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Regret region hypothesis testing using unbiased statistics--- Application to the left-retracted exponential distribution.

by

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Summary.

Hypothesis testings using unbiased statistics (or estimates) are neglected for a long time. Actually, hypothesis testings for mean and variance of the normal distribution are of this type. The tests for the regression coefficients of the regression analysis and F-tests in the experimental design are also of this type. In this paper we introduce an optimal hypothesis testing with the regret region derived from inverting the minimum random interval based on the unbiased statistic for the parameter obtained by using Lagrange's multiplier. As an example the tests for two parameters of the left-retracted exponential distribution are considered.

Key Words and Phrases: unbiased statistics, unbised estimates, hypothesis testing, minimum random interval, the logistic distribution, regret region, regret-probability function.

1. Introduction.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density $f(x|\theta)$ with a real parameter θ. Let $T(X_1, \ldots, X_n)$ be a statistic for θ. Let =: be the defining property. We call $T(X_1, \ldots, X_n)(=:T)$ an unbiased statistic (or estimate) for θ when $E(T)=\theta$. In this paper we first, in Sections 2 and 3, consider to test the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ (with known real θ_0). Let α be the real number such that $0<\alpha<1$. We call (U_1, U_2) a $(1-\alpha)$ random interval for the parameter γ if $P_\gamma[U_1<\gamma<U_2]=1-\alpha$. To get the optimal two-edged tests we find the minimum $(1-\alpha)$ random intervals for θ based on the unbiased statistic T and construct the regret regions derived from inverting this random intervals for θ_0. The name of the regret region D comes from the fact that we regret not rejecting H_0 if $T\in D$ and satisfy to reject H_0 if $T\notin D$. We define the regret-probability function by $\phi(\theta):=P_\theta(D)$, $\forall\theta$.

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Our optimal test is the one which makes $\phi(\theta)$ a concave function from below and maximizes $\phi(\theta)$ at θ_0 with $\phi(\theta_0)=1-\alpha$. The method introduced here also applies for testing the hypotheses H_0': $\theta \geqq \theta_0$ versus H_1': $\theta < \theta_0$ and for testing the hypotheses H_0'': $\theta \leqq \theta_0$ versus H_1'': $\theta > \theta_0$ as well. (See Sections 6 and 7.)

As the example we consider the left-retracted exponential distribution with the density

$$f(x|\theta,b)=b^{-1}e^{-(x-\theta)/b}, \qquad \text{for } \theta < x \tag{1}$$

where $-\infty<\theta<\infty$ and $b>0$. In Sections 2 and 6 we assume that b is known and in Sections 3 and 7 we deal with the tests for θ with unknown b. In Section 5 we introduce the two-edged test for (θ, b). In Section 4 we test unknown b. This family (1) of distributions has been used in many areas of application; for example, survival analysis, life testing and reliability theory et. al..

Based on i.i.d. observations $X_1, \ldots, X_n$ from (1) we consider, in Sections 2 and 3 to test the null hypothesis $H_0: \theta=\theta_0$ versus the alternative hypothesis H_1: $\theta \neq \theta_0$. In Section 2, to obtain the optimal two-edged test we find the minimum random interval for θ, using the unbiased statistic $Y:=\bar{X}-b$ $(=n^{-1}\sum_{i=1}^{n} X_i - b)$ for θ and construct the regret region derived from inverting this random interval for θ_0. In Section 3 we deal with the two-edged-test with unknown b. In Section 4 we consider the two-edged test for b. Let b_0 be a positive constant. In Section 5 we introduce the two-edged test for testing the hypotheses $H_0: \theta=\theta_0$, $b=b_0$ versus H_1: At least one equality in H_0 fails. In Section 6 we derive the optimal one-edged test for testing the hypothesis $H_0': \theta \geqq \theta_0$ versus the alternative hypothesis $H_1': \theta<\theta_0$, with known b. In Section 7 we deal with the one-edged tests for θ with unknown b.

2. The Two-edged test for θ with known b.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from (1) with known b. We consider the problem of testing the hypotheses $H_0: \theta=\theta_0$ versus $H_1: \theta \neq \theta_0$. We first derive necessary distributions to find the minimum random interval for θ.

As the statistic for θ we take $Y:=\bar{X}-b$. We can easily check $E(Y)=\theta$. Let $X_{(i)}$ be the i-th smallest observation such that $X_{(1)} \leqq X_{(2)} \leqq \ldots \leqq X_{(n)}$. We find the joint density of variables $W:=X_{(1)}+\dots+X_{(n)}\ (=X_1+\dots+X_n)$, $V:=X_{(1)}$, $Z_2=X_{(2)}$, $\dots$ $Z_{n-1}=X_{(n-1)}$ as follows:

$$g(w,v,z_2,z_3,\dots,z_{n-1}\mid\theta)=\begin{cases} n!b^{-n}e^{-(w-n\theta)/b}, & \text{for } \theta\leqq v\leqq z_2\leqq z_3\leqq\dots\leqq z_{n-1}\leqq w-\Sigma_{i=1} z_i \\ 0, & \text{otherwise.}\end{cases}$$

Integrating out z_2 through z_{n-1} from the above density we get the joint density of (W,V) as follows:

$$g(w,v\mid\theta)=\int_v^{(w-v)/(n-1)}\dots\int_{z_{n-3}}^{(w-\Sigma_{i=1}^{n-3}z_i)/3}\int_{z_{n-2}}^{(w-\Sigma_{i=1}^{n-2}z_i)/2} g(w,v,z_2,\dots,z_{n-1}\mid\theta)\,dz_{n-1}dz_{n-2}\dots dz_2$$

$$=\begin{cases}(n/\Gamma(n-1))b^{-n}e^{-(w-n\theta)/b}(w-nv)^{n-2}, & \text{for } \theta\leqq v\leqq w/n(<\infty) \\ 0, & \text{otherwise.}\end{cases}\qquad(2)$$

Taking the marginal density of W and furthermore, letting $t:=2(w-n\theta)/b$ we have the density of T so that

$$h_n(t)=(1/\Gamma(n))e^{-t/2}t^{n-1}2^{-n},\qquad \text{for } 0\leqq t;\ =0,\ \text{for } t<0.\qquad(3)$$

which is the chi-square density with 2n degrees of freedom.

Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the minimum $(1-\alpha)$ random interval for θ we want to minimize r_2-r_1 subject to

$$P_\theta[r_1<Y-\theta<r_2]=1-\alpha.\qquad(4)$$

But, by the transformation $t=2n(y+b-\theta)/b$ (4) is equivalent to

$$P[t_1<T<t_2]=1-\alpha \tag{5}$$

where $t_i:=2n(r_i+b)/b$ for $i=1,2$. Hence, we want to minimize t_2-t_1 subject to the condition (5). Let λ be a Lagrange's multiplier and define

$$L:=t_2-t_1-\lambda\{\int_{t_1}^{t_2} h_n(t)\,dt-1+\alpha\}. \tag{6}$$

Then, $\partial L/\partial t_1=0=\partial L/\partial t_2$ leads to

$$h_n(t_1)=h_n(t_2)\ (=\lambda^{-1}). \tag{7}$$

Taking t_1 and t_2 which satisfy (7) and $\partial L/\partial\gamma=0$, noticing that $t_1<T=2n(Y+b-\theta)/b<t_2$ and letting $t_3=bt_1/(2n)$ and $t_4=bt_2/(2n)$ we obtain the minimum $(1-\alpha)$ random interval for θ as follows:

$$(Y+b-t_4, Y+b-t_3). \tag{8}$$

Hence, by inverting (8) for θ_0 our regret region based on (W,V) becomes $\theta_0+t_3-b<Y<\theta_0+t_4-b$ and $V\geqq\theta_0$ where $Y=n^{-1}W-b$. Here, we emphasize the necessity of having the set $\{V\geqq\theta_0\}$ in the regret region. To check the optimality of this test we obtain the regret-probability function of the test as follows:

$$\phi(\theta)=P_\theta[\theta_0+t_3-b<Y<\theta_0+t_4-b \text{ and } V\geqq\theta_0]$$

$$=\begin{cases}(1-\alpha)\exp\{-n(\theta_0-\theta)/b\}, & \text{for } \theta<\theta_0\\ P[t_1-2n(\theta-\theta_0)/b<T<t_2-2n(\theta-\theta_0)/b], & \text{for } \theta_0\leqq\theta<\theta_0+t_3\\ P[0<T\leqq t_2-2n(\theta-\theta_0)/b], & \text{for } \theta_0+t_3\leqq\theta<\theta_0+t_4\\ 0, & \text{for } \theta_0+t_4\leqq\theta.\end{cases} \tag{9}$$

Hence, $d\phi(\theta)/d\theta>0$ for $\theta<\theta_0$; $d\phi(\theta)/d\theta=2nb^{-1}\{h_T(t_1-2n(\theta-\theta_0)/b)-h_T(t_2-2n(\theta-\theta_0)/b)\}$

<0 for $\theta_0 \leq \theta < \theta_0 + t_3$ because of (7) and (3), and $d\phi(\theta)/d\theta < 0$ for $\theta_0 + t_3 \leq \theta < \theta_0 + t_4$. Since $\phi'(\theta_0)=0$ by (7) and $\phi(\theta_0)=1-\alpha$, we have that $\phi(\theta) \leq 1-\alpha$ for real θ. Thus, the optimality of the test is proved.

In the next section we consider the two-edged test with unknown b.

3. The two-edged tests for θ with unknown b.

In this section we let b be unknown. We test the hypotheses $H_0: \theta=\theta_0$ versus $H_1: \theta \neq \theta_0$. Let W and V be as defined in Section 2. Since $(W-nV)/(n-1)$ is an unbiased statistic for b, we use the statistic

$$Z:=(n-1)(n^{-1}W-\theta)/(W-nV)$$

and furthermore the test statistic

$$S:=\{\log_e(n/(n-1)) + \log_e Z\}^{1/2} \qquad (9)$$

whose density has one peak. Since from (2) and a little calculation, the p.d.f. of Z is obtained by

$$g_Z(z)=(n-1)^n/\{n^{n-1}z^n\}, \qquad \text{for } (n-1)/n \leq z; =0, \text{ otherwise}, \qquad (10)$$

from the transformation (9) applied to (10) we obtain the density of S as follows:

$$g_S(s)=2(n-1)\exp\{-(n-1)s^2\}s, \quad \text{for } 0 \leq s; =0, \text{ for } s<0. \qquad (11)$$

Let r_1 and r_2 be positive numbers such that $r_1<r_2$,

$$\int_{r_1}^{r_2} g_S(s)\, ds=1-\alpha$$

and

$$g_S(r_1)=g_S(r_2).$$

For such r_1 and r_2 we have that $r_1<S<r_2$ or $\exp\{r_1^2\}<(W-n\theta)/(W-nV)<\exp\{r_2^2\}$.

Hence, the minimum $(1-\alpha)$ random interval for θ is

$$(n^{-1}(W-\exp\{r_2{}^2\}(W-nV)),n^{-1}(W-\exp\{r_1{}^2\}(W-nV))). \tag{12}$$

Thus, our regret region based on (W,V) for testing H_0 versus H_1 is as follows:

$$\{(W,V):\ n^{-1}(W-\exp\{r_2{}^2\}(W-nV))<\theta_0<n^{-1}(W-\exp\{r_1{}^2\}(W-nV)) \text{ and } \theta_0\leqq V\}.$$

In the next section we show two-edged tests for b.

4. The two-edged tests for b.

In this section we consider to test the hypotheses $H_0:b=b_0$ versus $H_1:b\neq b_0$.

We first assume that θ is known. Let W and V be as defined in Section 2. We use the test statistic $T:=2(W-n\theta)/b$. From Section 2 the density of T is given by (3). Taking positive numbers t_1 and t_2 which satisfy (5), (6) and (7) we obtain the minimum $(1-\alpha)$ random interval for b as follows:

$$((W-n\theta)/t_2,(W-n\theta)/t_1). \tag{13}$$

By inverting (13) for b_0 our regret region based on W becomes

$$n\theta+b_0t_1<W<n\theta+b_0t_2.$$

Secondly, we assume that θ is unknown. We consider another test based on the statistic $S:=2(W-nV)/b$. From (2) and a little computation we get the density $h_{n-1}(s)$ of S where $h_n(s)$ is given by (3). Hereafter, we proceed the same way as above. Taking positive numbers t_1 and t_2 which satisfy (5), (6) and (7) with T and n there replaced by S and n-1, respectively we obtain the minimum random interval for b as follows:

$$((W-nV)/t_2,(W-nV)/t_1). \tag{14}$$

Thus, by inverting (14) for b_0 the regret region of our test is as follows:

$$\{(W,V):\ (W-nV)/t_2 < b_0 < (W-nV)/t_1\}.$$

In the next section we introduce the two-edged test for (θ, b).

5. The two-edged test for (θ, b).

In this section we consider to test the hypothesis $H_0: \theta=\theta_0$, $b=b_0$ versus the alternative hypothesis H_1: At least one equality in H_0 fails. Let W and V be as defined in Section 2. We let $S:=(W-nV)^{1/2}$. To get the two-edged test for (θ, b) we use the statistic

$$Z:=2\sqrt{n-1}(n^{-1}W-\theta)/(\sqrt{b}S).$$

We find the density of Z, apply Lagrange's multiplier to get the minimum random interval for (θ, b) and from this random interval obtain the regret region based on (W, V) for the two-edged test with respect to (θ, b).

We first find the density of Z and show that this density of Z has one peak. Let $U:= n(V-\theta)/S$. To get the density of Z we first find the density of (S, V) from (2) of (W, V), then the density of (S, U) from the density of (S, V) and finally the density of (S, Z) with the transformation $Z=\{2\sqrt{n-1}/(n\sqrt{b})\}(S+U)$ from the density of (S, U). Obtained density of (S, Z) is as follows:

$$h_{S,Z}(s,z)=\begin{cases} (n/\{b^{n-1/2}\Gamma(n-1)\sqrt{n-1}\})s^{2n-2}\exp\{-nzs/(2\sqrt{(n-1)b})\}, & \text{for } 0\leq\{2\sqrt{n-1}s/(n\sqrt{b})\}\leq z \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Let

$$H_{2n-1}(z):=\int_{-\infty}^{z} h_{2n-1}(t)\,dt, \qquad \forall z.$$

Integrating out s from (15) we get the density of z as follows:

$$h_Z(z) = cz^{-(2n-1)} H_{2n-1}(n^2z^2/(n-1)), \qquad \forall z \geqq 0; \ =0, \quad \forall z<0,$$

where

$$c := (n-1)^{n-1} 2^{2n-1} \Gamma(2n-1)/\{n^{2n-2}\Gamma(n-1)\}.$$

Since $h_Z(0)=0=h_Z(-\infty)$, we show that there exists unique z (>0) such that $h_Z'(z)=0$. Now,

$$h_Z'(z) = c\{-(2n-1)z^{-2n}H_{2n-1}(n^2z^2/(n-1)) + \{2n^2/(n-1)\}z^{-(2n-2)} h_{2n-1}(n^2z^2/(n-1))\}.$$

Let $\xi(z) := (2n^2z^2/\{(n-1)(2n-1)\})h_{2n-1}(n^2z^2/(n-1)) = c_1 z^2 h_{2n-1}(n^2z^2/(n-1))$. We would like to show that there exists unique z such that $\xi(z) = H_{2n-1}(n^2z^2/(n-1))$. Since

$$\xi'(z) = c_1 z\{4n-2-n^2z^2/(n-1)\}h_{2n-1}(z)$$

and

$$dH_{2n-1}(n^2z^2/(n-1))/dz = 2n^2z/(n-1)h_{2n-1}(n^2z^2/(n-1)),$$

we have that $\xi'(z) > dH_{2n-1}(n^2z^2/(n-1))/dz$ (>0) for $0<z<\sqrt{(2n-1)(n-1)}/n$. Thus, there exists unique z(>0) such that $h_Z'(z)=0$. Therefore, $h_Z(z)$ is the density with one peak.

To find the minimum $(1-\alpha)$ random interval for (θ, b) we let r_1 and r_2 be positive numbers such that $r_1<r_2$ and want to minimize r_2-r_1 subject to

$$P_\theta[r_1<Z<r_2] = 1-\alpha. \tag{16}$$

In the same fashion as (5), (6) and (7) we obtain

$$h_Z(r_1) = h_Z(r_2). \tag{17}$$

Hence, the minimum $(1-\alpha)$ random interval for (θ, b) is as follows:

$$\{(\theta,b):\ r_1 < 2\sqrt{n-1}(n^{-1}W-\theta)/\sqrt{b(W-nV)} < r_2\},$$

where r_1 and r_2 satisfy (16) and (17). Thus, our regret region based on (W,V) becomes as follows:

$$\{(W,V):\ r_1 < 2\sqrt{n-1}(n^{-1}W-\theta_0)/\sqrt{b_0(W-nV)} < r_2 \quad \text{and } \theta_0 \leqq V\}.$$

In the next two sections we deal with the one-edged tests.

6. The one-edged test for θ with known b.

In this section we consider to test the hypotheses $H_0':\theta \geqq \theta_0$ versus $H_1':\theta < \theta_0$ based on a random sample $X_1, \ldots, X_n$ from (1) with known b. Let Y, W, V and T be as defined in Section 2. Let $t_6 := bt_5/(2n)$ where t_5 is given by

$$\int_{t_5}^{\infty} h_n(t)\,dt = 1-\alpha.$$

Here, $h_n(t)$ is given by (3). From Section 2 we can easily see that our test has the following regret region based on (W,V):

$$\{(W,V):\ \theta_0+t_6-b<Y \text{ and } \theta_0 \leqq V\}$$

where $Y:=n^{-1}W-b$. Let $g_{Y,V}(y,v|\theta)$ be the joint density of (Y,V). Then, from (2) and the relation $g_{Y,V}(y,v|\theta)=g(n(y+b),v|\theta)n$ we can easily get the regret-probability function of above test as follows:

$$\phi(\theta) = P_\theta[\theta_0+t_6-b<Y \text{ and } \theta_0 \leqq V]$$

$$= \begin{cases} (1-\alpha)\exp\{-n(\theta_0-\theta)/b\}, & \text{for } \theta<\theta_0 \\ P[t_5-2n(\theta-\theta_0)/b \leqq T], & \text{for } \theta_0 \leqq \theta < \theta_0+t_6 \\ 0, & \text{for } \theta_0+t_6 \leqq \theta. \end{cases}$$

Since $d\phi(\theta)/d\theta>0$ for $\theta<\theta_0+t_6$ and hence $\phi(\theta)\leqq 1-\alpha=\phi(\theta_0)$ for $\theta<\theta_0$, our test has optimal property.

In the next section we consider one-edged tests for θ with unknown b.

7. The one-edged tests for θ with unknown b.

We consider to test the hypotheses $H_0':\theta\geqq\theta_0$ versus $H_1':\theta<\theta_0$ based on a random sample $X_1, \ldots, X_n$ from (1) with unknown b. Let W and V be as in Section 2.

Let r_1 be a positive number such that

$$\int_{r_1}^{\infty} g_S(s)\, ds = 1-\alpha$$

where $g_S(s)$ is given by (11). From (12) in Section 3 we can easily see that the minimum $(1-\alpha)$ random interval for θ is obtained as follows:

$$(-\infty, n^{-1}(W-\exp\{r_1^2\}(W-nV))).$$

Hence, our regret region based on (W, V) is as follows:

$$\{(W,V):\ \theta_0<n^{-1}(W-\exp\{r_1^2\}(W-nV)) \text{ and } \theta_0\leqq V\}.$$

We next consider to test the hypotheses $H_0'':\theta\leqq\theta_0$ versus $H_1'':\theta>\theta_0$. Let r_2 be a positive number such that

$$\int_{0}^{r_2} g_S(s)\, ds = 1-\alpha.$$

From (13) in Section 3 the minimum $(1-\alpha)$ random interval for θ is as follows:

$$(n^{-1}(W-\exp\{r_2^2\}(W-nV)), \infty).$$

Hence, our regret region based on (W, V) is as follows:

$$\{(W,V):\ n^{-1}(W-\exp\{r_2^2\}(W-nV))<\theta_0\}.$$