

# Structural Analysis of Optimal Investment Strategy for Project Management via Real Option Approach

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December 11, 2003

## Abstract

ROA (Real Option Approach) has been recently drawing much attention of researchers and practitioners. In ROA, it is widely observed that alternative options for the project management are treated as financial options and the underlying uncertainty is evaluated accordingly. However, this approach requires the underlying asset of the project to be traded in a market since alternative options of the project cannot be replicated as financial options otherwise. In this paper, based on a broader definition of ROA by Yamamoto and Kariya [12], a Dynamic Programming approach is proposed. The optimal investment strategy is incorporated explicitly within the decision structure of the model, thereby extending the previous work by Huchzermeier and Loch [3] substantially. Structural properties of the optimal investment strategy are investigated in detail, establishing certain monotonicity properties of the optimal project value and the optimal investment amount. Some numerical results are also presented.

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# 1 Introduction

In evaluation of project values, one of the most prevalent methods is DCF (Discounted Cash Flow) where the future cash flow generated by a project, when it is completed as planned, is discounted to the present value using the capital cost as a discounting parameter. This discounted future cash flow is compared with the initial investment amount, yielding NPV (Net Present Value) of the project as the difference of the two. If  $NPV \geq 0$ , the project would be carried out, while it would be terminated when  $NPV < 0$ . Klammer [7] reported that only 15 % of U.S. companies employed DCF in 1959, but the percentage was increased to 57 % in 1970. Today almost all of U.S. companies use DCF for evaluation of project values, see e.g. Yamamoto and Kariya [12]. This approach, however, cannot explicitly incorporate uncertainty arising from development of the project.

In order to overcome this difficulty, ROA (Real Option Approach) has been recently drawing much attention of researchers and practitioners. Following Yamamoto and Kariya [12], ROA is defined in this paper as below:

- (1.1) Real option is the right of the management to explore alternative options in a management environment with high uncertainty.
- (1.2) ROV (Real Option Value) is the portion of the present project value representing the value of having alternative options.
- (1.3) A method to evaluate ROV is called ROA.

Typically alternative options include termination, deferral, expansion, contract, time to build, transfer, shutdown and restart, cancellation, market entry (Yamamoto and Kariya [12]), improvement (Huchzermeier and Loch [3]), exchange option (Lee and Paxson [9]), and growth option (Loch and Bode-Greuel [10]). ROA is superior to DCF when the degree of uncertainty is higher and/or various alternative options are available. In addition, ROA is more useful than DCF when the initial investment needed to carry out the project is larger than the discounted future cash flow. In this case, one has  $NPV < 0$  and the project would be terminated if DCF is employed. In reality, however, such a project itself may be traded in the market. The potential project value of this sort can be captured by ROA, but not by DCF.

Typical projects for which ROA is more attractive than DCF include: mining natural resource projects (Cortazer, Shwartz, and Casassus [2]), gas and electric projects (Yamamoto and Kariya [12]), infrastructure development

projects (Yamamoto and Kariya [12]), IT projects (Benarch [1], Kumer [8]), pharmaceuticals R&D project (Kellogg and Charnes [5], Trang, Takezawa, and Takezawa [11]), tree harvesting problems (Insley [4]), and lease projects (Kenyon and Tompaidis [6]).

In ROA, it is widely observed that alternative options for the project management are treated as financial options, and the underlying uncertainty is evaluated accordingly. In some literature, this specific approach is called ROA. In this paper, however, we stick to the original definition of (1.1) through (1.3), and the above approach is called “ the risk-neutral ” ROA. As discussed in a recent paper by Kellogg and Charnes [5], the risk-neutral ROA has the advantage of providing substantial flexibility in incorporating a variety of alternative options, and of eliminating the laborious evaluation of the capital cost which is replaced by the risk-free rate. However, the major draw-back of the risk-neutral ROA can be found in that it requires the underlying asset of the project to be traded in a market. Otherwise alternative options of the project cannot be replicated as financial options, destructing the foundation of this approach. Among the projects previously mentioned as those preferring ROA, only mining natural resource projects and gas and electric projects satisfy this condition. In order to eliminate the risk-neutral requirement, a recent paper by Huchzermeier and Loch [3] employed a DP (Dynamic Programming) approach, where the success probability of the project for each time stage does not have to be risk-neutral but arbitrary, and the capital cost is replaced by a risk-free rate as the discounting parameter. In the paper [3], however, investment costs are treated as functions of time, exogenous to the underlying decision structure.

The purpose of this paper is to develop a DP-based ROA for determining the optimal investment policy so as to maximize the expected present project value. Here investment costs are incorporated explicitly as a part of strategic decisions within the model. Probability of success at time  $t$  is treated as an increasing function of the investment amount to be decided at time  $t$ . A similar framework can be found in Kellogg and Charnes [5] but such probabilities are assumed to be constant there. Salvage values are also incorporated when the option to terminate the project is exercised or the project was forced to stop due to failure. Furthermore, such values are expressed as increasing functions of the estimated value of the project outcome at the occurrence of the stoppage. Thang, Takezawa, and Takezawa [11] treated option salvage values, but they were assumed to be constants. Introduction of salvage values upon failure is new. It will be shown that the optimal project value with options,  $V^*$ , is always larger than that without options,  $\hat{V}^*$ , so

that the optimal real option value  $ROV^* = V^* - \hat{V}^*$  is always nonnegative. Furthermore, both  $\hat{V}^*$  and  $V^*$  increase as the level of uncertainty involved in the successful completion of the project decreases. Similar monotonicity properties can be observed under more restrictive conditions for the optimal investment amount  $x^*$ .

The structure of this paper is as follows. In Section 2, a project management model based on ROA is formally introduced. Two associated DP problems are formulated in Section 3. It is shown that under certain conditions the unique optimal investment strategy exists for each of the DP problems. Sections 4 and 5 are devoted to establish structural properties of the optimal project value  $V^*$  and the optimal investment amount  $x^*$ . Numerical results are exhibited in Section 6, demonstrating the monotonicity properties of  $V^*$  and  $x^*$  established in the previous sections. Some concluding remarks are given in Section 7. Basic properties of certain concave functions are given in Appendix, which will play a key role in establishing the monotonicity properties.

## 2 Model Description

We consider a project management problem over  $T$  periods. Let  $S_0$  be an estimated value of the project outcome at time  $t = 0$ . This estimated value may increase or decrease as the project evolves. In managing this project toward the end of period  $T$ , one has an option to terminate the project at the beginning of each period  $t$ ,  $1 \leq t \leq T$ , with some salvage value. The decision criterion for this option will soon become clear. If it is decided to continue the project, the investment amount for this period should be determined. The project may be carried out successfully to the next period  $t + 1$  or may fail. The success probability may depend on the investment amount. When successful, the estimated value of the project outcome increases by a factor of  $u$  with probability  $p$  or decreases by a factor of  $d$  with probability  $1 - p$ , where  $0 < d < 1 < u$  and  $0 \leq p \leq 1$ . The former case is called an upward success and the latter is called a downward success. When the project fails in period  $t$ , the project is forced to stop with certain salvage value, which is different from the salvage value under the option of termination.

The decision structure described above can be expressed as a modified binary tree in the following manner. Suppose that period  $t$  is completed with  $k$  upward successes and  $t - k$  downward success at time  $t$ , and we are at the beginning of period  $t + 1$ . This state is denoted by  $(t, k)$ ,  $0 \leq k \leq t$ .

Let  $S_{t,k}$  be the estimated value of the project outcome at state  $(t, k)$  so that

$$(2.1) \quad S_{t,k} = S_0 u^k d^{t-k}, \quad 0 \leq k \leq t.$$

Similarly, we introduce:

$$(2.2) \quad \hat{V}_{t,k} : \text{the expected value of the project at state } (t, k) \\ \text{without option for termination,}$$

and

$$(2.3) \quad V_{t,k} : \text{the expected value of the project at state } (t, k) \\ \text{with option for termination.}$$

When the decision is made to terminate the project at state  $(t, k)$ , we denote the salvage value by  $V_{A:t+1,k}$ , i.e.

$$(2.4) \quad V_{A:t+1,k} : \text{the salvage value of the project} \\ \text{when it is decided to terminate it at state } (t, k).$$

The corresponding state is denoted by  $(A : t + 1, k)$ .

If the project is continued, the investment amount  $x_{t+1,k}$  is determined. How to determine  $x_{t+1,k}$  will be discussed in Section 3. Let  $\beta(x)$  be the probability that the project can be continued successfully for one period given that the investment amount for the period is  $x$ . Throughout the paper, we assume that, for  $x \geq 0$ ,

$$(2.5) \quad \beta(0) = 0, \quad 0 \leq \beta(x) \leq 1, \quad \beta'(x) > 0, \quad \beta''(x) < 0,$$

where  $\beta'(x) = \frac{d}{dx}\beta(x)$  and  $\beta''(x) = \frac{d^2}{dx^2}\beta(x)$ , which reflects the fact that the success probability increases as the investment amount increases with the effect of diminishing return. The project experiences an upward success with probability  $\beta(x_{t+1,k}) \cdot p$ . In this case, state moves from  $(t, k)$  to  $(t + 1, k + 1)$ . With probability  $\beta(x_{t+1,k}) \cdot (1 - p)$ , a downward success is realized and state moves from  $(t, k)$  to  $(t + 1, k)$ .

The project may be forced to stop because of failure with probability  $1 - \beta(x_{t+1,k})$ . In this case, the associated salvage value is denoted by  $V_{F:t+1,k}$ , i.e.

$$(2.6) \quad V_{F:t+1,k} : \text{the salvage value of the project} \\ \text{when it is forced to stop} \\ \text{because of failure starting from state } (t, k).$$

The corresponding state is denoted by  $(F : t + 1, k)$ . The structure of these state transitions is depicted in Figure 2.1.

**Remark 2.1** *If the investment amount for the  $t$ -th period should be discounted to the present value, we define  $x_{t,k} = y_{t,k}e^{-rt}$  where  $r$  is the risk-free interest rate and  $y_{t,k}$  is the actual amount to be invested at time  $t$ . Since there is one-to-one correspondence between  $x_{t,k}$  and  $y_{t,k}$ , only  $x_{t,k}$  will be considered from now on.*

**Remark 2.2** *In the literature concerning real options, the upward success probability  $p$  is often defined as the risk neutral probability. More specifically,  $p$  is determined by*

$$pu + (1 - p)d = e^r$$

so that

$$p = \frac{e^r - d}{u - d}$$

where  $0 \leq d \leq e^r$  and  $e^r \leq u$ . This assumption may be appropriate when the project value can be tied with a value determined through a market with no arbitrage, e.g. stock price. In this paper, we do not require this assumption and  $p$  can be any probability.

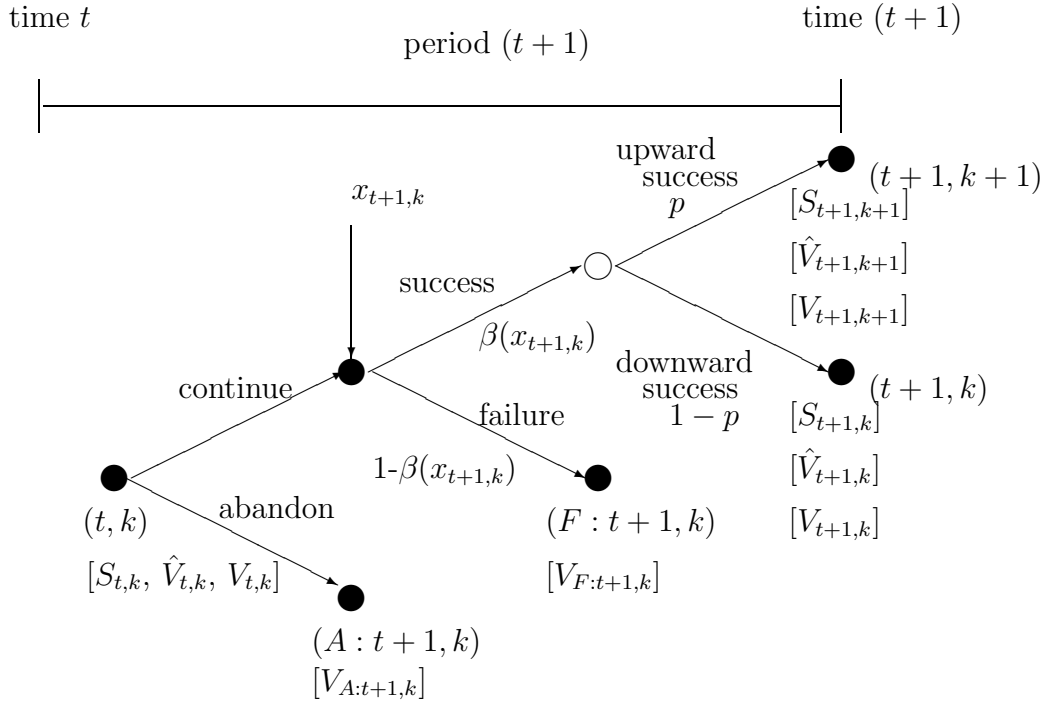


Figure 2.1 Modified Binary Tree Structure for One Period

### 3 Formulation of Optimal Investment Policy: Dynamic Programming Approach

From the modified binary tree in Figure 2.1, if the project continues with success to the final period  $T$ , the project value  $V_{T,k}$  at time  $T$  with  $k$  upward successes and  $T - k$  downward successes can be written as

$$(3.1) \quad \hat{V}_{T,k} = V_{T,k} = S_{T,k} - X_T = S_0 u^k d^{T-k} - X_T, \quad 0 \leq k \leq T,$$

where  $X_T$  is the operational cost needed for generating the cash flow from the completed project, see Remark 2.1. For the two salvation values  $V_{F:t+1,k}$  upon failure and  $V_{A:t+1,k}$  due to decision to terminate, we assume that both are functions of  $S_{t,k}$  and define

$$(3.2) \quad V_{F:t+1,k} = W_F(S_{t,k}); \quad V_{A:t+1,k} = W_A(S_{t,k}).$$

It is natural to assume that  $W_F(x)$  and  $W_A(x)$  are zero without investment and are strictly increasing and concave, i.e.,

$$(3.3) \quad \begin{aligned} W_F(0) &= 0, & W'_F(x) &> 0, & W''_F(x) &\leq 0; \\ W_A(0) &= 0, & W'_A(x) &> 0, & W''_A(x) &\leq 0. \end{aligned}$$

When no option for terminating the project is available, the corresponding expected project value given the investment amount  $\hat{x}_{t+1,k}$  satisfies the following backward recursive formula.

$$(3.4) \quad \hat{V}_{t,k} = \left\{ \beta(\hat{x}_{t+1,k}) \left( p \hat{V}_{t+1,k+1} + (1-p) \hat{V}_{t+1,k} \right) + \left( 1 - \beta(\hat{x}_{t+1,k}) \right) V_{F:t+1,k} \right\} e^{-r} - \hat{x}_{t+1,k}.$$

Accordingly the optimal investment policy without options should be determined so as to maximize  $\hat{V}_{0,0}$ . This problem can be formulated as the following DP problem. For notational convenience, let  $G(\lambda, A, B)$  be defined by

$$(3.5) \quad G(\lambda, A, B) = \lambda A + (1 - \lambda) B$$

where  $0 \leq \lambda \leq 1$ .

[ DP- $\hat{V}$  ]

$$(3.6) \quad \max_{[\hat{x}_{t,k}]} \hat{V}_{0,0}$$

subject to

(3.7)

$$\hat{V}_{t,k} = G\left(\beta(\hat{x}_{t+1,k}), G(p, \hat{V}_{t+1,k+1}, \hat{V}_{t+1,k}), V_{F:t+1,k}\right)e^{-r} - \hat{x}_{t+1,k}$$

*with*  $\hat{x}_{t+1,k} \geq 0, \quad 0 \leq k \leq t, \quad 0 \leq t \leq T-1;$

$$(3.8) \quad S_{T,k} = S_0 u^k d^{T-k}, \quad 0 \leq k \leq T;$$

$$(3.9) \quad \hat{V}_{T,k} = S_{T,k} - X_T, \quad 0 \leq k \leq T; \text{ and}$$

$$(3.10) \quad V_{F:t+1,k} = W_F(S_{t,k}), \quad 0 \leq k \leq t, \quad 0 \leq t \leq T-1.$$

Since  $G(p, \hat{V}_{t+1,k+1}, \hat{V}_{t+1,k})$  is the expected project value with success while  $V_{F:t+1,k}$  is the salvage value upon failure, we assume that

$$(3.11) \quad G(p, \hat{V}_{t+1,k+1}, \hat{V}_{t+1,k}) > V_{F:t+1,k}.$$

It should be noted that  $DP - \hat{V}$  can be solved recursively by finding

$$(3.12) \quad \hat{V}_{t,k}^* = \hat{f}_{t,k}(\hat{x}_{t+1,k}^*) = \max_{x \geq 0} \hat{f}_{t,k}(x);$$

$$\hat{f}_{t,k}(x) = G(\beta(x), \hat{A}_{t,k}, \hat{B}_{t,k}) - x$$

where

$$(3.13) \quad \hat{A}_{t,k} = G(p, \hat{V}_{t+1,k+1}^*, \hat{V}_{t+1,k}^*)e^{-r}; \quad \hat{B}_{t,k} = V_{F:t+1,k}e^{-r}$$

for  $0 \leq k \leq t$  and  $0 \leq t \leq T-1$ , starting with

$$(3.14) \quad \hat{V}_{T,k}^* = \hat{V}_{T,k} = S_{T,k} - X_T, \quad 0 \leq k \leq T.$$

With option for termination, the expected value of the project has to be compared with the salvage value of the project for termination at the beginning of each period. Hence the backward recursive formula for the expected project value  $V_{t,k}$  given the investment amount  $x_{t+1,k}$  should be

$$(3.15) \quad V_{t,k} = \max \left[ \left\{ \beta(x_{t+1,k}) \left( pV_{t+1,k+1} + (1-p)V_{t+1,k} \right) \right. \right. \\ \left. \left. + \left( 1 - \beta(x_{t+1,k}) \right) V_{F:t+1,k} \right\} e^{-r} - x_{t+1,k}, V_{A:t+1,k} \right].$$

The DP formulation then becomes:



[ DP-V ]

$$(3.16) \quad \max_{[x_{t,k}]} V_{0,0}$$

subject to

$$(3.17) \quad V_{t,k} = \max \left[ G\left(\beta(x_{t+1,k}), G(p, V_{t+1,k+1}, V_{t+1,k}), V_{F:t+1,k}\right) e^{-r} \right. \\ \left. - x_{t+1,k}, V_{A:t+1,k} \right] \\ \text{with } x_{t+1,k} \geq 0, \quad 0 \leq k \leq t, \quad 0 \leq t \leq T-1;$$

$$(3.18) \quad S_{T,k} = S_0 u^k d^{T-k}, \quad 0 \leq k \leq T;$$

$$(3.19) \quad V_{T,k} = S_{T,k} - X_T, \quad 0 \leq k \leq T;$$

$$(3.20) \quad V_{F:t+1,k} = W_F(S_{t,k}), \quad 0 \leq k \leq t, \quad 0 \leq t \leq T-1; \text{ and}$$

$$(3.21) \quad V_{A:t+1,k} = W_A(S_{t,k}), \quad 0 \leq k \leq t, \quad 0 \leq t \leq T-1.$$

Similar to (3.11), one has

$$(3.22) \quad G(p, V_{t+1,k+1}, V_{t+1,k}) > V_{F:t+1,k}.$$

In parallel with (3.12) through (3.14), one sees that  $DP - V$  can be solved recursively by finding

$$(3.23) \quad V_{t,k}^* = \max\{f_{t,k}(x_{t+1,k}^*), V_{A:t+1,k}\}; \\ f_{t,k}(x_{t+1,k}^*) = \max_{x \geq 0} f_{t,k}(x); \quad f_{t,k}(x) = G(\beta(x), A_{t,k}, B_{t,k}) - x$$

where

$$(3.24) \quad A_{t,k} = G(p, V_{t+1,k+1}^*, V_{t+1,k}^*) e^{-r}; \quad B_{t,k} = V_{F:t+1,k} e^{-r}$$

for  $0 \leq k \leq t$  and  $0 \leq t \leq T-1$ , starting with

$$(3.25) \quad V_{T,k}^* = V_{T,k} = S_{T,k} - X_T, \quad 0 \leq k \leq T.$$

For the option value at state  $(t, k)$ , we define

$$(3.26) \quad ROV_{t,k}^* = V_{t,k}^* - \hat{V}_{t,k}^*.$$

Of particular interest is to find the option value  $ROV_{0,0}^*$  and the associated optimal investment strategy  $(x_{t,k}^*)$  at the start of the project for the optimal investment policy problem.

We next show that, when it is decided to continue the project, the optimal investment amount can be determined uniquely under (2.5) for both  $DP - \hat{V}$  and  $DP - V$ , which facilitates the necessary DP computation substantially. We note from (3.11), (3.13), (3.22), and (3.24) that

$$(3.27) \quad \hat{A}_{t,k} > \hat{B}_{t,k}; \quad A_{t,k} > B_{t,k}; \quad \hat{B}_{t,k} = B_{t,k}.$$

**Theorem 3.1** *Suppose that the success probability function  $\beta(x)$  satisfies (2.5). Then, whenever it is decided to continue the project at  $(t, k)$ , both  $DP - \hat{V}$  and  $DP - V$  have the unique optimal investment amounts  $\hat{x}_{t+1,k}^*$  and  $x_{t+1,k}^*$  respectively for all  $k$ ,  $0 \leq k \leq t$ , and all  $t$ ,  $0 \leq t \leq T - 1$ .*

**Proof** From (2.5), (3.12), (3.27) and Lemma A.1, one sees that  $\hat{f}(x)$  is strictly concave. For  $DP - \hat{V}$ , if  $\hat{f}'(0) < 0$ , then  $\hat{f}'(x) < 0$ , since  $\hat{f}''(x) < 0$ , for all  $x \geq 0$  and the unique maximum value of  $\hat{f}(x)$  for  $x \geq 0$  is attained at  $x^* = 0$ . Otherwise, from (2.5), (3.12), (A.4) and (A.5),  $\hat{f}(x)$  has the unique maximum point  $\hat{x}_{t+1,k}^*$  determined by

$$(3.28) \quad \beta'(x_{t+1,k}^*) = (\hat{A}_{t,k} - B_{t,k})^{-1}.$$

Under the assumption that the project is continued at  $(t, k)$ , similar arguments can be repeated for  $DP - V$  with  $f(x)$  of (3.23).  $\square$

For  $DP - V$ , it may be worth noting that, when  $f'(x) < 0$  for all  $x \geq 0$ , one has

$$(3.29) \quad V_{t,k}^* = \max\{V_{F:t+1,k}e^{-r}, V_{A:t+1,k}\}$$

since  $x_{t+1,k}^* = 0$ ,  $\beta(0) = 0$  from (2.5), and  $f_{t,k}(0) = V_{F:t+1,k}$ . This means that, when  $f'_{t,k}(x) < 0$  for all  $x \geq 0$ , the project is continued with zero investment if and only if  $V_{F:t+1,k} > V_{A:t+1,k}$ , i.e. the salvage value due to failure is larger than the salvage value upon decision to terminate the project.

**Example 3.2** *Let  $\beta(x) = \frac{bx}{1+bx}$  with  $b > 0$ . Then  $\beta(x)$  satisfies the conditions in (2.5). One has from Theorem 3.1,*

$$(3.30) \quad \hat{x}_{t+1,k}^* = \frac{-1 + \sqrt{b(\hat{A}_{t,k} - B_{t,k})}}{b} \quad \text{for } DP - \hat{V}$$

and

$$(3.31) \quad x_{t+1,k}^* = \frac{-1 + \sqrt{b(A_{t,k} - B_{t,k})}}{b} \quad \text{for } DP - V.$$

This example will be used in Section 6 for numerical exploration.

## 4 Structural Properties of $\hat{V}^*$ and $V^*$

In this section, we derive various monotonicity properties of the expected project values  $\hat{V}_{t,k}^*$  and  $V_{t,k}^*$ . Furthermore, it is shown that  $ROV_{t,k}^*$  defined in (3.26) is always nonnegative. We first show that both  $\hat{V}_{t,k}^*$  and  $V_{t,k}^*$  increase as  $k$  increases, i.e. the more upward successes the project experiences, the larger the expected project value is.

**Theorem 4.1** *Let  $0 \leq k \leq t - 1$  for  $1 \leq t \leq T$ . Then:*

- a)  $\hat{V}_{t,k+1}^* > \hat{V}_{t,k}^*$
- b)  $V_{t,k+1}^* > V_{t,k}^*$

**Proof** We prove part a) by backward induction. For  $t = T$ , one sees from (3.1) that

$$\hat{V}_{T,k+1}^* - \hat{V}_{T,k}^* = S_{0,0} u^k d^{T-k-1} (u - d) > 0,$$

since  $0 < d < 1 < u$ .

Suppose part a) holds for  $t + 1$  and consider the case of  $t$ . Let  $\hat{A}_{t,k}$  and  $B_{t,k}$  be as in (3.13) and (3.27) respectively. From the induction hypothesis, one has

$$(4.1) \quad \begin{aligned} \hat{A}_{t,k+1} - \hat{A}_{t,k} &= \{p(\hat{V}_{t+1,k+2}^* - \hat{V}_{t+1,k+1}^*) + (1-p)(\hat{V}_{t+1,k+1}^* - \hat{V}_{t+1,k}^*)\} e^{-r} \\ &> 0 \end{aligned}$$

i.e.

$$(4.2) \quad \hat{A}_{t,k+1} > \hat{A}_{t,k}.$$

From (2.1) and  $0 < d < 1 < u$ , it can be readily seen that

$$(4.3) \quad S_{t,k+1} = S_0 u^{k+1} d^{t-(k+1)} > S_0 u^k d^{t-k} = S_{t,k}.$$

From the monotonicity of  $W_F(x)$  in (3.3) together with (3.2), (3.13) and (3.27), it then follows that

$$(4.4) \quad B_{t,k+1} = W_F(S_{t,k+1})e^{-r} > W_F(S_{t,k})e^{-r} = B_{t,k}.$$

Since  $\hat{V}_{t,k+1}^* = \hat{f}_{t,k+1}(\hat{x}_{t+1,k+1}^*)$  and  $\hat{V}_{t,k}^* = \hat{f}_{t,k}(\hat{x}_{t+1,k}^*)$ , part a) follows from (4.2), (4.4), and Lemma A.2.

For part b), we first note from (3.23) that

$$(4.5) \quad V_{t,k}^* = \max\{H_{t,k}, V_{A:t+1,k}\}$$

where

$$(4.6) \quad H_{t,k} = f_{t,k}(x_{t+1,k}^*) = G(\beta(x_{t+1,k}^*), A_{t,k}, B_{t,k}) - x_{t+1,k}^*,$$

and  $A_{t,k}$  and  $B_{t,k}$  are as in (3.24). Similarly to the case of part a), we prove by backward induction. Since  $V_{T,k}^* = \hat{V}_{T,k}^*$ , the case  $t = T$  follows from part a). Suppose part b) holds for  $t + 1$  and consider  $t$ . From (4.5), 4 cases should be examined separately. We first note that, as for part a), one has  $A_{t,k+1} > A_{t,k}$  and  $B_{t,k+1} > B_{t,k}$  so that from Lemma A.2

$$(4.7) \quad H_{t,k+1} > H_{t,k}.$$

From the monotonicity of  $W_A(x)$  in (4.7) and  $S_{t,k+1} > S_{t,k}$  from (4.3), one has

$$(4.8) \quad V_{A:t+1,k+1} = W_A(S_{t,k+1}) > W_A(S_{t,k}) = V_{A:t+1,k}.$$

Case 1:  $V_{t,k+1}^* = H_{t,k+1}$ ;  $V_{t,k}^* = H_{t,k}$

In this case, from (4.7), one has  $V_{t,k+1}^* = H_{t,k+1} > H_{t,k} = V_{t,k}^*$ .

Case 2:  $V_{t,k+1}^* = V_{A:t+1,k+1}$ ;  $V_{t,k}^* = H_{t,k}$

From (4.5) and (4.7), one sees that

$$V_{t,k+1}^* - V_{t,k}^* = V_{A:t+1,k+1} - H_{t,k} \geq H_{t+1,k} - H_{t,k} > 0.$$

Case 3:  $V_{t,k+1}^* = H_{t,k+1}$ ;  $V_{t,k}^* = V_{A:t+1,k}$

One sees from  $V_{t,k+1} = H_{t,k+1}$  and (4.5) that  $H_{t,k+1} \geq V_{A:t+1,k+1}$ . It then follows from (4.8) that

$$\begin{aligned} V_{t,k+1}^* - V_{t,k}^* &= H_{t,k+1} - V_{A:t+1,k} \\ &\geq V_{A:t+1,k+1} - V_{A:t+1,k} \\ &> 0. \end{aligned}$$

Case 4:  $V_{t,k+1}^* = V_{A:t+1,k+1}$ ;  $V_{t,k}^* = V_{A:t+1,k}$

In this case,  $V_{t,k+1}^* > V_{t,k}^*$  from (4.8). □

Parameters  $p, u$  and  $d$  represent the level of uncertainty involved in the successful completion of the project. The next theorem shows monotonicity of  $\hat{V}_{t,k}^*$  and  $V_{t,k}^*$  in terms of these parameters, i.e. the expected project value increases as the level of uncertainty decreases.

**Theorem 4.2** *For  $0 \leq k \leq t$  and  $0 \leq t \leq T - 1$ , the following statements hold.*

- a)  $\hat{V}_{t,k}^*$  is strictly increasing in  $p, u$ , and  $d$ .
- b)  $V_{t,k}^*$  is nondecreasing in  $p$ .
- c)  $V_{t,k}^*$  is strictly increasing in  $u$  and  $d$ .

**Proof** As before, we prove the theorem by backward induction for the case of  $p$ . Proofs for other cases are similar and omitted. Let  $p_1 > p_2$ . For  $i = 1, 2$ , we define

$$(4.9) \quad \hat{V}_{t,k}^*(\hat{x}_{i:t+1,k}^*, p_i) = \hat{f}_{t,k}(\hat{x}_{i:t+1,k}^*, p_i) = \max_{x \geq 0} \hat{f}_{t,k}(x, p_i)$$

where, from (3.12), (3.13) and (3.27),

$$(4.10) \quad \hat{f}_{t,k}(x, p_i) = G(\beta(x), \hat{A}_{t,k}(p_i), B_{t,k}) - x$$

and

$$(4.11) \quad \begin{aligned} \hat{A}_{t,k}(p_i) &= G(p_i, \hat{V}_{t+1,k+1}^*(\hat{x}_{i:t+2,k+1}^*, p_i), \hat{V}_{t+1,k}^*(\hat{x}_{i:t+2,k}^*, p_i))e^{-r}; \\ B_{t,k} &= V_{F:t+1,k}e^{-r}. \end{aligned}$$

Since  $\hat{V}_{T,k} = S_0 u^k d^{T-k}$  is independent of  $p$ , one has  $\hat{A}_{T-1,k}(p_1) > \hat{A}_{T-1,k}(p_2)$  from (A.16). It then follows from (4.9), (4.10), and Lemma A.5 that

$$(4.12) \quad \hat{V}_{T-1,k}^*(\hat{x}_{1:T,k}^*, p_1) > \hat{V}_{T-1,k}^*(\hat{x}_{2:T,k}^*, p_2).$$

Suppose (4.12) holds with  $t+1$  replacing  $T-1$  and consider  $t$ . From (A.16) and the induction hypothesis, one sees that  $\hat{A}_{t,k}(p_1) > \hat{A}_{t,k}(p_2)$ . Applying Lemma A.5 again, it then follows that (4.12) holds with  $t$  in place of  $T-1$ .

For the monotonicity of  $V_{t,k}^*$  in  $p$ , in parallel with (4.5) and (4.6), we introduce for  $i = 1, 2$ :

$$(4.13) \quad V_{t,k}^*(x_{i:t+1,k}^*, p_i) = \max\{H_{t,k}(x_{i:t+1,k}^*, p_i), V_{A:t+1,k}\}$$

where, from (3.12) and (3.13),

$$(4.14) \quad H_{t,k}(x_{i:t+1,k}^*, p_i) = f_{t,k}(x_{i:t+1,k}^*, p_i) = \max_{x \geq 0} f_{t,k}(x, p_i)$$

$$(4.15) \quad f_{t,k}(x, p_i) = G(\beta(x), A_{t,k}(p_i), B_{t,k}) - x$$

and

$$(4.16) \quad \begin{aligned} A_{t,k}(p_i) &= G(p_i, V_{t+1,k+1}^*(x_{i:t+2,k+1}^*, p_i), V_{t+1,k}^*(x_{i:t+2,k}^*, p_i))e^{-r}; \\ B_{t,k} &= V_{F:t+1,k}e^{-r}. \end{aligned}$$

It is clear that  $V_{T,k}^*(p_1) = V_{T,k}^*(p_2) = S_0 u^k d^{T-k} - X_T$ . From (3.24), one then has  $A_{T-1,k}(p_1) > A_{T-1,k}(p_2)$  from Theorem 4.1 and (A.16) so that  $H_{T-1,k}(x_{1:T-1,k}^*, p_1) > H_{T-1,k}(x_{2:T-1,k}^*, p_2)$  from Lemma A.5. Hence from (4.13),

$$(4.17) \quad V_{T-1,k}^*(x_{1:T,k}^*, p_1) > V_{T-1,k}^*(x_{2:T,k}^*, p_2).$$

Suppose (4.17) holds with  $t+1$  replacing  $T-1$ , and consider the case of  $t$ . From (A.16), (4.16) and the induction hypothesis, one has  $A_{t,k}(p_1) > A_{t,k}(p_2)$ . From Lemma A.5, this in turn leads to  $H_{t,k}(x_{1:t+1,k}^*, p_1) > H_{t,k}(x_{2:t+1,k}^*, p_2)$ . It then follows from (4.13) that (4.17) holds for  $t$ , completing the proof.  $\square$

Similar arguments lead to following theorem. Proof is omitted here.

**Theorem 4.3** *Let  $\beta = \beta(x, b) = \frac{bx}{1+bx}$  as in Example 3.2. Then  $\hat{V}_{t,k}^*$  is strictly increasing and  $V_{t,k}^*$  is nondecreasing in  $b$ .*

We next show that the optimal project value with options,  $V_{t,k}^*$ , is larger than that without options,  $\hat{V}_{t,k}^*$  so that the optimal option value  $ROV_{t,k}^*$  is nonnegative for all  $t, k$ .

**Theorem 4.4** For  $0 \leq k \leq t$  and  $0 \leq t \leq T$ ,  $ROV_{t,k}^* = V_{t,k}^* - \hat{V}_{t,k}^* \geq 0$ .

**Proof** One sees from (3.1) that  $V_{T,k} = \hat{V}_{T,k}$  and  $ROV_{T,k}^* = 0$ . Suppose  $ROV_{t+1,k}^* \geq 0$  and consider the case of  $t$ . By the induction hypothesis, one has  $V_{t+1,k} \geq \hat{V}_{t+1,k}$ . From (3.13), (3.24) and (A.16), this in turn implies that

$$(4.18) \quad A_{t,k} > \hat{A}_{t,k}.$$

It then follows from Lemma A.2, (4.5) and (4.6) that  $V_{t,k}^* \geq \hat{V}_{t,k}^*$  completing the proof.  $\square$

## 5 Structural Properties of $\hat{x}^*$ and $x^*$

In parallel with the preceding section, we establish similar monotonicity properties for the optimal invest amount without option for termination,  $\hat{x}_{t,k}^*$ , and that with option  $x_{t,k}^*$ . Our first theorem below shows that having the option for termination provides an incentive to invest more because the risk involved can be controlled better.

**Theorem 5.1** If it is decided to continue the project at state  $(t,k)$ , then  $x_{t+1,k}^* \geq \hat{x}_{t+1,k}^*$ ,  $0 \leq k \leq t$ ,  $0 \leq t \leq T - 1$ .

**Proof** From (4.18) together with (3.12), (3.27), and (4.6), Lemma A.3 implies that  $x_{t,k}^* \geq \hat{x}_{t,k}^*$ .  $\square$

We next derive the monotonicity property of  $\hat{x}_{t,k}^*$  and  $x_{t,k}^*$  in  $k$ , i.e. as more upward successes are experienced, the optimal investment amount increases. In contrast with the monotonicity results for  $\hat{V}_{t,k}^*$  and  $V_{t,k}^*$  given in Section 4, the proof for  $\hat{x}_{t,k}^*$  and  $x_{t,k}^*$  involves certain subtlety and the following two assumptions are needed.

### Assumption 5.2

a)  $W_F(x) = \alpha x$ ,  $\alpha > 0$

b)  $pu + (1 - p)d > 1$

For notational convenience, the first difference of a sequence  $(a_k)_{k=0}^\infty$  is denoted by

$$(5.1) \quad \Delta_k a_k = a_k - a_{k-1}, \quad k \geq 1.$$

**Theorem 5.3** *Under Assumption 5.2, one has for  $0 \leq k \leq t$ ,  $0 \leq t \leq T - 1$ :*

a)  $\Delta_k \hat{A}_{t,k} - \Delta_k B_{t,k} \geq 0$ ;  $\Delta_k A_{t,k} - \Delta_k B_{t,k} \geq 0$

b)  $x_{t+1,k}^*$  and  $\hat{x}_{t+1,k}^*$  are strictly increasing in  $k$ .

**Proof** We prove the theorem for  $\hat{x}_{t+1,k}^*$  by backward induction. The proof for  $x_{t+1,k}^*$  is similar and omitted here. Since  $\hat{A}_{t,k} > B_{t,k}$  from (3.27), one has, in particular,  $\hat{A}_{T-1,k} > B_{T-1,k}$ . From (2.1), (3.13), (3.14), and Assumption 5.2 a), it then follows that

$$\begin{aligned} \hat{A}_{T-1,k} - B_{T-1,k} &= G(p, \hat{V}_{T,k+1}, \hat{V}_{T,k}) - W_F(S_{T-1,k}) \\ &= p(S_{T,k} - X_T) + (1 - p)(S_{T,k} - X_T) - \alpha S_{T-1,k}, \end{aligned}$$

i.e.

$$(5.2) \quad \hat{A}_{T-1,k} - B_{T-1,k} = S_{T-1,k} \{pu + (1 - p)d - \alpha\} - X_T > 0.$$

This, in turn, implies that

$$(5.3) \quad pu + (1 - p)d - \alpha > 0.$$

By taking the first difference of both sides of (5.2) with respect to  $k$ , one then sees that

$$\begin{aligned} \Delta_k \hat{A}_{T-1,k} - \Delta_k B_{T-1,k} &= \{pu + (1 - p)d - \alpha\} \Delta_k S_{T-1,k} \\ &= \{pu + (1 - p)d - \alpha\} \left(1 - \frac{d}{u}\right) S_{T-1,k}. \end{aligned}$$

Since  $0 < d < u$ , from (5.3), this then leads to

$$(5.4) \quad \Delta_k \hat{A}_{T-1,k} > \Delta_k B_{T-1,k}.$$

This inequality (5.4) and Lemma A.4 imply that

$$(5.5) \quad \Delta_k \hat{x}_{T,k}^* > 0.$$



Suppose (5.4) and (5.5) hold true when  $T - 1$  is replaced by  $t + 1$  and  $T$  is replaced by  $t + 2$  respectively. We first show that

$$(5.6) \quad \Delta_k \hat{V}_{t+1,k}^* > \Delta_k B_{t+1,k}.$$

Noting  $\Delta_k[a_k b_k] = b_k \Delta_k a_k + a_{k-1} \Delta_k b_k$ , one sees from (3.5) and (3.12) that

$$\begin{aligned} & \Delta_k \hat{V}_{t+1,k}^* - \Delta_k B_{t+1,k} \\ &= \Delta_k [\beta(\hat{x}_{t+2,k}^*) \hat{A}_{t+1,k} + (1 - \beta(\hat{x}_{t+2,k}^*)) B_{t+1,k} - \hat{x}_{t+2,k}^*] - \Delta_k B_{t+1,k} \\ &= [\hat{A}_{t+1,k} - B_{t+1,k}] \Delta_k \beta(\hat{x}_{t+2,k}^*) - \Delta_k \hat{x}_{t+2,k}^* \\ & \quad + \beta(\hat{x}_{t+2,k-1}^*) [\Delta_k \hat{A}_{t+1,k} - \Delta_k B_{t+1,k}]. \end{aligned}$$

By the induction hypothesis of (5.4) replacing  $T - 1$  by  $t + 1$ , the last term in the above expression is positive so that

$$(5.7) \quad \Delta_k \hat{V}_{t+1,k}^* - \Delta_k B_{t+1,k} > [\hat{A}_{t+1,k} - B_{t+1,k}] \Delta_k \beta(\hat{x}_{t+2,k}^*) - \Delta_k \hat{x}_{t+2,k}^*.$$

From the induction hypothesis, one has  $\Delta_k \hat{x}_{t+2,k}^* > 0$ . From strict concavity of  $\beta(x)$  together with (3.28), it can be seen that

$$(5.8) \quad \beta'(\hat{x}_{t+2,k}^*) \Delta_k \hat{x}_{t+2,k}^* = \frac{\Delta_k \hat{x}_{t+2,k}^*}{\hat{A}_{t+1,k} - B_{t+1,k}} < \Delta_k \beta(\hat{x}_{t+2,k}^*).$$

Employing (5.8) in (5.7) then yields (5.6). Finally, one sees from (3.5) and (3.13) that

$$\begin{aligned} & (\Delta_k \hat{A}_{t,k} - \Delta_k B_{t,k}) e^r \\ &= G(p, \Delta_k \hat{V}_{t+1,k+1}, \Delta_k \hat{V}_{t+1,k}) - \Delta_k B_{t,k} \\ &> G(p, \Delta_k B_{t+1,k+1}, \Delta_k B_{t+1,k}) - \Delta_k B_{t,k}, \end{aligned}$$

where (5.6) is used to derive the last inequality. From Assumption 5.2 a) together with (3.2), it can be readily seen that

$$(5.9) \quad \begin{aligned} \Delta_k B_{t+1,k+1} &= \alpha u \left(1 - \frac{d}{u}\right) S_{t,k}; \\ \Delta_k B_{t+1,k} &= \alpha d \left(1 - \frac{d}{u}\right) S_{t,k}; \\ \Delta_k B_{t,k} &= \alpha \left(1 - \frac{d}{u}\right) S_{t,k}. \end{aligned}$$

Substituting (5.9) into the last inequality above, it follows that

$$(\Delta_k \hat{A}_{t,k} - \Delta_k B_{t,k})e^r > \alpha \left(1 - \frac{d}{u}\right) S_{t,k} \{pu + (1-p)d - 1\}.$$

Hence one concludes from Assumption 5.2 b) that

$$(5.10) \quad \Delta_k \hat{A}_{t,k} > \Delta_k B_{t,k}.$$

Using (5.10) and Lemma A.4 then yields  $\Delta_k \hat{x}_{t+1,k}^* > 0$ , completing the proof.  $\square$

For a function  $\xi(z)$  with  $z_1 > z_2$ , we define

$$(5.11) \quad \Delta_z \xi(z) = \xi(z_1) - \xi(z_2).$$

With this notation, the monotonicity properties of  $\hat{x}_{t,k}^*$  and  $x_{t,k}^*$  with respect to  $u$  and  $d$  can be shown as in the theorem below. The proof is almost identical to that of Theorem 5.3, and is omitted here.

**Theorem 5.4**

a) Let  $u_1 > u_2$  and suppose Assumption 5.2 is satisfied where  $u = u_2$ . Then

$$\min\{\Delta_u A_{t,k}(u), \Delta_u \hat{A}_{t,k}(u)\} > \Delta_u B_{t,k}(u)$$

and both  $x_{t,k}^*$  and  $\hat{x}_{t,k}^*$  are strictly increasing in  $u$ .

b) Let  $d_1 > d_2$  and suppose Assumption 5.2 is satisfied where  $d = d_2$ . Then

$$\min\{\Delta_d A_{t,k}(d), \Delta_d \hat{A}_{t,k}(d)\} > \Delta_d B_{t,k}(d)$$

and both  $x_{t,k}^*$  and  $\hat{x}_{t,k}^*$  are strictly increasing in  $d$ .

The monotonicity properties of  $\hat{x}_{t,k}^*$  and  $x_{t,k}^*$  with respect to  $p$  hold true without Assumption 5.2 as we prove next.

**Theorem 5.5** Let  $p_1 > p_2$ . Then

$$\min\{\Delta_p A_{t,k}(p), \Delta_p \hat{A}_{t,k}(p)\} > \Delta_p B_{t,k}(p)$$

and both  $x_{t,k}^*$  and  $\hat{x}_{t,k}^*$  are strictly increasing in  $p$ .

**Proof** We first note from (3.2) and (3.24) that  $B_{t,k} = W_F(S_{t,k})$  which is independent of  $p$ . Hence  $\Delta_p B_{t,k} = 0$ . On the other hand, one sees from (3.13), Theorem 4.2 a) and (A.16) that  $\Delta_p \hat{A}_{t,k} > 0 = \Delta_p B_{t,k}$  and  $\Delta_p A_{t,k} > 0 = \Delta_p B_{t,k}$ . Lemma A.4 then implies that  $\Delta_p \hat{x}_{t+1,k}^*(p) > 0$  and  $\Delta_p x_{t+1,k}^*(p) > 0$ , i.e. both  $\hat{x}_{t+1,k}^*$  and  $x_{t+1,k}^*$  are strictly increasing in  $p$ , proving the theorem.  $\square$

## 6 Numerical Results

In this section, we present numerical results to demonstrate the monotonicity properties derived in the previous two sections. Throughout this section, it is assumed that the success probability  $\beta(x)$  is of the form given in Example 3.2, and two salvage value functions  $W_A(x)$  for option to terminate and  $W_F(x)$  for failure are both linear. More specifically, we define:

$$(6.1) \quad \beta(x) = \frac{bx}{1+bx}, \quad b > 0,$$

$$(6.2) \quad W_A(x) = \alpha_A x, \quad \alpha_A > 0,$$

and

$$(6.3) \quad W_F(x) = \alpha_F x, \quad 0 < \alpha_F < \alpha_A.$$

As a basic model, we adopt parameter values specified in the table below. The monetary unit is one million yen and the time unit is one year. These parameter values are assumed throughout this section unless specified otherwise.

| Symbol     | Definition  | Value |
|------------|---|-------|
| $S_0$      | the estimated future cash flow of the project outcome   | 3,000 |
| $T$        | the planning horizon                                    | 5     |
| $p$        | the upward probability given success                    | 0.7   |
| $r$        | the risk-free interest rate                             | 0.1   |
| $u$        | the increasing rate given an upward success             | 1.25  |
| $d$        | the decreasing rate given a downward success            | 0.75  |
| $X_T$      | the operational cost                                    | 100   |
| $b$        | the success probability parameter in (6.1) <sup>†</sup> | 0.02  |
| $\alpha_A$ | the salvage value parameter in (6.2)                    | 0.25  |
| $\alpha_F$ | the salvage value parameter in (6.3)                    | 0.2   |

**Remark 6.1<sup>†</sup>** *This value is set so that  $\beta(x)$  would likely to lie between 0.7 and 0.95 for the basic model.*

Table 6.1 summarizes computational results for the basic model, while Table 6.2 exhibits those for a model similar to the basic model except that  $p$  is reduced from 0.7 to 0.55. We note that  $\hat{V}_{t,k+1}^* \geq \hat{V}_{t,k}^*$  and  $V_{t,k+1}^* \geq V_{t,k}^*$  as they should be from Theorem 4.1. This monotonicity is also observed for the investment amounts, i.e.  $\hat{x}_{t,k+1}^* \geq \hat{x}_{t,k}^*$  and  $x_{t,k+1}^* \geq x_{t,k}^*$ . This means that the investment amount increases as more upward successes are experienced.

Comparing Table 6.1 with Table 6.2, one finds that both  $\hat{V}_{0,0}$  and  $V_{0,0}^*$  decrease as  $p$  decreases, which is expected from Theorem 4.2. Furthermore it can be seen that the option value in Table 6.2 with  $p=0.55$  is 177, while that for the basic model with  $p=0.7$  is 0. This suggests that the risk potential of the project increases as  $p$  decreases, and the option for terminating the project becomes a viable alternative. The increase of the risk potential can also be observed in the fact that the investment amounts with  $p = 0.55$  are uniformly smaller than those with  $p = 0.7$ , i.e.  $x_{t,k}^*$  with  $p=0.55$  are less than  $x_{t,k}^*$  with  $p=0.7$  for all  $0 \leq k \leq t$ ,  $0 \leq t \leq T - 1 = 4$ .

In order to explore monotonicity properties of  $\hat{V}_{t,k}^*$  and  $V_{t,k}^*$ , further, numerical experiments are conducted by varying parameters  $p$  (Figure 6.3),  $u$  (Figure 6.4),  $d$  (Figure 6.5), and  $b$  (Figure 6.6). As we already observed, the risk potential increases as  $p$  decreases. From Example 3.2, one has  $\frac{\partial}{\partial b}\beta > 0$  so that the success probability  $\beta$  increases as  $b$  increases, which in turn leads to reduction of the risk potential. It is clear that increasing  $u$  or  $d$  results in larger  $\hat{V}_{0,0}^*$  and  $V_{0,0}^*$  and the risk potential decreases. Figures 6.7 through 6.10 exhibit similar monotonicity properties for  $\hat{x}^*$  and  $x^*$ . Assumption 5.2 b) is satisfied for  $p > 0.5$ ,  $u > 1.108$ , and  $d > 0.416$ . However the monotonicity of  $\hat{x}^*$  and  $x^*$  is observed outside these ranges. Indeed, for nonlinear concave functions  $W_A(x) = 100(1 + \log 0.25x)$  and  $W_F(x) = 1 + \log 0.2x$  tested in Figures 6.11 through 6.18, all of  $\hat{V}^*$ ,  $V^*$ ,  $\hat{x}^*$ , and  $x^*$  are monotone as before. This is expected for  $\hat{V}^*$  and  $V^*$  from Theorems 4.2 and 4.3, but the results for  $\hat{x}^*$  and  $x^*$  are outside the scope of Theorems 5.4 and 5.5.

Numerical results presented in this section suggest that the option value  $ROV_{0,0}^*$  increases as the risk potential increases. Theoretical proof for this conjecture is difficult and is being attempted.

| $t$ | $k$ | DP-V        |             | DP- $\hat{V}$     |                   | $ROV_{t,k}^*$ |
|-----|-----|-------------|-------------|-------------------|-------------------|---------------|
|     |     | $V_{t,k}^*$ | $x_{t,k}^*$ | $\hat{V}_{t,k}^*$ | $\hat{x}_{t,k}^*$ |               |
| 0   | 0   | 824         | 119         | 824               | 119               | 0             |
| 1   | 1   | 1474        | 199         | 1474              | 199               | 0             |
| 1   | 0   | 654         | 111         | 654               | 111               | 0             |
| 2   | 2   | 2510        | 288         | 2510              | 288               | 0             |
| 2   | 1   | 1225        | 189         | 1225              | 189               | 0             |
| 2   | 0   | 556         | 112         | 556               | 112               | 0             |
| 3   | 3   | 4034        | 386         | 4034              | 386               | 0             |
| 3   | 2   | 2144        | 275         | 2144              | 275               | 0             |
| 3   | 1   | 1088        | 188         | 1088              | 188               | 0             |
| 3   | 0   | 517         | 120         | 517               | 120               | 0             |
| 4   | 4   | 6166        | 492         | 6166              | 492               | 0             |
| 4   | 3   | 3498        | 368         | 3498              | 368               | 0             |
| 4   | 2   | 1943        | 271         | 1943              | 271               | 0             |
| 4   | 1   | 1045        | 195         | 1045              | 195               | 0             |
| 4   | 0   | 535         | 135         | 535               | 135               | 0             |

Table 6.1  $p = 0.7$

| $t$ | $k$ | DP-V        |             | DP- $\hat{V}$     |                   | $ROV_{t,k}^*$ |
|-----|-----|-------------|-------------|-------------------|-------------------|---------------|
|     |     | $V_{t,k}^*$ | $x_{t,k}^*$ | $\hat{V}_{t,k}^*$ | $\hat{x}_{t,k}^*$ |               |
| 0   | 0   | 750         | 49          | 573               | 39                | 176           |
| 1   | 1   | 1024        | 131         | 1024              | 131               | 0             |
| 1   | 0   | 563         | 57          | 471               | 57                | 91            |
| 2   | 2   | 1923        | 232         | 1923              | 232               | 0             |
| 2   | 1   | 929         | 145         | 929               | 145               | 0             |
| 2   | 0   | 423         | 77          | 423               | 77                | 0             |
| 3   | 3   | 3424        | 343         | 3424              | 343               | 0             |
| 3   | 2   | 1806        | 241         | 1806              | 241               | 0             |
| 3   | 1   | 908         | 162         | 908               | 162               | 0             |
| 3   | 0   | 427         | 100         | 427               | 100               | 0             |
| 4   | 4   | 5715        | 469         | 5715              | 469               | 0             |
| 4   | 3   | 3237        | 350         | 3237              | 350               | 0             |
| 4   | 2   | 1792        | 256         | 1792              | 256               | 0             |
| 4   | 1   | 959         | 184         | 959               | 184               | 0             |
| 4   | 0   | 488         | 126         | 488               | 126               | 0             |

Table 6.2  $p = 0.55$

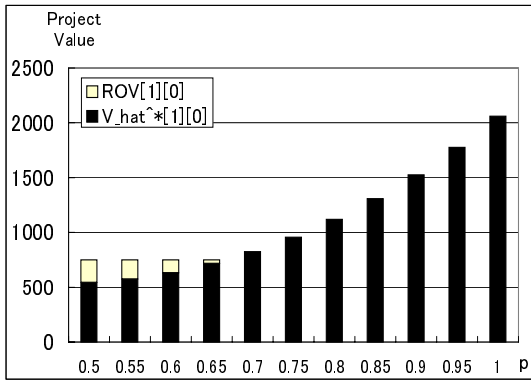


Figure 6.3

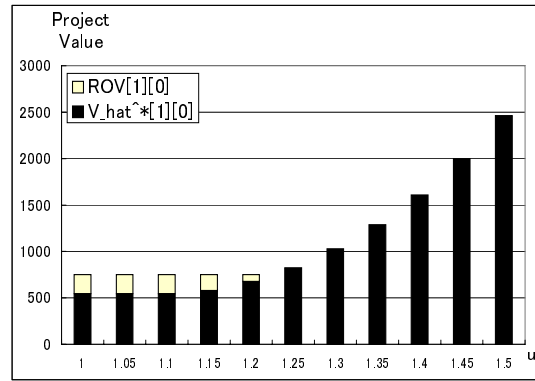


Figure 6.4

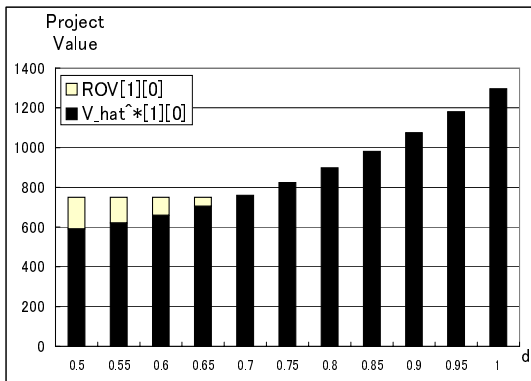


Figure 6.5

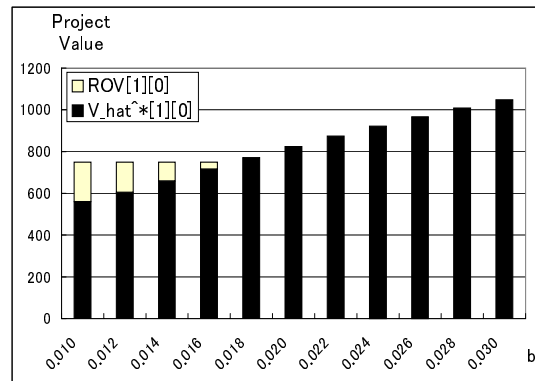


Figure 6.6

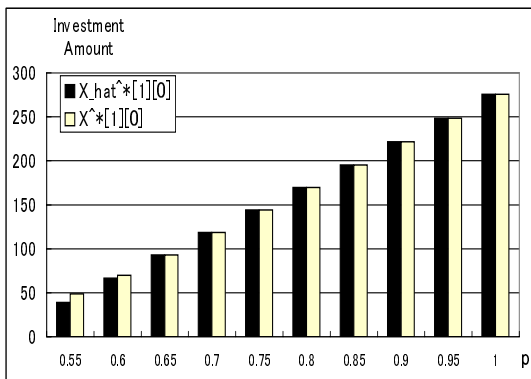


Figure 6.7

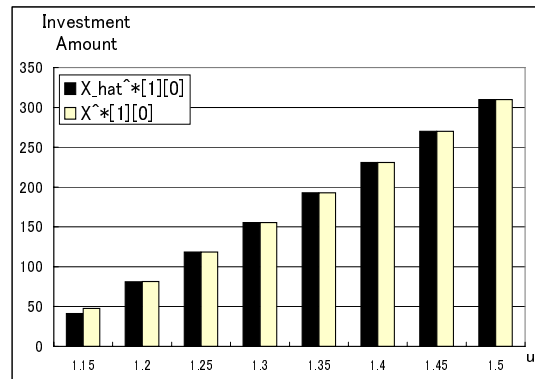


Figure 6.8

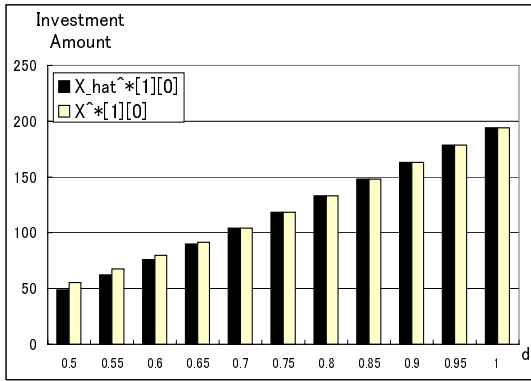


Figure 6.9

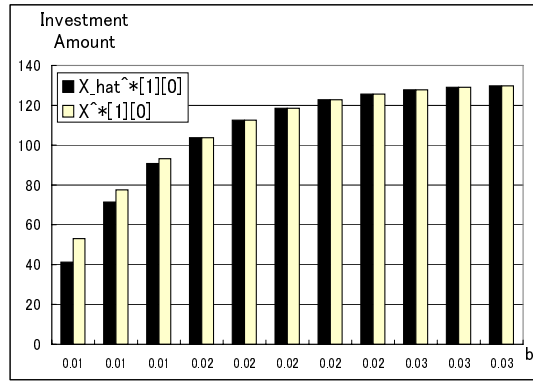


Figure 6.10

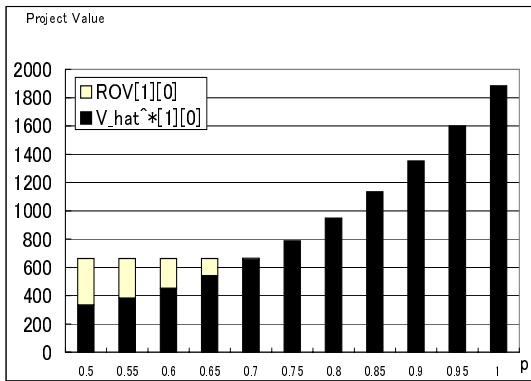


Figure 6.11

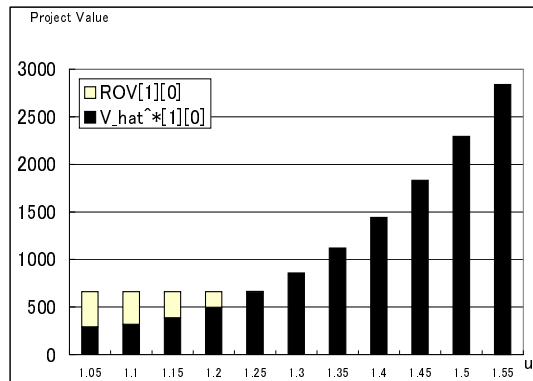


Figure 6.12

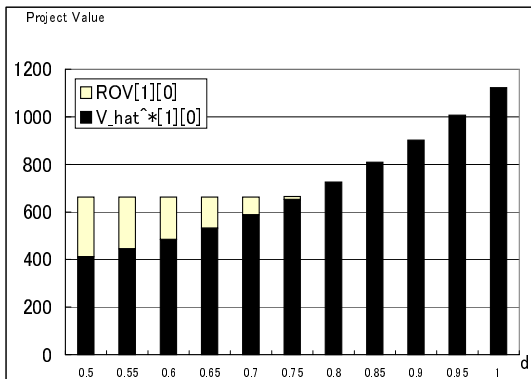


Figure 6.13

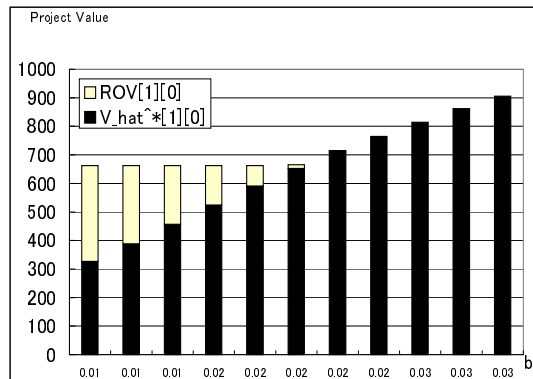


Figure 6.14

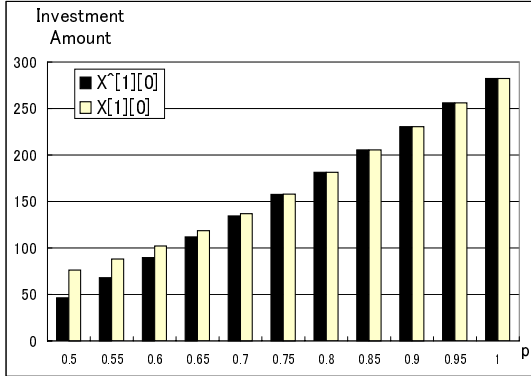


Figure 6.15

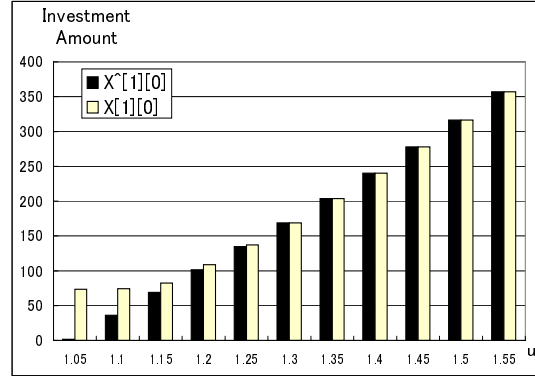


Figure 6.16

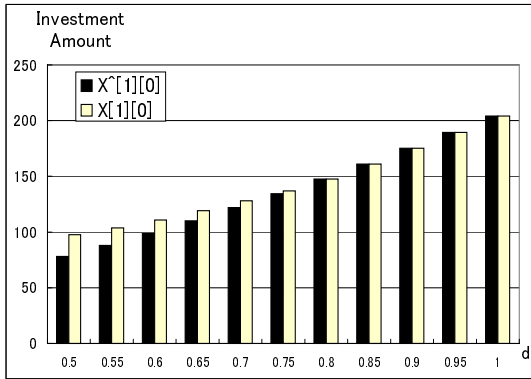


Figure 6.17

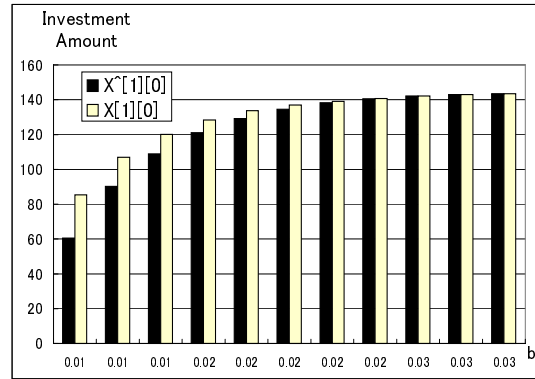


Figure 6.18

## 7 Concluding Remarks and Future Research

In this paper, two DP models have been developed to investigate the optimal investment policy ( $x_{t,k}^*$ ), the present value of the project ( $V_{0,0}^*$ ), and the option value  $ROV_{0,0}^*$  based on ROA (Real Option Approach) defined by Yamamoto and Kariya [12]. Structural properties of  $\hat{V}_{t,k}^*$  and  $V_{t,k}^*$  are examined analytically, proving the unique existence of optimal investment policy and yielding useful monotonicity results. Analytical results combined with extensive numerical experiments revealed the following observations.

- (7.1) Both the project value at time  $t$  with  $k$  upward successes,  $V_{t,k}^*$ , and the associated investment amount  $x_{t,k}^*$  increase as a function of  $k$ .



- (7.2) As the risk-potential of the project increases: 1)  $V_{0,0}^*$  decreases to the lower bound determined by the value of doing nothing from the very beginning, and then stays at the level; 2)  $\hat{V}_{0,0}^*$  decreases; and the option value  $ROV_{0,0}^*$  increases.

As for the future research, the following issues will be addressed in due course.

- (7.3) It will be attempted with best efforts to prove the monotonicity properties of  $ROV_{0,0}^*$  observed throughout the numerical experiments.
- (7.4) A capital constraint will be incorporated in the DP formulation.
- (7.5) An optimal resource allocation problem will be considered, where a central management allocates common resources to individual business units which manage various projects through the DP model developed in this paper.

#### Acknowledgement

This research has been supported in part by the research fund provided by Mizuho-DL Financial Technology.

## Appendix

In this appendix, we establish various lemmas which will provide useful tools for proving key theorems in the main text. We largely focus on structural properties of a function  $f$  defined by

$$(A.1) \quad \begin{aligned} f(x) &= G(\beta(x), A, B) - x \\ &= \beta(x)A + \{1 - \beta(x)\}B - x. \end{aligned}$$

We recall that  $\beta(x)$  is the probability that the project can be continued successfully for one period given that the investment amount for the period is  $x$ . It was assumed that, for  $x \geq 0$ ,

$$(A.2) \quad \beta(0) = 0, \quad 0 \leq \beta(x) \leq 1, \quad \beta'(x) > 0, \quad \beta''(x) < 0,$$

where  $\beta'(x) = \frac{d}{dx}\beta(x)$  and  $\beta''(x) = \frac{d^2}{dx^2}\beta(x)$ . We also define

$$(A.3) \quad f(x^*) = \max_{x \geq 0} f(x).$$

It can be readily seen that

$$(A.4) \quad f'(x) = \beta'(x)(A - B) - 1$$

and

$$(A.5) \quad f''(x) = \beta''(x)(A - B).$$

Because of (3.12) and (3.20), it is assumed that

$$(A.6) \quad A > B.$$

The next lemma is immediate from (A.2), (A.5), and (A.6).

**Lemma A.1**  *$f(x)$  is strictly concave in  $x$ .*

For notational convenience, let

$$(A.7) \quad f_i(x) = G(\beta(x), A_i, B_i) - x, \quad i = 1, 2$$

and

$$(A.8) \quad f_i(x_i^*) = \max_{x \geq 0} f_i(x).$$

We assume that  $x_i^*$  can be determined uniquely. As in (A.6), we assume that

$$(A.9) \quad A_i > B_i, \quad i = 1, 2.$$

The following lemmas then hold true.

**Lemma A.2** *If  $A_1 \geq A_2$  and  $B_1 \geq B_2$ , then  $f_1(x_1^*) \geq f_2(x_2^*)$ . Equality holds if and only if  $A_1 = A_2$  and  $B_1 = B_2$ .*

**Proof** We note from the definition of  $x_1^*$  in (A.8) that  $f_1(x_1^*) \geq f_1(x_2^*)$ . We can also see from the assumption that

$$\begin{aligned} f_1(x_2^*) - f_2(x_2^*) &= \beta(x_2^*)(A_1 - A_2) + (1 - \beta(x_2^*))(B_1 - B_2) \\ &\geq 0. \end{aligned}$$

It then follows that

$$\begin{aligned} &f_1(x_1^*) - f_2(x_2^*) \\ &= f_1(x_1^*) - f_1(x_2^*) + f_1(x_2^*) - f_2(x_2^*) \\ &\geq 0. \end{aligned}$$

It is clear that if  $A_1 \neq A_2$  or  $B_1 \neq B_2$ , then  $f_1(x_1^*) > f_1(x_2^*)$  from the uniqueness of  $x_1^*$ , and  $f_1(x_1^*) > f_2(x_2^*)$ , completing the proof.  $\square$

**Lemma A.3** *If  $A_1 \geq A_2$  and  $B_1 = B_2$ , then  $x_1^* \geq x_2^*$ . Equality holds if and only if  $A_1 = A_2$ .*

**Proof** We first note from (A.2), (A.5) and (A.6) that  $f$  is strictly concave. From (A.4), it can be readily seen that for  $i = 1, 2$ ,  $f'(x_i) = 0$  at  $x_i^*$  if and only if

$$(A.10) \quad \beta'(x_i^*) = \frac{1}{(A_i - B_i)}.$$

One then sees from  $A_1 \geq A_2$  and  $B_1 = B_2$  together with (A.9) that

$$\beta'(x_1^*) = \frac{1}{(A_1 - B_1)} \leq \frac{1}{(A_2 - B_2)} = \beta'(x_2^*).$$

The lemma now follows from the strict concavity of  $\beta(x)$  given in (A.2).  $\square$

**Lemma A.4** *If  $A_1 \geq A_2$ ,  $B_1 \geq B_2$  and  $A_1 - A_2 \geq B_1 - B_2$ , then  $x_1^* \geq x_2^*$ . Equality holds if and only if  $A_1 - A_2 = B_1 - B_2$ .*

**Proof** From (A.6), (A.9), and the assumption, one sees that  $A_1 - B_1 \geq A_2 - B_2$  and

$$\beta'(x_1) = \frac{1}{A_1 - B_1} \leq \frac{1}{A_2 - B_2} = \beta'(x_2).$$

The lemma then follows from (A.2).  $\square$

In order to observe monotonicity properties concerning  $p$ , we modify  $f(x)$  as

$$(A.11) \quad f(x, p, A, B) = G(\beta(x), G(p, A, B), C) - x$$

where

$$(A.12) \quad G(p, A, B) > C, \quad 0 \leq p \leq 1$$

and

$$(A.13) \quad f(x^*, p, A, B) = \max_{x \geq 0} f(x, p, A, B).$$

As for (A.7) and (A.8), we also define

$$(A.14) \quad f_i(x, p_i) = f(x, p_i, A_i, B_i)$$

and

$$(A.15) \quad f_i(x_i^*, p_i) = \max_{x \geq 0} f_i(x, p_i).$$

We assume that  $x^*$  in (A.13) or  $x^*$  in (A.15) can be determined uniquely.

**Lemma A.5** *If  $A_1 \geq A_2$ ,  $B_1 \geq B_2$ , and  $p_1 > p_2$ , then:*

**a)**  $f_1(x_1^*, p_1) > f_2(x_2^*, p_2)$

**b)**  $x_1^* > x_2^*$

**Proof** We note from (3.5) that  $G(\lambda, A_1, B_1) \geq G(\lambda, A_2, B_2)$ . Since  $\frac{\partial}{\partial \lambda} G(\lambda, A, B) = \lambda(A - B) > 0$  from (A.6), one has, for  $\lambda_1 > \lambda_2$ ,

$$(A.16) \quad G(\lambda_1, A_1, B_1) > G(\lambda_2, A_1, B_1) \geq G(\lambda_2, A_2, B_2).$$

Let  $\bar{A}_i = G(p_i, A_i, B_i)$  for  $i = 1, 2$ . From (A.12) and (A.16), one has  $\bar{A}_1 > \bar{A}_2 > C$ , for  $p_1 > p_2$ . It then follows from (A.11), (A.14), and (A.16) that

$$\begin{aligned}
 \text{(A.17)} \quad f_1(x, p_1) &= G(\beta(x), \bar{A}_1, C) - x \\
 &> G(\beta(x), \bar{A}_2, C) - x \\
 &= f_2(x, p_2).
 \end{aligned}$$

One has  $f_1(x_1^*, p_1) > f_1(x_2^*, p_1)$  from (A.15) and  $f_1(x_2^*, p_1) > f_2(x_2^*, p_2)$  from (A.17), so that  $f_1(x_1^*, p_1) > f_2(x_2^*, p_2)$ , proving part a).

From (A.9) and (A.16), one sees that

$$\beta'(x_1^*) = \frac{1}{\bar{A}_1 - C} < \frac{1}{\bar{A}_2 - C} = \beta'(x_2^*).$$

Part b) then follows from (A.2). □

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