma s + \rho, & \text{on} \quad \mathrm{V} \quad \cdots \ (3), \end{cases} \tag{C.2}$$

$$G(x, c') \;=\; \begin{cases} \gamma\big(\lambda(1-q)\beta T(x) + s\big) - x + \rho, & \text{on} \quad \mathrm{I} \quad \cdots \ (1'), \\ \gamma(\lambda\beta T(x) - c' + s) - x + \rho, & \text{on} \quad \mathrm{II} \cup \mathrm{III} \quad \cdots \ (2'), \\ -x + \gamma s + \rho, & \text{on} \quad \mathrm{IV} \cup \mathrm{V} \quad \cdots \ (3'). \end{cases} \tag{C.3}$$

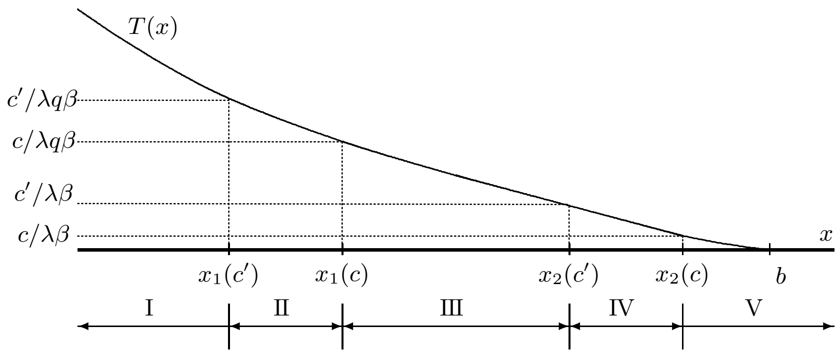

Figure 3.3: The relationship between $x_1(c'), x_1(c), x_2(c')$ and $x_2(c)$ where $\mathrm{I} = (-\infty, x_1(c')]$, $\mathrm{II} = \big(x_1(c'), x_1(c)\big]$, $\mathrm{III} = \big(x_1(c), x_2(c')\big)$, $\mathrm{IV} = \big[x_2(c'), x_2(c)\big)$, and $\mathrm{V} = [x_2(c), \infty)$

1. On the intervals I and V, we have $G(x,c) = G(x,c')$, respectively, from Eqs. (C.2 (1)) and (C.3 (1′)) and Eqs. (C.2 (3)) and (C.3 (3′)).
2. On the interval II, from Eqs. (C.2 (1)) and (C.3 (2′)) we get

$$\begin{aligned} G(x,c) - G(x,c') &= \gamma\lambda(1-q)\beta T(x) - \gamma\lambda\beta T(x) + \gamma c' \\ &= -\gamma\lambda q\beta T(x) + \gamma c' = -\gamma(\lambda q\beta T(x) - c') > 0 \end{aligned}$$

   due to $\lambda q\beta T(x) - c' < 0$ on $x_1(c') < x$, hence $G(x,c) > G(x,c')$.
3. On the interval III, from Eqs. (C.2 (2)) and (C.3 (2′)) we have

$$G(x,c) = \gamma(\lambda\beta T(x) - c + s) - x + \rho > \gamma(\lambda\beta T(x) - c' + s) - x + \rho = G(x,c').$$

4. On the interval IV, from Eqs. (C.2 (2)) and (C.3 (3′))we obtain

$$G(x,c) - G(x,c') = \gamma\big(\lambda\beta T(x) - c + s\big) - x + \rho - (-x + \gamma s + \rho) = \gamma(\lambda\beta T(x) - c) > 0$$

   due to $\lambda\beta T(x) - c > 0$ on $x < x_2(c)$.

From all the above, it eventually follows that $G(x,c) \geq G(x,c')$ for all x, that is $G(x,c)$ is nonincreasing in c for all x. Monotonicity of $G(x)$ in q can be proven in almost the same way as the above where it is to be noted that γ and ρ are strictly decreasing in q. The last assertion is evident from the fact that ρ is strictly decreasing in τ due to $\eta < 1$ and that $G(x)$ is strictly increasing in ρ.

(d) Proven in a similar way to that in the proof of (c) where it is to be noted that γ and ρ are strictly increasing in β. The last assertion is evident from the fact that ρ is strictly increasing in θ and that $G(x)$ is strictly increasing in ρ.

(e) Immediate from Eq. (5.1) since $\gamma > 0$. ▮

D. Lemma 5.4

(a) The former half is immediate from the facts that $T(x)$ is nonincreasing in x due to Lemma 5.2(a) and that both of $-x/\gamma\lambda(1-q)\beta$ and $-x/\gamma\lambda\beta$ are strictly decreasing in x. The latter half is evident from Lemma 5.2(f).

(b) The former half is evident from (a). Let $\lambda\beta T(\rho) > c$. Then since $T(0) \geq T(\rho)$ from Lemma 5.2(a) due to $\rho \geq 0$, noting $T(0) > 0$, we have

$$\begin{aligned} B_1(0) &= T(0) + (\gamma s + \rho)/\gamma\lambda(1-q)\beta > 0, \\ B_2(0) &= T(0) - c/\lambda\beta + (\lambda s + \rho)/\gamma\lambda\beta \geq T(\rho) - c/\lambda\beta + (\lambda s + \rho)/\gamma\lambda\beta \\ &= (\lambda\beta T(\rho) - c)/\lambda\beta + (\lambda s + \rho)/\gamma\lambda\beta > 0. \end{aligned}$$

Hence, x_1^* and x_2^* are positive for any $s \geq 0$.

(c) Clear from

$$\begin{aligned} B_1(x) - B_2(x) &= -(x - \gamma s - \rho)/\gamma\lambda(1-q)\beta + \big(x + \gamma(c-s) - \rho\big)/\gamma\lambda\beta \\ &= \big(-qx + \gamma qs + q\rho + \gamma(1-q)c\big)/\gamma\lambda(1-q)\beta = -q(x-\chi)/\gamma\lambda(1-q)\beta. \end{aligned}$$

(d) Let $x_2^* > \chi$. Then $0 = B_2(x_2^*) < B_2(\chi) = B_1(\chi)$ due to (a,c), hence $0 < B_1(\chi)$. Accordingly, since $B_1(x_1^*) = 0 < B_1(\chi)$, we obtain $x_1^* > \chi$ due to (a). Let $x_1^* > \chi$. Then $0 = B_1(x_1^*) < B_1(\chi) = B_2(\chi)$ due to (a,c), hence $0 < B_2(\chi)$. Accordingly, since $B_2(x_2^*) = 0 < B_2(\chi)$, we obtain $x_2^* > \chi$ due to (a). The latter half is proven by the contrapositions of the above results. ▮

Bibliography

[1] Feng, Y. and Xiao, B.: Optimal policies of yield management with multiple predetermined prices, *Operations Research*, Vol. 48(2) (2000), 332–343.

[2] Heyman, D. P.: Optimal operating policies for $M/G/1$ queueing systems, *Operations Research*, Vol. 16 (1968), 362–382.

[3] Ikuta, S.: A basic theory of customers selection problem (Japanese), *Journal of Industrial Management Association*, Vol. 1 (1971).

[4] Ikuta, S.: Optimal stopping problem with uncertain recall, *Journal of the O. R. Society of Japan*, Vol. 31(2) (1988), 145–170.

[5] Ikuta, S.: An integration of the optimal stopping problem and the optimal pricing problem - Model with no recall -, *Discussion paper,* No. 1084, University of Tsukuba, Institute of Policy and Planning Sciences, (2004).

[6] Kang, B. K.: Optimal stopping problem with recall cost, *European Journal of Operations Research*, Vol. 117 (1999), 222–238.

[7] Lippman, S. A. and Ross, S. M.: The streetwalker's dilemma: a job shop model, *SIAM J. Appl. Math.*, Vol. 20(3) (1971), 336–342.

[8] Low, D. W.: Optimal dynamic pricing policies for an $M/M/s$ queue, *Operations Research*, Vol. 22(3) (1974), 545–561.

[9] M. C. Fu, S. I. M. and Wang, I.: Monotone optimal policies for a transient queueing staffing problem, *Operations Research*, Vol. 48(2) (2000), 327–331.

[10] Miller, B.: A queueing reward system with several customer classes, *Management Science*, Vol. 16 (1969), 234–245.

[11] Ross, S. M.: *Introduction to Stochastic Dynamic Programming*, Academic Press, 1983.

[12] Saito, T.: Optimal stopping problem with controlled recall, *Probability in the Engineering and Informational Sciences*, Vol. 12 (1998), 91–108.

[13] Sakaguchi, M.: Dynamic programming of some sequential sampling design, *Journal of Mathematical Analysis and Applications*, Vol. 2 (1961), 446–466.

[14] Son, J. D.: Customers selection problem with idling profit, *Discussion paper,* No.1087, University of Tsukuba, Institute of Policy and Planning Sciences, (2004).

[15] Yoon, S. H. and Lewis, M. E.: Optimal pricing and admission control in a queueing system with periodically varying parameters, *Queueing Systems*, Vol. 47(3) (2004).

[16] You, P. S.: Sequential buying policies, *European Journal of Operational Research*, Vol. 120 (2000), 535–544.