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NUMERICAL EXPLORATION OF DYNAMIC BEHAVIOR OF THE ORNSTEIN-UHLENBECK PROCESS VIA EHRENFEST PROCESS APPROXIMATION

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Abstract

The Ornstein-Uhlenbeck (O-U) process is of practical importance in many application areas such as statistics, meteorology and financial engineering. While transition probabilities of the O-U process are readily accessible, quantifying its dynamic behavior is numerically cumbersome. The purpose of this paper is to develop numerical procedures for evaluating distributions of first passage times and the historical maximum of the O-U process via the Ehrenfest process approximation. It is shown that a sequence of Ehrenfest processes with appropriate scaling and shifting converges in law to the O-U process. Accordingly, first passage times and the historical maximum of the Ehrenfest process converge in law to those of the O-U process. Through analysis of the spectral structure of the Ehrenfest process, efficient numerical algorithms are developed, thereby providing effective approximation tools for capturing the dynamic behavior of the O-U process. Some numerical results are also exhibited.

Keywords: Ornstein-Uhlenbeck (O-U) Process, Ehrenfest Process, Dynamic Behavior, Convergence in Law, First Passage Times, Historical Maximum, Numerical Approximation

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0. Introduction

The Ornstein-Uhlenbeck (O-U) process $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ is a Markov diffusion process on the real continuum $-\infty < x < \infty$. Its probability density function $f(x,t) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[X_{\mathrm{OU}}(t) \leq x\right]$ is governed by the forward diffusion equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t) + \frac{\partial}{\partial x}\left[x\, f(x,t)\right]. \tag{0.1}$$

A basic function describing this process is the conditional transition density $g(x_0, x, \tau) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[X(t+\tau) \leq x | X(t) = x_0\right]$ given by

$$g(x_0, x, \tau) = \frac{1}{\sqrt{2\pi}\sqrt{1-\mathrm{e}^{-2\tau}}} \exp\left\{-\frac{\left(x - x_0\,\mathrm{e}^{-\tau}\right)^2}{2\left(1-\mathrm{e}^{-2\tau}\right)}\right\}, \quad -\infty < x < \infty. \tag{0.2}$$

Its stationary or ergodic density is given by

$$f_\infty(x) \stackrel{\text{def.}}{=} \lim_{t\to\infty} f(x,t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}, \quad -\infty < x < \infty. \tag{0.3}$$

The O-U process has many applications to statistics, including the studies of "goodness of fit" of a set of observations to a distribution function, see e.g. Anderson and Darling [1] and the studies of stopping time for sample sequences, see e.g. Armitage, McPherson and Rowe [2]. The process also plays an important role in meteorology describing random behavior of the temperature, see e.g. Keilson and Ross [11]. During the past three decades, the usefulness of the O-U process has been reinforced in the area of financial engineering where spot interest rates are often represented by the O-U process, see e.g. Vasicek [18]. As we saw in (0.2) and (0.3), the O-U process itself is quite tractable. However, simple related processes and random variables for capturing its dynamic behavior become numerically intractable.

The O-U process is often approximated by the Ehrenfest urn model where both the state space and the time axis are discretized, see e.g. Karlin and Taylor [8]. For capturing the dynamic behavior of the O-U process, however, this approach is rather cumbersome. In this paper, we propose to utilize the continuous time Ehrenfest process for approximating the O-U process where only the state space is discretized. The underlying spectral structure enables one to develop efficient numerical procedures for computing distributions of first passage times and the historical maximum of the O-U process, which are of considerable importance in applications.

A finite Markov chain in continuous time of practical interest arises from the sum of K independent identical chains $\{J_j(t) : t \geq 0\},\ j = 1, ..., K$, each on state space $\{0, 1\}$ governed by transition rates $\nu_{01} = \nu_{10} = \frac{1}{2}$. The Markov chain of interest

$$\left\{ N_K(t)\ :\ N_K(t) = \sum_{j=1}^{K} J_j(t),\ t \geq 0 \right\} \tag{0.4}$$

on state space $\{0, 1, ..., K\}$ is called the Ehrenfest process and has transition rates

$$\nu_{n,n+1} = \frac{1}{2}(K - n),\ 0 \leq n \leq K - 1,\ \text{and}\ \nu_{n,n-1} = \frac{1}{2}n,\ 1 \leq n \leq K. \tag{0.5}$$

Consequently the local growth rate of the variance is given by

$$\nu_{n,n+1} + \nu_{n,n-1} = \frac{K}{2}, \tag{0.6}$$

which is independent of n, and the local velocity is given by

$$\nu_{n,n+1} - \nu_{n,n-1} = \frac{K}{2} - n. \tag{0.7}$$

For the associated stationary chain $\{N_{KS}(t) : t \geq 0\}$, one has

$$\text{cov}\,[\,N_{KS}(t),\ N_{KS}(t+\tau)\,] = \frac{K}{4}\,\mathrm{e}^{-\tau}, \tag{0.8}$$

and asymptotic normality.

The O-U process is characterized by its Markov property, normal distribution, and exponential covariance function. Because of the properties of the Ehrenfest process specified in (0.5) through (0.8) together with its asymptotic normality, one then expects that a sequence of processes $\{X_V(t) : t \geq 0\}, V = 1, 2, 3, ...,$ defined by

$$X_V(t) = \sqrt{\frac{2}{V}}\,N_{2V}(t) - \sqrt{2V} \tag{0.9}$$

converges in law to the O-U process as $V \to \infty$. The purpose of this paper is to prove this convergence in law, to develop systematically the properties of $\{N_{2V}(t) : t \geq 0\}$ and to quantify its dynamic behavior numerically, which in turn provides a numerical foundation for capturing the dynamic behavior of the O-U process.

The structure of this paper is as follows. In Section 1, the spectral representation of the Ehrenfest process $\{N_{2V}(t) : t \geq 0\}$ is established following Karlin and McGregor [4, 5, 6, 7]. Section 2 summarizes the first passage time structure of birth-death processes from Keilson [10], and then that of the Ehrenfest process is studied in detail in Section 3. In particular, it is shown that the first passage time T_{0V} of $\{N_{2V}(t) : t \geq 0\}$ from 0 to V converges in law to the distribution conjugate to the extreme-value distribution. We also evaluate the historical maximum of $\{N_{2V}(t) : t \geq 0\}$. In Section 4, the convergence in law of $X_V(t)$ to $X_{\mathrm{OU}}(t)$ as $V \to \infty$ is proven for all $t \geq 0$ and some related results are obtained. Section 5 is devoted to development of numerical algorithms for evaluating transition probabilities, first passage times, and the historical maximum of the O-U process via the Ehrenfest process approximation. Numerical results are also exhibited, demonstrating speed and accuracy of the Ehrenfest process approximation procedure.

1. Spectral Representation of the Ehrenfest Process

We consider $2V$ independent and identical Markov chains $\{J_j(t) : t \geq 0\}$, $j = 1, ..., 2V$, in continuous time on $\{0, 1\}$ governed by the transition rate matrix

$$\underline{\underline{\nu}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{1.1}$$

The corresponding infinitesimal generator $\underline{\underline{Q}}$ is then given by

$$\underline{\underline{Q}} = -\underline{\underline{\nu}}_D + \underline{\underline{\nu}}; \quad \underline{\underline{\nu}}_D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{1.2}$$

Let $\underline{\underline{q}}(t) = [q_{ij}(t)]$, $0 \leq i, j \leq 1$, be the transition probability matrix of $\{ J_j(t) : t \geq 0 \}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\underline{\underline{q}}(t) = \underline{\underline{Q}}\,\underline{\underline{q}}(t)$. Since $\underline{\underline{q}}(0) = \underline{\underline{I}}$ which denotes the identity matrix, taking the Laplace transform of this matrix differential equation yields $s\,\widehat{\underline{\underline{q}}}(s) - \underline{\underline{I}} = \underline{\underline{Q}}\,\widehat{\underline{\underline{q}}}(s)$ or $\widehat{\underline{\underline{q}}}(s) = \left[s\underline{\underline{I}} - \underline{\underline{Q}}\right]^{-1}$ where $\widehat{\underline{\underline{q}}}(s) = \int_0^\infty \mathrm{e}^{-st}\underline{\underline{q}}(t)\,\mathrm{d}t$. From (1.2), one then finds that

$$\underline{\underline{q}}(t) = \begin{pmatrix} q_{00}(t) & q_{01}(t) \\ q_{10}(t) & q_{11}(t) \end{pmatrix} = \begin{pmatrix} f(t) & g(t) \\ g(t) & f(t) \end{pmatrix}, \tag{1.3}$$

where

$$f(t) = \frac{1}{2}\left(1 + \mathrm{e}^{-t}\right); \quad g(t) = \frac{1}{2}\left(1 - \mathrm{e}^{-t}\right). \tag{1.4}$$

For analytical convenience, we introduce two generating functions :

$$\alpha_0(t, u) \stackrel{\text{def.}}{=} q_{00}(t) + q_{01}(t)\,u = f(t) + g(t)\,u \tag{1.5}$$

and

$$\alpha_1(t,u) \stackrel{\text{def.}}{=} q_{10}(t) + q_{11}(t)\,u = g(t) + f(t)\,u. \tag{1.6}$$

Let $\{\, N_{2V}(t) : t \geq 0 \,\}$ be defined by

$$N_{2V}(t) \stackrel{\text{def.}}{=} \sum_{j=1}^{2V} J_j(t). \tag{1.7}$$

Then $\{\, N_{2V}(t) : t \geq 0 \,\}$ is a birth-death process on $\mathcal{N} = \{0, 1, ..., 2V\}$ governed by the upward transition rates λ_n and the downward transition rates μ_n, where

$$\lambda_n = \frac{1}{2}\,(2V - n)\,; \quad \mu_n = \frac{n}{2}, \quad n \in \mathcal{N}. \tag{1.8}$$

We note that

$$\nu_n \stackrel{\text{def.}}{=} \lambda_n + \mu_n = V, \quad n \in \mathcal{N}, \tag{1.9}$$

which is independent of state n. This birth-death process is called the Ehrenfest process, see e.g. Feller [3]. Let $\underline{\underline{P}}_{2V}(t) = [p_{2V:mn}(t)]\;\; m, n \in \mathcal{N}$ be the transition probability matrix of $\{\, N_{2V}(t) : t \geq 0 \,\}$. As in (1.5) and (1.6), we introduce the following generating functions :

$$\beta_m(t,u) = \sum_{k=0}^{2V} p_{2V:mk}(t)\,u^k, \quad m \in \mathcal{N}. \tag{1.10}$$

From the independence of $\{J_j(t) : t \geq 0\}$, one then has

$$\beta_m(t,u) = \alpha_0(t,u)^{2V-m}\,\alpha_1(t,u)^m = \{f(t) + g(t)\,u\}^{2V-m}\{g(t) + f(t)\,u\}^m. \tag{1.11}$$

In a series of papers [4, 5, 6, 7], Karlin and McGregor analyze the spectral representation of the transition probability matrix $\underline{\underline{P}}(t) = [p_{mn}(t)]$ for birth-death processes and use the results to evaluate various probabilistic quantities. More specifically, for a general birth-death process governed by upward transition rates $\lambda_n,\ n \geq 0$, and downward transition rates $\mu_n,\ n \geq 1$, the infinitesimal generator $\underline{\underline{Q}}$ associated with $\underline{\underline{P}}(t)$ has a vector eigenfunction $\underline{y}(x) = [y_n(x)]_{n \in \mathcal{N}}$ with eigenvalue $-x$, i.e.

$$\underline{\underline{Q}}\,\underline{y}(x) = -x\,\underline{y}(x). \tag{1.12}$$

Starting with $y_0(x) = 1$, this then leads to

$$\begin{cases} -\lambda_0\, y_0(x) + \lambda_0\, y_1(x) = -x\, y_0(x) \\ \mu_n\, y_{n-1}(x) - (\lambda_n + \mu_n)\, y_n(x) + \lambda_n\, y_{n+1}(x) = -x\, y_n(x), \quad n \geq 1 \end{cases} . \tag{1.13}$$

It then follows that $y_n(x)$ is a polynomial of degree n with leading coefficients $\left(\prod_{j=0}^{n-1} \lambda_j\right)^{-1}$. Let $\underline{f}(x,t)$ be a vector function defined by

$$\underline{f}(x,t) = \underline{\underline{P}}(t)\, \underline{y}(x). \tag{1.14}$$

From the Kolmogorov forward equation, one then finds that

$$\frac{\partial}{\partial t}\underline{f}(x,t) = \frac{\partial}{\partial t}\underline{\underline{P}}(t)\, \underline{y}(x) = \underline{\underline{P}}(t)\, \underline{\underline{Q}}\, \underline{y}(x) = -x\, \underline{\underline{P}}(t)\, \underline{y}(x)$$

so that from (1.14)

$$\frac{\partial}{\partial t}\underline{f}(x,t) = -x\, \underline{f}(x,t). \tag{1.15}$$

Since $\underline{f}(x, 0+) = \underline{y}(x)$, the vector partial differential equation (1.15) has the unique solution

$$\underline{f}(x,t) = \mathrm{e}^{-xt}\underline{y}(x). \tag{1.16}$$

Combining with (1.14), Equation (1.16) then implies that

$$\sum_{n \in \mathcal{N}} p_{mn}(t)\, y_n(x) = \mathrm{e}^{-xt} y_m(x), \quad m \in \mathcal{N}. \tag{1.17}$$

There exists a measure $\psi(x)$ on $[0, \infty)$ such that $\{y_n(x)\}_{n \in \mathcal{N}}$ becomes a set of orthogonal polynomials with respect to $\psi(x)$, see Karlin and McGregor [4]. Accordingly one has

$$\int_0^\infty y_m(x)\, y_n(x)\, \mathrm{d}\psi(x) = \frac{\delta_{mn}}{\pi_n}, \quad m, n \in \mathcal{N}, \tag{1.18}$$

where $\delta_{mn} = 1$ if $m = n$, $\delta_{mn} = 0$ if $m \neq n$, and $\pi_n = \prod_{j=0}^{n-1} \lambda_j / \prod_{j=1}^{n} \mu_j$, $n \geq 1$, with $\pi_0 = 1$.

From (1.14) and (1.17), $\{p_{mn}(t)\}_{m \in \mathcal{N}}$ may be recognized as the generalized Fourier coefficients of the m-th component $f_m(x,t)$ of $\underline{f}(x,t)$ associated with $\{y_n(x)\}_{n \in \mathcal{N}}$ and $\psi(x)$ for each $m \in \mathcal{N}$. Accordingly, one finds from (1.17) that

$$p_{mn}(t) = \pi_n \int_0^\infty \mathrm{e}^{-xt} y_m(x)\, y_n(x)\, \mathrm{d}\psi(x), \quad m, n \in \mathcal{N}. \tag{1.19}$$

For the case of the Ehrenfest process, these entities are identified, see Karlin and McGregor [6], as

$$y_n(x) = \frac{1}{\binom{2V}{n}} \sum_{j=0}^{2V} \binom{2V-x}{n-j} \binom{x}{j} (-1)^j, \quad n \in \mathcal{N}, \tag{1.20}$$

$$\mathrm{d}\psi(x) = \binom{2V}{n} 2^{-2V}, \quad x = 0, 1, ..., 2V, \tag{1.21}$$

and

$$p_{2V:mn}(t) = \frac{\binom{2V}{n}}{2^{2V}} \sum_{j=0}^{2V} \binom{2V}{j} y_m(j)\, y_n(j)\, \mathrm{e}^{-jt}. \tag{1.22}$$

In this case, the polynomials $\{y_n(x)\}_{n\in\mathcal{N}}$ are called the Krawtchouk polynomials. It is clear from the independence of $\{J_j(t) : t \geq 0\}$ that the ergodic distribution $\underline{e}^{\mathrm{T}}$ of $\{N_{2V}(t) : t \geq 0\}$ is given by

$$\underline{e} = [e_n]^{\mathrm{T}}_{n\in\mathcal{N}}\,; \quad e_n = \binom{2V}{n} 2^{-2V}, \quad n \in \mathcal{N}. \tag{1.23}$$

We note from (1.22) and (1.23) that

$$\lim_{t\to\infty} p_{2V:mn}(t) = \frac{\binom{2V}{n}}{2^{2V}} \binom{2V}{0} y_m(0)\, y_n(0) = e_n, \quad m, n \in \mathcal{N} \tag{1.24}$$

as expected.

2. First Passage Times and the Historical Maximum of General Birth-Death Processes

In this section, we first summarize, from Keilson [10], the first passage time structure of a general birth-death process $\{N(t) : t \geq 0\}$ governed by upward transition rates λ_n $(n \geq 0)$ and downward transition rates μ_n $(n \geq 1)$. Let T_{mn} be the first passage time of $\{N(t) : t \geq 0\}$ from state m to state n. Formally, we define

$$T_{mn} = \inf\{\, t : N(t) = n \,|\, N(0) = m \,\}. \tag{2.1}$$

Let $s_{mn}(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{P}\,[\,T_{mn} \leq \tau\,]$ and define the Laplace transform $\sigma_{mn}(s) = \mathrm{E}\left[\,\mathrm{e}^{-s\,T_{mn}}\,\right]$. For notational convenience, we denote $T_{m,m+1}$ by T^+_m, and $s^+_m(\tau)$ and $\sigma^+_m(s)$ are defined similarly. From the consistency relations, one has

$$\sigma^+_n(s) = \frac{\nu_n}{s + \nu_n} \left[\frac{\lambda_n}{\nu_n} + \frac{\mu_n}{\nu_n}\, \sigma^+_{n-1}(s)\, \sigma^+_n(s) \right], \quad n \geq 1,$$

where $\nu_n = \lambda_n + \mu_n$. This then yields

$$\sigma_n^+(s) = \lambda_n \left[s + \nu_n - \mu_n \, \sigma_{n-1}^+(s) \right]^{-1}, \quad n \geq 1; \quad \sigma_0^+(s) = \frac{\lambda_0}{s + \lambda_0}. \tag{2.2}$$

It is clear that

$$\sigma_{0n}(s) = \sigma_{0\,n-1}(s) \, \sigma_{n-1}^+(s), \quad n \geq 0. \tag{2.3}$$

From (2.2), it can be readily seen by induction that

$$\sigma_{0n}(s) = \frac{1}{g_n(s)}, \quad n \geq 1; \quad g_0(s) = 1. \tag{2.4}$$

where $g_n(s)$ is a polynomial of order n. It then follows from (2.2) that

$$g_{n+1}(s) = \frac{1}{\lambda_n} \left[(s + \nu_n) \, g_n(s) - \mu_n \, g_{n-1}(s) \right], \; n \geq 0, \tag{2.5}$$

where $g_{-1}(s) = 0$ and $g_0(s) = 1$. By comparing (2.5) with (1.13), it can be seen that polynomials $y_n(x)$ and $g_n(s)$ are related to each other by

$$y_n(x) = g_n(-x), \quad n \geq 0. \tag{2.6}$$

It should be noted from (2.3) and (2.4) that

$$\sigma_n^+(s) = \frac{g_n(s)}{g_{n+1}(s)}, \quad n \geq 0. \tag{2.7}$$

From (2.6), $\{g_n(s)\}$ are orthogonal polynomials. Accordingly the zeros of $g_n(s)$ are distinct, the zeros of any two successive polynomials interleave, and the zeros are all negative, see e.g. Szegö [17]. Consequently, from (2.7), $\sigma_n^+(s)$ can be written as

$$\sigma_n^+(s) = \sum_{j=0}^{n} r_{n+1,j} \, \frac{\alpha_{n+1,j}}{s + \alpha_{n+1,j}}, \tag{2.8}$$

where $-\alpha_{n+1,j}$ are the zeros of $g_{n+1}(s)$, $r_{n+1,j} = \lim_{s \to -\alpha_{n+1,j}} \frac{s+\alpha_{n+1,j}}{\alpha_{n+1,j}} \frac{g_n(s)}{g_{n+1}(s)} \geq 0$ and $\sum_{j=0}^{n} r_{n+1,j} = \sigma_n^+(0+) = 1$. This implies that $s_n^+(t)$ is a mixture of exponential densities and is completely monotone. The downward first passage times $T_{n,n-1}^- = T_n^-$ and T_{n0} can be treated similarly.

We next turn our attention to the historical maximum of $\{N(t) : t \geq 0\}$ in the time interval $[0, \theta]$ given that $N(0) = n_0$. More specifically, let $M(n_0, \theta)$ be defined as

$$M(n_0, \theta) = \max_{0 \leq t \leq \theta} \{ \, N(t) \, | \, N(0) = n_0 \, \} \, . \tag{2.9}$$

Then the following dual relation holds between $T_{n_0\,n+1}$ $(n_0 \le n)$ and $M(n_0, \theta)$.

$$F_{n_0\theta}(n) \overset{\text{def.}}{=} \mathrm{P}\,[\,M(n_0,\theta) \le n\,] = \mathrm{P}\,[\,T_{n_0\,n+1} > \theta\,] \overset{\text{def.}}{=} \overline{S}_{n_0\,n+1}(\theta). \tag{2.10}$$

Hence one has

$$F_{n_0\theta}(n) = \begin{cases} 0 & n < n_0 \\ \overline{S}_{n_0\,n+1}(\theta) & n \ge n_0. \end{cases} \tag{2.11}$$

For the corresponding stationary process $\{\,N_S(t) : t \ge 0\,\}$, the distribution function $F_\theta(n)$ of the historical maximum is then given by

$$F_\theta(n) = \sum_{m \le n} e_m\,\overline{S}_{m\,n+1}(\theta), \tag{2.12}$$

where $\underline{e}^{\mathrm{T}} = [\,e_m\,]$ is the ergodic distribution of $\{\,N(t) : t \ge 0\,\}$.

3. First Passage Times and the Historical Maximum of the Ehrenfest Process

As we saw in Section 1, the Ehrenfest process $\{N_{2V}(t) : t \ge 0\}$ is a birth-death process on $\mathcal{N} = \{0, 1, ..., 2V\}$ governed by transition rates λ_n and μ_n specified in (1.8). The recursive formula in (2.5) then becomes

$$g_{n+1}(s) = \frac{2}{2V - n}\left[\,(s+V)\,g_n(s) - \frac{n}{2}\,g_{n-1}(s)\,\right], \tag{3.1}$$

with $g_{-1}(s) = 0$ and $g_0(s) = 1$. From (1.20) and (2.6), $g_n(s)$ are given explicitly by

$$g_n(s) = \frac{1}{\binom{2V}{n}} \sum_{j=0}^{2V} \binom{2V+s}{n-j}\binom{-s}{j}(-1)^j, \quad 0 \le n \le 2V. \tag{3.2}$$

In order to evaluate the first passage times $s_{mn}(\tau)$ $(m < n)$ with corresponding Laplace transforms $\sigma_{mn}(s) = \sigma^+_m(s) \cdots \sigma^+_{n-1}(s) = g_m(s)/g_n(s)$ from (2.7), the zeros of $g_n(s)$ are needed. These zeros in turn enables one to quantify the historical maximum through (2.11). In principle, the zero search of $g_n(s)$ can be accomplished via a straightforward bisection approach since the zeros of $g_n(s)$ and $g_{n+1}(s)$ interleave because of the underlying orthogonality. In case of the Ehrenfest process, the amount of effort required for the zero search can be considerably reduced by the following properties.

Theorem 3.1. *Let* $h_n(s) = g_n(s - V),\ n \geq 0$. *Then* $h_n(s) = (-1)^n\, h_n(-s),\ n \geq 0$, *i.e.*

$$h_n(s)\ \textit{is} \begin{cases} \textit{odd} & \textit{when } n \textit{ is odd,} \\ \textit{even} & \textit{when } n \textit{ is even.} \end{cases}$$

Proof. Equation (3.1) can be rewritten in terms of $h_n(s)$ as

$$h_{n+1}(s) = \frac{2}{2V - n}\left[\, s\, h_n(s) - \frac{n}{2}\, h_{n-1}(s)\right], \quad n \geq 0, \tag{3.3}$$

with $h_{-1}(s) = 0$ and $h_0(s) = 1$. The result then follows by induction on n.

The next corollary is immediate from Theorem 3.1.

Corollary 3.1.

(a) *If* $g_n(-x) = 0$, *then* $g_n(x - 2V) = 0$.

(b) *If* n *is odd, then* $g_n(-V) = 0$.

Theorem 3.1 implies that the zeros of $h_n(s)$ are symmetric about 0 and, correspondingly from Corollary 3.1, the zeros of $g_n(x)$ are symmetric about $-V$. Hence we need to find only $\lceil (n-1)/2 \rceil$ zeros, where $\lceil x \rceil$ is the minimum integer which is greater than or equal to x. Furthermore, since $h_n(s)$ is either odd or even, there are only $1 + \lceil (n-1)/2 \rceil$ terms in each $h_n(s)$, while $g_n(s)$ has $(n+1)$ terms as can be seen from (3.2). Consequently the computational time of the zero search can be reduced approximately by a factor of 4. This property of the Ehrenfest process is due to the fact that the local growth rate is constant as specified in (0.6). Indeed, the results similar to Theorem 3.1 and Corollary 3.1 are available for general birth-death processes whenever $\nu_n = \lambda_n + \mu_n = \nu$ for all n.

We next show that $g_V(s)$ has negative odd integers as its root. A preliminary lemma is needed.

Lemma 3.1. *For* $m, n \in \mathcal{N}$, *one has* $g_n(-m) = g_m(-n)$.

Proof. Because of an elementary property of binomial coefficients, one sees that

$$\begin{aligned} \frac{\binom{2V-n}{m-j}\binom{n}{j}}{\binom{2V}{m}} &= \frac{(2V-n)!}{(m-j)!\,(2V-n-m+j)!} \cdot \frac{n!}{j!\,(n-j)!} \cdot \frac{m!\,(2V-m)!}{(2V)!} \\ &= \frac{\binom{2V-m}{n-j}\binom{m}{j}}{\binom{2V}{n}}, \end{aligned}$$

and the result follows from (3.2).

Theorem 3.2.

$$g_V(s) = \prod_{j=1}^{V} \frac{s+2j-1}{2j-1}$$

Proof. Corollary 3.1 b) states that $g_n(-V) = 0$ whenever $n \in \mathcal{N}$ is odd. Hence from Lemma 3.1, one has $g_V(-n) = g_n(-V) = 0$ whenever $n \in \mathcal{N}$ is odd. Since $g_V(s)$ is a polynomial of degree V, the theorem follows.

We are now in a position to evaluate the limiting behavior of T_{0V} as $V \to \infty$. For a random variable X with $F_X(x) = \mathrm{P}[\, X \le x\,]$, $-\infty < x < \infty$, suppose the corresponding Laplace transform $\varphi_X(s) = \mathrm{E}[\,\mathrm{e}^{-sX}\,] = \int_{-\infty}^{\infty} \mathrm{e}^{-sx}\,\mathrm{d}F_X(x)$ has the convergence strip containing the imaginary axis on the complex plane. Then the conjugate transform Y of X is defined as

$$F_Y(y) = \mathrm{P}[\, Y \le y\,] = \frac{\int_{-\infty}^{y} \mathrm{e}^{-sx}\,\mathrm{d}F_X(x)}{\varphi_X(s)}, \quad -\infty < y < \infty. \tag{3.4}$$

The reader is referred to Keilson [9] for more detailed discussions of the conjugate transform. The next theorem shows that T_{0V} with certain shifting and scaling converges in law to a conjugate transform of an extreme-value random variate.

Theorem 3.3. *Let Y be a random variable having the probability density function*

$$f_Y(\tau) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}\tau - \mathrm{e}^{-\tau}\right\}, \quad -\infty < \tau < \infty. \tag{3.5}$$

Then $2\,T_{0V} - \log V$ converges in law to Y as $V \to \infty$.

Proof. Let $Z = 2\,T_{0V} - \log V$. Then from Theorem 3.2 and (2.4), one sees that

$$\varphi_z(s) = \mathrm{E}[\,\mathrm{e}^{-sz}\,] = V^s\,\sigma_{0V}(2s) = V^s \prod_{j=1}^{V} \frac{2j-1}{2s+2j-1}.$$

By simple algebra, this then leads to

$$\varphi_z(s) = \frac{V^{\frac{1}{2}}}{2^{2V}} \binom{2V}{V} \left\{ V^{s-\frac{1}{2}} \prod_{j-1}^{V} \frac{j}{s-\frac{1}{2}+j} \right\}. \tag{3.6}$$

The factor inside the braces converges to $\Gamma(s+\frac{1}{2})$ as $V \to \infty$, while the rest converges to $\frac{1}{\Gamma(\frac{1}{2})} = \frac{1}{\sqrt{\pi}}$, i.e., $\varphi_z(s) \to \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2})}$ as $V \to \infty$. It is known that $\Gamma(s+1)$ is the Laplace transform of the extreme value distribution with p.d.f. $\exp\{-\tau - \mathrm{e}^{-\tau}\}$, $-\infty < \tau < \infty$, and thus the theorem follows.

We saw in (2.8) that the upward first passage time T_m^+ is a finite mixture of exponential variates for general birth-death processes. In case of the Ehrenfest process, the fact that the exit rate of each state $\nu_n = \lambda_n + \mu_n = V$ is constant enables one to show that T_n^+ can also be expressed as an infinite mixture of Gamma variates of odd order.

Theorem 3.4. *For the Ehrenfest process, T_n^+ and T_n^- $(n \in \mathcal{N})$ are infinite mixtures of Gamma variates of odd order with Laplace transforms $\Gamma(V, 2j+1)$, $j = 0, 1, 2, ...$*

Proof. The recursive formula for $\sigma_n^+(s)$ in (2.2) can be rewritten as

$$\sigma_n^+(s) = \frac{r_n^+\,\varepsilon(s)}{1 - r_n^-\,\varepsilon(s)\,\sigma_{n-1}^+(s)}; \quad \varepsilon(s) = \sigma_0^+(s) = \frac{V}{s+V},\ n \geq 1, \tag{3.7}$$

where $r_n^+ = 1 - \frac{n}{2V}$ is the probability of going up given exit from n and $r_n^- = \frac{n}{2V}$ is that of going down given exit from n. For $Re(s) > 0$, Equation (3.7) has a series expression

$$\sigma_n^+(s) = r_n^+\,\varepsilon(s) \sum_{j=0}^{\infty} \left\{ r_n^-\,\varepsilon(s)\,\sigma_{n-1}^+(s) \right\}^j$$

and the result follows by induction for $\sigma_n^+(s)$. For $\sigma_n^-(s)$, it suffices to note that $\sigma_n^+(s) = \sigma_{2V-n}^-(s)$, completing the proof.

4. Convergence of the Ehrenfest Process to the O-U Process

As we saw in (0.2), the state probability density of the O-U process $\{ X_{\mathrm{OU}}(t) \,:\, t \geq 0 \}$ with initial condition $X_{\mathrm{OU}}(0) = x_0$ is normally distributed with mean $x_0\,\mathrm{e}^{-t}$ and variance $1 - \mathrm{e}^{-2t}$ for any $t > 0$. The corresponding Laplace transform with respect to x is then given by

$$\gamma(x_0, s, t) = \exp\left\{ -x_0\,\mathrm{e}^{-t} s + \frac{1}{2}(1 - \mathrm{e}^{-2t})\, s^2 \right\}. \tag{4.1}$$

In this section, we show that the Ehrenfest process $N_{2V}(t)$ of (1.7) with suitable scaling and shifting converges in law to $X_{\mathrm{OU}}(t)$ as $V \to \infty$, for all $t > 0$.

Let $\{ X_V(t) \,:\, t \geq 0 \}$ be a stochastic process defined by

$$X_V(t) = \sqrt{\frac{2}{V}}\, N_{2V}(t) - \sqrt{2V}. \tag{4.2}$$

We note that $\{X_V(t) : t \geq 0\}$ has a discrete support on $\{r(0), ..., r(2V)\}$ where

$$r(n) = \sqrt{\frac{2}{V}}\, n - \sqrt{2V}, \quad n = 0, 1, ... \tag{4.3}$$

Clearly $r(n+1) - r(n) = \sqrt{\frac{2}{V}} \to 0$ as $V \to \infty$. In the following theorem, we prove that, when $N_{2V}(0)$ is chosen appropriately, $X_V(t)$ converges in law to $X_{\mathrm{OU}}(t)$ as $V \to \infty$. For notational convenience, we define

$$\eta_V(x) = \left\lceil \sqrt{\frac{V}{2}} x \right\rceil. \tag{4.4}$$

Theorem 4.1. *Let $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ be the O-U process with $X_{\mathrm{OU}}(0) = x_0, \ -\infty < x < \infty$. Let $\{X_V(t) : t \geq 0\}$ be as in (4.2) with $X_V(0) = \sqrt{\frac{2}{V}}\, \eta_V(x_0)$ where V is chosen large enough so that $-\sqrt{2V} \leq X_V(0) \leq \sqrt{2V}$. Then $X_V(t)$ converges in law to $X_{\mathrm{OU}}(t)$ for all $t,\ t \geq 0$, as $V \to \infty$.*

Proof. Let $\varphi_V(x_0, w, t) = \mathrm{E}\left[\mathrm{e}^{-wX_V(t)} \,\middle|\, X_V(0) = \sqrt{\frac{2}{V}}\, \eta_V(x_0)\right]$. One sees from (1.10) and (4.2) that

$$\varphi_V(x_0, w, t) = \mathrm{e}^{w\sqrt{2V}} \beta_{N_{2V}(0)}\left(t, \mathrm{e}^{-w\sqrt{\frac{2}{V}}}\right) \tag{4.5}$$

where $N_{2V}(0) = V + \eta_V(x_0)$. We wish to show that $\varphi_V(x_0, w, t) \to \gamma(x_0, w, t)$ as $\mathrm{V} \to \infty$. Equation (4.5) can be rewritten by (1.11) as

$$\begin{aligned} \varphi_V(x_0, w, t) \quad = \quad & \mathrm{e}^{w\sqrt{2V}} \left[\left\{f(t) + g(t)\, \mathrm{e}^{-w\sqrt{\frac{2}{V}}}\right\} \left\{g(t) + f(t)\, \mathrm{e}^{-w\sqrt{\frac{2}{V}}}\right\}\right]^V \\ & \times \left[\frac{g(t) + f(t)\mathrm{e}^{-w\sqrt{\frac{2}{V}}}}{f(t) + g(t)\mathrm{e}^{-w\sqrt{\frac{2}{V}}}}\right]^{\eta_V(x_0)}. \end{aligned} \tag{4.6}$$

Since $f(t) + g(t) = 1$, the first two factors on the right hand side of (4.6) can be written as

$$\mathrm{e}^{w\sqrt{2V}} \beta_V(t, \mathrm{e}^{-w\sqrt{\frac{2}{V}}}) = \left[1 + 2f(t)\, g(t) \left\{\cosh\left(w\sqrt{\frac{2}{V}}\right) - 1\right\}\right]^V. \tag{4.7}$$

For sufficiently small $|Re(w)|$, one has

$$f(t)\, g(t) \left|\cosh\left(w\sqrt{\frac{2}{V}}\right) - 1\right| < \frac{1}{2} \tag{4.8}$$

so that from (4.7),

$$\begin{aligned}\log\left[\mathrm{e}^{w\sqrt{\frac{2}{V}}}\beta_V(t,\mathrm{e}^{-w\sqrt{\frac{2}{V}}})\right] &= V\log\left[1+2\,f(t)\,g(t)\left\{\cosh\left(w\sqrt{\tfrac{2}{V}}\right)-1\right\}\right]\\ &= V\sum_{k=1}^{\infty}\frac{1}{k}\left\{2\,f(t)\,g(t)\right\}^k\left\{\cosh\left(w\sqrt{\frac{V}{2}}\right)-1\right\}^k.\end{aligned}$$

It then follows that

$$\log\left[\mathrm{e}^{w\sqrt{2V}}\beta_V(t,\mathrm{e}^{-w\sqrt{\frac{2}{V}}})\right]=\frac{1}{2}(1-\mathrm{e}^{-2t})\,w^2+o(V^{-1}). \tag{4.9}$$

The second factor on the right hand side of (4.6) can be rewritten as

$$\begin{aligned}\left[1-\frac{\{f(t)-g(t)\}\left(1-\mathrm{e}^{-w\sqrt{\frac{2}{V}}}\right)}{f(t)+g(t)\,\mathrm{e}^{-w\sqrt{\frac{2}{V}}}}\right]^{\eta_V(x_0)} &= \left[1-\frac{\mathrm{e}^{-t}w\sqrt{\frac{2}{V}}+O\left(V^{-1}\right)}{1+O\left(V^{-\frac{1}{2}}\right)}\right]^{\eta_V(x_0)}\\ &= \left[1-\frac{x_0\,\mathrm{e}^{-t}w\;+\;x_0\,O\left(V^{-\frac{1}{2}}\right)}{x_0\sqrt{\frac{V}{2}}\left\{1+O(V^{-\frac{1}{2}})\right\}}\right]^{x_0\sqrt{\frac{V}{2}}\left(\frac{\eta_V(x_0)}{x_0\sqrt{\frac{V}{2}}}\right)}.\end{aligned}$$

From (4.4), $\frac{\eta_V(x_0)}{x_0\sqrt{\frac{V}{2}}}\to 1$ as $V\to\infty$ while $(1+\frac{\beta}{\alpha})^\alpha\to\mathrm{e}^\beta$ as $\alpha\to\infty$. It then follows that

$$\left[\frac{g(t)+f(t)\,\mathrm{e}^{-w\sqrt{\frac{V}{2}}}}{f(t)+g(t)\,\mathrm{e}^{-w\sqrt{\frac{2}{V}}}}\right]^{\eta_V(x_0)}\to\exp\{-x_0\,\mathrm{e}^{-t}\,w\}\quad\text{as}\quad V\to\infty. \tag{4.10}$$

From (4.6), (4.9) and (4.10), one concludes that

$$\varphi_V(x_0,w,t)\to\exp\left\{x_0\,\mathrm{e}^{-t}\,w+\frac{1}{2}(1-\mathrm{e}^{-2t})\,w^2\right\}$$

as $V\to\infty$, completing the proof.

The next corollary is immediate from Theorem 4.1.

Corollary 4.1. *For any x_0, $x\in(-\infty,\infty)$, let $m=V+\eta_V(x_0)$ and $n=V+\eta_V(x)$. Then*

$$\sqrt{\frac{V}{2}}\,p_{2V:mn}(t)\;\to\;g(x_0,x,t)\ \text{as}\ V\to\infty$$

for all t, $t\ge 0$.

Corollary 4.1 may be seen alternatively in the following manner. Let $\left(H_{e_j}(x)\right)_{j=0}^{\infty}$ be the set of Hermite polynomials defined by the Rodrigues formula

$$H_{e_j}(x)=\mathrm{e}^{\frac{x^2}{2}}\,(-1)^j\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j\left(\mathrm{e}^{-\frac{x^2}{2}}\right),\quad j\ge 0, \tag{4.11}$$

where

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2}{2}} H_{e_i}(x)\, H_{e_j}(x)\, \mathrm{d}x = \delta_{ij}\, \sqrt{2\pi}\, j!. \tag{4.12}$$

The classical decomposition theorem, see e.g. Magnus, Oberhettinger and Soni [12], states that

$$\frac{1}{\sqrt{1-z^2}} \mathrm{e}^{-\frac{(x-yz)^2}{2(1-z^2)}} = \mathrm{e}^{-\frac{x^2}{2}} \sum_{j=0}^{\infty} H_{e_j}(x)\, H_{e_j}(y)\, \frac{z^j}{j!}. \tag{4.13}$$

Applying (4.13) to (0.2), one finds that

$$g(x_0, x, \tau) = \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} H_{e_n}(x_0)\, H_{e_n}(x)\, \frac{\mathrm{e}^{-n\tau}}{n!}. \tag{4.14}$$

From (1.22), (2.6) and Lemma 3.1, one obtains that

$$\sqrt{\frac{V}{2}}\, p_{2V:mn}(t) = \sqrt{\frac{V}{2}}\, \frac{\binom{2V}{n}}{2^{2V}} \sum_{j=0}^{2V} \binom{2V}{j} y_j(m)\, y_j(n)\, \mathrm{e}^{-jt}. \tag{4.15}$$

It is known, see e.g. Szegö [17], that

$$\lim_{V\to\infty} \sqrt{\binom{2V}{j}}\, y_j(m) = \frac{1}{\sqrt{j!}} H_{e_j}(x_0); \qquad \lim_{V\to\infty} \sqrt{\binom{2V}{j}}\, y_j(n) = \frac{1}{\sqrt{j!}} H_{e_j}(x).$$

The first factor $\sqrt{\frac{V}{2}}\frac{\binom{2V}{n}}{2^{2V}}$ in (4.15) converges to $\mathrm{e}^{-\frac{x^2}{2}}/\sqrt{\pi}$ as $V \to \infty$ from Starling formula and Corollary 4.1 follows.

It is natural to expect that a first passage time of $\{X_V(t) \,:\, t \geq 0\}$ also converges in law to the corresponding first passage time of $\{\, X_{\mathrm{OU}}(t) \,:\, t \geq 0\,\}$ as $V \to \infty$, which we prove next.

Theorem 4.2. *Let m and n be as in Corollary 4.1. Let $T_{r(m)r(n)} = \inf\{\,\tau : X_V(\tau) = r(n) \,|X_V(0) = r(m)\,\}$ and $T_{x_0\,x} = \inf\{\,\tau : X_{\mathrm{OU}}(\tau) = x \,|\, X_{\mathrm{OU}}(0) = x_0\,\}$. Then $T_{r(m)r(n)}$ converges in law to $T_{x_0\,x}$ as $V \to \infty$.*

Proof. Let $l(x_0, x, \tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{P}[\,T_{x_0\,x} \leq \tau\,]$ with $\lambda(x_0, x, s) = \int_0^\infty \mathrm{e}^{-s\tau} l(x_0, x, \tau)\, \mathrm{d}\tau = \mathrm{E}\left[\,\mathrm{e}^{-s\,T_{x_0\,x}}\,\right]$. From the consistency relations, one sees that

$$g(x_0, x, \tau) = \int_0^\tau l(x_0, x, \tau - y)\, g(x, x, y)\, \mathrm{d}y. \tag{4.16}$$

Taking the Laplace transform on both sides of (4.16) with respect to τ and solving for $\lambda(x_0, x, s)$, it follows that

$$\lambda(x_0, x, s) = \frac{\gamma(x_0, x, s)}{\gamma(x, x, s)}. \tag{4.17}$$

For the counter part of (4.16) for the Ehrenfest process $\{N_{2V}(t) : t \geq 0\}$, one has

$$p_{2V:mn}(t) = \int_0^t s_{mn}(t - y)\, p_{2V:nn}(y)\, \mathrm{d}y \tag{4.18}$$

where $s_{mn}(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{P}\left[T_{mn} \leq \tau\right]$ with $T_{mn} = \inf_{t \geq 0}\{N_{2V}(t) = n | N_{2V}(0) = m\}$. Let $\pi_{mn}(s) = \int_0^\infty \mathrm{e}^{-s\tau} p_{2V:mn}(\tau)\, \mathrm{d}\tau$ and $\sigma_{mn}(s) = \int_0^\infty \mathrm{e}^{-s\tau} s_{mn}(\tau)\, \mathrm{d}\tau = \mathrm{E}\left[\mathrm{e}^{-sT_{mn}}\right]$. Corresponding to (4.17), Equation (4.18) then yields that

$$\sigma_{mn}(s) = \frac{\pi_{mn}(s)}{\pi_{nn}(s)} = \frac{\sqrt{\frac{V}{2}}\pi_{mn}(s)}{\sqrt{\frac{V}{2}}\pi_{nn}(s)}. \tag{4.19}$$

Hence from (4.17), (4.19) and Corollary 4.1, one has $\sigma_{mn}(s) \to \lambda(x_0, x, s)$ as $V \to \infty$, i.e. T_{mn} converges in law to $T_{x_0,x}$ as $V \to \infty$. It is clear that $T_{mn} = T_{r(m)r(n)}$ almost surely, completing the proof.

Similarly, we can prove that the historical maximum defined in (2.9) also converges in law to that of the O-U process.

Theorem 4.3. *Let m be as in Corollary 4.1. Let $M(r(m), \theta) = \max_{0 \leq t \leq \theta}\{X_V(t) | X_V(0) = r(m)\}$ and $M(x_0, \theta) = \max_{0 \leq t \leq \theta}\{X_{\mathrm{OU}}(t) | X_{\mathrm{OU}}(0) = x_0\}$. Then $M(r(m), \theta)$ converges in law to $M(x_0\, \theta)$ as $V \to \infty$.*

Proof. For the O-U process, for $x > x_0$, one sees that $F_{x_0,\theta}(x) = \mathrm{P}[M(x_0, \theta) \leq x] = \mathrm{P}[T_{x_0,x} > 0] = \overline{S}_{x_0,x}(\theta)$. Hence

$$F_{x_0,\theta}(x) = \begin{cases} 0 & \text{if} \quad x < x_0 \\ \overline{S}_{x_0,x}(\theta) & \text{if} \quad x \geq x_0, \end{cases} \tag{4.20}$$

where $\overline{S}_{x_0,x_0}(\theta) \stackrel{\text{def.}}{=} \lim_{\Delta \to 0} \overline{S}_{x_0,x_0+\Delta}(\theta) = \mathrm{P}\left[X_{\mathrm{OU}}(\tau) \leq x_0,\ 0 \leq \tau \leq \theta \,|\, X_{\mathrm{OU}}(0) = x_0\right]$. The theorem then follows from Theorem 4.2 and (2.11).

Remark 4.1. Using (4.17), Keilson and Ross [11] tabulate distribution functions of first passage times of $\{X_{\mathrm{OU}}(t) : t \geq 0\}$. Their approach, however, involves elaborate zero searches on the complex plane for each value of x_0 and x separately. As we will see in the next section, our approach enables one to mechanize the underlying procedures once zeros of orthogonal polynomials are found. Because of this mechanization, the distribution of the historical maximum can also be computed efficiently.

5. Development of Algorithms and Numerical Results

We have seen that, when the initial state is arranged approximately, the stochastic process $X_V(t)$ derived from the Ehrenfest process $N_{2V}(t)$ converges in law to the O-U process $X_{\mathrm{OU}}(t)$ as $V \to \infty$ for all $t > 0$. First passage times and the historical maximum of $\{X_V(t) : t \geq 0\}$ also converges in law to those of $\{X_{\mathrm{OU}}(t) : t \geq 0\}$. In this section, we develop numerical algorithms for computing transition probabilities, first passage times, and the historical maximum of $\{X_V(t) : t \geq 0\}$ based on the theoretical results discussed in the previous sections. Numerical results are also exhibited, demonstrating the accuracy and efficiency of these algorithms.

Before going into the discussion of numerical algorithms, it is appropriate to summarize state conversions among $\{N_{2V}(t) : t \geq 0\}$, $\{X_V(t) : t \geq 0\}$ and $\{X_{\mathrm{OU}}(t) : t \geq 0\}$, see Table 5.1 below. We note that when the state of $\{N_{2V}(t) : t \geq 0\}$ moves from 0 to $2V$, the state of $\{X_V(t) : t \geq 0\}$ moves from $-\sqrt{2V}$ to $\sqrt{2V}$.

Table 5.1: State Conversions

Process	state conversion		State Space
	$x \in \mathbb{R} \to m \in \mathcal{N}$	$m \in \mathcal{N} \to x \in \mathbb{R}$	
$N_{2V}(t)$	$m = \eta_V(x) + V$	m	$\mathcal{N} = \{0, 1, ..., 2V\}$
$X_V(t) = \sqrt{\frac{2}{V}} N_V(t) - \sqrt{2V}$	$\sqrt{\frac{2}{V}} \eta_V(x)$	$r(m) = \sqrt{\frac{2}{V}} m - \sqrt{2V}$	$\{-\sqrt{2V}, ..., \sqrt{2V}\}$
$X_{\mathrm{OU}}(t)$	x	$x = r(m)$	$\mathbb{R} = (-\infty, \infty)$

Remark : $\eta_V(x) = \left\lceil \sqrt{\frac{V}{2}} x \right\rceil$.

5.1. Transition Probabilities and Tail Probabilities

Given x_0, x, t and V, the transition probability $p_{2V:mn}(t)$ can be computed by employing the state conversion in Table 5.1 and the discrete convolution algorithm based on (1.11). Formally one has from (1.11),

$$p_{2V:mn}(t) = \sum_{r=0}^{n} a_{m,r}(t)\, b_{m,n-r}(t) \tag{5.1}$$

where

$$a_{m,r}(t) = \binom{2V-m}{r} f(t)^{2V-m-r} g(t)^r; \quad b_{m,r}(t) = \binom{m}{r} f(t)^r g(t)^{m-r}. \tag{5.2}$$

From (4.2) and (5.1), the transition probability density function of $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ is then given by

$$g_V(m,n,t) = \sqrt{\frac{V}{2}}\, p_{2V:mn}(t) \tag{5.3}$$

where

$$m = \eta_V(x_0) + V; \quad n = \eta_V(x) + V \quad \text{with} \quad \eta_V(x) = \left\lceil \sqrt{\frac{V}{2}} x \right\rceil. \tag{5.4}$$

Accordingly, $g_V(m,n,t)$ approximates $g(x_0,x,t)$ of (0.2) through the state conversion determined by (5.4).

In Figure 5.1, values of $g(x_0,x,t) - g_V(m,n,t)$ are plotted for $x_0 = 0, \ -5 \leq x \leq 5, \ t = 1$, and $V = 10, 20, 30, 40, 45, 47, 48, 49, 50$, demonstrating the stochastic convergence of $X_V(t)$ to $X_{\mathrm{OU}}(t)$. We see that differences among $g_V(m,n,t)$ for $45 \leq V \leq 50$ are almost negligible. Figure 5.2 exhibits graphically $g(x_0,x,t)$ represented by solid curves and $g_V(x_0,x,t)$ marked by $+, \circ, *$ for $t = 1, 3, 5$ respectively with $x_0 = 0, \ -5 \leq x \leq 5$, and $V = 50$. For tail probabilities of $g(x_0,x,t)$ with respect to x, we define

$$\overline{G}(x_0,x,\tau) = \int_x^\infty g(x_0,y,\tau)\, \mathrm{d}y. \tag{5.5}$$

Values of $\overline{G}(x_0,x,\tau)$ can be computed fairly accurately with speed using the Laguerre transform. The reader is referred to Sumita [13], where 12 digit accuracy was achieved for such computations. More readily accessible references are Sumita and Kijima [15, 16]. In order to approximate $\overline{G}(x_0,x,\tau)$, a Simpson's method is employed, i.e.

$$\overline{G}_V(m,n,\tau) = \frac{1}{2} \sum_{k=n}^{2V-1} \{p_{2V:mk}(\tau) + p_{2V:m,k+1}(\tau)\} + p_{2V:m,2V}(\tau) \tag{5.6}$$

where the last term represents the approximation for $\overline{G}(x_0, \sqrt{2V}, \tau)$. Numerical results for $\overline{G}(x_0,x,\tau)$ and $\overline{G}_V(m,n,\tau)$ are depicted in Figures 5.3 and 5.4, corresponding to Figures 5.1 and 5.2.

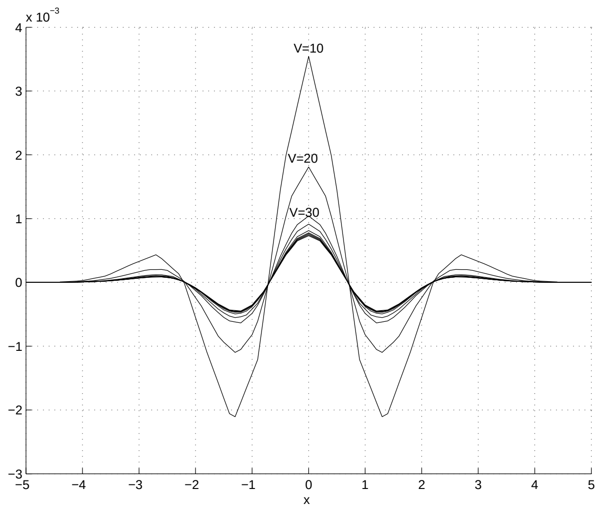

FIGURE 5.1: Difference of Transition Probabilities ($x_0 = 0, t = 1$)

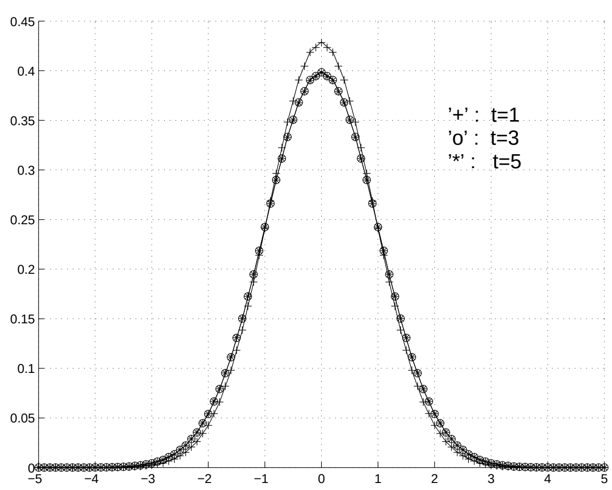

FIGURE 5.2: Transition Probabilities : the O-U Process vs the Ehrenfest Process ($x_0 = 0, V = 50$)

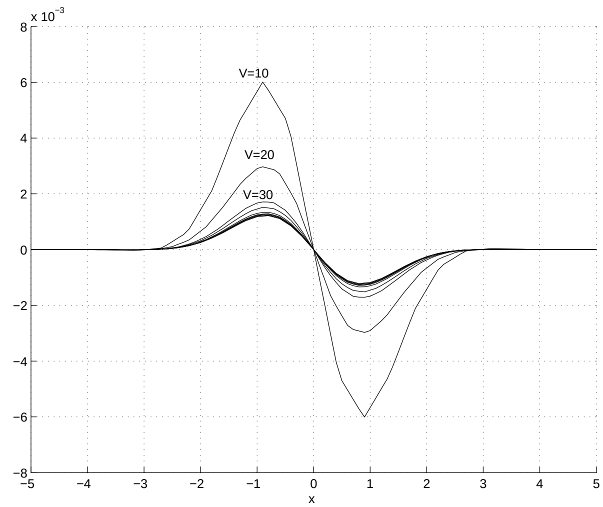

FIGURE 5.3: Difference of Tail Probabilities ($x_0 = 0, t = 1$)

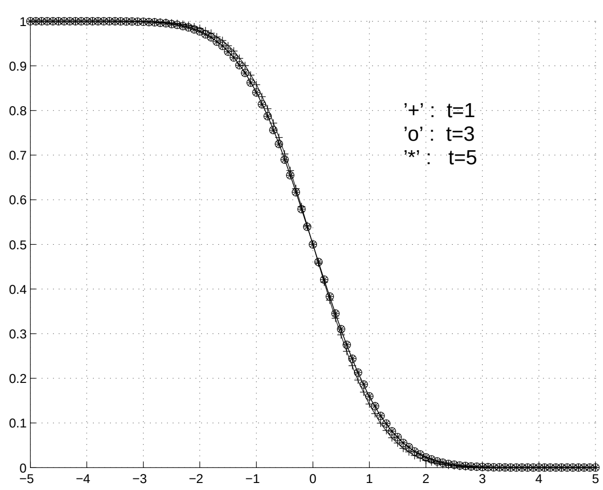

FIGURE 5.4: Tail Probabilities : the Ehrenfest Process vs the O-U Process ($x_0 = 0, V = 50$)

Algorithmic relationships discussed above are summarized in Table 5.2.

TABLE 5.2: Probability Conversions

Process	Transition Probability	Tail Probability
$N_{2V}(t)$	$p_{2V:mn}(t)$ via (5.1)	$\sum_{k=n}^{2V} p_{2V:mk}(t)$
$X_V(t) = \sqrt{\frac{2}{V}} N_V(t) - \sqrt{2V}$	$g_V(x_0, x, t) = \sqrt{\frac{V}{2}} p_{2V:mn}(t)$	$\overline{G}_V(m, n, \tau)$ in (5.6)
$X_{\rm OU}(t)$	$g(x_0, x, t)$	$\overline{G}(x_0, x, t) = \int_x^\infty g(x_0, y, t) \mathrm{d}y$

Remark : $\cdot$ $m = \eta_V(x_0) + V,\ n = \eta_V(x) + V,$

$\cdot$ $p_{2V:mn}(t) = \mathrm{P}\,[\,N_{2V}(t) = n \mid N_{2V}(0) = m\,]$

$\cdot$ $g(x_0, x, t) = \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{P}\,[\,X_{\rm OU}(t) \le x \mid X_{\rm OU}(0) = x_0\,]$
$= \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2t})}} \exp\left\{ -\frac{\left(x - x_0\,\mathrm{e}^{-t}\right)^2}{2(1-\mathrm{e}^{-2t})} \right\}$

5.2. Zeros of Orthogonal Polynomials for the Ehrenfest Process

In order to evaluate the first passage time densities $s_{mn}(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{P}[T_{mn} \le \tau],\ m < n,$ with corresponding Laplace transforms $\sigma_{mn}(s) = \sigma_m^+(s) \cdots \sigma_{n-1}^+(s) = g_m(s)/g_n(s)$ from (2.7), the zeros of $g_n(s)$ are needed. These zeros in turn enables one to evaluate the corresponding survival functions and the distribution of the historical maximum. For the Ehrenfest process, the zeros of $g_n(s)$ are related to those of $h_n(s)$ as specified in Theorem 3.1 and the computational burden can be reduced by a factor of 4. More specifically, one can write

$$\begin{cases} h_{2m}(s) = \sum_{j=0}^{m} w_{2m,2j}\, s^{2j}, \quad m \ge 0, \\ h_{2m+1}(s) = \sum_{j=0}^{m} w_{2m+1,2j+1}\, s^{2j+1}, \quad m \ge 0, \end{cases} \tag{5.7}$$

since $h_{2m}(s)$ is an even function and $h_{2m+1}(s)$ is an odd function from Theorem 3.1. It then follows from (3.3), for $m \ge 0$, that

$$\begin{cases} w_{2m,0} = -\frac{2}{2(V-m)+1} \left(m - \frac{1}{2} \right) w_{2m-2,0}, \\ w_{2m,2j} = \frac{2}{2(V-m)+1} \left\{ w_{2m-1,2j-1} - \left(m - \frac{1}{2} \right) w_{2m-2,2j} \right\}, \quad j = 1, ..., m-1, \\ w_{2m,2m} = \frac{2}{2(V-m)+1} w_{2m-1,2m-1}, \end{cases} \tag{5.8}$$

and

$$\begin{cases} w_{2m+1,2j+1} = \dfrac{w_{2m,2j} - m\, w_{2m-1,2j+1}}{V-m}, \quad j = 0, ..., m-1, \\ w_{2m+1,2m+1} = \dfrac{w_{2m,2m}}{V-m}, \end{cases} \tag{5.9}$$

where $h_0(s) = w_{0,0} = 1$.

We note that $h_{2m+1}(0) = 0$ for $m \geq 0$. Furthermore, $h_n(s) = 0$ if and only if $h_n(-s) = 0$ for all $n \geq 0$. Hence for both $h_{2m}(s)$ and $h_{2m+1}(s)$, it suffices to search m zeros in $(0, \infty)$. For $h_n(s)$ with $1 \leq n \leq 4$, the zeros can be obtained explicitly by solving the underlying equations. For higher values of n, a straightforward bisection method can be employed by exploiting the fact that zeros of $h_{n+1}(s)$ interleave those of $h_n(s)$. Let ξ_{nj} $(0 \leq j \leq n-1)$ be zeros of $h_n(s)$. For notational convenience, let $-\alpha_{nj}$ $(0 \leq j \leq n-1)$ be zeros of $g_n(s)$. From Theorem 3.1, one then has

$$\alpha_{nj} = V - \xi_{nj}, \quad 0 \leq j \leq n-1. \tag{5.10}$$

5.3. First Passage Times and the Historical Maximum

Let $T_{V:mn}$ $(m < n)$ be the first passage time of the Ehrenfest process $\{N_{2V}(t) : t \geq 0\}$ with probability density function $s_{V:mn}(\tau)$ and its Laplace transform $\sigma_{V:mn}(s)$. Since $\sigma_{V:mn}(s) = \sigma^+_{V:m}(s) \cdots \sigma^+_{V:n-1}(s)$, one has from (2.7) that

$$\sigma_{V:mn}(s) = \frac{g_m(s)}{g_n(s)} = c_{mn} \frac{\prod_{j=0}^{m-1} (s + \alpha_{mj})}{\prod_{j=0}^{n-1} (s + \alpha_{n,j})}; \quad c_{mn} = \frac{\prod_{j=0}^{n-1} \alpha_{nj}}{\prod_{j=0}^{m-1} \alpha_{mj}}. \tag{5.11}$$

As shown in Theorem 3.10 of Sumita and Masuda [14], $s_{V:mn}(\tau)$ is unimodal expressed as convolutions of completely monotone density functions. Since $\sigma_{V:mn}(s)$ is regular apart from singular points $-\alpha_{n,j}$, $0 \leq j \leq n-1$, Equation (5.11) can be rewritten as

$$\sigma_{V:mn}(s) = \sum_{j=0}^{n-1} A_{V:mn:j} \frac{\alpha_{nj}}{s + \alpha_{nj}}; \quad A_{V:mn:k} = \frac{\prod_{j=0}^{m-1} (1 - \frac{\alpha_{nk}}{\alpha_{mj}})}{\prod_{j=0, j \neq k}^{n-1} (1 - \frac{\alpha_{nk}}{\alpha_{nj}})}. \tag{5.12}$$

In real domain, Equation (5.12) leads to the probability function $s_{V:mn}(\tau)$ and its survival function $\overline{S}_{V:mn}(\tau) = \int_{\tau}^{\infty} s_{V:mn}(y)\,\mathrm{d}y$ given as

$$s_{V:mn}(\tau) = \sum_{j=0}^{n-1} A_{mn:j} \cdot \alpha_{nj}\mathrm{e}^{-\alpha_{nj}\tau}; \qquad \overline{S}_{V:mn}(\tau) = \sum_{j=0}^{n-1} A_{mn:j}\mathrm{e}^{-\alpha_{nj}\tau}. \tag{5.13}$$

Since $T_{V:mn}$ for $\{N_{2V}(\tau) : \tau \geq 0\}$ is, sample-path-wise, equal to $T_{r(m)r(n)}$ for $\{X_V(\tau) : \tau \geq 0\}$, $s_{V:mn}(\tau)$ and $\overline{S}_{V:mn}(\tau)$ provides approximations for $s_{x_0\,x}(\tau)$ and $\overline{S}_{x_0,x}(\tau)$ of $\{X_{\mathrm{OU}}(\tau) : \tau \geq 0\}$ from Theorem 4.2 with m and n as specified in Table 5.1. For $x_0 = 0$ and $x = 0.5, 1.0, 1.5, 2.0$, Figure 5.5 depicts $s_{V:mn}(\tau)$ with state conversion specified in Table 5.1, approximating $s_{x_0x}(\tau)$ with expected unimodality. Corresponding survival functions are plotted in Figure 5.6.

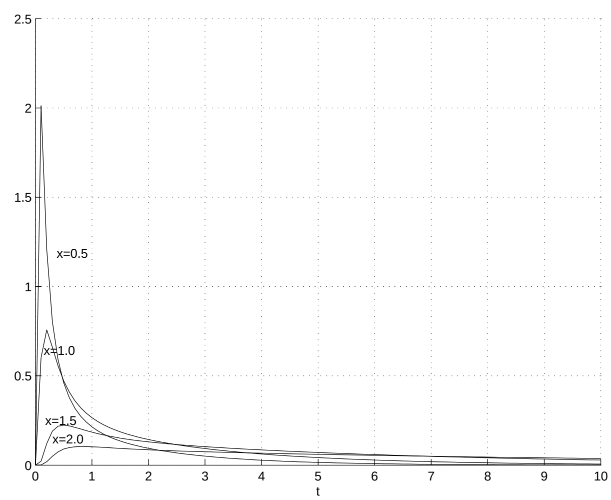

FIGURE 5.5: First Passage Time Density Functions ($V = 50$)

Let $M(x_0, \theta)$ be the historical maximum of $\{X_{\mathrm{OU}}(\tau) : \tau \geq 0\}$ in the time interval $[0, \theta]$. As in (4.20), its distribution function $F_{x_0,\theta}(x)$ has a dual relationship with the survival function $\overline{S}_{x_0,x}(\theta)$. Hence $F_{x_0,\theta}(x)$ can be approximated by $F_{V:m\theta}(n) = \overline{S}_{V:mn}(\theta)$ for $x_0 < x$, which implies $m < n$. When $x = x_0$, $\overline{S}_{x_0,x}(\theta) = \lim_{\Delta \to 0} \overline{S}_{x_0,x_0+\Delta}(\theta)$ can be approximated by $\overline{S}^{+}_{V:m}(\theta) = F_{V:m\theta}(m+1)$. With $m = \eta_V(x_0) + V$, and $n = \eta_V(x) + V$, $\overline{S}_{V:mn}(\theta)$ can be computed from (5.13). In Figure 5.7, $F_{V:m\theta}(n)$ are plotted for $x_0 = 0$, $\theta = 1, 3, 5$, and $V = 50$, where the stochastic ordering $T_{01} \prec T_{03} \prec T_{05}$ is observed as expected.

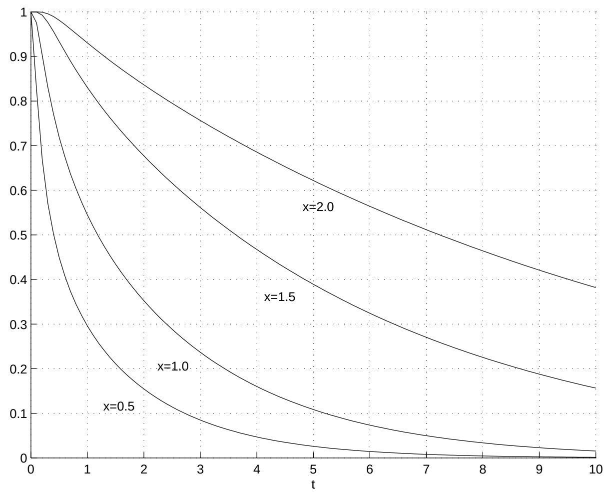

FIGURE 5.6: Survival Functions of First Passage Times ($V = 50$)

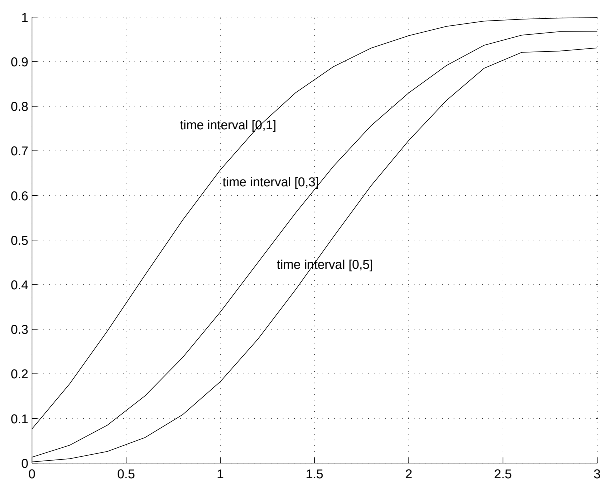

FIGURE 5.7: Distributions of the Historical Maximum ($V = 50$, $x_0 = 0$, $\theta = 1, 3, 5$)

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