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Optimal Hypothesis Testing under Unbiased
Estimates-- Applications to the Family of
Retracted Distributions (I).

by

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Applications to the Family of Retracted Distributions (I).

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§0. Summary.

First, I would like to say that I support Professor L. J. Savage (1972, Chap. 17, §2)'s statement concerning interval estimation.

Here, we consider hypothesis testing with the regret region derived from inverting the minimum random interval based on unbiased estimate for the parameter by using Lagrange's method.

As examples we consider two oblong (or uniform) distributions in the family of retracted distributions. (See Y. Nogami(2002 a,b,c).)

§1. The Oblong (or Uniform) $U[\theta+\delta_1, \theta+\delta_2)$ (δ_1, δ_2 : reals such that $\delta_1 < \delta_2$).
For simplicity, we let $\delta_1=0$ and $\delta_2=1$. The underlined density is

$$f(x|\theta) = \begin{cases} 1, & \text{for } \theta \leq x < \theta+1, & (-\infty < \theta < \infty) \\ 0, & \text{otherwise.} \end{cases}$$

We take a random sample X_1, \dots, X_n from $f(x|\theta)$ and test the null hypothesis $H_0: \theta = \theta_0$ versus the alternative hypothesis $H_1: \theta \neq \theta_0$ with a given real number θ_0 .

Part (I): Let $X_{(i)}$ be the i -th smallest observation of X_1, \dots, X_n . Let $V = X_{(1)}$, $W = X_{(n)}$ and $Y = (V+W-1)/2$. Since the p.d.f. of Y is given by

$$g_Y(y|\theta) = \begin{cases} n(1-2|y-\theta|)^{n-1}, & \text{for } -1/2 < y < 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

We can easily get $E(Y)=\theta$ and $\text{Var}(Y)=1/\{2(n+1)(n+2)\}$.

To find the density of $T^*=(Y-\theta)/\{\text{Var}(\theta)\}=\sqrt{2(n+1)(n+2)}(Y-\theta)$ we simplify the problem to find the density of $T=2n(Y-\theta)$ as follows:

$$h_T(t) = \begin{cases} 2^{-1}(1-|t|/n)^{n-1}, & -n < t < n \\ 0, & \text{otherwise.} \end{cases}$$

Let g be a real number with $0 < g < 1$. Let r_1, r_2, t_1 and t_2 be real numbers such that $r_1 < r_2$ and $t_1 < t_2$. Minimizing $t_2 - t_1$ subject to

$$P_\theta[r_1 < Y - \theta < r_2] = P[t_1 < T < t_2] = 1 - g,$$

we get by Lagrange's multiplier's method that $(t_2) = -t_1 = n(1 - g^{1/n})$ ($\rightarrow -\log_e g$ as $n \rightarrow \infty$) and $r_2 = -r_1 (=r) = (1 - g^{1/n})/2$ ($\rightarrow 0$ as $n \rightarrow \infty$).

Actually, the density of T^* converges to the density

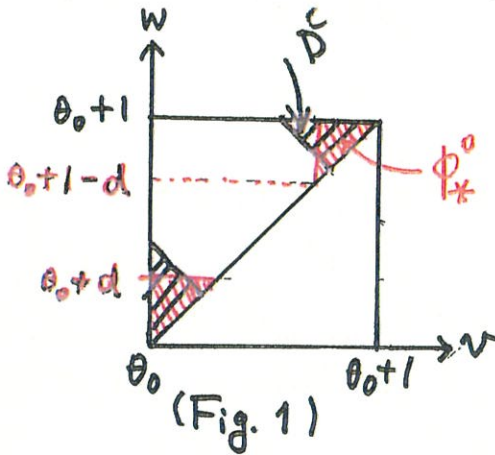
$$\psi(t) = 2^{-1/2} \exp\{-\sqrt{2}|t|\}, \quad -\infty < t < \infty$$

Hence, letting $T_0 = 2n(Y - \theta_0)$ we regret not rejecting H_0 if $-t < T_0 < t$ and satisfy to reject H_0 if $|T_0| \geq t$. We call $(-t, t)$ the regret region based on T_0 .

Part (II). Here, we consider the bibliographies; especially, S. K. Chatterjee and G. Chattopadhyay (1994). We first consider our non-regret region as follows:

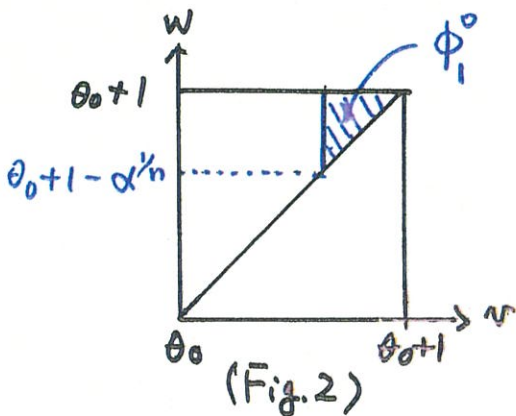
$$D^c = \{Y \leq \theta_0 - r, Y \geq \theta_0 + r, \theta_0 < V < W < \theta_0 + 1\} \cup \{V < \theta_0, W - V < 1\} \cup \{W \geq \theta_0 + 1, W - V < 1\}.$$

We call $\xi(\theta) = P_\theta(D)$ the regret probability function. (D^c is the



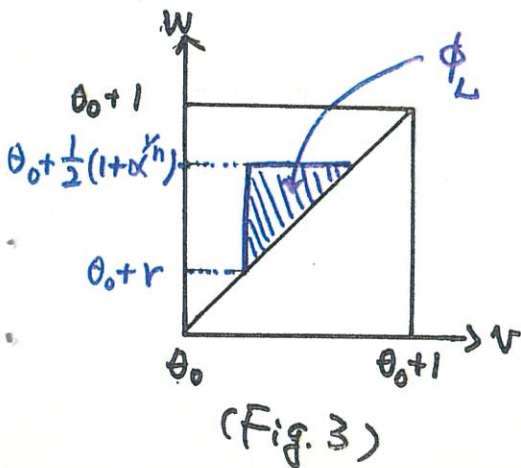
complement of D.) From p. 4 of Y. Nogami (2002b) $\xi(\theta)$ is symmetric at $\theta = \theta_0$, a concave function from below and $\xi(\theta) \leq 1 - \alpha = \xi(\theta_0)$, $\forall \theta$. (This property is also seen from Result 2.1 of S. K. Chatterjee and G. Chattopadhyay(1994).)

We return to S. K. Chatterjee and G. Chattopadhyay(1994). They consider the test of the hypotheses $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$. Let $\underline{x} = (x_1, \dots, x_n)$. In Result 2.2 they introduced the test



$$\phi_1^0(\underline{x}) = \begin{cases} 1, & \text{if } \underline{x} \notin [\theta_0, \theta_0+1]^n \text{ or} \\ & \theta_0+1 - \alpha^{1/n} < V < W < \theta_0+1, \\ 0, & \text{otherwise} \end{cases}$$

and compared with J. W. Pratt(1961)'s test $\phi_*^0(\underline{x})$ for $\theta > \theta_0$. We can see the difference of the structures of non-regret regions under H_0 for the tests D^c , ϕ_*^0 and ϕ_1^0 in Figures 1 and 2. In their Result 3.1 they showed unique locally best test



$$\phi_L(\underline{x}) = \begin{cases} 1, & \text{if } \underline{x} \in \mathcal{X} - [\theta_0, \theta_0+1]^n \text{ or} \\ & \theta_0+r < V < W < \theta_0 + (1 - \alpha^{1/n})/2, \\ 0, & \text{otherwise,} \end{cases}$$

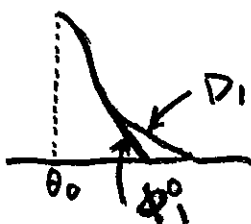
whose non-regret region appeared in Figure 3.

This is just like the regret region. Even if ϕ_L is the unique locally best test, practical statistician does not want to use this test.

This kind of curious phenomenon happens for this oblong (or uniform) distribution.

Remark: (i) We note that for $\theta_0 - \alpha^{1/n} < \theta < \theta_0$ $\xi(\theta) < E_\theta(1 - \phi_1^0(\underline{X}))$, but the other side of θ_0 the inequality is reversed.

(ii) Consider to test $H_0: \theta = \theta_0$ versus $H_1: \theta > \theta_0$. We can construct the regret region D_1 where $P_\theta(D_1) = 1 - \alpha$ and



$$P_\theta(D_1) = E_\theta(1 - \phi_1^0(\underline{X})), \quad \text{for } \theta_0 \leq \theta < \theta_0 + 1 - (2\alpha)^{1/n}$$

$$> E_\theta(1 - \phi_1^0(\underline{X})) = 0, \quad \text{for } \theta_0 + 1 - \alpha^{1/n} \leq \theta.$$

So, our test cannot overcome their ϕ_1^0 .

For the case of unknown $c = \delta_2 - \delta_1 (> 0)$, we omit the discussion here. (See e.g. Y. Nogami(2002b,c)).

§2. The Oblong (or Uniform) $U(0, \theta)$.

The underlined density is

$$f(x|\theta) = \begin{cases} \theta^{-1}, & \text{for } 0 < x < \theta, \\ 0, & \text{otherwise.} \end{cases} \quad (\theta > 0)$$

We take a random sample X_1, \dots, X_n from $f(x|\theta)$ and test the hypotheses $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$. Let $\theta^* = \log_e \theta$ and $Y = \log_e X$. The p.d.f. of Y is given by

$$g_Y(y) = \begin{cases} \exp\{y - \theta^*\}, & \text{for } 0 < y < \theta^* \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = \bar{Y} + 1 = n^{-1} \sum_{i=1}^n Y_i + 1$. Then, $E(U) = \theta^*$. We call (b_1, b_2) the $(1-\alpha)$ random interval for γ if $P_\gamma[b_1 < \gamma < b_2] = 1-\alpha$. Using the density $h_V(v)$ of $V = 2n(\theta^* + 1 - U)$ and minimizing $v_2 - v_1 (> 0)$ subject to

$$P[v_1 < V < v_2] = 1 - \alpha$$

we get by Lagrange's method the minimum $(1-\alpha)$ random interval $(U - 1 + v_1/(2n), U - 1 + v_2/(2n))$ for θ^* with v_1 and v_2 determined by

$$h_V(v_1) = h_V(v_2).$$

Letting $u_1 = \theta_0^* + 1 - v_2/(2n)$ and $u_2 = \theta_0^* + 1 - v_1/(2n)$, we get the regret region

$$D = \{u_1 < U < u_2\} \cap \{Y_{(n)} \leq \theta_0^*\} (= \{u_1 < U < u_2\} \cap \{X_{(n)} \leq \theta_0\}).$$

$\xi(\theta) = P_\theta(D)$ becomes a concave function from below and $\xi(\theta) \leq 1 - \alpha = \xi(\theta_0)$, $\forall \theta$.

This test is also applicable for the one-edged test of $H'_0: \theta \leq \theta_0$ versus $H'_1: \theta > \theta_0$ and gives uniformly least regret test. (See Y. Nogami(2002a).)

REFERENCES

- Chatterjee, S. K. & Chattopadhyay, G.(1994). On the nonexistence of certain optimal confidence sets for the rectangular problem., *Statist. & Prob. Let.* 21, 263-269
- Nogami, Y.(2002a). Hypothesis Testing based on Lagrange's Method: Application to the Uniform Distribution., Discussion Paper Series No. 995, Inst. of Policy & Planning Sciences, Univ. of Tsukuba
- (2002b). Hypothesis Testing based on Lagrange's Method: Application to the Uniform Distribution (II). Discussion Paper Series No. 1002, Inst. of Policy & Planning Sciences, Univ. of Tsukuba.
- (2002c). A Comment on Section 4 of D. P. 1002., Discussion Paper Series No. 1004, Inst. of Policy and Planning Sciences, Univ. of Tsukuba.
- Pratt, J. W.(1961). Review of Testing Statistical Hypotheses by E. L. Lehmann., *Jour. Amer. Statist. Assoc.* 56, 163-167.
- Savage, L. J.(1972). *The Foundations of Statistics.*, 2nd. ed., Dover, New York.