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Customers Selection Problem Where Multiple Customers Can Be Held

-Numerical Experiment-

by

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CUSTOMERS SELECTION PROBLEM WHERE MULTIPLE CUSTOMERS CAN BE HELD
— Numerical Experiment —

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Abstract
In the paper, we discuss, by use of some numerical experiments, a customer selection problem in which it is assumed that one or more customers can be held.

1 Introduction

In [2] we posed and examined the problem of selecting profitable orders to accept out of sequentially arriving ones in a custom production company, such as a shipbuilding company, an advertising agency, a consulting company, a design office, a construction firm, and so on. Let n denote the maximum permissible number of customers which can be held in the system at any instant. In [2] we strictly and in detail discussed the problem on the assumption of $n = 1$, which is changed into $n \geq 2$ in the present paper. When $n \geq 2$, it is known that a new problem of clarifying the properties of the optimal customer selection rule arises which can not be seen when $n = 1$; considerable difficulties can be expected in the analysis of the problem. So that, in the current paper we examine the properties through some numerical experiments, based on insights obtained from which we point a direction of the study of the problem. Now, in the paper, we only consider the customer-first case[1].

2 Model

The following general framework defines the model examined in the paper.

1. The model is defined as a discrete-time stochastic decision process with an infinite planning horizon. Let points in time be equally spaced on the axis of the planning horizon and let the time interval between successive points in time be called the *period.*
2. By $n \geq 1$ let us denote the maximum permissible number of orders that can be held in the system at any instance[2].
3. A customer appears with a known probability $p\,(0 < p \leq 1)$ by conducting a search activity for it, accompanied with some cost $c \geq 0$, called the *search cost.*
4. The introduction of the search cost inevitably leads us to the necessity of introduction of the decision between conducting and skipping the search at each point in time.
5. If there exists no order in the system for a period, some profit can be yielded from engaging in other economic activities for the period; let the profit $s \geq 0$ be called the *idling profit.* For example, consider a company producing current transfers where products with standard specification and products with special specification are manufactured; let the latter products be produced as custom production

[1] In [2], as to which of a customer and the system first offers the price of the order, the two cases of the customer-first case and the system-first case was considered. See [2] for their definitions.

[2] In [2] we assumed $n = 1$.

items. In such a company, if all the products with special specification accepted so far have been completed and there exists no products to produce, then the production of products with standard specification is started. In this case, the idling profit is yielded by the production of products with standard specification.

6. Prices offered by subsequently appearing customers, $w, w', \cdots$, are assumed to be independent identically distributed random variables having a known distribution function $F(w)$ with a finite expectation μ. Let $F(w)$ be either discrete or continuous where its probability (density) function is denoted by $f(w)$. Here, for certain given numbers a and b with $0 < a < b < \infty$ let us assume that

$$F(w) = 0, \quad w < a, \qquad 0 < F(w) < 1, \quad a \leq w < b, \qquad F(w) = 1, \quad b \leq w, \tag{2.1}$$

where $f(w) > 0$ on $[a, b]$.

7. An order in the system at a certain point in time is completed and goes out of the system up to the next point in time with the probability q $(0 < q < 1)$.
8. Let the discount factor be denoted by β; that is, a monetary value of one unit a period after is equivalent to that of β units at the present point in time; let $\beta < 1$ throughout the paper.
9. The objective is to find the optimal decision rule so as to maximize the total expected present discounted net profit gained over an infinite planning horizon, the total expected present discounted value of prices of orders accepted or placed plus the idling profits over the entire planning horizon minus the total expected present discounted value of search costs paid over the entire planning horizon.

For convenience in the later discussions, let us define

$$\alpha = p\beta\mu - c, \tag{2.2}$$

$$\lambda = (1 - \beta(1-q))^{-1} > 1, \tag{2.3}$$

where

$$\lambda > 1, \qquad 1 - \lambda q = \lambda(1-q)(1-\beta) > 0, \qquad 1 - \lambda q\beta = \lambda(1-\beta) > 0. \tag{2.4}$$

For expressional simplicity, by the symbols C, K, A and R let us denote the decisions of, respectively: continuing the search, skipping the search, accepting an order, and rejecting an order[3].

3 T-function

Let us define following function: For any real number x let us define

$$T(x) = \int_0^\infty \max\{w - x, 0\} dF(w), \tag{3.1}$$

called the *T-function* [1], which will be used to describe the optimal equation of our model and examine the properties of the optimal decision rule.

[3] We do not use S as a symbol representing "skipping the search" because it is often used as a symbol representing "stop the search"

4 Optimal Equation

Either if the search was skipped at the previous point in time or if no customer appears with probability $1-p$ regardless of having conducted the search at the previous point in time, it follows that no customer appears at the present point in time. For convenience, we shall refer to such a situation as "the system has a *fictitious customer* ϕ".

By $u(\phi, i)$ and $u(w, i)$ we shall denote the maximum of the total expected present discounted net profit starting from ,respectively, the state of having the fictitious customer ϕ and $i\,(0 \leq i \leq n)$ orders in the system and the state of having an appearing customer w and $i\,(0 \leq i < n)$ orders in the system; let us refer such situation to the states (ϕ, i) and $u(w, i)$. If no search is made over the entire planning horizon, then no customer appears at all the points in time, then the production line becomes idle for all the period over the planning horizon, implying that the total expected present discounted net profit becomes the total expected present discounted idling profit $s \geq 0$, hence it must follow that $u(\phi, i) \geq 0$. Now, for convenience in the later discussions, let us define

$$h_i = u(\phi, i) - u(\phi, i+1), \quad 0 \leq i < n. \tag{4.1}$$

Then, we get

$$u(\phi, 0) = \max \begin{cases} \mathtt{C} : \beta\big(p \int_0^\infty u(\xi, 0) dF(\xi) + (1-p) u(\phi, 0)\big) - c + s, \\ \mathtt{K} : \beta u(\phi, 0) + s, \end{cases} \tag{4.2}$$

$$u(\phi, i) = \max \begin{cases} \mathtt{C} : (1-q)\beta\big(p \int_0^\infty u(\xi, i) dF(\xi) + (1-p) u(\phi, i)\big) \\ \qquad + q\beta\big(p \int_0^\infty u(\xi, i-1) dF(\xi) + (1-p) u(\phi, i-1)\big) - c, \\ \mathtt{K} : (1-q)\beta u(\phi, i) + q\beta u(\phi, i-1), \end{cases} \quad 1 \leq i < n, \tag{4.3}$$

$$u(\phi, n) = \max \begin{cases} \mathtt{C} : (1-q)\beta u(\phi, n) + q\beta\big(p \int_0^\infty u(\xi, n-1) dF(\xi) + (1-p) u(\phi, n-1)\big) - c, \\ \mathtt{K} : (1-q)\beta u(\phi, n) + q\beta u(\phi, n-1), \end{cases} \tag{4.4}$$

$$u(w, i) = \max \begin{cases} \mathtt{A} : w + u(\phi, i+1), \\ \mathtt{R} : u(\phi, i), \end{cases} \geq w, \quad 0 \leq i < n. \tag{4.5}$$

Eq. (4.5) can be rearranged into

$$u(w, i) = \max\{w - h_i, 0\} + u(\phi, i), \quad 0 \leq i < n. \qquad \square \tag{4.6}$$

In Eq. (4.4) it should be noted that if an order is not be accepted, hence the current state $u(\phi, n)$ is not changed where the number of orders in the system reaches to the capacity n. Now, for convenience let us define

$$v(i) = \int_0^\infty u(w, i) dF(w), \quad 0 \leq i < n. \tag{4.7}$$

Here, from Eq. (4.5) we obtain

$$v(i) \geq m, \quad 0 \leq i < n. \tag{4.8}$$

Then, according to the definition Eq. (3.1), Eqs. (4.2) to (4.5) can be rewritten as follows (See Lemma A.2 for the unique existence of the solution of the below equations).

$$u(\phi,0) = \max\{p\beta v(0)+(1-p)\beta u(\phi,0)-c, \beta u(\phi,0)\}+s, \tag{4.9}$$

$$u(\phi,i) = \max\begin{cases}(1-q)\beta u(\phi,i)+q\beta\big(pv(i-1)+(1-p)u(\phi,i-1)\big)-c,\\ (1-q)\beta u(\phi,i)+q\beta u(\phi,i-1)\end{cases} \quad 1\le i<n, \tag{4.10}$$

$$u(\phi,n) = \max\begin{cases}(1-q)\beta u(\phi,n)+q\beta\big(pv(n-1)+(1-p)u(\phi,n-1)\big)-c,\\ \mathrm{K}:(1-q)\beta u(\phi,n)+q\beta u(\phi,n-1),\end{cases} \tag{4.11}$$

$$v(i) = T(h_i)+u(\phi,i), \quad 0\le i<n. \tag{4.12}$$

In the same procedure as in [2] we can finally get the following equations;

$$u(\phi,0) = \beta u(\phi,0)+\max\{p\beta T(h_0)-c,0\}+s \tag{4.13}$$

$$u(\phi,i) = \lambda q\beta u(\phi,i-1)+\lambda\max\{p(1-q)\beta T(h_i)+pq\beta T(h_{i-1})-c,0\}, \quad 1\le i<n, \tag{4.14}$$

$$u(\phi,n) = \lambda q\beta u(\phi,n-1)+\lambda\max\{pq\beta T(h_{n-1})-c,0\}. \tag{4.15}$$

5 Optimal Decision Rule

Theorem 5.1

(a) *Let* $\alpha\le 0$. *Then* $\mathrm{K}_{0\le i\le n}$.

(b) *Let* $\alpha>0$.

1. *If* $s=0$, *then* $\mathrm{C}_{1\le i<n}$ *and* h_i *is nondecreasing in* i *with* $h_i<b$ *for all* i.
2. *If* $s>0$, *then* h_i *is not always nondecreasing in* i.

Proof. (a,b1) See Appendix B. (b2) See the following Section 6). ∎

Theorem 5.1 tells us that

(a) If $\alpha\le 0$, i.e., $p\beta\mu\le c$, then skipping the search is always optimal, implying that not doing any business activities, in other words, not launching business is optimal. This can be intuitively understood from that the expected profit $p\beta\mu$ obtained by conducting the search does not compensate the search cost c paid to get an order.

(b) Let $\alpha>0$, implying that the expected profit $p\beta\mu$ obtained by conducting the search does compensate the search cost c paid to get an order. Accordingly, in this case, it can be expected that conducting the search yields positive net profit. If the idling profit $s=0$, then the expectation above comes true; however, when $s>0$, we have not yet gotten any theoretical results about whether the expectation comes true or not due to difficulty of its numerical analysis. So that, we examine this using some numerical experiments, involving the examinations of the optimal customer selection rule.

6 Numerical Experiment

Here, we demonstrate results of numerical experiments in which $F(w)$ is assumed to be the uniform distribution on $[0,1]$. Then, we have

$$T(x) = \begin{cases} \mu - x & \text{for} \quad x \le 0, \\ 0.5(1-x)^2 & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad x \ge 1. \end{cases}$$

Further, let $p = 0.85$, $q = 0.45$, $\beta = 0.95$, and $c = 0.05$.

(a) Case of $s = 0$.

In this case, let us have an eye to the fact that it can be proven in Theorem 5.1(b1) that the selection criterion h_i increases in the amount of backorder i as seen in the curve with $s = 0.00$ Figure 6.1. This implies the following: Suppose the amount of backorder is small. Then, if orders are not accepted, the risk of production process becoming idle will become large. In the case, an opportunity loss occurs for the reason that if there were the more amount of backorder, the loss from the realization of the risk could be avoided; we shall call the loss the first kind of opportunity loss. Accordingly, in order to avoid the loss, the system must accept any order however low its value may be, implying that the selection criterion must be set to be low when the amount of backorder is small.

On the contrary, let the amount of backorder be large. Then, since the delivery term of each arriving order becomes long on the average, the possibility of the order being rejected by its orderer will become large. This leads to the second kind of opportunity loss that if the amount of backorder were small, the arriving orders could have been accepted and profit could have been gained. This implies that when the amount of backorder is large, the selection criterion must be set to be high in order to increase the number of the orders accepted so as to avoid the opportunity loss. The above consideration means that the selection criterion should be set increasing function of the amount of backorder.

The monotonicity of the selection criterion brings about the following dynamic behavior for the movement of the amount of backorder. When the amount of backorder is small, since the selection criterion is low, the number of orders accepted becomes large; accordingly, the amount of backorder increases, hence it follows that the amount of backorder also increases. Since the selecting criterion becomes high as the amount of backorder increases, the number of orders accepted becomes small; therefore, the amount of backorder becomes small, hence it follows that the amount of backorder decreases. The above can be restated as follows; the smaller the amount of backorder may become, the stronger the force of making itself large may be. On the contrary, the larger the amount of backorder may become, the stronger the force of making itself small may be. Such movement of the amount of backorder looks just like the oscillation of pendulum. The pendulum always moves toward the vertical place, the most stable position for it.

The above consideration leads us to that the amount of backorder fluctuates with always being pulled toward an equilibrium point in a stochastic sense. The stabilization of the amount of backorder is also what management desire.

(b) Case of $s > 0$.

1. As seen in the curves of Figure 6.1, if $0 \le s \le 0.04$, then h_i is nondecreasing, and if $s > 0.04$, then it is convex in i. From the numerical results, we can get the following conjectures; there exist s^* and $i^*(s)$ such that if $s \le s^*$, then h_i is increasing in i, or else h_i is not increasing in i and that h_i is decreasing in $i \le i^*(s)$ and h_i is increasing in $i \ge i^*(s)$. Let us have an eye to the fact that the selection criterion h_i is not always in creasing in i the amount of backorders. If the amount

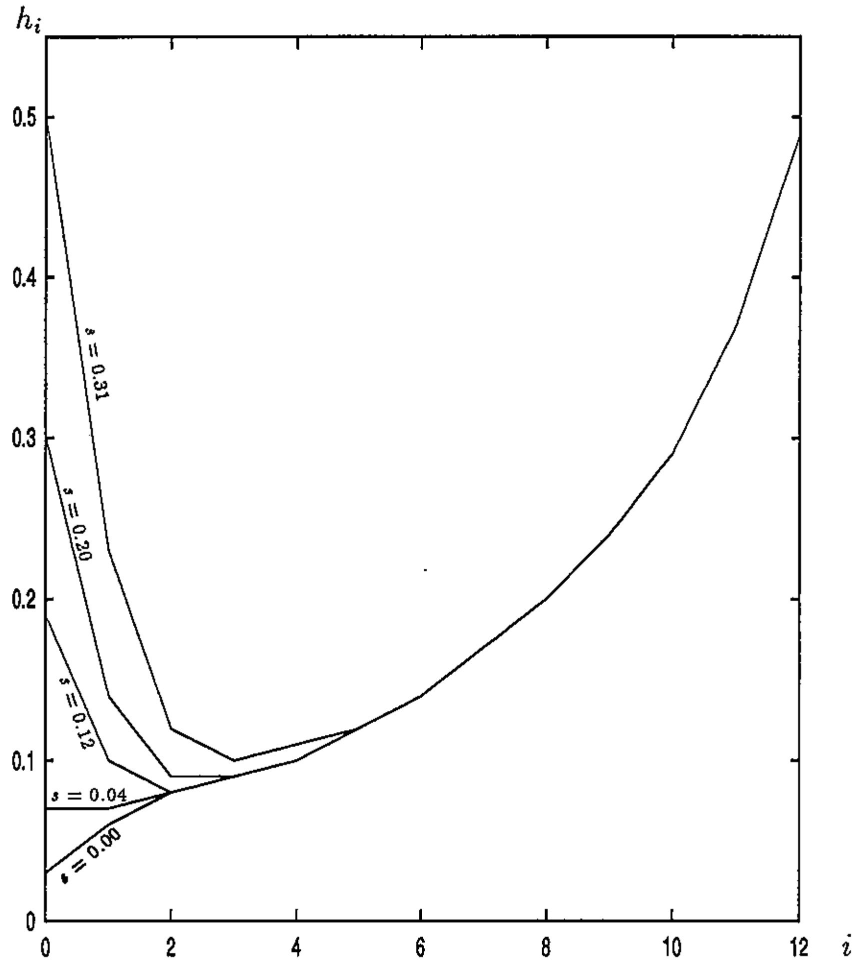

Figure 6.1: Graph of h_i.

of backorders are sufficiently small, then since the probability of production process being idle is large, the system enjoys the idling profit from rejecting unprofitable orders and making the process idle by setting the selection criterion high. Further, as the amount of backorder increases until $i = i^*(s)$, since the influence of idling profit on the selection criterion get weaker, the selection criterion comes back to the same as the case of $s = 0$. This implies that by setting the selection criterion low the system tries to avoid the first opportunity loss. When the amount of backorders are large, it is the same as the consideration of case of $s = 0$.

2. That the h_i takes such a shape stated above tells us the following. For an order value w there exists such $i(w) < i'(w)$ that if $i \leq i'(w)$, rejecting the order is optimal, if $i(w) < i \leq i'(w)$, accepting it is optimal, and if $i'(w) < i$, again rejecting it is optimal; that is, it follows that there exists double critical values in terms of i.

7 Suggested Future Study

The results of the above numerical experiment eventually lead us to suggest study of finding necessary and sufficient conditions on which there exist s^* and $i(w)^*$ as stated above.

Appendix – Proofs –

A Some Lemmas

Lemma A.1 *Properties of T-function*

(a) $T(x)$ *is nonincreasing on* $(-\infty,\infty)$*, strictly decreasing on* $(-\infty,b)$*, and convex on* $(-\infty,\infty)$.
(b) $T(x)\ge 0$ *on* $(-\infty,\infty)$.
(c) $T(x)\le m$ *on* $[0,\infty]$ *with* $T(0)=m$*,* $T(x)>0$ *on* $(-\infty,b)$*, and* $T(x)=0$ *on* $[b,\infty)$.
(d) $x+\nu T(x)$ *is nondecreasing in* x *if* $\nu\le 1$ *and strictly increasing in* x *if* $\nu<1$.

Proof. See [2]. ∎

Lemma A.2 *The system of equations* Eqs. (4.9) *to* (4.11) *has a unique solution.*

Proof. For any given vector $\boldsymbol{x}=(x_0,x_1,\cdots,x_n)'$ let us define the norm $\|\boldsymbol{x}\|=\max\{|x_0|,|x_1|,\cdots,|x_n|\}$. Further, by $D_i u$ let us denote the right hand sides of Eqs. (4.9) to (4.11), and let $\boldsymbol{D}u=(D_0u,D_1u,\cdots,D_nu)'$ and $\boldsymbol{u}=(u(\phi,0),u(\phi,1),\cdots,u(\phi,n))'$. Noting $\|\boldsymbol{x}\|\ge|x_i|$ for all i, from Eq. (4.5) we have

$$\begin{aligned}|v(i)-\hat{v}(i)| &= \left|\int_0^\infty \max\{w+u(\phi,i),u(\phi,i-1)\}dF(w)-\int \max\{w+\hat{u}(\phi,i),\hat{u}(\phi,i-1)\}dF(w)\right| \\ &\le \int_0^\infty \max\{|u(\phi,i)-\hat{u}(\phi,i)|,|u(\phi,i-1)-\hat{u}(\phi,i-1)|\}dF(w) \\ &\le \|\boldsymbol{u}-\hat{\boldsymbol{u}}\|.\end{aligned}$$

Accordingly, we obtain

$$\begin{aligned}|D_0u-D_0\hat{u}| &\le \max\left\{\begin{array}{l} p\beta|v(0)-\hat{v}(0)|+(1-p)\beta|u(\phi,0)-\hat{u}(\phi,0)|, \\ \beta|u(\phi,0)-\hat{u}(\phi,0)| \end{array}\right\} \\ &\le \max\{\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|,\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|\}=\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|.\end{aligned}$$

Similarly, we get $|D_iu-D_i\hat{u}|\le\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|$ for $0\le i\le n$. Thus, by definition we have $\|\boldsymbol{D}u-\boldsymbol{D}\hat{u}\|\le\beta\|\boldsymbol{u}-\hat{\boldsymbol{u}}\|$,implying that $\boldsymbol{D}u$ is a contraction mapping. Hence, the assertion holds. ∎

Lemma A.3

(a) *If* $\alpha>0$*, then* $u(\phi,i)>0$ *for all* i.
(b) *If* $s=0$*, then* C_0 *and* $h_0<b$.

Proof. (a) First, note $u(\phi,i)\ge 0$ for all i. Hence, from Eqs. (4.9) and (4.8) we have $u(\phi,0)\ge\beta(pv(0)+(1-p)u(\phi,0))-c+s\ge p\beta m-c=\alpha+s>0$. Suppose $u(\phi,i-1)>0$. Then, from Eq. (4.10) we can get $u(\phi,i)\ge(1-q)\beta u(\phi,i)+q\beta u(\phi,i-1)\ge q\beta u(\phi,i-1)>0$.

(b) Let $s=0$ and assume $p\beta v(0)+(1-p)\beta u(\phi,0)-c\le\beta u(\phi,0)$. Then, since $u(\phi,0)=\beta u(\phi,0)$ from Eq. (4.9), we have $\beta=1$ due to $u(\phi,0)>0$, which contradicts the assumption of $\beta<1$. Accordingly, it must be

$$u(\phi,0)=p\beta v(0)+(1-p)\beta u(\phi,0)-c=p\beta(v(0)-u(\phi,0))+\beta u(\phi,0)-c, \tag{A.1}$$

implying C_0. Rearranging this by using Eq. (4.12) with $i=0$ becomes

$$u(\phi,0) = (p\beta T(h_0)-c)/(1-\beta)>0. \tag{A.2}$$

Hence, $p\beta T(h_0)-c>0$, equivalently

$$T(h_0) > c/p\beta, \tag{A.3}$$

from which we have $h_0 < b$. ∎

Lemma A.4

(a) $u(\phi, i-1) \geq u(\phi, i)$ *for* $1 \leq i \leq n$.
(b) $h_i \geq 0$ *for all* i.

Proof.

(a) For convenience, let us define the following recurrent relation from Eqs. (4.5) and (4.9) to (4.11).

$$\begin{aligned}
u_t(\phi,0) &= \max\left\{\begin{array}{l} \beta(pv_{t-1}(0)+(1-p)u_{t-1}(\phi,0))-c, \\ \beta u_{t-1}(\phi,0) \end{array}\right\}, \\
u_t(\phi,i) &= \max\left\{\begin{array}{l} (1-q)\beta(pv_{t-1}(i)+(1-p)u_{t-1}(\phi,i)) \\ \qquad +q\beta(pv_{t-1}(i-1)+(1-p)u_{t-1}(\phi,i-1))-c, \\ (1-q)\beta u_{t-1}(\phi,i)+q\beta u_{t-1}(\phi,i-1) \end{array}\right\}, \ 1 \leq i < n, \\
u_t(\phi,n) &= \max\left\{\begin{array}{l} (1-q)\beta u_{t-1}(\phi,n) \\ \qquad +q\beta(pv_{t-1}(n-1)+(1-p)u_{t-1}(\phi,n-1))-c, \\ (1-q)\beta u_{t-1}(\phi,n)+q\beta u_{t-1}(\phi,n-1) \end{array}\right\}, \\
u_t(w,i) &= \max\{w+u_t(\phi,i+1), u_t(\phi,i)\}, \quad 0 \leq i < n.
\end{aligned}$$

The assertion is immediately proven by induction starting with $u_0(\phi, i-1) \geq u_0(\phi, i)$ for $1 \leq i \leq n$.

(b) Clear from Eq. (4.1) and (a). ∎

B Proof of Theorem 5.1

B.1 (a)

Assume $\alpha = p\beta\mu - c \leq 0$. Then, since $T(h_i) \leq \mu$ for all i from Lemmas A.4(b) and Lemma A.1(c), we have $0 \geq p\beta\mu - c \geq p\beta T(h_0) - c \geq pq\beta T(h_n) - c$, implying $\mathtt{K}_0$ and $\mathtt{K}_n$. Further,

$$\begin{aligned}
0 &\geq p(1-q)\beta(T(h_i)-\mu) + pq\beta(T(h_{i-1})-\mu) \\
&= p(1-q)\beta T(h_i) + pq\beta T(h_{i-1}) - p\beta c \geq p(1-q)\beta T(h_i) + pq\beta T(h_{i-1}) - c.
\end{aligned}$$ ∎

B.2 (b1)

To begin with, for convenience, let

$$Q_i = p(1-q)\beta T(h_i) + pq\beta T(h_{i-1}) - c, \quad 1 \leq i < n. \tag{B.1}$$

Then, Eq. (4.14) can be rewritten

$$u(\phi, i) = \lambda q\beta u(\phi, i-1) + \lambda \max\{Q_i, 0\}, \quad 1 \leq i < n. \tag{B.2}$$

• Proof of C_1: Assume $Q_1 \le 0$, i.e., $p(1-q)\beta T(h_1) + pq\beta T(h_0) - c \le 0$, hence $(1-q)T(h_1) + qT(h_0) - c \le c/p\beta$. Then, $u(\phi,1) = \lambda q\beta u(\phi,0)$ and $u(\phi,2) = \lambda^2 q^2 \beta^2 u(\phi,0) + \max\{Q_2, 0\}$ from Eq. (B.2) with $i = 1,2$. Accordingly, we get

$$\begin{aligned} h_1 - h_0 &= 2u(\phi,1) - u(\phi,0) - u(\phi,2) \\ &= 2\lambda q\beta u(\phi,0) - u(\phi,0) - \lambda^2 q^2 \beta^2 u(\phi,0) - \lambda \max\{Q_2, 0\} \\ &= -(1-\lambda q\beta)^2 u(\phi,0) - \lambda \max\{Q_2,0\} < 0 \end{aligned} \tag{B.3}$$

due to $1 > \lambda q\beta$ from Eq. (2.4) and $u(\phi,0) > 0$ from Lemma A.3(a). Thus, $h_1 < h_0$. Further, from the convexity of $T(h)$, the assumption, and Eq. (A.3) we get

$$T((1-q)h_1 + qh_0) \le (1-q)T(h_1) + qT(h_0) \le c/p\beta < T(h_0),$$

from which $(1-q)h_1 + qh_0 > h_0$. Hence, $h_1 > h_0$, which is a contradiction. Thus, $Q_1 > 0$, i.e., C_1.

• Proof of $h_0 \le h_1$: Accordingly, we obtain $u(\phi,0) \ge u(\phi,1) = \lambda q\beta u(\phi,0) + \lambda Q_1$, from which we get $(1-\lambda q\beta)u(\phi,0) = \lambda(1-\beta)u(\phi,0) \ge \lambda Q_1$, hence $(1-\beta)u(\phi,0) \ge Q_1$. Rearranging this inequality by substituting Eq. (A.2) into yields $p\beta T(h_0) - p(1-q)\beta T(h_1) - pq\beta T(h_0) \ge 0$, hence $T(h_0) \ge T(h_1)$. Now, since $h_0 < b$ from Lemma 5(b), if $h_1 < b$, then $h_0 \le h_1$, and if $h_1 \ge b$, then clearly $h_0 < h_1$. Accordingly, always $h_0 \le h_1$.

• Proof of C_2: Assume $Q_2 \le 0$. Then, we get

$$u(\phi,1) = \lambda q\beta u(\phi,0) + \lambda Q_1, \tag{B.4}$$

$$u(\phi,2) = \lambda q\beta u(\phi,1), \tag{B.5}$$

From which we obtain

$$h_1 = \lambda q\beta\big(u(\phi,0) - u(\phi,1)\big) + \lambda Q_1 = \lambda q\beta h_0 + \lambda pq\beta T(h_0) + \lambda p(1-q)\beta T(h_1) - \lambda c. \tag{B.6}$$

Assume $h_1 \ge b$, hence $T(h_1) = 0$. Then, from Eq. (B.6), Lemma A.1(d) we get

$$h_1 = \lambda q\beta(h_0 + pT(h_0)) - \lambda c < \lambda q\beta(b + pT(b)) - \lambda c = \lambda q\beta b - \lambda c \le \lambda q\beta b < b,$$

which is a contradiction. Hence, it must be $h_1 < b$. In the same way as in Eq. (B.3), we have $h_2 < h_1 < b$. From this and the assumption we have

$$0 \ge Q_2 > p(1-q)\beta T(h_1) + pq\beta T(h_1) - c > p\beta T(h_1) - c. \tag{B.7}$$

Since $h_0 \le h_1$, from lemma B.1 and Eq. (4.15) we obtain

$$h_1 \le \lambda q\beta(h_1 + pT(h_1)) + \lambda p(1-q)\beta T(h_1) - \lambda c = \lambda q\beta h_1 + \lambda p\beta T(h_1) - \lambda c,$$

hence,

$$0 \le (1-\lambda q\beta)h_1 \le \lambda p\beta T(h_1) - \lambda c. \tag{B.8}$$

Accordingly, we get $p\beta T(h_1) - c \geq 0$, which contradicts Eq. (B.7). Hence, $Q_2 > 0$, i.e., $\mathtt{C}_2$.

• Proof of $h_1 < b$: From the results obtained for $i = 1, 2$, we have

$$u(\phi, 1) = \lambda q\beta u(\phi, 0) + \lambda Q_1, \tag{B.9}$$

$$u(\phi, 2) = \lambda q\beta u(\phi, 1) + \lambda Q_2. \tag{B.10}$$

Assume $h_1 \geq b$, hence $T(h_1) = 0$. Then, Eqs. (B.9) and (B.10) can be rewritten as follows.

$$\begin{aligned} u(\phi, 1) &= \lambda q\beta u(\phi, 0) + \lambda pq\beta T(h_0) - \lambda c, \\ u(\phi, 2) &= \lambda q\beta u(\phi, 1) + \lambda p(1-p)T(h_2) - \lambda c, \end{aligned}$$

from which we get

$$h_1 = \lambda q\beta(h_0 + pT(h_0)) - \lambda p(1-q)\beta T(h_2). \tag{B.11}$$

Since $h_0 < b$ due to Lemma A.3(b), from Lemma A.1(d) we have $h_1 < \lambda q\beta b - \lambda p(1-p)\beta T(h_2) \leq \lambda q\beta b < b$, which is a contradiction. Thus, it must be $h_1 < b$.

• Proof of $h_1 \leq h_2$: From Eqs. (B.9) and (B.10) we also get

$$\begin{aligned} h_1 &= \lambda q\beta h_0 + \lambda(Q_1 - Q_2) = \lambda q\beta(h_0 + pT(h_0)) + \lambda p(1-2q)\beta T(h_1) - \lambda p(1-q)\beta T(h_2) \\ &= \lambda q\beta(h_0 + pT(h_0)) + \lambda p(1-2q)\beta T(h_1) - \lambda p(1-q)\beta T(h_2). \end{aligned}$$

Since $h_0 \leq h_1$, from Lemma A.1(d we obtain

$$h_1 \leq \lambda q\beta(h_1 + pT(h_1)) + \lambda p(1-2q)\beta T(h_1) - \lambda p(1-q)\beta T(h_2),$$

hence,

$$0 \leq (1 - \lambda q\beta)h_1 \leq \lambda p(1-q)\beta(T(h_1) - T(h_2)).$$

Accordingly, $T(h_2) \leq T(h_1)$. Since $h_1 < b$, if $h_2 < b$, then $h_1 \leq h_2$, and if $h_2 \geq h$ then clearly $h_1 < h_2$. Hence, always $h_1 \leq h_2$.

• Proof of $\mathtt{C}_{3 \leq i < n}$: The procedures of the above proof is:

$$h_0 < b \to \mathtt{C}_1 \to h_0 \leq h_1 \to \mathtt{C}_2 \to h_1 \leq h_2 \to h_1 < b.$$

In the quite same way as in this procedures we can successively prove

$$\mathtt{C}_3 \to h_2 \leq h_3 \to h_2 < b \to \cdots \to \mathtt{C}_{n-1} \to h_{n-2} \leq h_{n-1}.$$

From all the results above we have

$$u(\phi, n-1) = \lambda q\beta u(\phi, n-2) + \lambda(p(1-q)\beta T(h_{n-1}) + pq\beta T(h_{n-2}) - c),$$
$$u(\phi, n) = \lambda q\beta u(\phi, n-1) + \lambda \max\{pq\beta T(h_{n-1}) - c, 0\}.$$

Assume $h_{n-1} \geq b$, hence $T(h_{n-1}) = 0$. Then, we get

$$u(\phi, n-1) = \lambda q\beta u(\phi, n-2) + \lambda(pq\beta T(h_{n-2}) - c),$$
$$u(\phi, n) = \lambda q\beta u(\phi, n-1),$$

form which we obtain

$$h_{n-1} = \lambda q\beta(h_{n-2} + pT(h_{n-2})) - \lambda c.$$

Since $h_{n-2} < b$, from Lemma A.1(d) we obtain $b \leq h_{n-1} < \lambda q\beta(b + pT(b)) - \lambda c = \lambda q\beta b - \lambda c \leq \lambda q\beta b < b$, which is a contradiction. Accordingly, it must be $h_{n-1} < b$. Thus, the assertion is true. ∎

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References

[1] Ikuta, S.: K-function and optimal stopping problems, *Discussion paper*, No.705, University of Tsukuba, Institute of Policy and Planning Sciences, (1996).

[2] Jaedong, S.: Customers selection problem where only one customer is allowed to be held, *Discussion paper*, No.1032, University of Tsukuba, Institute of Policy and Planning Sciences, (2003).