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# Optimal Hypothesis Testing Under Unbiased Estimates——Application to the Logistic Distribution

by

Keiko Nogami

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Summary:

Hypothesis testings using unbiased estimates are neglected for a long time. Actually, hypothesis testings for mean and variance of the normal distribution are of this type. There are more tests in this direction. However, keen research for this direction has not been done for a long time. In this paper we introduce the hypothesis testings for a positional parameter of the logistic distribution using the sample median as an example.

## 1. Introduction.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density $f(x|\theta)$. Let $T(X_1, \ldots, X_n)$ be an estimate for $\theta$. We call $T(X_1, \ldots, X_n)$ an unbiased estimate (or statistic) for $\theta$ when $E(T(X_1, \ldots, X_n))=\theta$.

In this paper we consider to test the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ (with known real $\theta_0$) and introduce an optimal hypothesis testing under the unbiased estimate $T(X_1, \ldots, X_n)$, namely, the hypothesis testing with the regret region derived from inverting the minimum domain estimate based on the unbiased estimate for the parameter by using Lagrange's method. This method also applies both for testing the hypotheses $H'_0:\theta\geqq\theta_0$ versus $H'_1:\theta<\theta_0$ and for testing the hypotheses $H''_0:\theta\leqq\theta_0$ versus $H''_1:\theta>\theta_0$. As an example the logistic distribution with the density

$$f(x|\theta)=b^{-1}e^{-(x-\theta)/b}[1+\exp\{-(x-\theta)/b\}]^{-2}, \quad \text{for } -\infty<x<\infty \qquad (1)$$

provided that $-\infty<\theta<\infty$ and known $b>0$, is considered.

Throughout the paper it is enough to assume n is odd (i.e. $n=2m+1$ with m a nonnegative integer) because if n is even, then we discard one observation. Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We estimate $\theta$ by the sample median $Y\doteq X_{(m+1)}$ where $\doteq$ is the defining property. Then, we can easily check $E(Y)=\theta$.

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* Institute of Policy and Planning Sciences, University of Tsukuba, 1-1-1 Tennohdai, Tsukuba, Ibaraki, Japan; 305-8573, E-mail:nogami@sk.tsukuba.ac.jp

Let $\alpha$ be a real number such that $0<\alpha<1$. We call $(U_1, U_2)$ a $(1-\alpha)$ domain estimate for the parameter $\gamma$ if $P_\gamma[U_1<\gamma<U_2]=1-\alpha$. In the next section we find the minimum $(1-\alpha)$ domain estimate for $\theta$ using Y and derive the two-edged test for testing the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ by inverting the minimum $(1-\alpha)$ domain estimate for $\theta_0$. In Section 3 we introduce the regret-probability function of $\theta$ and see that this test has the optimal property. In Section 4 we consider the one-edged tests.

## 2. The two-edged test.

Let $Y \doteq X_{(m+1)}$. We first find the density of Y and obtain the minimum $(1-\alpha)$ domain estimate for $\theta$. Let $F(x|\theta)$ be the cdf of X. Then, by (1) we get

$$F(x) \doteq F(x|\theta)=\{1+e^{-(x-\theta)/b}\}^{-1}, \quad \text{for } -\infty<x<\infty. \tag{2}$$

Hence, the density of Y is of form

$$g_Y(y|\theta)=k(F(y))^m(1-F(y))^m f(y|\theta), \quad \text{for } -\infty<y<\infty \tag{3}$$

where

$$k=\Gamma(2m+2)/(\Gamma(m+1))^2.$$

Let $r_1$ and $r_2$ be real numbers such that $r_1<r_2$. To find the minimum $(1-\alpha)$ domain estimate for $\theta$ we want to minimize $r_2-r_1$ subject to

$$P_\theta[r_1<Y-\theta<r_2]=1-\alpha. \tag{4}$$

Let $\lambda$ be a Lagrange's multiplier. Using the transformation $W=F(Y)$ we define

$$L \doteq r_2-r_1-\lambda\{\int_{F(r_1+\theta)}^{F(r_2+\theta)} h_W(w)\, dw -1+\alpha\}$$

where $h_W(w)$ is the density of W given by

$$h_W(w)=kw^m(1-w)^m, \quad \text{for } 0<w<1. \tag{5}$$

Then, $\partial L/\partial r_1=0=\partial L/\partial r_2$ leads to

$$h_w(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=h_w(F(r_2+\theta))f(r_2+\theta\,|\,\theta)\quad(=\lambda^{-1}). \tag{6}$$

Let $\beta(\alpha/2)$ be the real number such that $0<\beta(\alpha/2)<1/2$ and

$$\int_0^{\beta(\alpha/2)} h_w(w)\,dw=\alpha/2.$$

Taking

$$F(r_1+\theta)=\beta(\alpha/2) \text{ and } F(r_2+\theta)=1-\beta(\alpha/2) \tag{7}$$

we obtain by (2) that $r_2=-r_1(=r)$ where

$$r=b\log_e\{(1-\beta(\alpha/2))/\beta(\alpha/2)\}. \tag{8}$$

Since we have that $h_w(F(-r+\theta))=h_w(F(r+\theta))$ and $f(-r+\theta\,|\,\theta)=f(r+\theta\,|\,\theta)$, when $r_2=-r_1=r$ (6) and $\partial L/\partial\lambda=0$ are satisfied. Therefore, in view of (4) the minimum $(1-\alpha)$ domain estimate for $\theta$ is given by

$$(Y-r, Y+r). \tag{9}$$

We now find the two-edged test. Let $y_1=\theta_0-r$ and $y_2=\theta_0+r$ with r given by (8). To test the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ we invert the domain estimate (9) for $\theta_0$ and obtain the regret region $(y_1, y_2)$ based on Y; we regret not rejecting $H_0$ if $y_1<Y<y_2$ and satisfy to reject $H_0$ if $Y\leqq y_1$ or $y_2\leqq Y$.

In the next section we show the optimal property of this test.

### 3. The regret-probability function of $\theta$.

Let $T=T(X_1, \ldots, X_n)$ be the unbiased statistic to test the hypotheses $H_0$ versus $H_1$. Let D be the regret region which we regret not rejecting $H_0$ if $T\in D$ and satisfy to reject $H_0$ if $T\notin D$. Let us call $\psi(\theta)=P_\theta(D)$ the regret-probability function of $\theta$. It is desirable that $\psi(\theta)$ is a concave function from below and maximized at $\theta_0$ (i.e. $\psi(\theta)\leqq\psi(\theta_0)$, $\forall\theta$) which is one optimal property of $\psi(\theta)$.

Let $y_1$ and $y_2$ be defined as in Section 2. Define the regret-probability function

$$\phi(\theta)=\int_{y_1}^{y_2} g_Y(y|\theta)\,dy, \quad \text{for } -\infty<\theta<\infty \tag{10}$$

where $g_Y(y|\theta)$ is given by (3). We show above optimal property of $\phi(\theta)$. From the construction $\phi(\theta_0)=1-\alpha$. To show the optimal property of $\phi(\theta)$ we show that $[d\phi(\theta)/d\theta]_{\theta=\theta_0}=0$ and $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}<0$. Since from the construction the equality (6) with $r_2=-r_1=r$ and $\theta=\theta_0$ is satisfied, it follows from (3) and (7) that $g_Y(y_1|\theta_0)=g_Y(y_2|\theta_0)$. Hence,

$$[d\phi(\theta)/d\theta]_{\theta=\theta_0}=g_Y(y_1|\theta_0)-g_Y(y_2|\theta_0)=0.$$

We finally show the following theorem:

Theorem. When $n=m+1$ and $0<\beta(\alpha/2)<2^{-1}$,

$$[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}<0. \tag{11}$$

Proof. From (10) we have that

$$[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}=[dg_Y(y_1|\theta)/d\theta]_{\theta=\theta_0} - [dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}. \tag{12}$$

By (3) we also have that

$$dg_Y(y|\theta)/d\theta=kmf(y|\theta)(dF(y)/d\theta)(F(y))^{m-1}(1-F(y))^{m-1}(1-2F(y)) \tag{13}$$

$$+k(F(y))^m(1-F(y))^m(df(y|\theta)/d\theta).$$

Since $df(y|\theta)/d\theta=b^{-2}e^{-(y-\theta)/b}(1-e^{-(y-\theta)/b})(1+e^{-(y-\theta)/b})^{-3}$, we have that

$$[df(y_2|\theta)/d\theta]_{\theta=\theta_0}=b^{-1}(1-2\beta(\alpha/2))f(y_2|\theta_0)=-[df(y_1|\theta)/d\theta]_{\theta=\theta_0}.$$

From (7) and (2) we also have that $F(y_1|\theta_0)=\beta(\alpha/2)=1-F(y_2|\theta_0)$ and $f(y_1|\theta_0)=$

$f(y_2|\theta_0)=b^{-1}\beta(\alpha/2)(1-\beta(\alpha/2))$. Applying these and the fact that $dF(y)/d\theta=-f(y|\theta)$ to (13) leads to

$$[g_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}=kb^{-2}(1-\beta(\alpha/2))^{m+1}(\beta(\alpha/2))^{m+1}(1-2\beta(\alpha/2))(m+1)\ (>0)$$

and $[dg_Y(y_1|\theta)/d\theta]_{\theta=\theta_0}=-[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}$. Therefore, in view of (12), (11) holds. ∎

4. The one-edged tests.

We consider to test the hypotheses $H'_0:\theta\geq\theta_0$ versus $H'_1:\theta<\theta_0$ based on a random sample $X_1, \ldots, X_n$ from the density (1) with known b (>0).

As in Section 2 let $n=2m+1$ (m:a nonnegative integer), $Y\doteq X_{(m+1)}$ and $r_1$ and $r_2$ be the real numbers such that $r_1<r_2$. We want to minimize $r_2-r_1$ subject to

$$P_\theta[r_1<Y-\theta<r_2]=1-2\alpha.$$

Let $\beta(\alpha)$ be the number such that $0<\beta(\alpha)<1$ and

$$\int_0^{\beta(\alpha)} h_W(w)\ dw=\alpha.$$

Taking $F(r_1+\theta)=\beta(\alpha)$ and $F(r_2+\theta)=1-\beta(\alpha)$ we obtain $r_2=-r_1=r$ where

$$r=b\ \log_e\{(1-\beta(\alpha))/\beta(\alpha)\}.$$

To test the hypotheses $H'_0:\theta\geq\theta_0$ versus $H'_1:\theta<\theta_0$ we take the regret region $(\theta_0-r,\infty)$ based on Y.

Let $\xi(\theta)=P_\theta[\theta_0-r<Y]$. Then, since

$$\xi(\theta)=\int_{\theta_0-r}^{\infty} g_Y(y|\theta)\ dy, \quad \text{for } -\infty<\theta<\infty$$

and hence

$$d\xi(\theta)/d\theta=d\{\int_{\theta_0-\theta-r}^{\infty} g_Y(z+\theta|\theta)\ dz\}/d\theta=g_Y(\theta_0-r|\theta)>0,\ \forall\theta,$$

it follows that $\xi(\theta)\geq1-\alpha$ $(=\xi(\theta_0))$ for $\theta\geq\theta_0$ and $\xi(\theta)<1-\alpha$ for $\theta<\theta_0$, which is the natural optimal property of the regret-probability function $\xi(\theta)$ of the one edged test.

For testing the hypotheses $H''_0:\theta\leqq\theta_0$ versus $H''_1:\theta>\theta_0$ we can take the regret region $(-\infty,\theta_0+r)$ based on Y. Letting $\xi_1(\theta)=P_\theta[Y<\theta_0+r]$ we get $d\xi_1(\theta)/d\theta=-g_Y(\theta_0+r|\theta)<0$, $\forall\theta$. Hence, $\xi_1(\theta)\geqq1-\alpha(=\xi_1(\theta_0))$ for $\theta\leqq\theta_0$ and $\xi_1(\theta)<1-\alpha$ for $\theta>\theta_0$ which is also the natural optimal property of $\xi_1(\theta)$.

5. Remark.

We can consider two more cases; one is the case where $\theta$ is known and b is unknown and the other is the case where $\theta$ and b are both unknown. The analyses for these two cases are in the process and will appear soon.