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Utility Function and Superneutrality of Money on the Transition Path in a Monetary Optimizing Model

by

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Abstract

Fischer [2] has demonstrated, using the constant relative risk aversion family of utility functions, that superneutrality does not generally hold on the transition paths to the Sidrauski [5] steady state, and that on such paths capital accumulation is faster the higher the growth rate of money supply. This note examines the dependence of these propositions by Fischer upon the form of utility functions.

The now-classical monetary optimizing growth model of Sidrauski [5] has recently revived much attention of monetary economists. This model is of particular interest in the literature of money and growth as it exhibits the superneutrality of money[1/] in the long-run steady state within the framework in which infinitely-lived families maximize an intertemporal utility function. The superneutrality of money in turn is known to play an essential role in relation to the proposition (claimed by new classical school such as Sargent and Wallace [4]) that anticipated monetary policy is ineffective when rational expectations are imposed upon market-clearing macroeconomic models.[2/]

Although it has been known that slight modifications of the original Sidrauski framework invalidate the superneutrality result in the long-run steady state[3/] (suggesting that superneutrality results only from the most abstract framework of Sidrauski), a recent paper by Fischer [2] has shed a new light on the monetary superneutrality issue. Namely, relying upon the special family of utility functions, he demonstrated that superneutrality does not generally hold, even within the original framework of Sidrauski, on the transition paths to the long-run steady state. The special family of utility functions Fischer considers are those with constant relative risk aversion:

$$u(c, m) = \frac{(c^{\alpha} m^{\beta})^{1-R}}{1 - R}, \quad \alpha, \beta, R > 0; \ \alpha + \beta \leq 1,$$

where c= per capita consumption and m= per capita real balances.

The two main propositions obtained by Fischer are as follows.

(i) Superneutrality does not hold on transition paths except for the case of unitary risk aversion (R = 1) for which the utility function becomes logarithmic

$$u(c, m) = \alpha \ln c + \beta \ln m.$$

(ii) Again except for the case of R = 1, the rate of capital accumulation is faster the higher the growth rate of money supply.

The purpose of this note is to examine the dependence of these propositons upon the assumption of the form of utility function since Fischer's analyses do not to the full extent explore it, although briefly mentioned in his concluding comment.

As for the first proposition, it is easy to extend Fischer's analysis to obtain that a sufficient condition for the superneutrality result on transition paths is that utility function is separable in c and m, that is the cross partial derivatives are null, $u_{12} = u_{21} = 0$. The logarithmic utility function is a special case which satisfies this condition. It is also easy to demonstrate that, among the constant relative risk aversion family of utility functions, the condition R = 1 is at the same time necessary for the superneutrality result.[4/] However, the derivation of necessary conditions for general utility functions is a difficult task and it is not the main aim of this note.

The second proposition is what Fischer observes yet indescribable in intuitive economic words.[5/] Since the result bears an important implication in actual policy makings, it is worthwhile to examine whether or not it is also valid for other utility functions (except for super-neutral cases). We approach this problem by presenting a "counter example" in which the rate of capital accumulation turns out to be slower the higher the growth rate of money supply.

Section 1 reproduces the Sidrauski-Fischer world with a special form of utility function which, in Section 2, serves as a counter example to the second proposition of Fischer. Although the counter example considered in this note may rely upon too special a case, it is in itself an interesting model because it is capable of capturing an interesting feature of the relation between capital accumulation and inflation. Namely, in Section 3, we discuss a possibility that the moderate or creeping inflation stimulates investment while much higher inflation instead represses investment.

1. The Model

The basic setup of the model is described in Sidrauski [5] and in Fischer [2]. The only difference seen between these two models is the assumption on the expectation-formation hypothesis; Sidrauski uses the traditional adaptive expectation hypothesis while Fischer uses the rational expectation hypothesis or, as the model is deterministic, perfect foresight.

An infinitely-lived family, whose membership grows at a constant rate of n, maximizes a discounted sum of utilities

$$\int_0^\infty u(c,\ m)e^{-\delta t}\ dt,$$

subject to the budget constraint

(1) $\dot{k} + nk + \dot{m} + nm = f(k) + x - \Pi m - c.$

where k is the per capita capital stock, f(k) is per capita income exhibiting neoclassical properties of production function, Π is the expected rate of price inflation, and x is the per capita rate of real money transfers which are regarded by each family to be independent of its own holdings of money.[6/]

Both Sidrauski and Fischer assume that instantaneous utility function u(.) is concave, with u_1, $u_2 > 0$ and u_{11}, $u_{22} < 0$, and satisfies some additional conditions which ensure that both consumption and real balances are normal goods. In this note, we assume that c and m are perfect "complements" with the utility function written as

(2) $u(c,\ m) = u(z);\quad u' > 0,\ u'' < 0,$

where $z = \min [\frac{1}{\gamma} c, m]$. This implies that the desired consumption-real balances ratio, γ, remains unaltered irrespective of econoimc conditions. Therefore, this is an extreme case since, for general utility functions, this ratio should change as economic conditions change.

Denoting by L the current value Lagrangian of the problem and q the multiplier

$$L = u(m) + q[f(k) + x - \Pi m - \gamma m - \dot{k} - nk - \dot{m} - nm],$$

the first-order conditions for an optimal path, other than the budget constraint (1), are

$$(3) \qquad \dot{q} = [n + \delta - f'(k)]q,$$

$$(4) \qquad \dot{q} = -u'(m) + (n + \delta + \Pi + \gamma)q,$$

which yield

$$(5) \qquad u'(m) = [f'(k) + \Pi + \gamma]q.$$

Moving from micro to macroeconomic analysis, we must first have

$$(6) \qquad \frac{\dot{m}}{m} = \theta - \frac{\dot{p}}{p} - n,$$

where $\frac{\dot{p}}{p}$ is the actual rate of price inflation and θ is a constant growth rate of money. Following Fischer, we shall work with rational expectations (or perfect foresight) equilibrium paths on which the expected and actual rates of price inflation coincide:

$$(7) \qquad \frac{\dot{p}}{p} = \Pi.$$

In addition, there is the economy wide constraint that the full employment output $f(k)$ equals aggregate demand: consumption, γm, and investment, $\dot{k} + nk$, that are derived from (1) and any two equations out of (3), (4), and (5):

(8) $$f(k) = \gamma m + \dot{k} + nk.$$

In the long-run steady state, $\dot{k} = \dot{m} = \dot{q} = 0$. Therefore, from (3), we obtain the superneutrality result for capital-labor ratio

(9) $$f'(k^*) = n + \delta,$$

where an asterisk denotes the steady-state value. Then, from (8), we obtain that even real money balances are independent of θ in the long-run steady state,

(10) $$m^* = \frac{1}{\gamma}\,[f(k^*) - nk^*].$$

This feature is stronger than the Sidrauski superneutrality result in which real balances are inversely related to θ. However, this is obviously a necessary requirement for superneutrality because otherwise consumption is affected by θ. In the present model, only q^* varies inversely as θ changes since we have

(11) $$q^* = \frac{u'(m^*)}{\delta + \theta + \gamma}\ .$$

The growth path of the economy can be represented by equations (3), (6) whose $\frac{\dot{p}}{p}$ is substituted out by use of (5) and (7), and (8). Then, linearizing around the steady state, we obtain

(12) $$\begin{bmatrix} \dot{q} \\ \dot{m} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -q^*f'' \\ \frac{u'm^*}{q^{*2}}\,, & -\frac{u''m^*}{q^*}\,, & f''m^* \\ 0 & -\gamma & \delta \end{bmatrix} \begin{bmatrix} q - q^* \\ m - m^* \\ k - k^* \end{bmatrix},$$

where derivatives are evaluated at the steady state.

The characteristic equation of the coefficient matrix of (12), say A, is written as

(13) $$\psi(\lambda,\ \theta) = -\lambda^3 + (\delta - \frac{u''m^*}{q^*})\lambda^2 + m^*(\frac{\delta u''}{q^*} - \gamma f'')\lambda + \frac{\gamma u'f''m^*}{q''} = 0.$$

There is a unique negative root for (13) so that the long-rum steady state is a saddle point. In order to show this, we apply the same reasoning as used by Fischer[8/] since

$$\det A = \frac{\gamma u'f''m^*}{q^*} < 0,$$

and

$$\text{trace } A = \delta - \frac{u''m^*}{q^*} > 0.$$

2. Changes in the Growth Rate of Money

To examine whether or not the rate of capital accumulation is affected by the growth rate of money supply, we have to examine whether or not the unique negative root of the dynamic system, say λ^*, is affected by θ. This is because we can write, by the definition of the characteristic root,

$$\dot{k}_t = -\lambda^*(k^* - k_t), \tag{14}$$

and, at any $k_t < k^*$, $\dot{k}$ is greater the greater is $(-\lambda^*)$. (See also Appendix.) Thus, we compute $\frac{d\lambda^*}{d\theta}$.

First, we see that the sign of $\frac{d\lambda^*}{d\theta}$ is the same as that of $\frac{\partial\psi}{\partial\theta}$ because

$$\frac{d\lambda^*}{d\theta} = -\frac{\partial\psi/\partial\theta}{\partial\psi/\partial\lambda^*}$$

and $\frac{\partial\psi}{\partial\lambda^*} < 0$ as $\lambda^* < 0$ and $\psi(0, \theta) < 0$. The direct computation yields

$$\frac{\partial\psi}{\partial\theta} = \frac{m^*}{q^{*2}}\left[(\lambda^{*2} - \delta\lambda^*)u'' - \gamma u'f''\right]\frac{\partial q^*}{\partial\theta}, \tag{15}$$

where, from (11), $\frac{\partial q^*}{\partial\theta} < 0$. But, from (13):

$$(\lambda^{*2} - \delta\lambda^*)(\lambda^* + \frac{u''m^*}{q^*}) + \gamma f''m^*(\lambda^* - \frac{u'}{q^*}) = 0,$$

so that the expression in the blacket of (15) equals

$$-\gamma f''\lambda^*(u' + u''m^*)/(\lambda^* + \frac{u''m^*}{q^*}),$$

which takes the same sign as $(u' + u''m^*)$. Therefore, we finally obtain

$$\text{sign}\left.\frac{d\dot{k}}{d\theta}\right|_{k<k^*} = -\text{sign}\frac{d\lambda^*}{d\theta} = \text{sign}(u' + u''m^*). \tag{16}$$

Expression (16) implies that the speed of capital accumulation is the faster (or slower) as the growth rate of money supply increase when $(u' + u''z)$ is positive (or negative); and the superneutrality result on transition paths holds if (and only if[9/]) $u' + u''z = 0$, or utility function (2) is logarithmic

(17) $$u(z) = \alpha \ell n z + \text{const.}$$

When $u(z)$ belongs to the constant relative risk aversion family:[10/]

(18) $$u(z) = \frac{z^{\alpha(1-R)}}{1 - R}, \quad \alpha > 0, \quad R > 1 - \frac{1}{\alpha},$$

we obtain

$$u' + u''z = \alpha^2 (1 - R) z^{\alpha(1-R)-1},$$

so that

$$\frac{d\lambda^*}{d\theta} \lesseqgtr 0 \quad \text{as} \quad R \lesseqgtr 1.$$

Namely, the rate of capital accumulation is faster (or slower) the higher the growth rate of money supply when the degree of relative risk aversion is smaller (or greater) than unity. The case of $R > 1$ therefore provides counter examples to the second proposition of Fischer.

3. Inflation and Capital Accumulation

The effect of the (expected) rate of inflation on investment is often said to be stimulative if inflation is moderate or creeping while repressive if the inflation rate is much higher. The standard economic theories have not been successful in explaining such phenomena, as they typically conclude the definite positive relation between inflation and investment. In this section, we attempt to explain the alleged change in relationship under the assumption that utility function (2) is of the constant relative risk aversion family (18).

For this purpose, we wish to show that the rate of price inflation is positively related to the growth rate of money supply on the transition path to the steady state. Unfortunately, however, this may not always be true in the present model because the rate of change of m, $\frac{\dot{m}}{m}$, is positively (or negatively) related to θ when R is smaller (or greater) than unity. Nonetheless, it is always possible to imagine a range of $R < 1$ even where

$$\frac{d\Pi}{d\theta} = \frac{d}{d\theta}\left(\frac{\dot{p}}{p}\right) = 1 - \frac{d}{d\theta}\left(\frac{\dot{m}}{m}\right) > 0. \quad (19)$$

(See Appendix for the formal discussion on the above argument.)

Confining ourselves to the range of R in which (19) is satisfied, and moving from the "reduced form" to "structural" relationship, we can state that capital accumulation increases (or decreases) as the rate of (expected) inflation increases if R is smaller (or greater) than unity. In other words, when the degree of relative risk aversion is smaller (or greater) than unity, the correlation between investment and the (expected) rate of inflation is positive (or negative).

If we introduced stochastic factors in the model and if we were successfully able to relate the degree of relative risk aversion to the variability of inflation rates, which is often positively related to the level of inflation rate, we could explain the structural change in the relation between the (expected) rate of inflation and investment. Namely, when the inflation rate is lower the case of $R < 1$ is appropriate with the result of stimulating investment, while when it is much higher the case of $R > 1$ is more appropriate and in such a case investment is decreased.

Appendix

We note that (14) yields

$$k_t = k^* - (k^* - k_0)e^{\lambda^* t}.$$

Therefore

$$\frac{dk}{d\theta} = -(k^* - k)t\frac{d\lambda^*}{d\theta}.$$

We should set t=0 when we implement a change in θ, so that $\frac{dk}{d\theta} = 0$. (This is equivalent to examining the effect of θ for a given k as k cannot change instantaneously.) Then, from (14), we have

$$\text{(A1)} \qquad \frac{d\dot{k}}{d\theta} = -(k^* - k)\frac{d\lambda^*}{d\theta},$$

and from (8)

$$\text{(A2)} \qquad \frac{dm}{d\theta} = -\frac{1}{\gamma}\frac{d\dot{k}}{d\theta}.$$

As the third row of matrix equation (12), we have

$$\dot{k} = \gamma(m^* - m) - \delta(k^* - k),$$

which, by use of (14), yields

$$\text{(A3)} \qquad \gamma(m^* - m) = (\delta - \lambda^*)(k^* - k),$$

so that $m < m^*$ when $k < k^*$.

By the definition of the characteristic root, we can write

$$\dot{m} = -\lambda^*(m^* - m).$$

Thus, making use of (A1), (A2), and (A3), we obtain

$$\text{(A4)} \qquad \frac{d}{d\theta}\left(\frac{\dot{m}}{m}\right) = \frac{1}{m^2}\left[-m(m^* - m)\frac{d\lambda^*}{d\theta} + \lambda^* m^* \frac{dm}{d\theta}\right]$$

$$= \frac{1}{\gamma m^2}\left[-\delta m + \lambda^*(m + m^*)\right](k^* - k)\frac{d\lambda^*}{d\theta}.$$

Expression (A4) takes, for given $k < k^*$, the opposite sign to the sign of $\frac{d\lambda^*}{d\theta}$, and, from (A1), it takes the same sign as $\frac{d\dot{k}}{d\theta}$.

Note

1. The definition of superneutrality is that real variables such as capital-labor ratio and real consumption are independent of the rate of price inflation and thereby of the growth rate of money supply.
2. See Tobin's comment to Sargent [3] and Fischer [1].
3. See Fischer [2] for references.
4. These proofs are straightforward enough for us to omit reporting.
5. He first reasons as follows: "[T]he higher the growth rate of money, the higher the inflation rate, the lower the value of real balances, and thus, given the capital stock, of wealth, and therefore the lower the rate of consumption. this cannot be quite right because it also implies that the nonneutrality would hold for the logarithmic utility function, even though we know ... that it does not." Then he goes on: "Presumably, the effect on capital accumulation results from the influence of holdings of real balances on the marginal utility of consumption, ..., though even here there is no simple connection, since u_{12} changes sign as the coefficient R moves through unity." Thus he concludes: "In brief, a convincing intuitive explanation of the basic result is not yet available." (Fischer [2], pp. 1438 - 39).
6. If x is transfered proportionally to the holdings of real balances, $x = \theta m$, superneutrality holds irrespectively of the form of utility function even on transition paths.
7. Note that, in the original Sidrauski model, real balances in the utility function are a proxy for the services derived from holding real balances. Therefore, to be precise, it is consumption-money

service ratio that remains constant.

8. The determinant gives the product of the three root; det A < 0 implies that there are either three negative roots (two of which may be complex with real parts negative) or one. A necessary condition for stability of the system requires that trace A < 0. However, since it on the contrary is positive, the system is unstable and there is at least one positive root. But as we know that the system has either zero or two positive roots, it has two. Therefore, there is a unique negative root.
9. The "only if" part is also straightforward.
10. It should be obvious that the logarithmic utility function (17) is a special case of (18) with R = 1.

References

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