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Customers Selection Problem
Where Only One Customer Is Allowed to Be Held

by

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CUSTOMERS SELECTION PROBLEM WHERE ONLY ONE CUSTOMER IS ALLOWED TO BE HELD

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Abstract

The paper deals with a customer selection problem where it is assumed that a customer with a value appears at each point in time with a known probability and a customer in service at any point in time goes out of the system at the next point in time with a known probability. In the problem, the number of orders that can be held at any instant should be assumed to be $n \geq 1$; however, in the paper we exhaustively examine the case of $n = 1$ for the reason that the analysis of the case includes a distinctive aspects which can not be seen in the case of $n \geq 2$. Further, we consider the two cases, customer-first and system-first cases, as to which of a customer and system first declares the price of order. Prices offered by subsequently appearing customers in the customer-first case and the maximum permissible ordering prices of subsequently appearing customers in the system-first case are assumed to be random variables. Properties of optimal decision rule maximizing the total expected present discounted net profit gained over infinite planning horizon are examined and clarified.

1 Introduction

This paper deals with the problem of selecting profitable orders to accept out of sequentially arriving ones in a custom production company, such as a shipbuilding company, an advertising agency, a consulting company, a design office, a construction firm, and so on.

Since any order to a custom production company has an appointed date of delivery, if the company has too many backorders at that time, however profitable the order may be, it might be possible to be unable to accept the order for the reason that it can not be completed up to the appointed date even if it were accepted. In this case, if an adequate allowance were kept in production lines by having rejected less profitable orders in advance, the company could enjoy accepting the profitable order. In other words, the profit from the order must be regarded as a kind of opportunity loss in a sense that the profit gained if it could be accepted could be lost due to the little allowance in production lines caused by having not controlled the acceptance of orders in advance. This implies that keeping an appropriate level of allowance in production lines by rightly selecting orders to accept in advance might yield larger long run profit. The above consideration eventually leads us to the necessity of formulating rational orders selection policy that makes the right selection of order to accept possible. The purpose of the paper is to propose a model for the problem and examine the properties of the optimal orders selection policy so as to maximize the expected long-run net profit. A very few investigations have been made so far [1] [6] for the problem.

Section 2 defines a model of the problem stated above and Section 3 that follows defines some functions and examines their properties, which will be used in the analyses in the subsequent sections. In Section 4 the optimal equation of the model is derived and in Section 5 the properties of the optimal decision rule are clarified, based on which the optimal decision rule is prescribed in Section 6. Section 7 summarizes the conclusions obtained in the previous sections and Section 8 suggests some subjects of study to be tackled in the future.

2 Model

The model examined in the paper is defined by the general framework below.

1. The model is defined as a discrete-time stochastic decision process with an infinite planning horizon. Let points in time be equally spaced on the axis of the planning horizon, and let the time interval between successive points in time be called the period.
2. In general, by n let us denote the maximum permissible number of orders that can be held in the system at any instance. For the reason that the analysis in the case of $n = 1$ includes a peculiar mathematical treatment which can not be seen in the case of $n \geq 2$, in the paper we exhaustively consider the case of $n = 1$; further reason for excluding the latter case will be stated in Section 8. The case of $n = 1$ reflects a situation that a customer requires for convenience of the customer the start of processing the order soon after the contract is concluded; such a situation will be occurs in seller's market. Here, note that the assumption of $n = 1$ implies that any customer appearing when there exists an order in the system cannot be accepted however profitable the order may be and that if a customer appears when there exists no order in the system, a negotiation starts between both sides.
3. A customer appears with a known probability $p\,(0 < p \leq 1)$ by conducting a search activity accompanied with some cost $c \geq 0$, called the search cost.
4. The introduction of the search cost inevitably leads us to the necessity of introduction of the decision between conducting and skipping the search at each point in time.
5. If there exists no order in the system for a period, some profit can be yielded from engaging in other economic activities for the period; let the profit $s \geq 0$ be called the idling profit. For example, consider a company producing current transfers where products with standard specification and products with special specification are manufactured; let the latter products be produced as custom production items. In such a company, if all the products with special specification accepted so far have been completed and there exists no products to produce, the production of products with standard specification is started. In this case, the idling profit is yielded by the production of products with standard specification.
6. As to which of a customer and the system (a company) first offers the price of the order, we consider the following two cases:

   i. the *customer-first case* in which a customer first offers the price w, judging from which the system decides whether or not to accept it. Prices offered by subsequently appearing customers, $w, w', \cdots$, are assumed to be independent identically distributed random variables having a known distribution function $F(w)$ with a finite expectation μ.

   ii. the *system-first case* in which the system first offers the price z, judging from which the customer decides whether or not to place the order to the company. Each appearing customer is assumed to have a maximum permissible ordering price w, implying that if $w > z$, the customer is willing to place the order to the system (then, the ordering price is z), or else not. Then, let the maximum permissible ordering prices of subsequently appearing customers, $w, w', \cdots$, be independent identically distributed random variables having a known distribution function $F(w)$ with a finite expectation μ. Then, if the system offers a price z to an appearing customer, the probability of the customer placing the order to the system is given by

$$p(z) = \Pr\{w > z\} = 1 - F(z). \tag{2.1}$$

In any of the customer-first case and the system-first case, let $F(w)$ be either discrete or continuous where its probability (density) function is denoted by $f(w)$ (See Figure 2.1). Here, for certain given numbers a and b with $0 < a < b < \infty$ let us assume that

$$F(w) = 0, \quad w < a, \qquad 0 < F(w) < 1, \quad a \leq w < b, \qquad F(w) = 1, \quad b \leq w, \tag{2.2}$$

where $f(w) > 0$ on $[a, b]$. If $F(w)$ is discrete, let a and b be both integers with $b > a$ and its mass points be defined on integers, i.e., $|w| = 0, 1, \cdots$. Then, clearly $0 < a < \mu < b$, and $F(w)$ is strictly increasing on $[a, b]$, hence $p(z)$ is strictly decreasing on $[a, b]$.

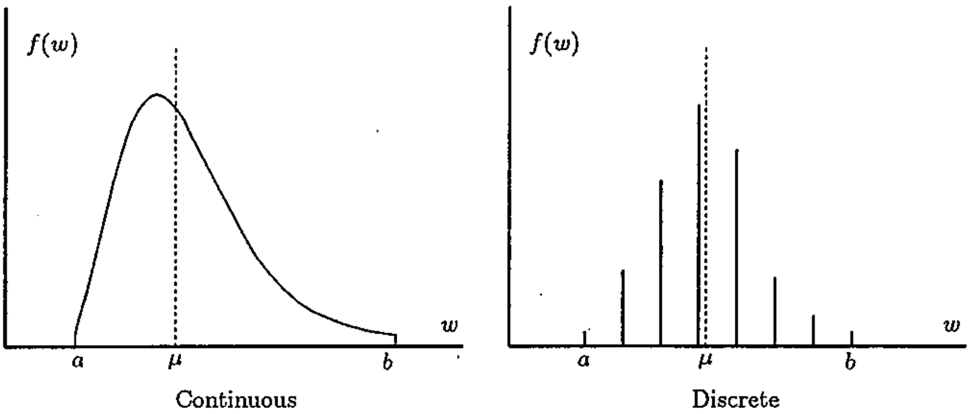

Figure 2.1: Probability (density) function $f(w)$

Now, in the system-first case it can be easily seen that

$$p(z) \begin{cases} = 1, & z < a \quad \cdots(1), \\ < 1, & a \leq z \quad \cdots(2), \\ > 0, & z < b \quad \cdots(3), \\ = 0, & b \leq z \quad \cdots(4). \end{cases} \tag{2.3}$$

7. With the probability $q\,(0 < q < 1)$, called the service completion probability, an order in the system at a certain point in time is completed and goes out of the system up to the next point in time.
8. Let the discount factor be denoted by β; that is, a monetary value of one unit a period after is equivalent to that of β units at the present point in time; let $\beta < 1$ throughout the paper.
9. The objective is to find the optimal decision rule so as to maximize the total expected present discounted net profit gained over an infinite planning horizon, the total expected present discounted value of (1)prices of orders accepted or placed plus (2)the idling profits minus (3)the total expected present discounted value of search costs.

For convenience in the later discussions, let us define

$$\lambda = (1 - \beta(1-q))^{-1} > 1, \tag{2.4}$$

$$\delta = c\lambda(1-q)/q + \lambda s \tag{2.5}$$

where

$$\lambda > 1, \qquad 1 - \lambda q = \lambda(1-q)(1-\beta) > 0, \qquad 1 - \lambda q\beta = \lambda(1-\beta) > 0. \tag{2.6}$$

3 T, A, B, and G-functions

Here, let us define four functions and examine their properties, which will be used to describe the optimal equation of our model and examine the properties of the optimal decision rule. For convenience of analyzing both of the system-first and customer-first cases in the same logic, let us define

$$m = \begin{cases} \mu & \text{for the customer-first case,} \\ \max_z p(z)z & \text{for the system-first case.} \end{cases} \tag{3.1}$$

Here, note that $m > 0$ since $\mu > a > 0$ and $\max_z p(z)z \geq p(a)a > 0$ due to Eq. (2.3 (2)). For expressional simplicity, by the symbols C, K, A and R let us denote the decisions of, respectively: continuing the search, skipping the search, accepting an order, and rejecting an order[1].

3.1 T-Function

For any real number x let us define

$$T(x) = \begin{cases} \int_0^\infty \max\{w - x, 0\} dF(w) & \text{for the customer-first case,} \\ \max_z p(z)(z - x) & \text{for the system-first case,} \end{cases} \tag{3.2}$$

called the *T-function* [3] [7]. In the system-first case, by $z(x)$ let us designate the *smallest* z attaining the maximum of $p(z)(z-x)$ on $z \in (-\infty, \infty)$ for a given x if it exists; i.e.,

$$z(x) = \min\{ z \mid p(z)(z-x) \geq p(z')(z'-x), \quad -\infty < z < \infty\}.$$

Lemma 3.1

(a) $a \leq z(x)$ *for all* x.
(b) *If* $x < b$, *then* $x < z(x) < b$, *and if* $x \geq b$, *then* $z(x) = b$.
(c) $z(x)$ *is nondecreasing in* x.

Proof. See Appendix A. ∎

Lemma 3.2 *In both of the customer-first and system-first cases we have:*

(a) $T(x)$ *is nonincreasing on* $(-\infty, \infty)$, *strictly decreasing on* $(-\infty, b)$, *and convex on* $(-\infty, \infty)$.
(b) $T(x) \geq 0$ *on* $(-\infty, \infty)$.
(c) $T(x) \leq m$ *on* $[0, \infty]$ *with* $T(0) = m$, $T(x) > 0$ *on* $(-\infty, b)$, *and* $T(x) = 0$ *on* $[b, \infty)$.
(d) *For any given* $\gamma > 0$ *the equation* $T(x) = \gamma$ *has a unique solution, less than* b.
(e) $\lim_{x\to\infty} T(x) = 0$ *and* $\lim_{x\to-\infty} T(x) = \infty$.
(f) $x + \nu T(x)$ *is nondecreasing in* x *if* $\nu \leq 1$ *and strictly increasing in* x *if* $\nu < 1$.
(g) $\lim_{x\to\infty} x + \nu T(x) = \infty$, *and if* $\nu < 1$, *then* $\lim_{x\to-\infty} x + \nu T(x) = -\infty$.

Proof. See Appendix B. ∎

3.2 A-Function

For any h let

$$A = p\beta T(h) - c, \qquad \tilde{A} = pq\beta T(h) - c,$$

called the A and $\tilde{A}$*-functions*, respectively. Regarding h as a function of s, let us represent the two functions

$$A(s) = p\beta T(h(s)) - c, \qquad \tilde{A}(s) = pq\beta T(h(s)) - c, \tag{3.3}$$

and by s^* and $\tilde{s}^*$ let us denote the *smallest* solutions of $A(s) = 0$ and $\tilde{A}(s) = 0$, respectively, if they exist, i.e.,

$$s^* = \min\{s \mid A(s) = 0\}, \qquad \tilde{s}^* = \min\{s \mid \tilde{A}(s) = 0\}. \tag{3.4}$$

[1] We do not use S as a symbol representing "skipping the search" because it is often used as a symbol representing "stop the search"

3.3 B-Function

Let us define

$$B_1(x) = T(x) - (x - \lambda s)/\lambda p(1-q)\beta, \tag{3.5}$$

$$B_2(x) = T(x) - (x + \lambda(c - s))/\lambda p\beta, \tag{3.6}$$

called the *B-function*. Then, by x_1^* and x_2^* let us denote the solutions of, respectively, $B_1(x) = 0$ and $B_2(x) = 0$, if they exist, i.e.,

$$B_1(x_1^*) = 0, \qquad B_2(x_2^*) = 0.$$

Regarding $H_1(x)$ and $H_2(x)$ as functions of s, we denote the solutions x_1^* and x_2^* as $x_1^*(s)$ and $x_2^*(s)$.

Lemma 3.3

(a) *$B_1(x)$ and $B_2(x)$ are both strictly decreasing in x where $B_1(x) > (<)\ 0$ and $B_2(x) > (<)\ 0$ for any sufficiently small (large) x.*
(b) *x_1^* and x_2^* are both uniquely exist, which are positive if $p\beta m > c$.*
(c) *$\delta > (= (<))\, x \Longleftrightarrow B_1(x) > (= (<))\, B_2(x)$ where $B_1(\delta) = B_2(\delta)$.*
(d) *$B_1(\delta) > (= (<))\, 0 \Longleftrightarrow x_1^* < (= (>))\, x_2^*$.*
(e) *If $\delta = x_2^*$ or $\delta = x_1^*$, then $\delta = x_1^* = x_2^*$.*
(f) *$x_1^* > x_2^*\, (x_1^* < x_2^*) \Longleftrightarrow x_2^* < x_1^* < \delta\, (x_2^* > x_1^* > \delta)$.*
(g) *If $x_1^* < (>)\, x_2^*$, then $\delta \notin [x_1^*, x_2^*]\, (\delta \notin [x_2^*, x_1^*])$.*
(h) *$x_2^* > \delta \Longleftrightarrow x_1^* > \delta$ and $x_2^* \le \delta \Longleftrightarrow x_1^* \le \delta$.*

Proof. See Appendix C. ∎

3.4 G-Function

For any given x let us define

$$G(x) = \lambda\big(\max\{p\beta T(x) - c, 0\} - \max\{pq\beta T(x) - c, 0\}\big) - x + \lambda s, \tag{3.7}$$

called the *G-function*, and by h^* let us denote the solution of $G(x) = 0$ if it exists, i.e.,

$$G(h^*) = 0.$$

Then, we have

$$G(s) = \lambda\big(\max\{p\beta T(s) - c, 0\} - \max\{pq\beta T(s) - c, 0\}\big) + (\lambda - 1)s \tag{3.8}$$

Note 3.1 *Throughout the paper, when discussing the function $G(x)$ in the relationship with parameter $\eta = (p, q, \beta, c, s)$, we describe it as $G(x|\eta)$.*

Lemma 3.4

(a) *$G(x)$ is strictly decreasing in x.*
(b) *$G(x) < (>)\, 0$ for any sufficiently large (small) x.*
(c) *$G(x|\eta)$ is nonincreasing in c and strictly decreasing in q for all x.*
(d) *$G(x|\eta)$ is nondecreasing in p and strictly increasing in β for all x.*
(e) *$G(x|\eta)$ is strictly increasing in s for all x.*

Proof. See Appendix D. ∎

4 Optimal Equation

Either if the search was skipped at the previous point in time or if no customer appears with probability $1-p$ regardless of having conducted the search at the previous point in time, it follows that no customer appears at the present point in time. For convenience, we shall refer to such a situation as "the system has a *fictitious customer* ϕ".

In any of the customer-first case and the first-first case, by $u(\phi,0)$ $\big(u(\phi,1)\big)$ we shall denote the maximum of the total expected present discounted net profit starting from the state of having the fictitious customer ϕ and no order in the system (the fictitious customer ϕ and an order in the system); let us refer such situation to the state $(\phi,0)$ $\big((\phi,1)\big)$. If no search is made over the entire planning horizon, then no customer appears at all the points in time, then the production line becomes idle for all the period over the planning horizon, implying that the total expected present discounted net profit becomes the total expected present discounted idling profit $s \geq 0$, hence it must follow that $u(\phi,0) \geq 0$ and $u(\phi,1) \geq 0$. Now, for convenience in the later discussions, let us define

$$h = u(\phi,0) - u(\phi,1). \tag{4.1}$$

1. *Customer-first case*: By $u(w,0)$ let us denote the maximum of the total expected present discounted net profit starting with an appearing customer w and no order in the system. Then, we get

$$u(\phi,0) = \max\begin{cases} \text{C}: \beta(p\int_0^\infty u(\xi,0)dF(\xi) + (1-p)u(\phi,0)) - c + s, \\ \text{K}: \beta u(\phi,0) + s, \end{cases} \tag{4.2}$$

$$u(\phi,1) = \max\begin{cases} \text{C}: (1-q)\beta u(\phi,1) + q\beta(p\int_0^\infty u(\xi,0)dF(\xi) + (1-p)u(\phi,0)) - c, \\ \text{K}: (1-q)\beta u(\phi,1) + q\beta u(\phi,0), \end{cases} \tag{4.3}$$

$$u(w,0) = \max\left\{\begin{array}{l} \text{A}: w + u(\phi,1) \\ \text{R}: u(\phi,0) \end{array}\right\} \geq w. \tag{4.4}$$

Eq. (4.4) can be rearranged into

$$u(w,0) = \max\{w-h, 0\} + u(\phi,0). \quad \square \tag{4.5}$$

2. *System-first case*: By $u(1,0)$ let us denote the maximum of the total expected present discounted net profit starting with an appearing customer and no order in the system. Then, we have

$$u(\phi,0) = \max\begin{cases} \text{C}: \beta\big(pu(1,0) + (1-p)u(\phi,0)\big) - c + s, \\ \text{K}: \beta u(\phi,0) + s, \end{cases} \tag{4.6}$$

$$u(\phi,1) = \max\begin{cases} \text{C}: (1-q)\beta u(\phi,1) + q\beta\big(pu(1,0) + (1-p)u(\phi,0)\big) - c, \\ \text{K}: (1-q)\beta u(\phi,1) + q\beta u(\phi,0), \end{cases} \tag{4.7}$$

$$u(1,0) = \max_z\{p(z)\big(z + u(\phi,1)\big) + (1-p(z))u(\phi,0)\} \geq \max_z p(z)z = m. \tag{4.8}$$

Eq. (4.8) can be rearranged into

$$u(1,0) = \max_z p(z)(z-h) + u(\phi,0). \quad \square$$

Now, for convenience let us define

$$v(0) = \begin{cases} \int_0^\infty u(w,0)dF(w) & \text{for the customer-first case,} \\ u(1,0) & \text{for the system-first case.} \end{cases} \tag{4.9}$$

Here, noting Eq. (3.1), from Eqs. (4.4) and (4.8) we obtain

$$v(0) \geq m. \tag{4.10}$$

Then, according to the definition Eq. (3.2), Eqs. (4.2) to (4.4) and Eqs. (4.6) to (4.8) can be rewritten by the identical equations below (See Lemma 5.1 for the unique existence of the solution of the below equations).

$$u(\phi,0) = \max\{p\beta v(0) + (1-p)\beta u(\phi,0) - c, \beta u(\phi,0)\} + s, \tag{4.11}$$

$$u(\phi,1) = \max\left\{ \begin{array}{l} (1-q)\beta u(\phi,1) + q\beta(pv(0) + (1-p)u(\phi,0)) - c \\ (1-q)\beta u(\phi,1) + q\beta u(\phi,0) \end{array} \right\}, \tag{4.12}$$

$$v(0) = T(h) + u(\phi,0). \tag{4.13}$$

Eqs. (4.11) and (4.12) can be rearranged into, respectively,

$$u(\phi,0) = \beta u(\phi,0) + \max\{p\beta(v(0) - u(\phi,0)) - c, 0\} + s, \tag{4.14}$$

$$u(\phi,1) = (1-q)\beta u(\phi,1) + q\beta u(\phi,0) + \max\{pq\beta(v(0) - u(\phi,0)) - c, 0\}, \tag{4.15}$$

Further, noting the definition of λ given by Eq. (2.4), from Eq. (4.15) we obtain

$$u(\phi,1) = \lambda q\beta u(\phi,0) + \lambda \max\{pq\beta(v(0) - u(\phi,0)) - c, 0\}. \tag{4.16}$$

Using Eq. (4.13), we can rearrange Eqs. (4.14) and (4.16) as follows, respectively,

$$u(\phi,0) = (\max\{p\beta T(h) - c, 0\} + s)/(1-\beta), \tag{4.17}$$

$$u(\phi,1) = \lambda q\beta u(\phi,0) + \lambda \max\{pq\beta T(h) - c, 0\}. \tag{4.18}$$

From Eqs. (4.1) and (4.18) we have

$$\begin{aligned} h &= u(\phi,0) - \lambda q\beta u(\phi,0) - \lambda \max\{pq\beta T(h) - c, 0\} \\ &= \lambda(1-\beta)u(\phi,0) - \lambda \max\{pq\beta T(h) - c, 0\}. \end{aligned} \tag{4.19}$$

Rearranging Eq. (4.19) by substituting Eq. (4.17) yields

$$h = \lambda \max\{p\beta T(h) - c, 0\} - \lambda \max\{pq\beta T(h) - c, 0\} + \lambda s. \tag{4.20}$$

Using G-function defined by Eq. (3.7), we can rewrite Eq. (4.20) as follows.

$$G(h) = 0. \tag{4.21}$$

Now, regarding h as a function of s, i.e., $h = h(s)$, then using $A(s)$ and $\tilde{A}(s)$-function defined by Eq. (3.3), we can rewrite Eq. (4.17) and (4.18) as follows.

$$u(\phi,0) = (\max\{A(s), 0\} + s)/(1-\beta), \tag{4.22}$$

$$u(\phi,1) = \lambda q\beta u(\phi,0) + \lambda \max\{\tilde{A}(s), 0\}. \tag{4.23}$$

5 Analyses

Lemma 5.1 *The system of* Eqs. (4.11) *to* (4.13) *has a unique solution.*

Proof. See Appendix E. ∎

Lemma 5.2

(a) $h \geq 0$.
(b) h^* *uniquely exists with* $h^* = h \geq s$.
(c) *If* $s \geq b/\lambda$, *then* $h \geq b$.
(d) *Let* $s < b/\lambda$.
  1 $h < b$.
  2 *If* $G(a) \geq 0$, *then* $a \leq h$, *or else* $h < a$.
(e) h *is nonincreasing in* c *and strictly decreasing in* q.
(f) h *is nondecreasing in* p *and strictly increasing in* β.
(g) $h(s)$ *is strictly increasing in* s *with* $\lim_{s\to\infty} h(s) = \infty$ *and* $\lim_{s\to-\infty} h(s) = -\infty$.

Proof. Below, note that $p\beta T(x) \geq pq\beta T(x)$ for any x due to Lemma 3.2(b).

(a) Evident from $\lambda s \geq 0$ and Eq. (4.20).

(b) The unique existence of h^* is immediate from Lemma 3.4(a,b), hence $h = h^*$ from Eq. (4.21). Since $\lambda > 1$, we have $G(s) \geq 0$ from Eq. (3.8), hence $h^* \geq s$.

(c) Since $G(b) = -b + \lambda s$ from Lemma 3.2(c), if $s \geq b/\lambda$, then $G(b) \geq 0$, implying $h \geq b$.

(d) Let $s < b/\lambda$.

(d1) Quite the same way as (c).

(d2) Evident from Lemma 3.4(a).

(e,f) Immediate from Lemma 3.4(c,d).

(g) The former half is immediate from Lemma 3.4(e). Suppose $h(s)$ converges to a finite $\tilde{h}$ as $s \to \infty$. Then $h(s) < \tilde{h}$ for any s, hence $G(\tilde{h}|s) < G(h(s)|s) = 0$ due to Lemma 3.4(a) and the definition of $h(s)$. Accordingly, $\lim_{s\to\infty} G(\tilde{h}|s) \leq 0$. However, $\lim_{s\to\infty} G(\tilde{h}|s) = \infty$ from Eq. (3.7), which is a contradiction. Thus, $h(s)$ must diverge to ∞ as $s \to \infty$. Similarly also proven for $\lim_{s\to-\infty} h(s) = -\infty$. ∎

Let s_b be the solution of $h(s) = b$, which uniquely exists from Lemma 5.2(g).

Lemma 5.3

(a) $A(s)$ *and* $\tilde{A}(s)$ *are strictly decreasing on* $(-\infty, s_b)$ *with* $A(s) > \tilde{A}(s)$ *on* $(-\infty, s_b)$ *and* $A(s) = \tilde{A}(s) = -c$ *on* $[s_b, \infty)$.
(b) *If* $c > 0$, *then* $A(s) > (<)\, 0$ *and* $\tilde{A}(s) > (<)\, 0$ *for any sufficiently small (large)* s.

Proof. (a) First, for any $s < s' < s_b$ we get $h(s) < h(s') < h(s_b) = b$ due to Lemma 5.2(g), hence $A(s) > A(s')$ and $\tilde{A}(s) > \tilde{A}(s')$ from Lemma 3.2(a) and the definition of $A(s)$ and $\tilde{A}(s)$ in Eq. (3.3). Secondly, for any $s < s_b$ we have $h(s) < h(s_b) = b$ due to Lemma 5.2(g), hence $T(h(s)) > 0$ due to Lemma 3.2(a), leading to $A(s) > \tilde{A}(s)$. Finally, for $s_b \leq s$ we get $b = h(s_b) \leq h(s)$, hence $A(s) = \tilde{A}(s) = -c$ due to Lemma 3.2(c).

(b) Let $c > 0$. Note that $\lim_{s\to-\infty} A(s) = \lim_{s\to-\infty} \tilde{A}(s) = \infty$ from Lemmas 3.2(e) and 5.2(g). Further, for $s_b \leq s$ we have $b = h(s_b) \leq h(s)$, hence $A(s) = \tilde{A}(s) = -c < 0$ due to Lemma 3.2(c). ∎

From Lemma 5.3 we can depict Figure 5.2.

Lemma 5.4

(a) *Let* $c = 0$. *Then* $A(s) \geq 0$ *and* $\tilde{A}(s) \geq 0$.
(b) *Let* $c > 0$.

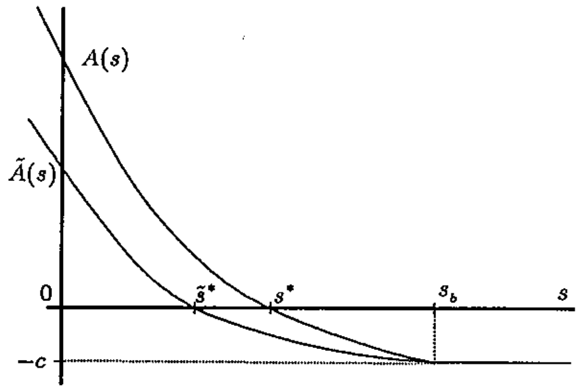

Figure 5.2: Graphs of $A(s)$ and $\tilde{A}(s)$

1 *s^* and $\tilde{s}^*$ uniquely exist with $\tilde{s}^* < s^* < s_b$.*
2 *If $s^* \le s$, then $A(s) \le 0$ and $\tilde{A}(s) \le 0$.*
3 *If $s < s^*$, then $A(s) > 0$.*
4 *If $\tilde{s}^* \le s$, then $\tilde{A}(s) \le 0$.*
5 *If $s < \tilde{s}^*$, then $A(s) > 0$ and $\tilde{A}(s) > 0$.*

Proof. (a) If $c = 0$, then $A(s) = p\beta T(h(s)) \ge 0$ and $\tilde{A}(s) = pq\beta T(h(s)) \ge 0$ due to Lemma 3.2(b).

(b) Let $c > 0$.

(b1) The unique existence of s^* and $\tilde{s}^*$ are immediate from Lemma 5.3 where $A(s) > \tilde{A}(s) = -c < 0$ (see Figure 5.2). The latter half is evident from $A(s) > \tilde{A}(s)$ on $(-\infty, s_b)$.

(b2-b5) Evident from Lemma 5.3(a). ∎

For the convenience in the later discussions, let us regard s^* and $\tilde{s}^*$ as functions of c, i.e., $s^*(c)$ and $\tilde{s}^*(c)$. Then, by c^* and $\tilde{c}^*$ let us denote the solution of $s^*(c) = 0$ and $\tilde{s}^*(c) = 0$, respectively, if they exist, i.e.,

$$s^*(c^*) = 0, \qquad \tilde{s}^*(\tilde{c}^*) = 0.$$

Lemma 5.5

(a) *$s^*(c)$ and $\tilde{s}^*(c)$ are strictly decreasing in c.*
(b) *$s^*(0) = \tilde{s}^*(0) = b/\lambda > 0$.*
(c) *$c^* = p\beta m$ and $\tilde{c}^* = qx_1^*(0)/\lambda(1-q)$.*
(d) *$\tilde{c}^* \le pq\beta m < c^*$.*

Proof. (a) From Eq. (3.3) we obtain $0 = A(s^*(c)) = p\beta T(h(s^*(c))) - c$, i.e., $c = p\beta T(h(s^*(c)))$. Now, for any $c < c'$ we have $0 \le p\beta T(h(s^*(c))) = c < c' = p\beta T(h(s^*(c')))$, from which we get $h(s^*(c)) > h(s^*(c'))$ due to Lemma 3.2(a). Hence $s^*(c) > s^*(c')$ from Lemma 5.2(g). Similarly also proven for $\tilde{s}^*(c)$.

(b) Let $c = 0$. Then, from Eq. (3.3) we have

$$0 = A(s^*(0)) = \tilde{A}(\tilde{s}^*(0)) = p\beta T(h(s^*(0))) = pq\beta T(h(\tilde{s}^*(0))),$$

hence $h(s^*(0)) \ge b$ and $h(\tilde{s}^*(0)) \ge b$ due to Lemma 3.2(c). From Eq. (3.7) we get $0 = G(h(s^*(0))) = -h(s^*(0)) + \lambda s^*(0)$. Hence, $\lambda s^*(0) = h(s^*(0)) \ge b$, from which $s^*(0) \ge b/\lambda > 0$. Now, since $T(h(b/\lambda)) \ge 0$ due to Lemma 3.2(b), from Eq. (3.7) we have $0 = G(h(b/\lambda)) = \lambda p(1-q)\beta T(h(b/\lambda)) - h(b/\lambda) + b$, from which $h(b/\lambda) - b = \lambda p(1-q)\beta T(h(b/\lambda)) \ge 0$ due to Lemma 3.2(b), hence $h(b/\lambda) \ge b$. Accordingly,

$A(b/\lambda) = p\beta T(h(b/\lambda)) = 0$ due to Lemma 3.2(c). Noting that s^* is the *smallest* solution of $A(s) = 0$, we get $s^*(0) = b/\lambda$. The proof of $\tilde{s}^*(0) = b/\lambda$ is the same as the above.

(c) First, from Eqs. (3.3) and (3.4) we have

$$c = p\beta T(h(s^*(c))) \geq pq\beta T(h(s^*(c))), \tag{5.1}$$

$$c = pq\beta T(h(\tilde{s}^*(c))). \tag{5.2}$$

Since $h(s^*(c))$ is the solution of Eq. (3.7) due to Lemma 5.2(b), we get

$$\begin{aligned} 0 = G(h(s^*(c))) &= \lambda(\max\{p\beta T(h(s^*(c)) - c, 0\} - \max\{pq\beta T(h(s^*(c))) - c, 0\}) - h(s^*(c)) + \lambda s^*(c) \\ &= -h(s^*(c)) + \lambda s^*(c), \end{aligned}$$

hence $h(s^*(c)) = \lambda s^*(c)$ for any c. Accordingly, $h(s^*(c^*)) = \lambda s^*(c^*) = 0$ due to $s^*(c^*) = 0$, hence $T(h(s^*(c^*))) = m$ from Lemma 3.2(c). Therefore, from Eq. (5.1) we have $c^* = p\beta m$.

Next, rearranging Eq. (3.7) with $x = h(\tilde{s}^*(c))$ by substituting (5.2) yields

$$\begin{aligned} 0 = G(h(\tilde{s}^*(c)) &= \lambda(\max\{p\beta T(h(\tilde{s}^*(c)) - c, 0\} - \max\{pq\beta T(h(\tilde{s}^*(c))) - c, 0\}) - h(\tilde{s}^*(c)) + \lambda\tilde{s}^*(c) \\ &= \lambda \max\{p\beta T(h(\tilde{s}^*(c))) - pq\beta T(h(\tilde{s}^*(c))), 0\} - h(\tilde{s}^*(c)) + \lambda\tilde{s}^*(c) \\ &= \lambda\{p(1-q)\beta T(h(\tilde{s}^*(c))), 0\} - h(\tilde{s}^*(c)) + \lambda\tilde{s}^*(c) \\ &= \lambda p(1-q)\beta T(h(\tilde{s}^*(c))) - h(\tilde{s}^*(c)) + \lambda\tilde{s}^*(c), \end{aligned}$$

from which

$$T(h(\tilde{s}^*(c)) = (h(\tilde{s}^*(c)) - \lambda\tilde{s}^*(c))/\lambda p(1-q)\beta.$$

Since $\tilde{s}^*(\tilde{c}^*) = 0$ by the definition, the above equation with $c = \tilde{c}^*$ can be rewritten

$$T(h(\tilde{s}^*(\tilde{c}^*)) = h(\tilde{s}^*(\tilde{c}^*)))/\lambda p(1-q)\beta, \tag{5.3}$$

which can be rearranged into $B_1(h(\tilde{s}^*(\tilde{c}^*))) = 0$ due to Eq. (3.5). Accordingly, it follows that $h(\tilde{s}^*(\tilde{c}^*)) = x_1^*(0)$, which is the unique solution of $B_1(x) = 0$ due to Lemma 3.3(b). Therefore, we have

$$T(h(\tilde{s}^*(\tilde{c}^*))) = T(x_1^*(0)) = x_1^*(0)/\lambda p(1-q)\beta. \tag{5.4}$$

From Eqs. (5.4) and (5.2) with $c = \tilde{c}^*$ we have $\tilde{c}^* = pq\beta T(h(\tilde{s}^*(\tilde{c}^*))) = qx_1^*(0)/\lambda(1-q)$.

(d) Since $T(x_1^*(0)) \geq 0$, we have $x_1^*(0)/\lambda p(1-q)\beta \geq 0$ from Eq. (5.4), hence $x_1^*(0) \geq 0$. Accordingly, from Lemma 3.2(c) we obtain $m \geq T(x_1^*(0)) = x_1^*(0)/\lambda p(1-q)\beta = (qx_1^*(0)/\lambda(1-q))/pq\beta = \tilde{c}^*/pq\beta$, that is, $\tilde{c}^* < pq\beta m < p\beta m = c^*$. ∎

Lemma 5.6

(a) *Let $c^* > c$.*

1 $u(\phi, 0) > 0$ *and* $u(\phi, 1) > 0$.

2 $s^* > 0$.

3 *Let* $s \leq s^*$.

  i *If* $\tilde{s}^* \leq s$, *then* $h = x_2^* \leq \delta$.

  ii *If* $s < \tilde{s}^*$, *then* $h = x_1^* > \delta$.

(b) *Let $c^* \le c$. Then $A(s) \le 0$ and $\tilde{A}(s) \le 0$.*

Proof. (a) Let $c^* > c$.

(a1) Since $u(\phi,0) \ge 0$ and $u(\phi,1) \ge 0$, from Eqs. (4.11) and (4.10) and Lemma 5.5(c) we have $u(\phi,0) \ge p\beta v(0) + (1-p)\beta u(\phi,0) - c + s \ge p\beta m - c + s = c^* - c + s > 0$ due to the assumption of $s \ge 0$. Then, from Eq. (4.16) we get $u(\phi,1) \ge \lambda q\beta u(\phi,0) > 0$.

(a2) Let $s = 0$. Assume $p\beta v(0) + (1-p)\beta u(\phi,0) - c \le \beta u(\phi,0)$. Then $u(\phi,0) = \beta u(\phi,0)$ from Eq. (4.11), leading to $\beta = 1$ due to $u(\phi,0) > 0$ from (a1), which contradicts the assumption of $\beta < 1$. Accordingly, it must be $p\beta v(0) + (1-p)\beta u(\phi,0) - c > \beta u(\phi,0)$, which can be rearranged into $0 < p\beta(v(0) - u(\phi,0)) - c = p\beta T(h(0)) - c = A(0)$ from Eq. (4.13), implying $s^* > 0$.

(a3) Let $s \le s^*$. Then, from Eq. (4.22) and Lemma 5.4(b3) we obtain

$$u(\phi,0) = (A(s)+s)/(1-\beta) = (p\beta T(h) - c + s)/(1-\beta). \tag{5.5}$$

(a3i) If $\tilde{s}^* \le s$, since $\tilde{A}(s) \le 0$ due to Lemma 5.4(b3), from Eq. (4.23) we have $u(\phi,1) = \lambda q\beta u(\phi,0)$. Accordingly, $h = u(\phi,0) - u(\phi,1) = (1-\lambda q\beta)u(\phi,0) = \lambda(1-\beta)u(\phi,0)$ due to Eq. (2.6). Rearranging this by substituting Eq. (5.5) yields $h = \lambda p\beta T(h) - \lambda(c-s)$, hence $T(h) - \big(h + \lambda(c-s)\big)/\lambda p\beta = 0$, i.e., $B_2(h) = 0$ due to Eq. (3.6). This implies that h defined by Eq. (4.1) is given by x_2^*, which is the unique solution of $B_2(x) = 0$ due to Lemma 3.3(b), i.e., $h = x_2^*$, hence

$$T(h) = T(x_2^*) = \big(x_2^* + \lambda(c-s)\big)/\lambda p\beta. \tag{5.6}$$

Now, from the assumption of $\tilde{s}^* \le s$ we obtain $0 \ge \tilde{A}(s) = pq\beta T(h) - c$. Rearranging the inequality by substituting Eq. (5.6) produces $0 \ge qx_2^*/\lambda - c(1-q) - qs$, hence $x_2^* \le c\lambda(1-q)/q + \lambda s = \delta$.

(a3ii) If $s < \tilde{s}^*$, then $\tilde{A}(s) > 0$ due to Lemma 5.4(b5), hence, from Eq. (4.23) we get $u(\phi,1) = \lambda q\beta u(\phi,0) + \lambda\tilde{A}(s) = \lambda q\beta u(\phi,0) + \lambda pq\beta T(h) - \lambda c$. Accordingly, noting Eq. (2.6), we have

$$\begin{aligned} h &= u(\phi,0) - u(\phi,1) \\ &= (1-\lambda q\beta)u(\phi,0) - \lambda pq\beta T(h) + \lambda c \\ &= \lambda(1-\beta)u(\phi,0) - \lambda pq\beta T(h) + \lambda c. \end{aligned} \tag{5.7}$$

Rearranging Eq. (5.7) by substituting Eq. (5.5) produces $h = \lambda p(1-q)\beta T(h) + \lambda s$, from which $T(h) - (h - \lambda s)/\lambda p(1-q)\beta = 0$, i.e., $B_1(h) = 0$ due to Eq. (3.5). This implies that h defined by Eq. (4.1) is also given by x_1^*, which is the unique solution of $B_1(x) = 0$ due to Lemma 3.3(b), i.e., $h = x_1^*$, hence

$$T(h) = T(x_1^*) = (x_1^* - \lambda s)/\lambda p(1-q)\beta. \tag{5.8}$$

Now, from the assumption of $s < \tilde{s}^*$, we have $\tilde{A}(s) = pq\beta T(h) - c > 0$. Rearranging this inequality by substituting Eq. (5.8) yields $q(x_1^* - \lambda s)/\lambda(1-q) - c > 0$, equivalently, $x_1^* > c\lambda(1-q)/q + \lambda s = \delta$.

(b) Let $c^* \le c$, i.e., $p\beta m \le c$ due to Lemma 5.5(c). Now, since $h \ge 0$ from Lemma 5.2(a), we have $T(h) \le m$ from Lemma 3.2(c). Hence, $0 \ge p\beta m - c \ge p\beta T(h) - c \ge pq\beta T(h) - c$, i.e., $A(s) \le 0$ and $\tilde{A}(s) \le 0$. ∎

6 Optimal Decision Rule

The following theorem prescribes the optimal decision rule.

Theorem 6.1

(a) *Let $c = 0$. Then* C_0 *and* C_1.

(b) *Let $c > 0$.*

  1 *If $c^* \leq c$, then* K_0 *and* K_1.

  2 *If $c < c^*$, then*

   i *If $s^* \leq s$, then* K_0 *and* K_1.

   ii *If $s > s^*$, then*

    A C_0.

    B *If $x_1^* \leq (>)\delta$, then $K_1(C_1)$, or if $x_2^* > (\leq)\delta$, then $C_1(K_1)$.*

    C *If $B_1(\delta) \leq (>)0$, then $K_1(C_1)$, or if $B_1(\delta) > (\leq)0$, then $C_1(K_1)$.*

Proof. (a,b1,b2i,b2iiA) Immediate from Lemmas 5.4(a), 5.6(b), 5.4(b2), and 5.4(b3), respectively.

(b2iiB) We immediately obtain the following relationships from Lemmas 3.3(h), 5.4(b4,(b5), and the contrapositions of Lemma 5.6(a3i,a3ii).

$$x_1^* > \delta \overset{\text{Lemma 3.3(h)}}{\iff} x_2^* > \delta \overset{\text{Lemma 5.6(a3i)}}{\implies} s < \tilde{s}^* \overset{\text{Lemma 5.4(b5)}}{\implies} \tilde{A}(s) > 0 \implies C_1$$

$$x_2^* \leq \delta \underset{\text{Lemma 3.3(h)}}{\iff} x_1^* \leq \delta \underset{\text{Lemma 5.6(a3ii)}}{\implies} \tilde{s}^* \leq s \underset{\text{Lemma 5.4(b4)}}{\implies} \tilde{A}(s) \leq 0 \implies K_1$$

(b2iiC) Clear from Figure 3.4 and (b2iiB). ∎

Theorem 6.2 *In the customer-first case we have*

(a) *$z(h)$ is nondecreasing in p, β, and s and nonincreasing in q and c.*

(b) *If $s < b/\lambda$, then $h < z(h) < b$.*

(c) *If $s \geq b/\lambda$, then $z(h) = b$.*

Proof. (a) Since h is nondecreasing in p, β, and s and nonincreasing in q and c from Lemma 5.2(e,f,g), the monotonicities of $z(h)$ is immediate from Lemma 3.1(c).

(b) Evident from Lemmas 5.2(d1) and 3.1(b).

(c) Immediate from Lemmas 5.2(c) and 3.1(b). ∎

7 Conclusions and Considerations

A. *Optimal decision rules.* The most important conclusions obtained in the paper are the statements in Theorem 6.1, which can be summarized as in Table 7.1.

Table 7.1: Summary of optimal decision rules

c	s	δ	State $(\phi, 0)$	State $(\phi, 1)$
$c = 0$			C	C
$0 < c < c^*$	$s < s^*$	$\delta < x_1^*$	C	C
		$x_1^* \leq \delta$	C	K
	$s^* \leq s$		K	K
$c^* \leq c$			K	K

C: continue the search, K: skip the search

B. *Relationships of the optimal decision rules with parameters.* Depicting the optimal decision rule prescribed in Table 7.1 in the relationship with the search cost c and the idling profit s, we have Figure 7.3 in which both $s^*(c)$ and $\tilde{s}^*(c)$ are strictly decreasing in c with $c^* = p\beta m$, $\tilde{c}^* = qx_j^\circ/\lambda(1-q)$ (Lemma 5.5(c)), $\tilde{s}^*(c) < s^*(c)$ for $c > 0$ (Lemma 5.4(b1)), and $\tilde{s}^*(0) = s^*(0) = b/\lambda$ (Lemma 5.5(b)).

The three regions $\Omega(\mathtt{K},\mathtt{K})$, $\Omega(\mathtt{C},\mathtt{K})$, and $\Omega(\mathtt{C},\mathtt{C})$ in Figure 7.3 correspond to the optimal decisions of, respectively, "skipping in both states $(\phi, 0)$ and $(\phi, 1)$", "continuing in state $(\phi, 0)$ and skipping in state $(\phi, 1)$", and "continuing in both states $(\phi, 0)$ and $(\phi, 1)$". Further, the figure tells us the following two points:

1. When the search cost c or the idling profit s is sufficiently large, skipping the search becomes optimal in both of states $(\phi, 0)$ and $(\phi, 1)$, implying, respectively, that it is profitable to avoid the search cost through skipping the search or that it becomes profitable to enjoy the idling profits with making the process empty by skipping the search and not obtaining orders.
2. When the search cost c or the idling profit s is sufficiently small, continuing the search becomes optimal in both of states $(\phi, 0)$ and $(\phi, 1)$, implying, respectively, that it is reasonable to enjoy the profit from an order obtained through conducting the search or that it does not become profitable even if making the process empty by skipping the search.

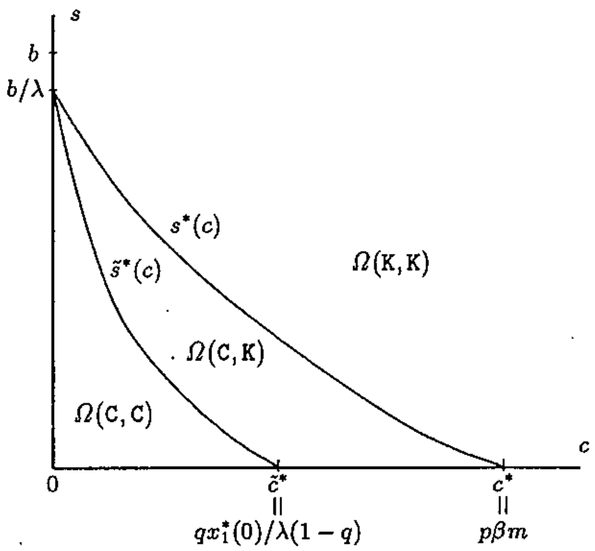

Figure 7.3: Three regions encircled by the functions $\tilde{s}^*(c)$ and $s^*(c)$ and the axes c, s

C. *Properties of h.*

1. In the *customer-first case* the optimal selection criterion, on which the system decides whether to accept an appearing customer or not, is given by h, and in the *system-first case* the optimal price, on which an appearing customer decides whether to place his order to the system or not, is given by the function $z(h)$. Further, it is only on the regions $\Omega(\mathtt{C},\mathtt{K}) \cup \Omega(\mathtt{C},\mathtt{C})$ that the decisions stated above are to be made.
2. The h is given by the unique solution h^* of the equation $G(x) = 0$, i.e., $h = h^*$; refer Lemma 5.2 for the properties of the h.
3. Let $(s, c) \in \Omega(\mathtt{C},\mathtt{C}) \cup \Omega(\mathtt{C},\mathtt{K})$, hence $h < b$ (Lemma 5.2(d1)). Then, the optimal decisions can be prescribed as follows.
   1) In the *customer-first case,* if $w > h$, an order with value w appearing after having conducted the search is accepted, or else rejected.
   2) In the *system-first case,* the optimal price $z(h)$ is on the interval (h, b)(Lemma 6.2(b)).

D. *The monotonicities of h and z(h) in the parameters* (Lemmas 5.2(e, g) and 6.2(a)).

1. *Both h and $z(h)$ are nondecreasing in p, β, and s.* This implies that the larger the customer appearing probability p, the discount factor β, and the idling profit s may be, it is reasonable to accept orders with higher values in the customer-first case and to offer higher prices in the system-first case, and vice versa.
2. *Both h and $z(h)$ are nonincreasing in q and c.* This implies that the higher the service completion probability q and the search cost c may be, the inverse of the above can be said, i.e., it is reasonable to accept orders, even if their values are smaller, in the customer-first case and to offer smaller prices in the system-first case, and vice versa.

8 Future Studies

In order to make the model posed in this paper more realistic, some points must be investigated; especially, the following two points are challenging and should be necessarily examined.

1. We have assumed so far that $n = 1$, i.e., more than one customer can not be held at any instant. However, we should also examine the case of $n \geq 2$ from a practical viewpoint. Nevertheless, for the reason that the mathematical treatment for $n = 1$ has some distinctive way of analysis which can not be seen in the analysis of the case of $n \geq 2$, we exclude the case of $n \geq 2$ in the present paper. In fact, in the latter case, h may depend on the number i of orders in the system, i.e, h_i; accordingly, a problem of examining the monotonicity of h_i in i arises. Now, as seen in some numerical experiments conducted in [4], we obtained a result that there may exist an $i^* \geq 2$ such that h_i is strictly decreasing on $i \leq i^*$ and strictly increasing on $i^* \leq i$. Here, another problem arises of finding the necessary and sufficient conditions on which such an i^* exists; this is left as a subject of study in the future.
2. Thus far we have implicitly assumed that a customer once rejected can not be solicited and accepted in the future. The future availability of rejected customer, that is assumed in usual models of optimal stopping problems [9] [2] [5] [8], should be also introduced in our model, which will become an important as well as interesting subject of study from a practical viewpoint.
3. Here, let us look back at our procedures of analysis. First, m and $T(x)$ are defined by Eqs. (3.1) and (3.2), and next $v(0)$ by Eq. (4.9). Then, we saw that Eqs. (4.4) and (4.8) composing the optimal equation of both cases can be expressed by the identical Eq. (4.13). By the above procedures it eventually follows that the optimal equation of both cases can be expressed by quite the same equation. In addition to this, it was proven that T-function defined by the same symbol for both cases has the same properties as shown in Lemma 3.2. It is from all the above that we can examine together both of customer-first and system-first cases by the same framework of analysis. Now, noting that there also exists the concept of customer (offerer) and system in optimal stopping problems, the above suggests that we could define the customer-first case and system-first case for almost all model of optimal stopping problems and that it may be possible to obtain almost the same conclusions for both cases without adding any new analysis to conventional ones, the strict examination of this is to be left as one of future studies.

Appendix – Proofs –

A. Lemma 3.1

(a) $p(z)(z - x) = z - x$ for $z < a$ due to Eq. (2.3 (1)), which is strictly increasing in $z < a$ for any x. Hence $a - \varepsilon \leq z(x)$ for any x and any infinitesimal $\varepsilon > 0$. If $a > z(x)$, then $a - \varepsilon' > z(x)$ for any infinitesimal $\varepsilon' > 0$, which is a contradiction, hence it must be $a \leq z(x)$ for any x.

(b) Let $x < b$. First, clearly $p(z)(z-x) \leq 0$ for $z \leq x$. Then, $p(z)(z-x) > 0$ for $x < z < b$ due to Eq. (2.3 (3)) and $p(z)(z-x) = 0$ for $b \leq z$ due to Eq. (2.3 (4)), hence $x < z(x) < b$. Let $x \geq b$. Then, $p(z)(z-x) < 0$ for $z < b$ due to Eq. (2.3 (3)) and $p(z)(z-x) = 0$ for $b \leq z$ due to Eq. (2.3 (4)), hence $z(x) = b$.

(c) For any $\varepsilon > 0$ we have

$$
\begin{aligned}
T(x+\varepsilon) &= \max_z p(z)\big(z-(x+\varepsilon)\big) = p(z(x+\varepsilon))\big(z(x+\varepsilon)-(x+\varepsilon)\big) \\
&= p(z(x+\varepsilon))\big(z(x+\varepsilon)-x\big) - p(z(x+\varepsilon))\varepsilon \\
&\leq T(x) - p(z(x+\varepsilon))\varepsilon \\
&= p(z(x))\big(z(x)-x\big) - p(z(x+\varepsilon))\varepsilon \\
&= p(z(x))\big(z(x)-(x+\varepsilon)\big) + \varepsilon\big(p(z(x)) - p(z(x+\varepsilon))\big) \\
&\leq T(x+\varepsilon) + \varepsilon\big(p(z(x)) - p(z(x+\varepsilon))\big),
\end{aligned}
$$

from which $0 \leq p(z(x)) - p(z(x+\varepsilon))$, that is, $p(z(x)) \geq p(z(x+\varepsilon))$. Since $a \leq z(x) \leq b$ and $a \leq z(x+\varepsilon) \leq b$ due to (a,b) and since $p(z)$ is strictly decreasing on $[a,b]$, it eventually follows that $z(x) \leq z(x+\varepsilon)$. This completes the proof. ∎

B. Lemma 3.2

1. *Customer-first case:*

(a) The former half is immediate from the fact that $\max\{w-x, 0\}$ is nondecreasing in x on $(-\infty, \infty)$. The latter half can be shown as follows. Consider any x_1 and x_2 such that $x_1 < x_2 < b$. Then, since $F(x_2) < 1$ due to Eq. (2.2), we have

$$
\begin{aligned}
T(x_1) - T(x_2) &= \int_{x_1}^{\infty} (w-x_1)dF(w) - \int_{x_2}^{\infty} (w-x_2)dF(w) \\
&\geq \int_{x_2}^{\infty} (w-x_1)dF(w) - \int_{x_2}^{\infty} (w-x_2)dF(w) \\
&= (x_2-x_1)(1-F(x_2)) > 0 \qquad \text{(B.1)}
\end{aligned}
$$

Next, the convexity is immediate from the fact that $\max\{w-x, 0\}$ is convex in x on $(-\infty, \infty)$ for any w.

(b) Immediate from the fact that $\max\{w-x, 0\} \geq 0$ for any w and x.

(c) For any $x \geq 0$ we have $T(x) \leq T(0) = \mu = m$ due to (a). If $x < b$, then $T(x) > T(b) \geq 0$ due to (a,b). If $b \leq x$, then $\max\{w-x, 0\} = 0$ for $a \leq w \leq b$, hence $T(x) = 0$.

(d) Immediate from (a,c).

(e) The former half is immediate from (c). If $x < a$, then $\max\{w-x, 0\} = w-x$ for $a \leq w \leq b$, hence $T(x) = m - x$, thus $\lim_{x \to -\infty} T(x) = \infty$.

(f) For any $x_1 > x_2$, from Eq. (B.1) we have

$$x_1 + \nu T(x_1) - x_2 - \nu T(x_2) = (x_1 - x_2) + \nu\big(T(x_1) - T(x_2)\big)$$
$$\geq (x_1 - x_2) + \nu(x_2 - x_1)(1 - F(x_2))$$
$$= (x_1 - x_2)\big(1 - \nu(1 - F(x_2))\big) \cdots (1^*).$$

If $\nu \leq 1$, then $(1^*) \geq 0$, and if $\nu < 1$, then $(1^*) > 0$; accordingly, the assertion holds.

(g) The former half is evident from (e). If $x < a$, then since $T(x) = m - x$, we get $x + \nu T(x) = x + \nu(m - x) = (1 - \nu)x + \nu m$. Hence, if $\nu < 1$, then $\lim_{x \to -\infty} x + \nu T(x) = -\infty$. □

2. *System-first case:*

(a) The nonincreasingness is immediate from the fact that $z - x$ is nondecreasing in x on $(-\infty, \infty)$. If $x < b$, then $z(x) < b$ due to Lemma 3.1(b), hence $p(z(x)) > 0$ from Eq. (2.3 (3)). Accordingly, if $x_1 < x_2 < b$, since $p\big(z(x_2)\big) > 0$, we get $T(x_2) = p(z(x_2))(z(x_2) - x_2) < p(z(x_2))(z(x_2) - x_1) \leq T(x_1)$, hence the assertion holds.

(b) Immediate from the fact that $p(b)(b - x) = 0$ due to Eq. (2.3 (4)).

(c) The proof of the former half is almost the same as the one of (c) in the customer-first case. If $x < b$, then $p(z)(z - x) > 0$ for $x < z < b$ due to Eq. (2.3 (3)), hence $T(x) > 0$. If $x \geq b$, since $z(x) = b$ from Lemma 3.1(b), we get $T(x) = p(b)(b - x) = 0$ due to Eq. (2.3 (4)).

(d) Immediate from (a, c).

(e) The former half is immediate from (c). Since $T(x) \geq p(a - \varepsilon)(a - \varepsilon - x) = a - \varepsilon - x$ for any infinitestimal $\varepsilon > 0$ due to Eq. (2.3 (1)), we have $\lim_{x \to -\infty} T(x) \geq \lim_{x \to -\infty}(a - \varepsilon - x) = \infty$.

(f) For any $x_1 < x_2$ we obtain

$$x_1 + \nu T(x_1) - x_2 - \nu T(x_2) = (x_1 - x_2) + \nu\big(\max_z p(z)(z - x_1) - \max_z p(z)(z - x_2)\big)$$
$$\leq (x_1 - x_2) + \nu \max_z p(z)(x_2 - x_1)$$
$$= (x_1 - x_2) + \nu(x_2 - x_1) = (x_1 - x_2)(1 - \nu) \cdots (2^*),$$

from which, if $\nu \leq 1$, then $(2^*) \leq 0$, and if $\nu < 1$, then $(2^*) < 0$, hene the assertions are true.

(g) The former half is immediate from (e). Let $x < b$. Then, since $z(x) < b$ from Lemma 3.1(b), we get $p(z(x)) > 0$ due to Eq. (2.3 (3)). Now, we obtain

$$x + \nu T(x) = x + \nu p(z(x))(z(x) - x)$$
$$= \big(1 - \nu p(z(x))\big)x + \nu p(z(x))z(x).$$

If $\nu < 1$, since $1 - \nu p(z(x)) > 0$ we have $\lim_{x \to -\infty} \big(x + \nu T(x)\big) = -\infty$. ∎

C. Lemma 3.3

(a) The former half is immediate from the facts that $T(x)$ is nonincreasing in x due to Lemma 3.2(a) and that both of $-x/\lambda p(1-q)\beta$ and $-x/\lambda p\beta$ are strictly decreasing in x. The latter half is evident from Lemma 3.2(e).

(b) The former half is evident from (a). Let $p\beta m > c$. Then

$$B_1(0) = T(0) + s/p(1-q)\beta = m + s/p(1-q)\beta > 0,$$
$$B_2(0) = T(0) - (c - s)/p\beta = m - (c - s)/p\beta = (p\beta m - c + s)/p\beta > 0$$

due to $m > 0$ and the assumption $s \geq 0$. Hence, it follows that x_1^*, x_2^*, $x_1^*(0)$, and $x_2^*(0)$ are all positive.

(c) Clear from

$$\begin{aligned} B_1(x) - B_2(x) &= -(x - \lambda s)/\lambda p(1-q)\beta + \big(x + \lambda(c-s)\big)/\lambda p\beta \\ &= \big(-qx + \lambda qs + \lambda(1-q)c\big)/\lambda p(1-q)\beta = -q(x-\delta)/\lambda p(1-q)\beta. \end{aligned}$$

(d-h) Immediately from the fact that the functions $B_1(x)$ and $B_2(x)$ can be depicted as in Figure 3.4 due to (a) to (c). ∎

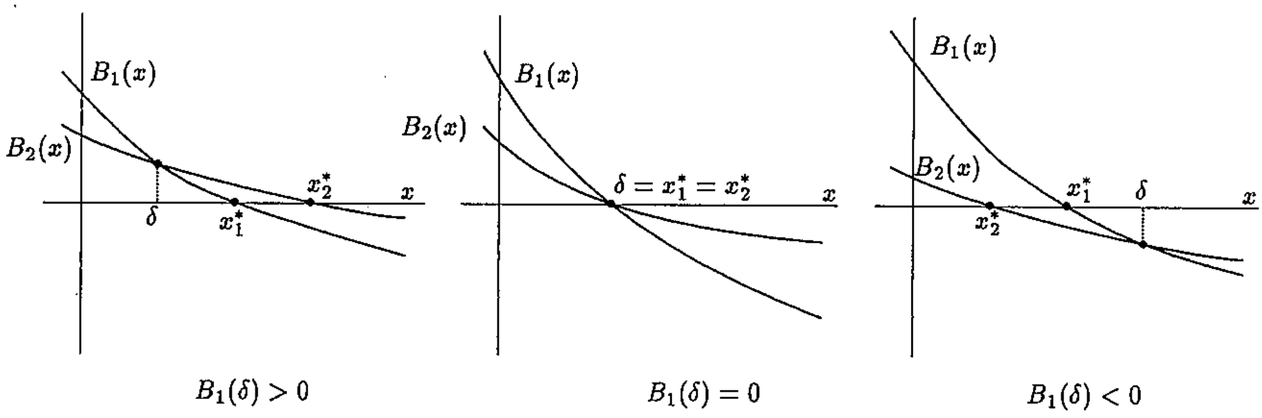

Figure 3.4: Graphs of $B_1(x)$ and $B_2(x)$.

D. Lemma 3.4

(a) The function $G(x)$ can be rearranged into

$$G(x) = \lambda \max\{p\beta T(x) - c, 0\} - \max\{x + \lambda pq\beta T(x) - \lambda c, x\} + \lambda s, \tag{D.1}$$

the first term of the right hand side of which is nonincreasing in x from Lemma 3.2(a). Since $1 > \lambda q\beta \geq \lambda pq\beta$ from Eq. (2.6), it follows that $x + \lambda pq\beta T(x) - \lambda c$ is strictly increasing in x from Lemma 3.2(f), hence the entire right hand side of Eq. (D.1) is strictly decreasing in x.

(b) Applying Lemma 3.2(e) to Eq. (3.7) leads to

$$\lim_{x\to\infty} G(x) = \lambda\big(\max\{-c,0\} - \max\{-c,0\}\big) - \lim_{x\to\infty} x + \lambda s = -\infty$$

Since $G(x) \geq -\max\{x + \lambda pq\beta T(x) - \lambda c, x\} + \lambda s$ from Eq. (D.1), applying Lemma 3.2(e,g) to Eq. (D.1) yields

$$\lim_{x\to-\infty} G(x) \geq -\max\{\lim_{x\to-\infty} (x + \lambda pq\beta T(x)) - \lambda c, \lim_{x\to-\infty} x\} + \lambda s = \infty.$$

(c) For any given $c > 0$ let $x_1(c)$ and $x_2(c)$ be the solution of $T(x) = c/pq\beta$ and $T(x) = c/p\beta$, respectively. Then, since $c/pq\beta > c/p\beta > 0$, clearly both of $x_1(c)$ and $x_2(c)$ uniquely exit from Lemma 3.2(d) where $x_1(c) < b$ and $x_2(c) < b$. In addition, since $T(x_1(c)) > T(x_2(c))$, we have $x_1(c) < x_2(c)$. Now, let $c' = c + \varepsilon$ for any infinitestimal $\varepsilon > 0$, hence $c' > c$. Then, $x_1(c') < x_1(c) < x_2(c') < x_2(c)$ (see Figure 4.5). Below, let us examine the rélationship of $G(x)$ and c. First, Eq. (3.7) for each of c and c' can be rewritten as follows, respectively.

$$G(x|c) = \begin{cases} \lambda\big(p(1-q)\beta T(x)+s\big)-x, & \text{on} \quad \text{I} \cup \text{II} \quad \cdots \ (1), \\ \lambda(p\beta T(x)-c+s)-x, & \text{on} \quad \text{III} \cup \text{IV} \quad \cdots \ (2), \\ -x+\lambda s, & \text{on} \quad \text{V} \quad \cdots \ (3), \end{cases} \tag{D.2}$$

$$G(x|c') = \begin{cases} \lambda\big(p(1-q)\beta T(x)+s\big)-x, & \text{on} \quad \text{I} \quad \cdots \ (1'), \\ \lambda(p\beta T(x)-c'+s)-x, & \text{on} \quad \text{II} \cup \text{III} \quad \cdots \ (2'), \\ -x+\lambda s, & \text{on} \quad \text{IV} \cup \text{V} \quad \cdots \ (3'). \end{cases} \tag{D.3}$$

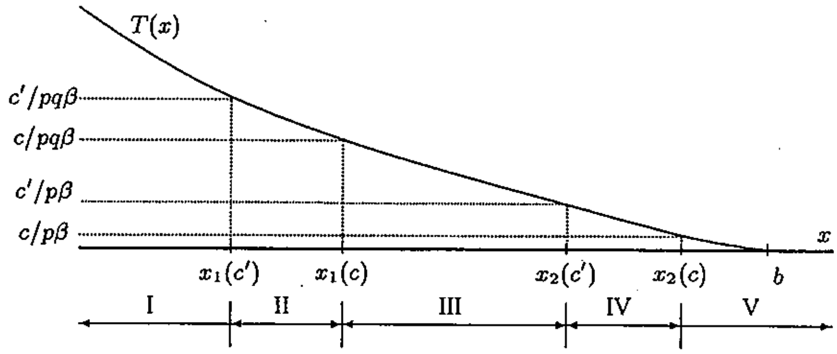

Figure 4.5: The relationship between $x_1(c'), x_1(c), x_2(c')$ and $x_2(c)$ where $\text{I} = (-\infty, x_1(c')]$, $\text{II} = \big(x_1(c'), x_1(c)\big)$, $\text{III} = \big(x_1(c), x_2(c')\big)$, $\text{IV} = \big[x_2(c'), x_2(c)\big)$, and $\text{V} = [x_2(c), \infty)$

1. On the intervals I and V, we have $G(x|c) = G(x|c')$, respectively, from Eqs. (D.2 (1)) and (D.3 (1')) and Eqs. (D.2 (3)) and (D.3 (3')).
2. On the interval II, from Eqs. (D.2 (1)) and (D.3 (2')) we get

$$\begin{aligned} G(x|c) - G(x|c') &= \lambda(p(1-q)\beta T(x)+s) - x - \big(\lambda(p\beta T(x)-c'+s)-x\big) \\ &= \lambda p(1-q)\beta T(x) - \lambda p\beta T(x) + \lambda c' \\ &= -\lambda pq\beta T(x) + \lambda c' = -\lambda(pq\beta T(x) - c') > 0 \end{aligned}$$

due to $pq\beta T(x) - c' < 0$ on $x_1(c') < x$, hence $G(x|c) > G(x|c')$.

3. On the interval III, from Eqs. (D.2 (2)) and (D.3 (2')) we have

$$G(x|c) = \lambda(p\beta T(x) - c + s) - x > \lambda(p\beta T(x) - c' + s) - x = G(x|c').$$

4. On the interval IV, from Eqs. (D.2 (2)) and (D.3 (3'))we obtain

$$G(x|c) - G(x|c') = \lambda\big(p\beta T(x) - c + s\big) - x - (-x + \lambda s) = \lambda(p\beta T(x) - c) > 0$$

due to $p\beta T(x) - c > 0$ on $x < x_2(c)$.

From all the above, it eventually follows that $G(x|c) \geq G(x|c')$ for all x, that is, $G(x|c)$ is nonincreasing in c for all x. The latter half can be proven in the almost same way as the above where it is to be noted that λ is strictly decreasing in q.

(d) Proven in the similar way to in the proof of (c) where it is to be noted that λ is strictly increasing in β.

(e) Immediate from Eq. (3.7) since $\lambda > 0$. ∎

E. Lemma 5.1

For any given vector $x = (x_0, x_1)'$ let us define the norm $\|x\| = \max\{|x_0|, |x_1|\}$. Further, by D_0u and D_1u let us denote the right hand sides of Eqs. (4.11) and (4.12), and let $Du = (D_0u, D_1u)'$ and $u = (u(\phi,0), u(\phi,1))'$. Since $\|x\| \geq |x_i|$ for $i = 0, 1$, using the definition of $v(0)$, from Eqs. (4.4) we have

$$|v(0) - \hat{v}(0)| \leq \int_0^\infty \max\{|u(\phi,1) - \hat{u}(\phi,1)|, |u(\phi,0) - \hat{u}(\phi,0)|\} dF(w) = \|u - \hat{u}\|, \tag{E.1}$$

and from (4.8) we get

$$\begin{aligned} |v(0) - \hat{v}(0)| &\leq \max_z\{p(z)|u(\phi,1) - \hat{u}(\phi,1)| + (1-p(z))|u(\phi,0) - \hat{u}(\phi,0)|\} \\ &\leq \max_z\{p(z)\|u - \hat{u}\| + (1-p(z))\|u - \hat{u}\|\} = \|u - \hat{u}\|. \end{aligned}$$

Accordingly, we obtain

$$\begin{aligned} |D_0u - D_0\hat{u}| &\leq \max\left\{\begin{array}{l} p\beta|v(0) - \hat{v}(0)| + (1-p)\beta|u(\phi,0) - \hat{u}(\phi,0)|, \\ \beta|u(\phi,0) - \hat{u}(\phi,0)| \end{array}\right\} \\ &\leq \max\{\beta\|u - \hat{u}\|, \beta\|u - \hat{u}\|\} = \beta\|u - \hat{u}\|. \end{aligned}$$

Similiarly, we get $|D_1u - D_1\hat{u}| \leq \beta\|u - \hat{u}\|$. Thus $\|Du - D\hat{u}\| \leq \beta\|u - \hat{u}\|$, implying that Du is a contraction mapping. Accordingly, the assertion holds. ∎

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