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(Correction)
Optimal Hypothesis Testing under Unbiased Estimates—
Applications to the Family of Retracted Distributions (I).

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§0. Summary.

Here, we consider hypothesis testing with the regret region derived from inverting the minimum random interval based on unbiased estimate for the parameter by using Lagrange's method. As examples we consider two oblong (or uniform) distributions in the family of retracted distributions.(See Y. Nogami(2002 a,b,c).)

The expressions and definitions in the author's previous works (Y. Nogami(2002 a,b,c)) must be altered in the same expressions as those appeared in this paper.

§1. The Oblong (or Uniform) $U[\theta+\delta_1,\theta+\delta_2)$ (δ_1,δ_2: reals such that $\delta_1<\delta_2$).

For simplicity, we let $\delta_1=0$ and $\delta_2=1$. The underlined density is

$$f(x|\theta)=\begin{cases}1, & \text{for } \theta<x<\theta+1, \quad (-\infty<\theta<\infty)\\ 0, & \text{otherwise.}\end{cases}$$

We take a random sample $X_1, \ldots ,X_n$ from $f(x|\theta)$ and test the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ with a given real number θ_0.

Part (I): Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots ,X_n$. Let $V=X_{(1)}$, $W=X_{(n)}$ and $Y=(V+W-1)/2$. Since the p.d.f. of Y is given by

$$g_Y(y|\theta)=\begin{cases}n(1-2|y-\theta|)^{n-1}, & \text{for } -1/2<y<1/2,\\ 0, & \text{otherwise.}\end{cases}$$

We can easily get $E(Y)=\theta$ and $\mathrm{Var}(Y)=1/\{2(n+1)(n+2)\}$.

To find the density of $T^{*}=(Y-\theta)/\sqrt{\mathrm{Var}(\theta)}=\sqrt{2(n+1)(n+2)}(Y-\theta)$ we simplify the problem to find the density of $T=2n(Y-\theta)$ as follows:

$$h_T(t)=\begin{cases}2^{-1}(1-|t|/n)^{n-1}, & -n<t<n\\ 0, & \text{otherwise.}\end{cases}$$

Let α be a real number with $0<\alpha<1$. Let r_1, r_2, t_1 and t_2 be real numbers such that $r_1<r_2$ and $t_1<t_2$. Minimizing t_2-t_1 subject to

$$P_\theta[r_1<Y-\theta<r_2]=P[t_1<T<t_2]=1-\alpha,$$

we get by Lagrange's multiplier's method that $(t=)t_2=-t_1=n(1-\alpha^{1/n})$ $(\to -\log_e\alpha$ as $n\to\infty)$ and $r_2=-r_1(=r)=(1-\alpha^{1/n})/2$ $(\to 0$ as $n\to\infty)$.
Actually, the density of T^{*} converges to the density

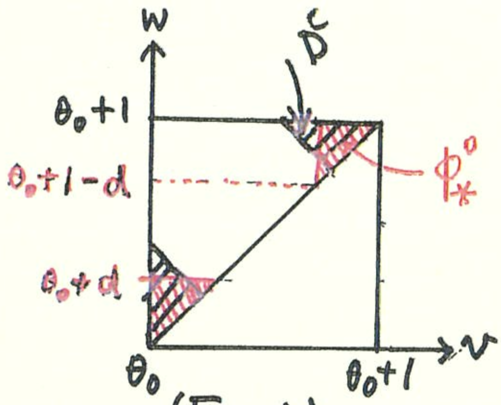
$$\phi(t)=2^{-1/2}\exp\{-\sqrt{2}|t|\},\qquad -\infty<t<\infty.$$

Hence, letting $T_0=2n(Y-\theta_0)$ we regret not rejecting H_0 if $-t<T_0<t$ and satisfy to reject H_0 if $|T_0|\geqq t$. We call $(-t,t)$ the regret region based on T_0.

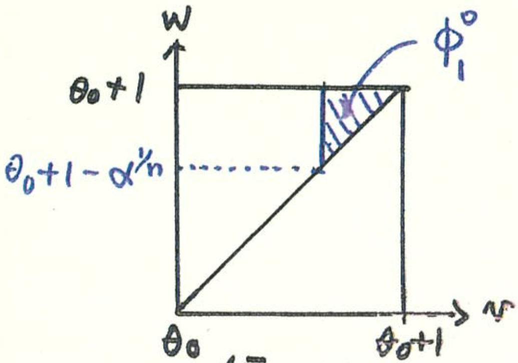
Part (II). Here, we consider the bibliographies; especially, S. K. Chatterjee and G. Chattopadhyay(1994). We first consider our non-regret region as follows:

$$D^c=\{Y\leqq\theta_0-r, Y\geqq\theta_0+r, \theta_0<V<W<\theta_0+1\}\cup\{V<\theta_0, W-V<1\}\cup\{W\geqq\theta_0+1, W-V<1\}.$$

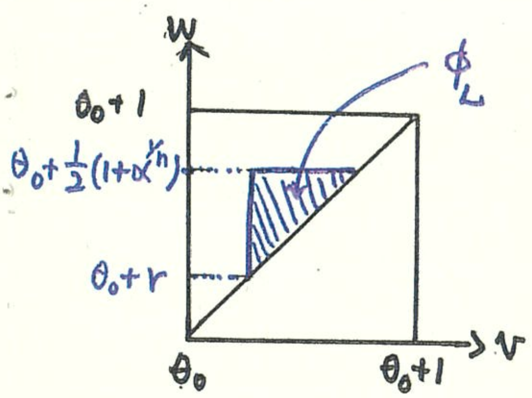
We call $\xi(\theta)=P_\theta(D)$ the regret probability function. (D^c is the



(Fig. 1)



(Fig. 2)



(Fig. 3)

complement of D.) From p. 4 of Y. Nogami (2002b) $\xi(\theta)$ is symmetric at $\theta=\theta_0$, a concave function from below and $\xi(\theta)\leq 1-\alpha = \xi(\theta_0)$, $\forall\theta$. (This property is also seen from Result 2.1 of S. K. Chatterjee and G. Chattopadhyay(1994).)

We return to S. K. Chatterjee and G. Chattopadhyay(1994). They consider the test of the hypotheses $H_0:\theta=\theta_0$ versus $H_1: \theta\neq\theta_0$. Let $\underline{x}=(x_1, \dots, x_n)$. In Result 2.2 they introduced the test

$$\phi^0{}_1(\underline{x}) = \begin{cases} 1, & \text{if } \underline{x}\notin[\theta_0,\theta_0+1]^n \text{ or } \\ & \theta_0+1-\alpha^{1/n}<V<W<\theta_0+1, \\ 0, & \text{otherwise} \end{cases}$$

and compared with J. W. Pratt(1961)'s test $\phi^0{}_*(\underline{x})$ for $\theta>\theta_0$. We can see the difference of the structures of non-regret regions under H_0 for the tests D^c, $\phi^0{}_*$ and $\phi^0{}_1$ in Figures 1 and 2. In their Result 3.1 they showed unique locally best test

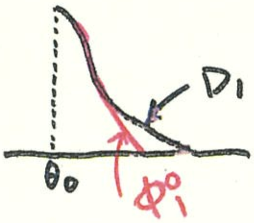
$$\phi_L(\underline{x}) = \begin{cases} 1, & \text{if } \underline{x}\in \mathfrak{X}-[\theta_0,\theta_0+1]^n \text{ or } \\ & \theta_0+r<V<W<\theta_0+(1-\alpha^{1/n})/2, \\ 0, & \text{otherwise,} \end{cases}$$

whose non-regret region appeared in Figure 3.

This is just like the regret region. Even if ϕ_L is the unique locally best test, practical statistician does not want to use this test.

This kind of curious phenomenon happens for this oblong (or uniform) distribution.

Remark: (i) We note that for $\theta_0-\alpha^{1/n}<\theta<\theta_0$ $\xi(\theta)<E_\theta(1-\phi^0{}_1(\underset{\sim}{X}))$, but the other side of θ_0 the inequality is reversed.
(ii) Consider to test $H_0:\theta=\theta_0$ versus $H_1:\theta>\theta_0$. We can construct the regret region D_1 where P_θ $(D_1)=1-\alpha$ and



$$P_\theta(D_1)=E_\theta(1-\phi^0{}_1(\underset{\sim}{X})), \qquad \text{for } \theta_0\leq\theta<\theta_0+1-(2\alpha)^{1/n}$$

$$>E_\theta(1-\phi^0{}_1(\underset{\sim}{X}))=0, \qquad \text{for } \theta_0+1-\alpha^{1/n}\leq\theta.$$

So, our test cannot overcome their $\phi^0{}_1$.

For the case of unknown $c=\delta_2-\delta_1(>0)$, we omit the discussion here. (See e.g. Y. Nogami(2002b,c)).

§2. The Oblong (or Uniform) U(0,θ).

The underlined density is

$$f(x|\theta)=\begin{cases}\theta^{-1}, & \text{for } 0<x<\theta, \qquad (\theta>0)\\ 0, & \text{otherwise.}\end{cases}$$

We take a random sample X_1, ... ,X_n from $f(x|\theta)$ and test the hypothses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$. Let $\theta^*=\log_e\theta$ and $Y=\log_e X$. The p.d.f. of Y is given by

$$g_Y(y)=\begin{cases}\exp\{y-\theta^*\}, & \text{for } 0<y<\theta^* \\ 0, & \text{otherwise.}\end{cases}$$

Let $U=\overline{Y}+1=n^{-1}\Sigma_{i=1}^{n}Y_i+1$. Then, $E(U)=\theta^*$. We call (b_1,b_2) the $(1-\alpha)$ random interval for γ if $P_\gamma[b_1<\gamma<b_2]=1-\alpha$. Using the density $h_V(v)$ of $V=2n(\theta^*+1-U)$ and minimizing $v_2-v_1(>0)$ subject to

$$P[v_1<V<v_2]=1-\alpha$$

we get by Lagrange's method the minimum $(1-\alpha)$ random interval $(U-1+v_1/(2n),U-1+v_2/(2n))$ for θ^* with v_1 and v_2 determined by

$$h_V(v_1)=h_V(v_2).$$

Letting $u_1=\theta_0{}^*+1-v_2/(2n)$ and $u_2=\theta_0{}^*+1-v_1/(2n)$, we get the regret region

$$D=\{u_1<U<u_2\}\cap\{Y_{(n)}\leqq\theta_0{}^*\}(=\{u_1<U<u_2\}\cap\{X_{(n)}\leqq\theta_0\}).$$

$\xi(\theta)=P_\theta(D)$ becomes a concave function from below and $\xi(\theta)\leqq 1-\alpha=\xi(\theta_0)$, $\forall\theta$.

This test is also applicable for the one-edged test of $H'_0:\theta\leqq\theta_0$ versus $H'_1:\theta>\theta_0$ and gives uniformly least regret test. (See Y. Nogami(2002a).)

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