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A NOTE ON VAN DER HEYDEN'S VARIABLE DIMENSION ALGORITHM FOR THE LINEAR COMPLEMENTARITY PROBLEM

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The purpose of this note is to provide an interpretation of Van der Heyden's variable dimension algorithm [6] for the linear complementarity problem in terms of Eaves and Scarf's unifying model [1] for the system of piecewise linear equations. As in Kojima and Yamamoto [2,3] a pair of manifolds is introduced and a new manifold with a higher dimension produced from the pair plays an essential role in the discussion below. The following interpretation will contribute to developing a new algorithm for the fixed point problem as well as the nonlinear version of Van der Heyden's one.

Given an $n \times n$ matrix M and an n-vector q the linear complementarity problem (abbreviated by LCP in the sequel) is to find vectors s and z such that

$$s - M z = q, \quad z, s \geqq 0 \quad \text{and} \quad s^T z = 0.$$

Let $g^j(x_j)$ and $f: R^n \longrightarrow R^n$ be defined such that

$$g^j(x_j) = \begin{cases} m^j \cdot x_j & \text{if } x_j > 0, \\ e^j \cdot x_j & \text{otherwise,} \end{cases}$$

$$f(x) = \sum_{j=1}^{n} g^j(x_j) + q,$$

where m^j and e^j are the j-th columns of the given matrix M and the $n \times n$ identity matrix, respectively. Then LCP is known to be equivalent to solving the system of piecewise linear equations:

(1) $$f(x) = 0, \quad x \in R^n.$$

For $1 \leqq k \leqq n$ if $x = (x_1, x_2, \cdots, x_k, 0, \cdots, 0)^T \in R^n$ satisfies the first k equations of the system (1), we say that x solves the k-problem. It is clear that the k-vector $(x_1, x_2, \cdots, x_k)^T$ provides a solution of the k-dimensional LCP defined by the k-th leading principal minor $M^{(k)}$ of M and $q^{(k)} = (q_1, q_2, \cdots, q_k)^T$. Van der Heyden's algorithm solves a sequence of k-problems, the dimension k, however, is not necessarily monotonic. The similar idea can be found in Lemke [4] and Reiser [5].

For $\alpha \in \{+1,-1\}$ and $1 \leqq k \leqq n$ let

$$X_k^\alpha = \{ x \in R^n:\ \alpha \cdot x_k \geqq 0,\ \ x_j = 0\ (\forall j > k)\ \},$$
$$Y_k^\alpha = \{ y \in R^n:\ \alpha \cdot y_k \geqq 0,\ \ y_j = 0\ (\forall j < k)\ \},$$
$$X_k = X_k^{+1} \cup X_k^{-1},$$

and

$$Y_k = Y_k^{+1} \cup Y_k^{-1}.$$

In the sequel we employ the convention that for any $\alpha \in \{+1,-1\}$

$$X_0^\alpha = Y_{n+1}^\alpha = \{0\},$$

where 0 denotes the origin of R^n. Define Z_k^α be the Cartesian product of X_k^α and Y_k^α : $\quad Z_k^\alpha = X_k^\alpha \times Y_k^\alpha$,

and L be the collection of all Z_k^α ($1 \leqq k \leqq n,\ \alpha \in \{+1,-1\}$). Then we see that L is a finite collection of (n+1)-cells no two of which have a relatively interior point in common. We also obtain the following lemmata from the definitions. For a convex subset $X \subseteq R^m$ we denote by ∂X the set of boundary points of X with respect to the affine subspace spanned by X and for a collection L of (n+1)-cells by ∂L the collection of all n-(pseudo) faces of L.

<u>Lemma 1.</u>

$$\partial Z_k^\alpha = (X_k^\alpha \times Y_{k+1}) \cup (X_{k-1} \times Y_k^\alpha).$$

proof. If we note that for any $1 \leqq k \leqq n$ and $\alpha \in \{+1,-1\}$

$$X_{k-1} \subseteq \partial X_k^\alpha, \qquad Y_{k+1} \subseteq \partial Y_k^\alpha,$$

then we immediately have

$$\partial Z_k^\alpha \supseteq (X_k^\alpha \times Y_{k+1}) \cup (X_{k-1} \times Y_k^\alpha).$$

To show the reverse relation let $(x, y) \in \partial Z_k^\alpha$. Then either $x \in \partial X_k^\alpha$ or $y \in \partial Y_k^\alpha$. If $x \in \partial X_k^\alpha$, x lies in X_{k-1}^β for some $\beta \in \{+1,-1\}$, and we have $(x, y) \in X_{k-1}^\beta \times Y_k^\alpha \subseteq X_{k-1} \times Y_k^\alpha$. In the case where $y \in \partial Y_k^\alpha$ we also have $(x, y) \in X_k^\alpha \times Y_{k+1}^\beta \subseteq X_k^\alpha \times Y_{k+1}$ for some $\beta \in \{+1,-1\}$. Q.E.D.

Since for $i \leqq j$ $X_i^\alpha \times Y_j^\beta$ is an (n+1)-(j-i)-dimensional cell, we shall call each n-dimensional cell $X_k^\alpha \times Y_{k+1}^\beta$ ($0 \leqq k \leqq n$) an n-pseudo face of L.

Lemma 2. For any $0 \leqq k \leqq n$ $X_k^\alpha \times Y_{k+1}^\beta \subsetneq Z_i^\gamma \in L$ implies that Z_i^γ is either Z_k^α or Z_{k+1}^β. Thus every n-pseudo face lies in at most two (n+1)-cells of L.

proof. From Lemma 1 it is straightforward to see that Z_k^α and Z_{k+1}^β contain $X_k^\alpha \times Y_{k+1}^\beta$. To see that no other cells of L have $X_k^\alpha \times Y_{k+1}^\beta$ suppose that $X_k^\alpha \times Y_{k+1}^\beta \subsetneq Z_i^\gamma = X_i^\gamma \times Y_i^\gamma$. Then $X_k^\alpha \subsetneq X_i^\gamma$ and $Y_{k+1}^\beta \subsetneq Y_i^\gamma$ together imply that $k \leqq i \leqq k+1$. Hence we obtain that Z_i^γ is either Z_k^α or Z_{k+1}^β. Q.E.D.

Let us use $|L|$ to denote the union of all cells of L.

Corollary 3.

$$|\partial L| = (\{0\} \times R^n) \cup (R^n \times \{0\}).$$

proof. From the lemma above we see that all n-pseudo faces each of which lies in exactly one (n+1)-cell of L are

$$X_0 \times Y_1^{+1},\ X_0 \times Y_1^{-1},\ X_n^{+1} \times Y_{n+1} \text{ and } X_n^{-1} \times Y_{n+1}.$$

This together with the fact that $X_n^{+1} \cup X_n^{-1} = Y_1^{+1} \cup Y_1^{-1} = R^n$ implies the desired result. Q.E.D.

Now define the piecewise linear map h defined on $|L|$ be

$$h(x, y) = f(x) + y.$$

In order to utilize the unifying model for systems of piecewise linear equations by Eaves and Scarf [1] we shall provide a refinement M of L so that each cell of M may be a piece of linearity of h (h is not linear on all Z_k^α). Let T^k ($1 \leqq k \leqq n$) be the set of k-vectors consisting of +1's and -1's. For $1 \leqq k \leqq n$ and $t \in T^k$ define an (n+1)-cell

$$Z_k(t) = \left\{ (x, y): \begin{array}{ll} t_j \cdot x_j \geqq 0 \quad (\forall j \leqq k), & x_j = 0 \ (\forall j > k) \\ t_k \cdot y_k \geqq 0, & y_j = 0 \ (\forall j < k) \end{array} \right\}.$$

Then since x_j ($1 \leqq j \leqq n$) does not alternate its sign while staying in $Z_k(t)$, h(x, y) is linear on each cell $Z_k(t)$. Furthermore it is clear from the definition that $Z_i(t)$ and $Z_j(t')$ have no relatively interior points in common whenever $(i,t) \neq (j,t')$ and that

$$\mathcal{M} = \{ Z_k(t) : t \in T^k, 1 \leqq k \leqq n \}$$

is a refinement of $\mathcal{L}$ with $|\mathcal{M}| = |\mathcal{L}|$. Define S be the set of zeroes of $h: |\mathcal{M}| \to R^n$:

$$S = \{ (x, y) \in |\mathcal{M}| : h(x, y) = 0 \} .$$

Then the next lemma follows directly from Corollary 3.

Lemma 4. $(x, y) \in S \cap |\partial \mathcal{M}|$ if and only if (x, y) is either $(0, -q)$ or $(x, 0)$ such that x provides a solution of LCP.

Here we introduce the nondegenerate assumption.

Assumption 5. The system of linear equations

$$s - M z = q$$

is nondegenerate, i.e. every solution of the system has at least n nonzero components.

It is straightforward to see that if Assumption 5 is satisfied, the solution set S intersects with no faces of $\mathcal{M}$ with dimension lower than n, i.e. zero is a nondegenerate value of $h: |\mathcal{M}| \to R^n$ (see Eaves and Scarf [1] for the definition of nondegenerate value and further results). Therefore the set S turns out to be a disjoint union of 1-dimensional subdivided manifolds each of which is either a path or a loop. Let S^0 be the connected component of S having the boundary point $(0, -q)$ of $\mathcal{M}$. Then S^0 forms a path. If, in addition, S^0 is bounded, it has another boundary point $(x, 0)$ of $\mathcal{M}$ which as Lemma 4 shows provides a solution of LCP. We shall see the following assumption on the matrix M introduced by Van der Heyden [6] ensures the boundedness of S^0.

Assumption 6. Every principal submatrix M^+ of the matrix M has the property that there exists no positive vector z^+ such that the last coordinate of M^+z^+ is nonnegative and the remaining ones are zero.

Lemma 7. If the given matrix M satisfies Assumption 6, $S^0 \cap Z_k(t)$ is bounded for any $1 \leqq k \leqq n$ and any $t \in T^k$.

proof. Suppose on the contrary that $S^0 \cap Z_k(t)$ is unbounded for some $1 \leqq k \leqq n$ and some $t \in T^k$. Since $h(x, y)$ is linear on $Z_k(t)$ we find an unbounded ray in $S^0 \cap Z_k(t)$ whose direction $(\Delta x, \Delta y) \neq 0$ satisfies

$$M \Delta x^+ + I \Delta x^- + I \Delta y = 0$$

and

(2) $$(\Delta x, \Delta y) \in Z_k(t),$$

where Δx^+ (resp. Δx^-) is an n-vector whose i-th component is $\max\{0, \Delta x_i\}$ (resp. $\min\{0, \Delta x_i\}$) and I is the $n \times n$ identity matrix. Let

$$P = \{ i : \Delta x_i > 0, \ 1 \leqq i \leqq n \},$$

then P is not empty since otherwise we would have $\Delta x^- + \Delta y = 0$, which together with (2) implies that $\Delta x = \Delta y = 0$. This is a contradiction. If we note that $P \subseteq \{1,2,\cdots,k\}$, we have

(3) $$M^i \Delta x^+ = 0, \qquad (\forall i \in P \backslash \{k\})$$

(4) $$M^k \Delta x^+ + \Delta y_k = 0,$$

where M^i denotes the i-th row vector of M. If $\Delta x_k > 0$, then $\Delta y_k \geqq 0$ and from (4) $M^k \Delta x^+ \leqq 0$. This is contrary to the assumption on M. If $\Delta x_k \leqq 0$, (3) alone contradicts to the assumption. Q.E.D.

Thus under two assumptions 5 and 6 we can find a solution of LCP by tracing the path S^0 form the initial boundary point (0, -q) to another boundary point of $\mathcal{M}$.

In what follows we shall investigate n-pseudo faces we encounter in tracing the path S^0. Let $(x, y) \in S^0 \cap (X_k^\alpha \times Y_{k+1}^\beta)$ for some $1 \leqq k \leqq n$ and some $\alpha, \beta \in \{+1,-1\}$. Then it is clear that the k-vector $(x_1,x_2,\cdots,x_k)^T$ consisting of the first k components of x solves the k-problem. Thus every time we cross an n-pseudo face, we obtain a solution of a subproblem of LCP. As we have seen in Lemma 1 each (n+1)-cell Z_k^α of $\mathcal{L}$ has four distinct n-pseudo faces: $X_k^\alpha \times Y_{k+1}^{+1}$, $X_k^\alpha \times Y_{k+1}^{-1}$, $X_{k-1}^{+1} \times Y_k^\alpha$, $X_{k-1}^{-1} \times Y_k^\alpha$. We say that the n-pseudo face $X_k^{\alpha} \times Y_{k+1}^{\beta}$ (resp. $X_{k-1}^{\beta} \times Y_k^{\alpha}$) is opposite to $X_k^{\alpha} \times Y_{k+1}^{-\beta}$ (resp. $X_{k-1}^{-\beta} \times Y_k^{\alpha}$). If $\alpha = -1, S^0 \cap Z_k^{\alpha}$ has the following nice property.

Lemma 8. If $\alpha = -1$ and $S^0 \cap Z_k^{\alpha} \neq \phi$, then $S^0 \cap Z_k^{\alpha}$ is a disjoint union of linear line segments each of which connects two n-pseudo faces of Z_k^{α} which are not opposite to each other.

proof. Suppose $S^0 \cap Z_k^{\alpha}$ has a point $(x, y) \in X_{k-1}^{\beta} \times Y_k^{-1}$ for some $\beta \in \{+1,-1\}$. Then by Assumption 5

$$x = (x_1, x_2, \cdots, x_{k-1}, 0, 0, \cdots, 0),$$

$$y = (0, 0, \cdots, 0, y_k, y_{k+1}, \cdots, y_n),$$

$$x_j \neq 0 \quad (\forall j \leqq k-1), \qquad y_j \neq 0 \quad (\forall j \geqq k),$$

and especially $y_k < 0$. Let

$$x' = (x_1, x_2, \cdots, x_{k-1}, y_k, 0, \cdots, 0),$$

$$y' = (0, 0, \cdots, 0, 0, y_{k+1}, \cdots, y_n),$$

then clearly $(x', y') \in X_k^{-1} \times Y_{k+1}^{\gamma}$ for $\gamma = \text{sign } y'_{k+1} = \text{sign } y_{k+1}$ and $\lambda (x, y) + (1 - \lambda)(x', y') \in S^0 \cap Z_k^{\alpha}$ for any $0 \leqq \lambda \leqq 1$. Q.E.D.

We illustrate Z_k^{-1} surrounded by four n-pseudo faces of L in Figure 1, where S^0 is depicted by the darkened line. It is noteworthy that we can continue tracing the path S^0 by repeating the same procedure in the proof of the above lemma until the first nonzero component of y' becomes positive.

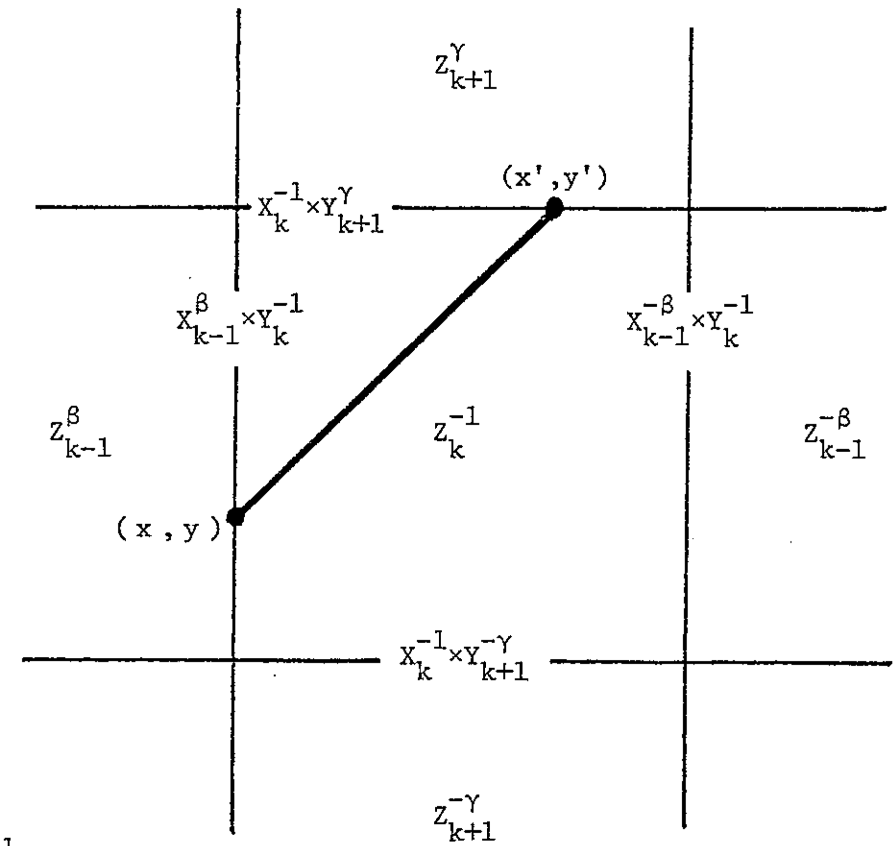
Now consider the case where the path S^0 intersects with some Z_k^{+1}. Since Z_k^{+1} is not a piece of linearity of h, $S^0 \cap Z_k^{+1}$ possibly returns to another point in the starting n-pseudo face. Then one of the following cases occurs:

(i) $S^0 \cap Z_k^{+1}$ connects two points of an n-pseudo face of Z_k^{+1},

(ii) $S^0 \cap Z_k^{+1}$ connects two opposite n-pseudo faces of Z_k^{+1},

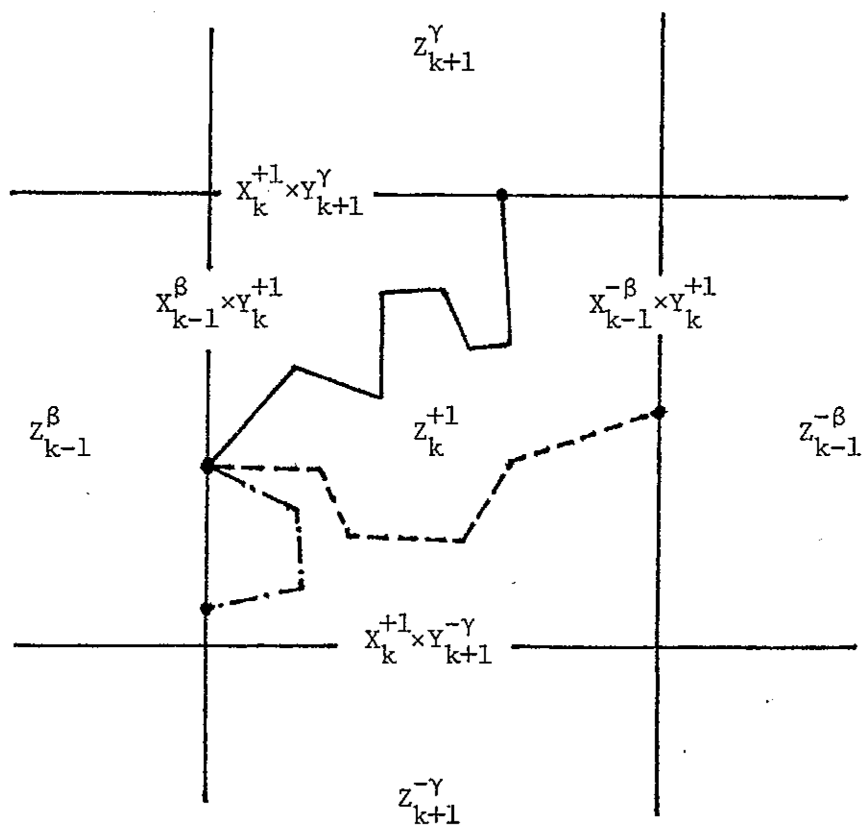
(iii) $S^0 \cap Z_k^{+1}$ connects two n-pseudo faces of Z_k^{+1} which are not opposite to each other.

If either (i) or (ii) occurs, two end points of $S^0 \cap Z_k^{+1}$ provide two distinct solutions of the same subproblem of LCP, either (k-1)- or k-problem. If (iii) occurs, one end point corresponds to the solution of the (k-1)-problem and the other to the k-problem. In this way the dimension of the subproblem being solved changes as we trace the path S^0. Figure 2 summarizes the above three cases.



Z_{k+1}^{γ}
(x',y')
$X_k^{-1} \times Y_{k+1}^{\gamma}$
$X_{k-1}^{\beta} \times Y_k^{-1}$
$X_{k-1}^{-\beta} \times Y_k^{-1}$
Z_{k-1}^{β}
Z_k^{-1}
$Z_{k-1}^{-\beta}$
(x , y)
$X_k^{-1} \times Y_{k+1}^{-\gamma}$
$Z_{k+1}^{-\gamma}$

Fig. 1.



Z_{k+1}^{γ}
$X_k^{+1} \times Y_{k+1}^{\gamma}$
$X_{k-1}^{\beta} \times Y_k^{+1}$
$X_{k-1}^{-\beta} \times Y_k^{+1}$
Z_{k-1}^{β}
Z_k^{+1}
$Z_{k-1}^{-\beta}$
$X_k^{+1} \times Y_{k+1}^{-\gamma}$
$Z_{k+1}^{-\gamma}$

Fig. 2.

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