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by

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**Abstract**

We examine accuracy of an approximation of the nonlinear term structure of interest rates, which is obtained by applying local linear approximation to a generally specified process of the short-rate. We use various underlying short-rate models that are well studied in the literature to see the accuracy, which is measured by comparing the approximate model with benchmark calculated by the Monte Carlo method. We also examine accuracy of the model that is extended to a multi-variate case. The numerical experiments show that yields produced by the model are close to those by the Monte Carlo method over the reasonable range of underlying states.

Keywords: term structure, nonlinear, local linear approximation, multi-variate, Monte Carlo

# 1 Introduction

The short-rate is a key variable for contingent claim valuation including the term structure of interest rates and many researchers have examined its stochastic behavior. In particular, Chan, Karolyi, Longstaff and Sanders (hereafter CKLS) (1992) considers a process of the short-rate $r$ with a diffusion term of the form $\sigma r^{\gamma}$, so called a constant-elasticity-variance (CEV) model. The estimate of $\gamma$ is around 1.5, which is significantly different from 0.5 assumed in Cox, Ingersoll and Ross (hereafter CIR) (1985). Thus, the short-rate models in Vasicek (1977) ($\gamma = 0$) as well as CIR are strongly rejected against the CEV model with unconstrained $\gamma$. This may suggest that the descriptive power of these closed-form term structure models is also poor. Similar results are obtained by, for example, Broze et al. (1995), Tse (1995), Dahlquist (1996), and Nowman (1997). In addition, Aït-Sahalia (1996) finds that nonlinearity in the drift is the key to matching a parametric model with the marginal density implied by actual data. Though statistical techniques for estimation are different, Conley et al. (1997), and Stanton (1997) also report the nonlinear drift.

In order to value the term structure accurately, it is important to model the short-rate consistently with its observed behavior. However, tractable models cannot be obtained in such settings. Recently, Takamizawa and Shoji (2001, 2002) propose approximations of the nonlinear term structure that is consistent with stochastic behavior of the short-rate examined in the above studies. They derive an analytical model by applying local linear approximation to the stochastic differential equation for a short-rate process. If the approximate model is as accurate as some numerical techniques for valuation, its tractability is a significant advantage. Therefore, we investigate accuracy of local linear approximation for modeling the nonlinear term structure of interest rates. The main contributions of this paper are as follows.

I. accuracy of the approximate model is systematically investigated under several short-rate models which are well studied in the literature. These include not only the CEV models such as Brennan and Schwartz (1979), CKLS (1992), and Conley et al. (1997), but also more general models with nonlinear drift terms such as Aït-Sahalia (1996).

II. accuracy of the approximate model which is extended to a multi-variate case is examined.

These are not considered sufficiently in Takamizawa and Shoji (2001, 2002). In fact, only particular short-rate models are used in these papers; a short-rate model with a cubic drift and a square-root volatility in the former paper and the same CEV model as CKLS in the latter. Thus, the accuracy of the approximate model is not systematically investigated by using several short-rate models. Moreover, they assume that the short-rate is a single state variable for modeling the term structure. Thus, multi-variate models are not provided.

We measure accuracy by comparing yields produced by the approximate model with those calculated by the Monte Carlo method, which is one of the most popular numerical tools for contingent-claim valuation. We present in this paper the extension of the approximate model originally derived in a uni-variate case to a multi-variate counterpart. In contrast with a uni-variate case where no particular restrictions on dynamics of a state variable (the short-rate) are required, we have to impose some restrictions on state processes in a multi-variate case even though much more flexibility is assured than existing closed-form term structure models. We show admissible specification for the approximate model by utilizing as guides the intensive studies on affine models by Duffie and Kan (1996), and Dai and Singleton (2000). This is because in deriving the approximate model, we need to specify state processes in such a way that by applying the local linear approximation these processes become similar to those specified in affine models. Accuracy in the multi-variate case is examined by an illustrative two-factor model.

The rest of the paper is organized as follows. Section 2 provides brief explanation about the local linear approximation and derives the analytical model of the nonlinear term structure. Section 3 examines accuracy of the approximate term structure under several short-rate models. Section 4 presents extension to the multi-variate case and examines its accuracy. Section 5 concludes.

## 2 Derivation of analytical approximation for the nonlinear term structure

The short-rate $r$ in the risk neutral measure is given by the solution to the following stochastic differential equation (SDE):

$$dr = \mu(r,t)dt + \sigma(r,t)\, dW_t \,, \tag{1}$$

where $\{W_t\}$ is Brownian motion in $\mathbf{R}$, and where the drift and diffusion functions are defined as follows; $\mu : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$, $\sigma : \mathbf{R}_+ \times [0,T] \to \mathbf{R}_+$. We assume that both $\mu$ and $\sigma$ are well specified so that the solution to (1) exists for an arbitrary initial state. Given information at time $t$, applying the local linear approximation to the process $\{r_u : t \leq u \leq t + \Delta\}$ leads to the following SDE:

$$d\tilde{r}_u = \{a_2(t)\tilde{r}_u + a_1(t)u + a_0(t)\}\, du + \sqrt{b_2(t)\tilde{r}_u + b_1(t)u + b_0(t)}\, dW_u \,, \tag{2}$$

where

$$a_2(t) \equiv \frac{\partial \mu}{\partial r}(r,t) \,,$$

$$
\begin{aligned}
a_1(t) &\equiv \tfrac{1}{2}\frac{\partial^2 \mu}{\partial r^2}(r,t)\sigma^2(r,t) + \frac{\partial \mu}{\partial t}(r,t)\ , \\
a_0(t) &\equiv \mu(r,t) - a_2(t)r_t - a_1(t)t\ , \\
b_2(t) &\equiv \frac{\partial \sigma^2}{\partial r}(r,t)\ , \\
b_1(t) &\equiv \tfrac{1}{2}\frac{\partial^2 \sigma^2}{\partial r^2}(r,t)\sigma^2(r,t) + \frac{\partial \sigma^2}{\partial t}(r,t)\ , \\
b_0(t) &\equiv \sigma^2(r,t) - b_2(t)r_t - b_1(t)t\ .
\end{aligned}
$$

Note that $\{\tilde{r}_u : t \le u \le t+\Delta\}$ represents the approximated process and that $\tilde{r}_t = r_t$ holds by construction. Let $\tilde{P}(r,t,t+\Delta)$ be discount bond price under the process (2) at time $t$ with maturity date $t+\Delta$. Then, $\tilde{P}$ satisfies the following partial differential equation:

$$
\begin{aligned}
&\tfrac{1}{2}\left\{b_2(t)\tilde{r}_u + b_1(t)u + b_0(t)\right\}\frac{\partial^2 \tilde{P}}{\partial \tilde{r}^2}(\tilde{r},u,t+\Delta) \\
&+ \ \left\{a_2(t)\tilde{r}_u + a_1(t)u + a_0(t)\right\}\frac{\partial \tilde{P}}{\partial \tilde{r}}(\tilde{r},u,t+\Delta) \\
&+ \ \frac{\partial \tilde{P}}{\partial u}(\tilde{r},u,t+\Delta) - \tilde{r}\tilde{P}(\tilde{r},u,t+\Delta) = 0\ ,
\end{aligned} \tag{3}
$$

with the boundary condition $\tilde{P}(\tilde{r},t+\Delta,t+\Delta) = 1$. Since all the coefficients of partial derivatives of $\tilde{P}$ become linear functions of $\tilde{r}$, equation (3) has the following analytical solution:

$$
\tilde{P}(r_t,\Delta) = A(\Delta)\exp(-B(\Delta)r_t)\,, \tag{4}
$$

where

$$
B(\Delta) = \frac{2(e^{\gamma\Delta}-1)}{g(\Delta)}\ ,
$$

$$
A(\Delta) = \left[\frac{2\gamma e^{k_1\Delta/2}}{g(\Delta)}\right]^{\frac{2}{b_2^2}(b_2\mu(r,t)-a_2\sigma^2(r,t)+b_1)} \times (2\gamma)^{2k_3\Delta/b_2^2}
$$

$$
\times \exp\left\{\frac{1}{b_2}(\sigma^2(r,t)-b_2r)(\Delta - B(\Delta)) + \frac{\Delta^2}{2b_2^2}(b_1b_2 + k_1k_3)\right\}
$$

$$\times \exp\left\{-\frac{2k_3}{b_2^2}\int_0^\Delta \ln g(u)du\right\} ,$$

$$\begin{aligned}
\gamma &= (2b_2 + a_2^2)^{0.5} , \\
g(\Delta) &= k_1 e^{\gamma\Delta} + k_2 , \\
k_1 &= \gamma - a_2 , \\
k_2 &= \gamma + a_2 , \\
k_3 &= b_2 a_1 - b_1 a_2 .
\end{aligned}$$

Note that convergence of $\tilde{P}(r_t, \Delta)$ to the true price is guaranteed; as $\Delta \to 0$, we have $A(\Delta) \to 1$ and $B(\Delta) \to 0$ so that $\tilde{P}(r_t, \Delta) \to 1$ holds. This is simply because $A(\Delta)$ and $B(\Delta)$ are the solutions to the ordinary differential equations with boundary conditions $A(0) = 1$ and $B(0) = 0$, which satisfy $\tilde{P}(r_t, 0) = 1$. In the next section, we examine accuracy of the approximate model at finite maturities.

## 3 Accuracy of the approximate term structure model

The order of approximate error for $E_t[|\tilde{r}_{t+\Delta} - r_{t+\Delta}|^p]$ is proven to be $O(\Delta^{2p})$ in $L^p$ space; see Shoji (1998). However, the order of approximate error for $\tilde{P}(r_t, \Delta)$, or equivalently its yield to maturity, $\tilde{Y}(r_t, \Delta) \equiv \frac{-1}{\Delta} \ln \tilde{P}(r_t, \Delta)$, is not yet examined formally.

Let $Y(r, t, t + \Delta)$ be the true (benchmark) yield of a discount bond under the process (1). Then, we measure accuracy by calculating

$$|\tilde{Y}(r_t, \Delta) - Y(r_t, t, t + \Delta)| \equiv \left|\tilde{Y}(r_t, \Delta) - \frac{-1}{\Delta} \ln E_t\left[\exp\left\{-\int_t^{t+\Delta} r_u du\right\}\right]\right| . \quad (5)$$

The benchmark yields are obtained by the Monte Carlo method; we generate paths of the short-rate in the risk neutral measure of the size 50,000 with a discretization step 1/480, which corresponds to approximately two observations per trading day. The antithetic variable technique is used to reduce sampling errors. In the numerical experiments to follow, we present differences before taking absolute values since they are more informative.

We consider several short-rate models taken from Brennan and Schwartz (1979), CKLS (1992), Takamizawa and Shoji (2001), Conley et al. (1997), and Aït-Sahalia (1996):

$$
\begin{aligned}
\text{(Brennan \& Schwartz)} \quad dr &= (\alpha_1 r + \alpha_0)dt + \sqrt{\beta_2 r^2}\, dZ_t\,, \\
\text{(CKLS)} \quad dr &= (\alpha_1 r + \alpha_0)dt + \sqrt{\beta_2 r^{\beta_3}}\, dZ_t\,, \\
\text{(Takamizawa \& Shoji)} \quad dr &= (\alpha_3 r^3 + \alpha_2 r^2 + \alpha_1 r + \alpha_0)dt + \sqrt{\beta_1 r}\, dZ_t\,, \\
\text{(Conley et al.)} \quad dr &= (\alpha_2 r^2 + \alpha_1 r + \alpha_0 + \alpha_{-1} r^{-1})dt + \sqrt{\beta_2 r^{\beta_3}}\, dZ_t\,, \\
\text{(Aït-Sahalia)} \quad dr &= (\alpha_2 r^2 + \alpha_1 r + \alpha_0 + \alpha_{-1} r^{-1})dt + \sqrt{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\, dZ_t\,.
\end{aligned}
$$

It is noted that these models can be classified into three as follows; (Brennan & Schwartz) and (CKLS) belong to a class with the linear drift and CEV (CEV-LD), (Takamizawa & Shoji) and (Conley et al.) belong to a class with the nonlinear drift and CEV (CEV-ND), and (Aït-Sahalia) belongs to a class with the non-CEV.

We assume for simplicity that the market price of risk is constant in each model, say $\lambda$. Then, to preclude arbitrage opportunities, the risk premium is proportional to the diffusion term. In the risk neutral measure, the drift term in each model is replaced with $\mu(r) - \lambda\sigma(r)$, where $\mu(r)$ and $\sigma(r)$ are the drift and diffusion terms, respectively, in the actual measure.

We prepare for two different sets of parameter values to obtain robust results. Panel A of table 1 presents the first set of parameters that are estimated by using weekly data on Treasury-bill yields. The estimation scheme consists of two steps. The first step is to estimate all the parameters except $\lambda$ by using time-series data on the short-rate, which is proxied by the three-month yield from 1/8/1954 to 12/28/2001 (2504 observations). We apply the Euler discretization technique to continuous-time models and estimate parameters by the maximum likelihood method. [1] While the Euler method is known to be crude, it is useful enough to achieve our purpose here for obtaining reasonable parameter values. The second step is to estimate $\lambda$ by using cross-sectional data on six-month and twelve-month yields, which are also weekly sampled from 7/17/1959 to 8/24/2001 (2198 observations). We derive under each short-rate process the approximate term structure model, which contains the unknown parameter $\lambda$. By introducing measurement errors that are independently and identically distributed, we match the model with yield data so as to minimize the errors. We also employ the maximum likelihood method.

Panel B of table 1 presents the second set of parameters that are obtained from other papers, except $\lambda$ which is estimated in the same way as just mentioned above. The sources are shown at the bottom row. It is noted that proxy variables, sample periods, and estimation methods are different among papers. For example, CKLS (1992) uses monthly data on one-month Treasury-bill yield from Jun 1964 to December 1989 and employ the generalized method of moments after discretizing continuous-time models by the Euler scheme. Takamizawa and Shoji (2001) uses monthly data on three-month Treasury-bill yield from January 1971 to December 1993 and also use the Euler discretization scheme for estimation by the maximum likelihood method. Aït-Sahalia (1999) uses monthly data on the Federal fund rate from January 1963 to December 1998 and estimates parameters by the maximum likelihood method in which the technique for analytical approximation to the density function is utilized. In contrast with these studies, Aït-Sahalia (1996) uses daily data on seven-day Eurodollar deposit rate from 6/1/1973 to 2/25/1995 and estimates parameters by minimizing distance between the marginal density implied by a parametric model and that estimated from data by the nonparametric technique. The notable differences between the first and second parameter sets are as follows; in the second case, (i) for (Brennan & Schwartz), (CKLS) and (Takamizawa & Shoji), degree of mean reversion is strengthened, (ii) for (Aït-Sahalia), degree of mean reversion at high interest-rate levels is weakened, and (iii) for (CKLS) and (Conley et al.), the estimates of $\beta_3$ become more than double, so that the local variance is more elastic to the level of $r$. Therefore, our interest lies in whether the accuracy is confirmed regardless of the differences between the two parameter sets.

We value the term structure corresponding to various levels of $r$. We choose $r = (0.03, 0.06, 0.12)$. The middle value is roughly the sample average of three-month Treasury-bill yield used for estimation, and the low and high values are half and double the middle value, respectively. The following maturity points $\tau$ are chosen; two-week (2W), one-month (1M), three-month (3M), six-month (6M), twelve-month (12M) and two-year (2Y). The reason we limit the maturity range up to two years is that in modeling the term structure by a single state variable, *i.e.*, the short-rate, it seems sufficient to explain the short-end of the term structure. The accuracy of the approximate term structure at the long end is examined in a multi-variate case in Section 4.

Table 2 shows differences of yields in basis point (bp) between the approximate model and the Monte Carlo method for the first parameter set. We report the results according to the three classes. First, under (CEV-LD), there are no noticeable deviations even at $\tau = 2Y$; the maximum deviation in this class is 0.3 bp under (Brennan & Schwartz) at $r = 0.12$ and $\tau = 2Y$. Second, under (CEV-ND), deviations at $\tau = 12M$ are small except for (Takamizawa & Shoji, -2.5 bp) at $r = 0.12$. Deviations are not ignorable at $r = 0.12$ and $\tau = 2Y$; (Takamizawa & Shoji, -13.4 bp) and (Conley et al., -3.3 bp). On the other hand, when the short-rate is low, deviations under both short-rate models are negligible even at $\tau = 2Y$. Moreover, the deviation at $r = 0.06$ and $\tau = 2Y$ under (Conley et al., 1.4 bp) is not large. Third, under (Aït-Sahalia), while the deviation at $r = 0.12$ and $\tau = 12M$ is -2.9 bp, deviations at other levels of $r$ are within 0.5 bp. At $\tau = 2Y$, however, deviations are large particularly at $r = 0.12$. From Table 2, we can deduce that the approximate model deviates from the benchmark by the Monte Carlo method under short-rate models with nonlinear drift. More specifically, comparing the results between (Takamizawa & Shoji) and (Conley et al.), the greater is the degree of mean reversion at high interest-rate levels, the larger is the deviation at $r = 0.12$. It is also noted that the approximate model underestimates yields at $r = 0.12$ under these short-rate models. This indicates that the approximate model has tendency to overvalue the degree of mean reversion at high interest-rate levels, so that the yield predicted by the approximate model at $\tau = 2Y$ is closer to the mean value than the yield by the Monte Carlo method.

Table 3 presents differences of yields between the approximate model and the Monte Carlo method for the second parameter set. While the overall pattern of results is the same as provided in Table 2, the two points that differ are worth noting. First, deviations at $\tau = 2Y$ are larger under (CKLS), (Takamizawa & Shoji) and (Conley et al.). Since the second parameter set implies faster mean reversion for these short-rate models, the aforementioned tendency for the approximate model to overvalue the mean reversion works more strongly in the second case. Second, deviations under (Aït-Sahalia) are negligible even at $\tau = 2Y$. This result is just the converse of the former result; almost no mean reversion is implied at high interest-rate levels so that the tendency of the approximate model has no actual effect.

As a summary for the uni-variate case, the approximate model is accurate at the short-end of the term structure up to one-year maturity, which is an appropriate range for the uni-variate model to explain. While some caution is needed when we use nonlinear drift models that exhibit faster mean reversion at high interest-rate levels, [2] yields produced by the approximate model are close to those by the Monte Carlo method. In fact, by using linear drift models, deviations are negligible even at two-year maturity.

# 4 Extension to a multi-variate model and analysis on its accuracy

In contrast with the uni-variate case, we have to impose some restrictions on dynamics of state-variable processes in order to derive the analytical model of the nonlinear term structure. We present the admissible specification for a vector state-variable process $X$ in $\mathbf{R}^n$, which is the solution to the following SDE:

$$dX = \mu(X,t)dt + \Sigma\sigma(X,t)dW_t\,, \tag{6}$$

where $\{W_t\}$ is a vector of orthogonal Brownian motions in $\mathbf{R}^n$, and where the drift and diffusion coefficients satisfy the following; $\mu : \mathbf{R}^n \times [0,T] \rightarrow \mathbf{R}^n$ , $\Sigma : n \times n$ matrix of constants, and $\sigma : n \times n$ diagonal matrix with the $i$th element $\sigma_i : \mathbf{R}^n \times [0,T] \rightarrow \mathbf{R}$. The specification for the diffusion term is similar with that given in affine term structure models. In fact, Duffie and Kan (1996), and Dai and Singleton (2000) specify $\sigma\sigma'$ to be a diagonal matrix with the $i$th element $\sigma_i^2$ linear in $X$ for all $i$. Then, We can apply the local linear approximation to $\mu(X,t)$ and $\sigma_i^2(X,t)$ $(i = 1, ..., n)$, and the partial differential equation valuing a discount bond has coefficients all of which are linear in $X$ and $t$ as before. We note that in order to derive the analytical model, it is also required that the short-rate is a linear function of $X$. That is,

$$r = w_0 + w'X\,. \tag{7}$$

We present an illustrative two-factor model and examine its accuracy. We consider the following SDE of state-variable processes:

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \left\{ \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} \alpha_{10} \\ \alpha_{20} \end{pmatrix} \right\} dt + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}, \tag{8}$$

with $r = w_0 + w_1 X_1 + w_2 X_2$. We assume the linear drift because no plausible empirical findings about nonlinear drift in a multi-variate case have been reported. We also assume that the risk premiums are zero or that they are included in each of the drift terms. This model has the advantage in that the following two properties are compatible; one is time-varying volatilities which are attributed to both $X_1$ and $X_2$, and the other is the instantaneous correlation between the two. Note that affine models cannot accommodate both properties together; see Dai and Singleton (2000). Since the drift terms are linear, we apply the local linear approximation to the instantaneous variances $X_i^2\,(i = 1, 2)$ given information at time $t$. Then, we obtain the following SDE of $\{\tilde{X}_u\,; t \le u \le t + \Delta\}$:

$$\begin{pmatrix} d\tilde{X}_{1,u} \\ d\tilde{X}_{2,u} \end{pmatrix} = \left\{ \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{pmatrix} \begin{pmatrix} \tilde{X}_{1,u} \\ \tilde{X}_{2,u} \end{pmatrix} + \begin{pmatrix} \alpha_{10} \\ \alpha_{20} \end{pmatrix} \right\} dt$$

$$+ \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sqrt{b_1(t)\tilde{X}_{1,u} + c_1(t)u + d_1(t)} & 0 \\ 0 & \sqrt{b_2(t)\tilde{X}_{2,u} + c_2(t)u + d_2(t)} \end{pmatrix} \begin{pmatrix} dW_{1,u} \\ dW_{2,u} \end{pmatrix}, \tag{9}$$

where

$$b_i(t) \equiv 2X_{i,t}\,,$$

$$
\begin{aligned}
c_i(t) &\equiv \sigma_{i1}^2 X_{1,t}^2 + \sigma_{i2}^2 X_{2,t}^2 , \\
d_i(t) &\equiv -X_{i,t}^2 - c_i(t)t \qquad (i = 1, 2) ,
\end{aligned}
$$

and where by construction $X_t = \tilde{X}_t$ holds. Under the process (9), we assume the functional form of the solution for discount bond price to be

$$
\tilde{P}(\tilde{X}_1, \tilde{X}_2, u, t + \Delta) = \exp(-A(u, t + \Delta) - B_1(u, t + \Delta)\tilde{X}_1 - B_2(u, t + \Delta)\tilde{X}_2) . \quad (10)
$$

Then, we have the following system of ordinary differential equations for $A, B_1$ and $B_2$:

$$
\begin{aligned}
\frac{dB_i(u, t + \Delta)}{du} &= \tfrac{1}{2} b_i(t) \left\{ \sigma_{1i}^2 B_1(u, t + \Delta)^2 + \sigma_{2i}^2 B_2(u, t + \Delta)^2 + 2\sigma_{1i}\sigma_{2i} B_1(u, t + \Delta) B_2(u, t + \Delta) \right\} \\
&\quad -\alpha_{ii} B_i(u, t + \Delta) - w_i = 0 \qquad (i = 1, 2) , \quad (11) \\
\frac{dA(u, t + \Delta)}{du} &= \sum_{i=1}^{2} \tfrac{1}{2}(c_i(t)(u - t) - \tilde{X}_{i,t}^2) \\
&\quad \times \left\{ \sigma_{1i}^2 B_1(u, t + \Delta)^2 + \sigma_{2i}^2 B_2(u, t + \Delta)^2 + 2\sigma_{1i}\sigma_{2i} B_1(u, t + \Delta) B_2(u, t + \Delta) \right\} \\
&\quad - \sum_{i=1}^{2} \alpha_{i0} B_i(u, t + \Delta) - w_0 = 0 , \quad (12)
\end{aligned}
$$

with the boundary conditions $B_i(t+\Delta, t+\Delta) = 0$ $(i = 1, 2)$ and $A(t+\Delta, t+\Delta) = 0$. The numerical integration to (11) and (12) on $u \in [t, t+\Delta]$ provides $A(\Delta)$ and $B_i(\Delta)$ $(i = 1, 2)$, which are now substituted into (10) to obtain $\tilde{P}(X_{1,t}, X_{2,t}, \Delta)$.

In order to calculate benchmark yields by the Monte Carlo method, we generate paths of the state variables of the size 25,000 with a discretization step 1/480, and also use the antithetic variable technique to reduce the sampling errors. We regard $X_1$ and $X_2$ as the level and slope factors, respectively. In particular, we measure the level by long yield and the slope by the difference between long and short yields. Parameter values are

$$
\begin{aligned}
(\alpha_{11}, \alpha_{10}, \sigma_{11}, \sigma_{12}) &= (-0.09, 0.007, 0.13, 0) , \\
(\alpha_{22}, \alpha_{20}, \sigma_{21}, \sigma_{22}) &= (-0.50, 0.007, 0.01, 0.73) , \\
(w_0, w_1, w_2) &= (0, 1.07, -0.45) . \quad (13)
\end{aligned}
$$

They are roughly the estimates by using actual time-series data (for estimation we set $\sigma_{12} = w_0 = 0$ in advance). Note that these estimates are reasonable in that the first factor $X_1$ exhibits slower mean reversion whereas the second factor $X_2$ has faster one, which is reported by Duffie and Singleton (1997), and Duffee (1999).

We value the term structure at the following pairs of realized states; $(X_1, X_2) = (0.04, 0.01)$, (0.08, 0.00), (0.08, 0.015), (0.08, 0.03), (0.12, -0.02), (0.12, 0.02), (0.12, 0.04). The first pair corresponds to the low interest-rate level with the moderate slope. The next three pairs correspond to the moderate level with flat, moderate and steep slopes, respectively. The last three pairs correspond to the high level with downward, moderate and steep slopes, respectively. The maturity points $\tau$ we consider are two-week (2W), one-month (1M), three-month (3M), six-month (6M), twelve-month (12M), two-year (2Y), three-year (3Y), five-year (5Y), seven-year (7Y) and ten-year (10Y).

Table 4 presents differences of yields in basis point between the approximate model and the Monte Carlo method in the two-factor framework. The approximate model is accurate over the entire range of maturities. In particular, deviations at $\tau$ = 10Y are well within 1 bp in most of the cases. Exceptions are the cases of (0.08, 0.03) at $\tau$ = 10Y and (0.12, 0.04) at $\tau$ = 7Y and 10Y, both of which correspond to the steep-slope case. However, even in these cases deviations are within -0.6 bp up to $\tau$ = 7Y and $\tau$ = 5Y, respectively.

# 5 Conclusion

We examine accuracy of local linear approximation for modeling the nonlinear term structure of interest rates proposed by Takamizawa and Shoji (2001, 2002). We compare yields produced by the approximate model with those by the Monte Carlo method. The accuracy is examined in both uni-variate and multi-variate cases. Numerical experiments show that up to one-year maturity which is a reasonable range for uni-variate models to explain, the approximate model is accurate although some caution is needed when we use nonlinear drift models at high interest rate levels. Under the linear drift models, deviations from the benchmark by the Monte Carlo method are very small at two-year maturity. In the multi-variate case, we provide admissible specification for deriving the analytical model of the nonlinear term structure. We also provide the illustrative two-factor model which has the linear drift terms and flexible covariance structure. The accuracy is confirmed over the entire range of maturities up to ten years. Therefore, in terms of accuracy, the approximate model examined in this paper is reliable for valuing the term structure. In terms of numerical tractability, the model has much more advantage. The model can be utilized for inferring state processes (including those of default events) implicit in the (credit-risky) term structure, which is left for future research.

## Endnote

1 The first step of the estimation scheme for (Aït-Sahalia) is slightly altered. We first estimate drift parameters by OLS. Then, we employ the maximum likelihood method to estimate diffusion parameters by utilizing squared residuals from OLS.

2 Pritsker (1998), and Chapman and Pearson (2000) point out that nonlinearity in the drift is not a compelling feature. Therefore, the deviations produced under nonlinear drift models may not be a serious problem practically.

## Reference


Aït-Sahalia, Y., 1996, Testing continuous-time models of the spot interest rate, *Review of Financial Studies* 9, 385-426.

Aït-Sahalia, Y., 1999, Transition densities for interest rate and other nonlinear diffusions, *Journal of Finance* 54, 1361-1395.

Brennan, M. J., and E. S. Schwartz, 1979, A continuous time approach to the pricing of bonds, *Journal of Banking and Finance* 3, 133-155.

Broze, L., O. Scaillet, and J. Zakoïan, 1995, Testing for continuous-time models of the short-term interest rate, *Journal of Empirical Finance* 2, 199-223.

Chan, K. C., G. A. Karolyi, F. A. Longstaff, and A. B. Sanders, 1992, An empirical comparison of alternative models of the short-term interest rate, *Journal of Finance* 47, 1209-1227.

Chapman, D. A., and N. D. Pearson, 2000, Is the short rate drift actually nonlinear?, *Journal of Finance* 55, 355-388.

Conley, T. G., L. P. Hansen, E. G. J. Luttmer, and J. A. Scheinkman, 1997, Short-term interest rates as subordinated diffusions, *Review of Financial Studies* 10, 525-577.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385-407.

Dahlquist, M., 1996, On alternative interest rate processes, *Journal of Banking and Finance* 20, 1093-1119.

Dai, Q., and K. J. Singleton, 2000, Specification analysis of affine term structure models, *Journal of Finance* 55, 1943-1978.

Duffee, G. R., 1999, Estimating the price of default risk, *Review of Financial Studies* 12, 197-226.

Duffie, D., and R. Kan, 1996, A yield-factor model of interest rates, *Mathematical Finance* 6, 379-406.

Duffie, D. and K. J. Singleton, 1997, An econometric model of the term structure of interest-rate swap yields, *Journal of Finance* 52, 1287-1321.

Nowman, K. B., 1997, Gaussian estimation of single-factor continuous time models of the term structure of interest rates, *Journal of Finance* 52, 1695-1706.

Pritsker, M., 1998, Nonparametric density estimation and tests of continuous time interest rate models, *Review of Financial Studies* 11, 449-487.

Shoji, I., 1998, Approximation of continuous time stochastic processes by a local linearization method, *Mathematics of Computation* 67, 287-298.

Stanton, R., 1997, A nonparametric model of term structure dynamics and the market price of interest rate risk, *Journal of Finance* 52, 1973-2002.

Takamizawa, H., I. Shoji, 2001, Approximation of nonlinear term structure models, *Journal of Derivatives* 8, 44-51.

Takamizawa, H., and I. Shoji, 2002, Modeling the term structure of interest rates with general short-rate models, forthcoming in *Finance & Stochastics*.

Tse, Y. K., 1995, Some international evidence on the stochastic behavior of interest rates, *Journal of International Money and Finance* 14, 721-738.

Vasicek, O., 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177-188.

| | Brennan -Schwartz | CKLS | Takamizawa -Shoji | Conley et al. | Aït-Sahalia |
|---|---|---|---|---|---|
| | | | Panel A: Estimates by using actual data on T-bills | | |
| $\alpha_{-1}$ | - | - | - | 0.00017 | 0.00034 |
| $\alpha_0$ | 0.0073 | 0.0062 | 0.0146 | -0.0168 | -0.0360 |
| $\alpha_1$ | -0.1409 | -0.1114 | -0.9122 | 0.5122 | 0.9820 |
| $\alpha_2$ | - | - | 17.17 | -4.1201 | -6.9402 |
| $\alpha_3$ | - | - | -92.94 | - | - |
| $\beta_0$ | - | - | - | - | 0.00025 |
| $\beta_1$ | - | - | 0.0030 | - | -0.0112 |
| $\beta_2$ | 0.0674 | 0.0101 | - | 0.0102 | 0.2636 |
| $\beta_3$ | 2.0000 | 1.4161 | - | 1.4182 | 2.2012 |
| $\lambda$ | -0.3430 | -0.3750 | -0.3414 | -0.3747 | -0.3255 |

| | | | Panel B: Estimates from original papers | | |
|---|---|---|---|---|---|
| $\alpha_{-1}$ | - | - | - | 0.0007 | 0.00013 |
| $\alpha_0$ | 0.0242 | 0.0408 | 0.0549 | -0.0347 | -0.0046 |
| $\alpha_1$ | -0.3142 | -0.5921 | -2.4550 | 0.6760 | 0.0433 |
| $\alpha_2$ | - | - | 33.70 | -4.0590 | -0.1143 |
| $\alpha_3$ | - | - | -136.4 | - | - |
| $\beta_0$ | - | - | - | - | 0.00011 |
| $\beta_1$ | - | - | 0.0086 | - | -0.0019 |
| $\beta_2$ | 0.1185 | 1.6704 | - | 0.6747 | 0.0097 |
| $\beta_3$ | 2.0000 | 3.0000 | - | 3.0000 | 2.0730 |
| $\lambda$ | -0.0682 | -0.2413 | -0.0822 | -0.1704 | -0.9336 |
| source | CKLS (1992) | CKLS (1992) | Takamizawa -Shoji (2001a) | Aït-Sahalia (1999) | Aït-Sahalia (1996) |

**Table 1. Parameter values**

Panel A displays parameter values estimated by using data on Treasury-bill yields. The estimation scheme consists of two steps. The first step is to estimate all the parameters except $\lambda$ by using time-series data on three-month yield, which is sampled weekly form 1/8/1954 to 12/28/2001 (2504 obs.). The second step is to estimate $\lambda$ by using cross-sectional data on six-month and twelve-month yields from 7/17/1959 to 8/24/2001 (2198 obs.). In both steps, the maximum likelihood method is employed. Panel B displays parameters obtained form other papers with their sources at the bottom row.

| | 2W | 1M | 3M | 6M | 12M | 2Y |
|---|---|---|---|---|---|---|
| $r$ | Panel A: Brennan & Schwartz model (CEV-LD) | | | | | |
| 0.03 | -0.06 | -0.06 | -0.06 | -0.05 | -0.05 | -0.08 |
| 0.06 | -0.04 | -0.03 | -0.06 | -0.02 | -0.03 | 0.00 |
| 0.12 | 0.00 | 0.00 | -0.01 | -0.05 | 0.01 | 0.28 |
| $r$ | Panel B: CKLS model (CEV-LD) | | | | | |
| 0.03 | -0.07 | -0.07 | -0.06 | -0.05 | -0.05 | -0.03 |
| 0.06 | -0.05 | -0.05 | -0.05 | -0.04 | -0.10 | -0.04 |
| 0.12 | -0.01 | -0.02 | -0.02 | 0.01 | 0.00 | -0.05 |
| $r$ | Panel C: Takamizawa & Shoji model (CEV-ND) | | | | | |
| 0.03 | -0.03 | -0.03 | -0.04 | -0.03 | -0.01 | 0.04 |
| 0.06 | -0.07 | -0.07 | -0.06 | -0.02 | 0.29 | 3.30 |
| 0.12 | 0.02 | 0.02 | -0.03 | -0.37 | -2.52 | -13.42 |
| $r$ | Panel D: Conley et al. model (CEV-ND) | | | | | |
| 0.03 | -0.04 | -0.04 | -0.03 | -0.05 | 0.10 | 0.64 |
| 0.06 | -0.07 | -0.07 | -0.08 | -0.03 | 0.16 | 1.40 |
| 0.12 | 0.05 | 0.06 | 0.03 | -0.04 | -0.53 | -3.25 |
| $r$ | Panel E: Aït-Sahalia model | | | | | |
| 0.03 | 0.00 | 0.00 | -0.02 | -0.02 | -0.13 | -1.09 |
| 0.06 | -0.08 | -0.08 | -0.06 | 0.05 | 0.49 | 4.54 |
| 0.12 | 0.03 | 0.06 | 0.04 | -0.42 | -2.89 | -19.92 |

**Table 2. Differences of yields between the approximate model and the Monte Carlo method: the first parameter set**

The table shows differences of yields between the approximate model and the Monte Carlo method, expressed in basis point (bp). We replicate 50,000 paths of the short-rate in the equivalent martingale measure with a discretization step 1/480. Parameters of each short-rate model are shown in Panel A of Table 1. We evaluate yields at the following maturity points; two-week (2W), one-month (1M), three-month (3M), six-month (6M), twelve-month (12M) and two-year (2Y).

| $r$ | 2W | 1M | 3M | 6M | 12M | 2Y |
|---|---|---|---|---|---|---|
| | Panel A: Brennan & Schwartz model (CEV-LD) | | | | | |
| 0.03 | -0.16 | -0.15 | -0.13 | -0.18 | -0.17 | -0.11 |
| 0.06 | -0.07 | -0.07 | -0.06 | -0.14 | -0.14 | 0.20 |
| 0.12 | 0.11 | 0.14 | 0.14 | -0.06 | 0.10 | 0.05 |
| $r$ | Panel B: CKLS model (CEV-LD) | | | | | |
| 0.03 | -0.25 | -0.26 | -0.27 | -0.31 | -0.45 | -1.31 |
| 0.06 | -0.10 | -0.11 | -0.12 | -0.12 | -0.15 | 0.06 |
| 0.12 | 0.19 | 0.21 | 0.22 | -0.02 | 0.59 | 0.75 |
| $r$ | Panel C: Takamizawa & Shoji model (CEV-ND) | | | | | |
| 0.03 | -0.10 | -0.11 | -0.07 | 0.05 | 1.10 | 4.63 |
| 0.06 | -0.01 | -0.01 | -0.01 | 0.13 | 1.02 | 10.29 |
| 0.12 | -0.12 | -0.11 | -0.07 | -0.12 | -2.41 | -28.23 |
| $r$ | Panel D: Conley et al. model (CEV-ND) | | | | | |
| 0.03 | -0.07 | -0.06 | -0.07 | -0.09 | -0.36 | -1.94 |
| 0.06 | -0.05 | -0.05 | -0.05 | -0.05 | 0.08 | 1.12 |
| 0.12 | 0.02 | -0.01 | 0.04 | -0.16 | -0.68 | -5.77 |
| $r$ | Panel E: Aït-Sahalia model | | | | | |
| 0.03 | -0.07 | -0.07 | -0.07 | -0.08 | -0.09 | -0.22 |
| 0.06 | -0.05 | -0.05 | -0.04 | -0.04 | -0.05 | -0.11 |
| 0.12 | -0.04 | -0.04 | -0.04 | -0.04 | -0.03 | -0.02 |

**Table 3. Differences of yields between the approximate model and the Monte Carlo method: the second parameter set**

The table shows differences of yields between the approximate model and the Monte Carlo method, expressed in basis point (bp). We replicate 50,000 paths of the short-rate in the equivalent martingale measure with a time step 1/480. Parameters of each short-rate model are shown in Panel B of Table 1. We evaluate yields at the following maturity points; two-week (2W), one-month (1M), three-month (3M), six-month (6M), twelve-month (12M) and two-year (2Y).

| $\tau$ | $(X_1, X_2)$ = (level, slope) | | | | | | |
|---|---|---|---|---|---|---|---|
| | (0.04, 0.01) | (0.08, 0.00) | (0.08, 0.015) | (0.08, 0.03) | (0.12, −0.02) | (0.12, 0.02) | (0.12, 0.04) |
| 2W | -0.25 | 0.31 | 0.00 | -0.31 | 1.10 | 0.26 | -0.16 |
| 1M | -0.25 | 0.30 | 0.00 | -0.29 | 1.08 | 0.27 | -0.17 |
| 3M | -0.25 | 0.29 | -0.02 | -0.25 | 1.03 | 0.28 | -0.12 |
| 6M | -0.24 | 0.26 | -0.02 | -0.18 | 0.94 | 0.29 | -0.11 |
| 12M | -0.22 | 0.19 | -0.05 | -0.06 | 0.74 | 0.30 | -0.09 |
| 2Y | -0.12 | 0.12 | -0.08 | 0.16 | 0.53 | 0.31 | 0.12 |
| 3Y | -0.07 | 0.10 | -0.11 | 0.20 | 0.45 | 0.25 | 0.06 |
| 5Y | -0.12 | 0.15 | -0.14 | 0.07 | 0.41 | -0.03 | -0.61 |
| 7Y | -0.15 | 0.23 | -0.10 | 0.28 | 0.35 | -0.37 | -1.65 |
| 10Y | -0.21 | 0.27 | -0.46 | -1.43 | -0.01 | -0.91 | -3.70 |

**Table 4. Differences of yields in the two-factor case**

The table shows differences of yields between the approximate model and the Monte Carlo method, expressed in basis point (bp). The approximate model is now extended to a two-factor case, which is given by (10) - (12). For Monte Carlo simulations, we replicate 25,000 paths of the state variables according to the SDE (8) with a discretization step 1/480. Parameters are given by (13). Seven pairs of $(X_1, X_2)$ are considered; the first (second) element corresponds to the level (slope) factor. Maturity points at which yields are evaluated are two-week (2W), one-month (1M), three-month (3M), six-month (6M), twelve-month (12M), two-year (2Y), three-year (3Y), five-year (5Y), seven-year (7Y) and ten-year (10Y).