No. 100 (81-1)

Foundations of the Economic Theory of Location

- Transport Distance v.s. Substitution

by

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January 1981

This paper partly contains and in many respects extends the results obtained in the ISEP DP. No.93(80-31).
The author has been benefited from helpful suggestions by my colleagues K. Asako, Y. Kanemoto and A. Okabe, and also from stimulating conversations with Professors N. Sakashita and H. Watanabe. The responsibility for any error which may remain is solely of mine.

ABSTRACT

We are interested in optimal choice of location, from economization view point, by an economic agent, in the triangle space which any three representable sites initially endowed of markets would make. We shall see; the agent is faced, in his making an optimal choice, with two contradicting factors which he should take into account. One factor attributes its origin to the fact that the aggregate transport distance, which is strictly convex on the space, takes its minimum in the interior. The other so does to the neoclassical substitution property, that is, the property that a cheaper transport cost commodity will be substituted for a more expensive one. Additional but significant factor, that plays a critical role in the locational decision of the agent, is a transport cost structure in terms of transportation rates on output and inputs, which is an *ad hoc* assumption in the framework. In order to be more specific, we shall raise simple but illuminating special cases, so that each of the three factors appears and either a market site (one vertex) or an interior point of the triangle space will be found as the agent's optimal location. These cases are, respectively, unified in two general theorems, each with an *ad hoc* set of assumptions, proving an interior point or an extreme point of location to be optimal.

1. Introductory Remarks and Summary

1.1. Historical Remarks

A locational decision problem was first proposed, at least in a way that came to attention, by Weber (1929). The location theory was developed by Isard (1956) for the economics of spatial location, but, it was Moses who tried, in the way that came to wide attention by economists as well as urban or regional scientists, to integrate theory of production with the theory of location.

Moses (1958) examined local implications of input-substitution, spatial as well as technological, on a firm's locational decision. For a long time, the theory continued to be developed for Moses'production model in the strict sense of a model in which the neoclassical theory of production incorporates transportation costs of inputs and output. The followers are notably Sakashita (1967), Woodward (1973), Mathur (1979) and others, in a linear (one-dimensional) space of transportation, and Bradfield (1971), Khalili, Mathur, and Bodenhorn (1973, 1974), Miller and Jensen (1978) and others, in the triangle (two-dimensional) space.

In one demensional case, they established a somewhat surprising fact that a cost-minimizing firm would only locate at one of the fixed sites of input or output markets and never at an intermediate one of the fixed ones. This result is due to technological substitution property, concavity of transport cost rates, and the one-dimensionality.

In case two representable input markets and one representable output market are situated at the vertices of a triangle space, they established that (a) the firm would never choose an intermediate location on the line joining the output market to either of the two markets, (b) when the firm is constrained to remain at a fixed distance from a market site, the firm continues to stay for any level of output at the point of location, at which its cost function for a level of output takes its minimum in the space, if and only if the firm's production is governed by homotheticity with respect to scale, (c) when the firm can choose any point in the space, it continues to stay for any level of output, at the optimal point for a level of output, if and only if production is governed by constant returns to scale, and (d) with various returns to scale production, the firm would move towards (away from) the output market under increasing (decreasing) returns to scale, etc..

Most recently, Eswaran, Kanemoto and Ryan (1980), proposed a dual approach to both the one and two-dimensional location theories, and derived additional comparative static results and examined the effects on the locational decision of exogeneous changes in market prices or transportation rates.

In connection with whether or how the firm locates at an interior point or an extreme (one of the three vertices) point of the triangle space, little literature has been available at present. Every author assume where the optimal location is, for example, an interior point in the triangle space.

In what follows, we shall investigate and solve by applying the fundamental mathematical method, those basic questions yet unsolved, left open, and integrate new findings with those conjectural and theoretical assertions in the literature available of locational choices. The process of the reformulation will also claim a quick survay in character, taking up with the existing literatures, and concerns the comparative evaluations of these studies.

1.2. A Theoretical Framework for Locational Choice; Summaries.

We are interested in optimal choice of location by an economic agent in a triangle space which any three representable sites initially endowed of markets would make. Take for example, as the agent, a firm operating a neoclassical technology of production and finding an optimal location for production. From the economics's view point of production and transport costs, the cost minimizing firm would have an intuitive observation of spatial structure as well as production, that is; in its making an optimal choice, the firm is confronted with two contradicting factors; one attributable to the fact that the total transport distance, which is strictly convex on the space, takes its minimum distance in the interior, and, the other to the substitution property of a neoclassical production technology.

In using factors (inputs) of production, a cheaper transport cost factor of production will be substituted for a more expensive one, so that the cheaper one will intensively be used. The former will contribute to its making a locational choice among interior points whereas, the latter to a choice among the three extreme points (market sites). The third factor is, of course, convexity or concavity property of transport costs. This seems to be an *ad hoc* assumption, but, will play an important role in the locational decision making. Specifically, we shall pursue this intuition in the present analysis through the following sections, so that we shall see its theoretical validity and elaboration.

In section 2, an abstract location problem will be posed as a cost minimization problem, incorporating transport costs of product (output), input factors and materials. Let us designate by x the distance between the firm's location and a fixed market site (as which we shall take the output market site), and by α the angle which measures a direction from the output market site towards the firm's location. Then, the location is always shown as the variable pair (x, α) in the space. With the usual assumptions (smooth structures) on production and transport costs, the total cost $C^*(., .)$ varies continuously with pair (x, α) and takes its minimum and miximum somewhere in the space. We shall see this first in Section 3. Then, we shall deal with the sufficient conditions for a theorem, well known in a linear space case; see Sakashita etc., that all interior (intermediate) locations are excluded by the firm in the triangle case. We observe, thus, the fact that the convexity of the cost function is not sufficient for an optimal interior location, whereas, the concavity or monotonicity is sufficient for an extreme point to be optimal. This stems from that the convexity may imply the monotonicity, hence, imply an optimal extreme point in the space of candidate locations. However, we confirm that the concavity condition holds without any further condition only if the three sites make a linear space, but, not always in case of a triangle space; see

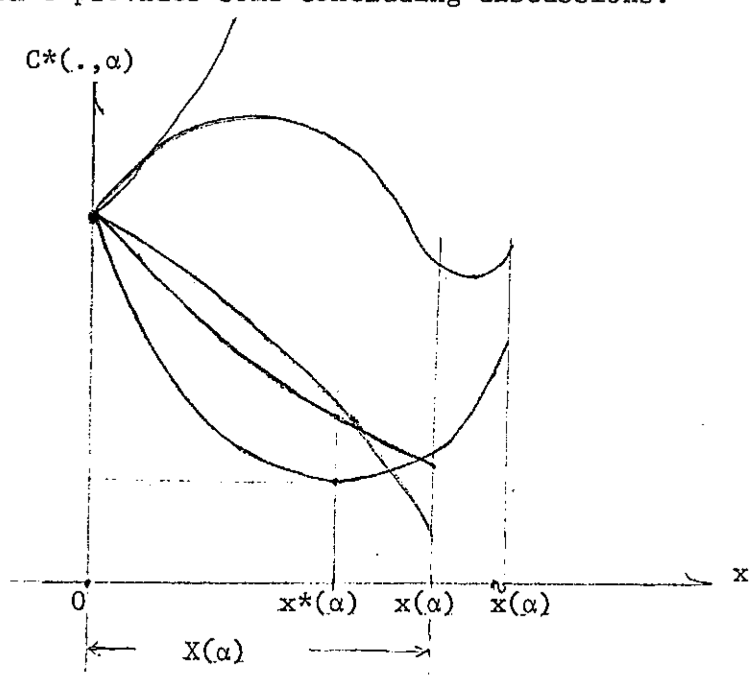
Section 3.3. In fact, the concavity or monotonicity of the cost function can not be contained on the whole space if it is a triangle space: In Section 4, we shall define a <u>(total) transport distance</u> (function) as the aggregated sum of ranges from the firm's location to the three fixed sites. This is a real valued function of the two location variables; distance x and angle α. We examine then the global properties of the transport distance function in the space. The properties we shall find out are, (i) the transport distance, along which the firm must convey output for sales, inputs for production, is strictly convex on the triangle space, continuously varies with the location variables (x, α), and (ii) it takes its minimum in the <u>interior</u> of the space, while, in the <u>linear</u> location space, the transport distance is always <u>constant</u> over the location variables. The set of the property (i) and (ii), we shall call <u>the transport distance property</u>, while, the <u>factor substitution</u> property we shall call <u>the neoclassical (technological) property</u>.

In Section 5, sufficient conditions for an optimal interior location will be established and with them, it will be examined whether the cost function takes its minimum in the interior. We shall call this interior location case the <u>interior point location (theorem)</u>. We shall observe here; (i) in this general flamework, the concavity of the cost function (C*(.)) with respect to the space X × A at all (x, α)'s can not be obtained; see Proposition 5 and 6, (ii) irrespective of whether the cost function C*(., α) for each α ∈ A, is concave or convex with respect to X at any x, the cost function C*(., .) takes its minimum at an intermediate or interior point of the space, provided that a set of ad hoc but appropriate conditions is presumed, and (iii) under this set of conditions, it can not be concave on X for some $\alpha \in A$ and hence, the concavity of $C^*(., \alpha)$ for each α, can be presumed, only when at least one of the conditions for the interior location is violated, as in case the firm finds one of the vertices optimal; see Remark 5, 6. We shall presume, in section 6, that the given production function is of a Leontief type, so that no substitution prevails among factor-inputs. We wish here to specify more clearly characteristics of transport cost and spatial structure, which would let the firm choose an interior or extreme point.

As the matter of fact, some stronger results are obtained for the interior point location (Theorem 1), associated with the Leontief technology, in which case the spatial property (ii) reveals and the only source, from which concavity of the cost function can be obtained, is an ad hoc assumption on the set of transport cost functions $R_0(.)$, $R_i(.)$, i = 1, 2, 3; see Corollary 1, 2, 3, and Remark 7. Illuminating special cases are raised to exemplify an extreme or interior optimal location and to see contents of the above results easily.

In Section 7, first, we shall examine the properties of the optimal locations which a class of homthetic or homogenious production functions can provide. These include those which the various authors found in their frameworks, concerning the single optimal locations with respect to the level of output. Secondly, the general substitutive technology, will be reconsidered from the point of view of the extreme point location. The negative substitution effect provides a decreasing effect in the marginal cost $\partial C^*(x, \alpha)/\partial u$; $u = x, \alpha$, with respect to each location variable. In general, the neoclassical property would contribute to any concavity property the cost function may have, only with respect to distance x; see Proposition 5, 6. We shall establish the extreme point location as a counterpart of the interior point location. In fact, the alternative sufficient conditions imposed, in Theorem 2, on the transport cost structure are the counterparts of those which are imposed in Theorem 1, Corollary 1-4, and, the concavity or even convexity assumption will imply the monotonicity in all cases except one. The exceptional one, which is most interesting, only implies one of the market sites optimal; see (vii) in Theorem 2, and may be compared to Case 1 for the one-dimensional case.

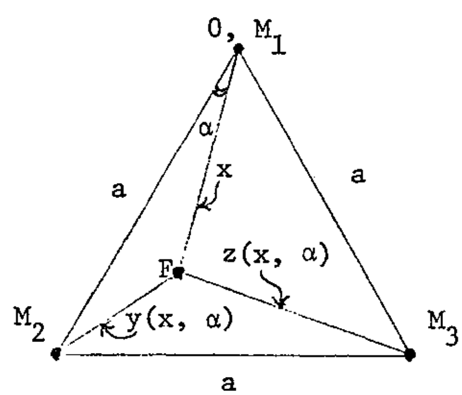
Section 8 provides some concluding discussions.



2. The Abstract Location Problem

An abstract location problem can in general be set out as follows; Given transport cost functions and production function of the types below characterized. Also given and fixed a geographical distribution of output and input markets. Then, from the economics' view point, what can we say about the firm's optimal choice of location for production?

We presume that the output market and a first input market are situated at the same place (site), and hence we assume the four markets; one output market and three input markets thus located, in general, form a triangle distribution, which can dipict, as in Figure 0, the location problem. We make the triangle regular (equilateral) without losing any generality. The whole problem now reduces to: Where does the firm find its production location F in the space including the regular triangle; from the point of view of cost minimization ?



$\overline{M_i M_j} = a, \; i \neq j.$

Figure 0: $\overline{O\,F} = x$, $\overline{F\,M} = y$, $\overline{F\,M} = z$.

2.1 The Cost Minimizing Location

The cost minimization problem, incorprating transport costs is set out as;

$$\text{(P.I.)} \qquad \min_{(M_1,\, M_2,\, M_3;\, x,\, \alpha)} \quad C = c(M_1, M_2, M_3; x, \alpha)$$

subject to

$$Q \leqq f(M_1, M_2, M_3), \quad Q > 0, \quad M_1 \geqq 0, \quad M_2 \geqq 0, \quad M_3 \geqq 0,$$

where $c(.) = R_0(x)\, Q + \sum_{i=1}^{3} P_i(x, \alpha)\, M_i$, and M_i, $i = 1, 2, 3$, x, α, and Q, denote i th input, distance (range) between output market and the firm's production location, its direction angle, quantity of output, respectively, and transport costs of output R_0, and of input P_i, $i = 1, 2, 3$, may depend on range x eventually, that is, they are real valued functions of range x, while P_i, $i = 2, 3$ depends also on angle α. We shall see this dependency in full detail later.

2.2 Transport Cost Functions

We presume that the output transport cost function $R_0(.)$, the input transport cost functions $R_i(.)$, $i = 1, 2, 3$, take the forms,

$$\text{(A. 1)} \qquad R_0(x) = r_0(x)\, x$$

$$R_i(u_i) = P_i(u_i) - \bar{P}_i = r_i(u_i) u_i \quad i = 1, 2, 3, \quad \bar{P}_i;\ \text{given},$$

where range variable u_i represents the distance between the firm's locaton which is <u>variable</u> and the i th input market which is <u>fixed</u>, and, $\bar{P}_i$, $i = 1, 2, 3$, is the (<u>F.O.B.</u>) price established at the i th input market, so that $P_i(u_i)$ may be interpreted as a <u>C.I.F.</u> price.

We shall assume that transport cost functions are continuously twice differentiable in (non-negative) ranges.

Take any transport cost as positive (expensive) and also take it as a non decreasing function in range, so that, for each $x > 0$,

(A. 2)1 $\quad R_0(x) \;(\geqq) 0, \quad R_0'(x) = r_0'(x)\, x + r_0(x) \geqq 0,$

and for each $u_i > 0$, $i = 1, 2, 3,$

(A. 2)2 $\quad R_i(u_i) > 0, \quad P_i'(u) = r_i'(u_i)\, u_i + r_i(u_i) \geqq 0.$

We also do not presume the marginal transport cost is increasing in range, that is, we assume, except otherwise stated,

(A. 3) $\quad R_0''(x) \leqq 0,$

$\quad P_i''(u_i) \leqq 0, \quad i = 1, 2, 3.$

We now easily see that, for each location variable pair (x, α),

(1) $\quad u_i = u(x, \alpha_i),\; P_i(u_i) = P_i(u(x, \alpha_i)),\; r_i(u_i) = r_i(u(x, \alpha_i)),$

$\quad i = 2, 3,$

where, for $i = 2, 3$, $u(x, \alpha_i) = \sqrt{(x - a \cos \alpha_i)^2 + a^2 \sin^2 \alpha_i}$, $\alpha_2 = \alpha$, $\alpha_3 = 60° - \alpha$.

In Figure 0, we see $u(x, \alpha_2) = y(x, \alpha)$, and $u(x, \alpha_3) = z(x, \alpha)$.

3. Global Properties of the Optimal Production Location

Given Q and $\bar{P}_i$, $i = 1, 2, 3$. Then, the solutions to the problem (P. I) for each fixed pair (x, α), are $\Phi_i(P_1(x), P_2(x, \alpha), P_3(x, \alpha), Q) = M_i(x, \alpha)$, $i = 1, 2, 3$, and the optimal cost C^* for each pair (x, α) may be denoted and expressed by

$$(2) \qquad C^* = C^*(x, \alpha) = c(M_1(x, \alpha), \; M_2(x, \alpha), \; M_3(x, \alpha); x, \alpha)$$

$$= R_0(x)Q + \sum_{i=1}^{3} P_i(x, \alpha) \, M_i(x, \alpha).$$

We assume here, except otherwise stated, that production function f well defined on an input factor space (a non negative orthant in a euclidean space), possesses a Hessian matrix, bordered with its first derivatives, which is everywhere invertible so that its inverse is negative semidefinite. Then, input demand functions (optimal input factors), generated from an extremization of the cost; that is, (P. I.), are continuously differentiable in respect to pair (x, α), provided that the C.I.F. prices of inputs; under assumptions, the transport cost functions are continuously differentiable in (x, α). We shall hereinafter call this production function f a neoclassical one. See Debreu (1972), and Kusumoto (1973) for the continuous differentiability. Now, we may view $M_i(x, \alpha)$ as a continuous function in (x, α), hence, we may also see the optimal cost function $C^*(.)$ continuously depends on the location variables (x, α). We wish to see how the firm's cost $C^*(.)$ varies when the pair (x, α) varies in the space of candidate locations.

Lemma 0 (Existence of the Solution):

Since $C^*(.)$ varies continuously with pair (x, α), it takes a maximum and a minimum on the triangle space, if the space is bounded (and closed).

First, conditions for the exclusion of all interior locations from the interior of the space will be examined.

3.1. Conditions for an Extreme Point Location

Now, suppose $C^*(., .)$ is a concave function in $X \times A$, where $X = [\, x \in R;\ 0 \leqq x \leqq a(\sqrt{3}/2)/\sin(60° + \alpha)]$ and $A = [\ ; 0° \leqq \alpha \leqq 60°]$. Suppose $C^*(.)$ is not constant in $X \times A$, as it is. Then, there exists a pair (x, α) in $X \times A$ such that $C^*(x, \alpha)$ is strictly larger than $C^*(x^*, \alpha^*) = \min C^*(x', \alpha');\ (x', \alpha') \in X \times A$. Take any $(s, \beta) \in (X \times A)°$ and for some $(t, \gamma) \in X \times A$, we can take a positive number λ such that $0 < \lambda < 1$, and $(s, \beta) = (\lambda x + (1 - \lambda) t, \lambda \alpha + (1 - \lambda)\gamma)$. This and the concavity imply $C^*(s, \beta) \geqq \lambda C^*(x, \alpha) + (1 - \lambda) C^*(t, \gamma) > C^*(x^*, \alpha^*)$. Reminding that (s, β) is arbitrarily taken from the interior of the space $X \times A$, we may establish a lemma, which is very important for our analysis and will repeatedly be used. Let a point (x^*, α^*) be such that $C^*(x^*, \alpha^*) \leqq C^*(x, \alpha)$, for every $(x, \alpha) \in X \times A$. Then,

Lemma 1: If the cost function $C^*(.)$ is not constant and concave in $X \times A$, then, for it to attain its minimum on $X \times A$, the point (x^*, α^*) must not be an interior point of $X \times A$; i.e. $(x^*, \alpha^*) \notin (X \times A)°$, where $(X \times A)°$ is the interior of $X \times A$.

Lemma 2: If the cost function $C^*(.)$ is not constant and convex in the space $X \times A$, then, the point $(\bar{x}, \bar{\alpha})$, at which the cost takes its maximum over the space, must not be an interior point. However, the point (x^*, α^*), at which it takes its minimum, may be an interior point or a boundary point.

A direct application of Lemma 1 will lead to the sufficient condition.

Proposition 1: In order that all interior points are excluded by the firm, it suffices to show that the cost function $C^*(.)$ is concave with respect to the space $X \times A$ at any point of location. Further, if $C^*(.)$ is not constant over the space, then, all interior points of location are excluded.

This proposition, however, will be found out to be less important; see Proposition 5 and 6.

Whether or not it is concave, we still have sufficient conditions for an extreme point which is optimal. Let $C^*(.)$ be Gateaux differentiable on $X \times A$. A necessary condition for $C^*(.)$ to have an extremum at $(x^\circ, \alpha^\circ) \in (X \times A)$ is that, for each $v = (v_1, v_2)$ such that $v_1^2 + v_2^2 = 1$,

$$(2) \qquad \frac{dC^*(x^\circ + hv_1, \alpha^\circ + hv_2)}{dh}\Big|_{h=0} = 0$$

$$(3) \qquad \frac{\partial C^*(x^\circ, \alpha^\circ)}{\partial x} v_1 + \frac{\partial C^*(x^\circ, \alpha^\circ)}{\partial \alpha} v_2 = 0.$$

Proposition 2:1 In order for an extreme point of the space to be optimal it is sufficient that the necessary condition (2) or (3) for an extremum is not satisfied in the interior $(X \times A)^\circ$ of the space (if the cost function is not constant).

2:2 The monotonicity of the cost function C((.), which is not constant, is sufficient for an extreme point to be optimal.

Obviously, Proposition 2.2 implies Proposition 2.1. Proposition 2, also is less interesting for our succeeding analysis, which will be found out.

Proposition 1-2, in fact, deal with somewhat strong conditions to establish that all interior locations in the space are excluded, but, not always the locations, which are intermediate along the line joining a market to a neighboring market in the space. We shall say this case weak.

Proposition 3: Each condition imposed on the cost $C^*(.)$, which appears in Proposition 1-2, is sufficient for the interior locations to be weakly excluded in the space, if it holds with respect only to distance variable $x \in X$, for each direction angle $\alpha \in A$.

Since the cost function $C^*(.)$ is continuously differentiable in x, the conditions in Proposition 3 can be given in terms of the first and second partial derivatives of $C^*(.)$ with respect to X at each x.

$$(4) \quad \frac{\partial C^*(x, \alpha)}{\partial x} = R_0'(x)Q + \{P_1'(x)\ M_1(x, \alpha) + \sum_{i=2}^{3} \frac{\partial P_i(x, \alpha)}{\partial x} M_i(x, \alpha)\} \leqq 0, \text{ or, } \geqq 0 \ ;$$

$$(5) \quad \frac{\partial^2 C^*(x, \alpha)}{\partial x^2} = R_0''(x)\ Q + \{P_1''(x)\ M_1 + \sum_{i=2}^{3} \frac{\partial^2 P_i(x, \alpha)}{\partial x^2} M_i(x, \alpha)\} + \{P_1'(x) \frac{\partial M_1(x, \alpha)}{\partial x} + \sum_{i=2}^{3} \frac{\partial P_i(x, \alpha)}{\partial x} \frac{\partial M_i(x, \alpha)}{\partial x}\} \leqq 0.$$

Remark 0: The condition (5), i.e. the concavity of $C^*(., \alpha)$, for each α, is necessary for the concavity of $C^*(.)$ with respect to the space X x A, and the condition (4), i.e. the monotonicity of $C^*(., \alpha)$, for each α, necessary for the monotonicity of $C^*(.)$ with respect to X x A. Hence, if the concavity (resp. monotonicity) fails to hold with respect to X, then, so does the concavity (resp. monotonicity) with respect to X x A.

Observe that, in the right hand side of (5), the last term may be called the substitution effect (term), whereas, the first two the transport cost effect .

The substitution effect takes, for each (x, α), non-positive value, if production function f is neoclassical. This effect may be interpreted below. Denote this by $S_x(x, \alpha)$, then, we can write, for a real valued function Φ_i in prices P, such that $\Phi_i(P(x, \alpha)) = M_i(x, \alpha)$, $P(x, \alpha) = (P_1(x, \alpha), P_2(x, \alpha), P_3(x, \alpha))$, and $P_1(x, \alpha) = P_1(x)$;

$$(6) \quad S_x(x, \alpha) = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial P_i(x, \alpha)}{\partial x} \frac{\partial \Phi_i(P(x, \alpha))}{\partial P_j} \frac{\partial P_j(x, \alpha)}{\partial x},$$

where $\sum_{j=1}^{3}\{\partial \Phi_i(P(x, \alpha)) / \partial P_j\}\{\partial P_j(x, \alpha)/ \partial x\} = \partial M_i(x, \alpha)/ \partial x$.

Remark 1: When we consider a change in distance x, hence prices $P(x, \alpha)$ for each α, which is such that it leaves the firm on the same isoproduct level, we can always say that the new collection of inputs purchased and transported must have a higher value in terms of the old transport prices than the old collection of inputs had. For the old collection was the only collection of inputs on this isoproduct level which was available to the firm at the old transport prices. Similarly, the old collection of inputs must have a higher value in terms of the new transport prices than the new collection of inputs has. This is the sense in which the most generalized change in transport prices must be set up a change in demands for inputs in the opposite direction. This is an interpretation of the negative $S_x(x, \alpha)$.

A 3 by 3 matrix, whose typical element $\partial\Phi_i/\partial P_j$, is negative semidefinite and equal to a submatrix of the inverse of the bordered Hessian of f.

If we assume that the transport cost functions are all concave w. r. t. X, at any $x \in X$, then, it is sufficient for (5) to hold for each (x, α) that,

$$(7) \quad \partial P_i^2(x, \alpha)/ \partial x^2 \leqq 0, \; i = 1,2,3.$$

This, however, hardly holds, in general. We shall see this in what follows.

3.2 Global Properties of Transport Cost Function in Location Variables

Here, we shall see that transport cost (price), which is continuously twice differentiable on $(X \times A)^{\circ}$, is not always concave with respect to X for each $\alpha \in A$, but, can be (strictly) convex if the assumption (A. 3) is specified so that $P_i(.)$ is a convex function. If $R_i''(u_i) = 0$, $i = 2,3$, then, $P_i(.)$, $i = 2,3$, are (strictly) convex with respect to X at any x in X, hence, leading to a contradiction to (7). We have:

Lemma 3 The signs of $\partial^2 P_i(x, \alpha)/\partial x^2$, $i = 2, 3$, are in general indeterminate, except in special cases, such as $P_i''(u) = 0$, in which case it is positive (or zero if $\alpha = 0°$).

This exception, which is, however, a fundamental case, implies the existence of a difficulty feature, which is proper to a triangle case, in excluding all interior locations, while, in a linear case, the assumption (A. 3) implies to establish the exclusion; see section 3.3.

Variations of the transport costs with respect to x will be listed below, among which we shall see the proof of Lemma 3:

Let $u = u_i$, then,

$$\frac{\partial P_i}{\partial x} = P_i'(u)\,\frac{\partial u}{\partial x}, \tag{8}$$

$$\frac{\partial^2 P_i}{\partial x^2} = P_i''(u)\left\{\frac{\partial u}{\partial x}\right\}^2 + \frac{P_i'(u)}{u}\left\{1 - \left\{\frac{\partial u}{\partial x}\right\}^2\right\} \tag{9}$$

$$= \left\{P_1''(u) - \frac{P_i'(u)}{u}\right\}\left\{\frac{\partial u}{\partial x}\right\}^2 + \frac{P_i'(u)}{u}, \tag{9'}$$

where $P_i = P_i(u(x, \alpha))$, $i = 2, 3$, and

$$(10) \qquad \frac{\partial u}{\partial x} = \frac{(x - a \cos\alpha_i)}{u}, \quad \alpha_2 = \alpha, \quad \alpha_3 = 60° - \alpha,$$

and

$$(11) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{u}\{1 - (\frac{\partial u(x, \alpha_i)}{\partial x})^2\} \geqq 0,$$

since $\quad |\frac{\partial u}{\partial x}| \leqq 1.$

Under the assumption (A. 1) - (A. 3), in view of (2), if $\alpha \leqq 30°$, then,

$$(12) \qquad \frac{\partial P_2(x, \alpha)}{\partial x} \leqq 0, \quad \text{for any } x \text{ such that } 0 \leqq x \leqq a \cos \alpha,$$

$$(13) \qquad \frac{\partial P_3(x, \alpha)}{\partial x} \leqq 0, \quad \text{for any } x \text{ such that } 0 \leqq x \leqq a \cos(60° - \alpha), \text{ or,}$$

$$> 0 \quad \text{otherwise.}$$

Let $u(x, \alpha_2) = y(x, \alpha)$ and $u(x, \alpha_3) = z(x, \alpha)$, then,

$$(14) \qquad \frac{\partial u(x, \alpha_2)}{\partial x} = \frac{\partial y(x, \alpha)}{\partial x} \leqq 0, \quad \text{if} \quad 0 \leqq x \leqq a \cos \alpha,$$

$$(15) \qquad \frac{\partial u(x, \alpha_3)}{\partial x} = \frac{\partial z(x, \alpha)}{\partial x} \leqq 0, \quad \text{if} \quad 0 \leqq x \leqq a \cos(60° - \alpha), \text{ or,}$$

$$> 0 \quad \text{otherwise.}$$

Since $y(x, \alpha)$ and $z(x, \alpha)$ can be symmetrically treated, it follows that the same properties of P_i, $i = 2, 3$, with respect to x are obtained for the case in which $\alpha \geqq 30°$.

Suppose here, as a very special case which is compatible with (A. 2)2, $P_i'(u) = 0$, $i = 1,2,3$. Then, every price including transport cost (price) is constant, and, the location problem will disappear. This assumption thus will be made only when it is for some i but not all i.

In view of (9) (10) and (11), (A. 2)2 and (A. 3) together will imply the first term in the right hand side of (9) is negative while the second positive, except in case $P_i'(u) = 0$, or $P_i''(u) = 0$, which completes a proof of Lemma 3.

3.3 Special Case in which an Extreme Point of the Space is Optimal.

However, we shall first point out special cases, in which any intermediate location for production is excluded weakly.

Case 1 includes as a special case the perfect substitution in which case any two of factors are not used for production.

Case 1: Let $M_3 = 0$, $M_1 \geqq 0$, $M_2 \geqq 0$ in (P.I). Then, the cost C^* for each (x, α) is;

(2) $$C^*(x, \alpha) = R_0(x)\ Q + P_1(x)\ M_1(x, \alpha) + P_2(x, \alpha)\ M_2(x, \alpha),$$

where note that $x + y(x, \alpha) \geqq a = x + y(x, 0)$, hence $y(x, \alpha) \geqq y(x, 0) \geqq 0$ and by monotonicity of $r_i(.)$, $r_2(y(x, \alpha)) \geqq r\ (y(x, 0)) \geqq 0$, and therefore, for all $\alpha \geqq 0$,

(16) $$P_2(x, \alpha) = P_2 + r_2(y(x, \alpha))\ y(x, \alpha)$$
$$\geqq P_2 + r_2(y(x, 0))\ y(x, 0) = P_2(x, 0),$$

and

(17) $$C^*(x, \alpha) \geqq C^*(x, 0).$$

Thus, for each x $(0 \leqq x \leqq a)$ and for $\alpha = 0$, it suffices to show the inequality (12) or (13) to hold. In view of (3)-(6), we may have $\frac{\partial y(x, 0)}{\partial x} = \frac{1}{y(x, 0)}\ (x - a) = -1$, hence $\frac{\partial^2 y(x, 0)}{\partial x^2} = 0$, $\frac{\partial P_2(x, 0)}{\partial x} = -\ P_2'(y(x, 0)) \leqq 0$, and,

$$\frac{\partial^2 P_2(x, 0)}{\partial x^2} = P_2''(y(x, 0)) \leqq 0.$$ Since $P_1''(x) \leqq 0$ by assumption (A. 3), this proves (5) holds and completes the proof.

This one-dimensional case, which has been analysed by Sakashita (1 967) with more restrictive assumptions, by Mathur (1979) and Higano with the weaker assumptions, can be easily generalized in many input markets; see Higano(1980) and also Woodward (1973).

Case 2: (i) $R_0(x)$ = constant, and $R_1(x)$ = constant, (ii) $R_2(y)$ = constant, or (iii) $R_3(z)$ = constant. Each case is a trivial one to see it is essentially the same as Case 1. For example, suppose case (i) holds, we can choose a location pair $(x(\beta), \beta)$, along $\overline{M_1M_2}$ in order to produce Q with $M_i(x, \alpha)$ i = 2, 3, so that $C^*(x, \alpha) \geqq C^*(x(\beta), \beta)$, $y(x, \alpha) \geqq y(x(\beta), \beta)$, $z(x, \alpha) \geqq z(x(\beta), \beta)$ and $y(x(\beta), \beta) + z(x(\beta), \beta) = a$. Hence, an interior point (x, α) can not be optimal. Among all such pairs on the segment $\overline{M_1M_2}$, we may choose, as an optimal solution, either of $(x(0), 0)$ or $(x(60°), 60°)$, by applying the result of Case 1 here.

Case 1-2 may lead to a remark, which refers to a characteristic feature essential to the triangle space including as a special case a linear space.

Remark 2: In an essentially one-dimensional case, where $\alpha = 0$ and the site M_2 and M_3 are identical, a (total) transport distance; $d(x, \alpha) = x + y(x, \alpha) + z(x, \alpha)$, can be viewed as constant, i.e. $d(x, 0) = x + y(x, 0) = a$, $z(x, 0) = 0$, whereas, in the two-dimensional case, the total distance $d(x, \alpha)$ is not constant and continuously varies with location pair (x, α), so that it attains a maximum and a minimum on a (closed and) bounded set X x A.

Furthermore, the transport distance function d(.) is strictly convex on the interior of X, from (11), and we may apply Lemma 2 to this convex function to see that the location point, attaining its minimum distance, may possibly be at a boundary point of X, or in the interior of it. Whether or not in the interior must be examined and this motivates a further investigation. Before proceeding to that, we make a trivial but useful lemma:

Lemma 4 :1 For each direction angle α e A, the transport distance x + d(x, α) for our market distribution attains its minimum at the output market, at which markets are agglomerated.

2 For each direction angle α e A, the transport distance d(x, α) attains its maximum at a boundary of the space X x A.

To see the first remark, in Figure 1, observe that a $\leqq$ x + y(x, α) and a $\leqq$ x + z(x, α). Hence, 2a $\leqq$ 2x + y(x, α) + z(x, α), with equality when x = 0. The second follows from Lemma 2.

4. Global Properties of Transport Distance Function

We shall examine here whether the distance function d(.) achieves its minimum distance at an interior point of the triangle space X x A. The necessary condition for a minimum distance in the interior X° for each α is that, for some $\bar{x}$ e X°

(18) $\partial d(\bar{x}, \alpha) / \partial x = 0$ for each α e A.

To inquire into whether or not (18) holds, we shall list up functional relations of the distance function d(.) on X for each α.

(19) $d(x, \alpha) = x + y(x, \alpha) + z(x, \alpha) > 0$

where the total transport distance is $x + d(x, \alpha)$ for our market distribution.

(20) $$\frac{\partial d(x, \alpha)}{\partial x} = 1 + \frac{\partial y(x, \alpha)}{\partial x} + \frac{\partial z(x, \alpha)}{\partial x},$$

(21) $$\frac{\partial^2 d(x, \alpha)}{\partial x^2} = \frac{\partial^2 y(x, \alpha)}{\partial x^2} + \frac{\partial^2 z(x, \alpha)}{\partial x^2} > 0.$$

Now, we like to investigate what are the values of $d(., \alpha)$, the signs of the first derivatives (20), at each $x \in X$ such that $x = 0$, $a/2$, a, $\frac{\sqrt{3}}{2}a$ and $x(\alpha)$, where $x(\alpha) = \{(\sqrt{3}/2)/\sin(60° + \alpha)\}a$.

Use (1) and (10), then, we can directly calculate and get those value and signes, and fill up tables below with them. Hence, the graphs of $d(., \alpha)$ on X for each $\alpha \in A$ can be illustrated as in Figure 2. Make use of the symmetry property of an equilateral triangle, i.e. that of $y(x, \alpha)$ and $z(x, 60°-\alpha)$ with respect to the plane [$(x, \alpha) \in X \times A$; $\alpha = 30°$, we have a result on the function d(.), which can be summarized in a lemma:

Lemma 5: For each direction angle $\alpha \in A$, $\alpha \leq 30°$, the transport distance $d(., \alpha)$ attains its minimum at an interior point $(\bar{x}(\alpha), \alpha)$.

The interior points, at which the distance $d(., \alpha)$ attains its minimum for each $\alpha \in A$, are located along the linear segment whose boundaries are $(\frac{1}{2}a, 0)$ and $(\frac{\sqrt{3}}{3}a, 30°)$. Thus, the locus of such minimizing points ; $x(\alpha)$'s for all location angles $\alpha \in A$, consists of two linear segments, one with $(\frac{1}{2}a, 0)$ and $(\frac{\sqrt{3}}{3}a, 30°)$ as its boundaries, and another with $(\frac{\sqrt{3}}{3}a, 30°)$ and $(\frac{1}{2}a, 60°)$ as its boundaries.

We can also prove the folowing lemma, which ~~seems~~ intuitively true, and is illustrated below in Figure 2: See Appendix B.

Lemma 6 : The distance function d(.) attains its minimum at the gravity point $G((\sqrt{3}/3)a, 30°)$ on the triangle $X \times A$.

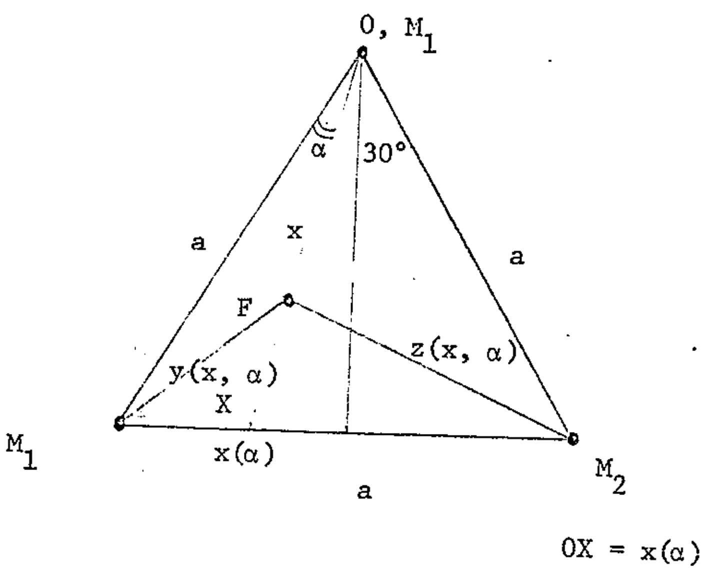


Figure 1

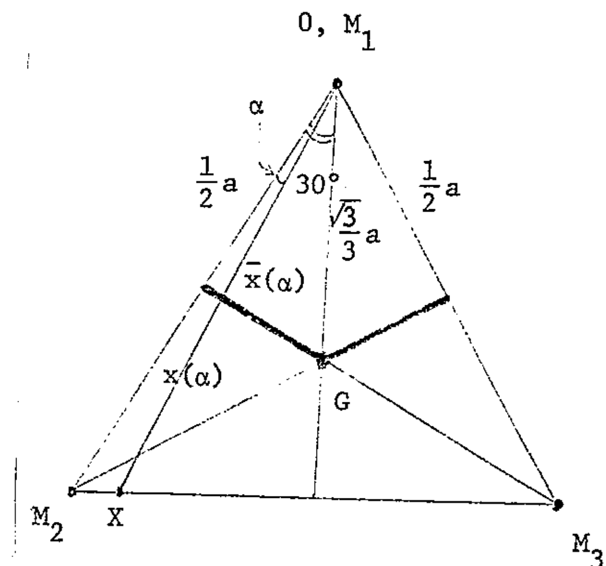


Figure 2

The variable (total) transport distance, along which the firm must carry its output and inputs, takes its minimum in the interior. This is a spatial property that it must take into account on its making a locational decision. The properties of transport distance d(.) on $X \times A$ is the simply aggregated sum of those of transport ranges x, y(x, .) and z(x, .) on $X \times A$. Combined with the cost structure given by (A. 1-3), the properties of d(.) determine for each $\alpha \in A$ the properties of transport costs (prices) $P_i(., \alpha)$ $i = 2,3$, with respect to X at x; see (9), hence, determine the transport cost effect in (5).

5. Condition for an Interior Location to be Chosen.

The last remark may suggest that a weak interior point should be very possible to be chosen. The possibility depends on the structural properties of transport costs as well as transport distance (ranges). We would like to establish a condition for an interior location to be chosen (henceforth, we shall say the interior point location. Lemma 3 says the transport cost (price) function $P_i(., \alpha)$ $i = 2,3$, tends to be convex (in fact, if $P_i''(u) \geqq 0$, $i = 2,3$, then, it is convex) on X.

First, however, we must observe that, even if the substitution effects $S_x(x,\alpha)$ would be so small (relatively to the transport cost effects) that the (total) cost function $C^*(.)$ can be convex, the (strict) convexity by itself can not always be enough for the interior location to obtain, while the concavity by itself can establish its exclusion argument. This is due to the fact that the convexity may imply the monotonicity hence imply the extreme point location. See Proposition 1,2,3 and Lemma 2.

5.1 A Condition for an Interior Location

In what follows, assume $\partial C^*(0, \alpha)/\partial x < 0$, and $\partial C^*(x(\alpha), \alpha)/\partial x > 0$. Let $Y_\alpha = [\, x \in X;\ 0 < x \leqq x(\alpha),\ \partial C^*(x, \alpha)/\partial x \geqq 0]$, for each $\alpha \in A$. $\partial C^*(.)/\partial x : X \to R$ is continuous on X so that Y is closed in X. $Y_\alpha \neq \emptyset$ since by assumption $x(\alpha) \in Y_\alpha$. Take the infimum of Y_α so that $\underline{x} = \inf x;$ $x \in Y_\alpha$, that is, $\underline{x} = \min x;$ $x \in Y_\alpha$, to which an appropriate sequence in Y_α converges, so that $\partial C^*(\underline{x}, \alpha)/\partial x \geqq 0$. Note by assumption $0 \notin Y_\alpha$. Thus, $0 < \underline{x}$. Let $x^\nu = (1 - (1/\nu))\underline{x}$, for each $\nu = 1,2,3,\ldots$, so that $\lim x = \underline{x}$. Then, $0 \leqq x^\nu < \underline{x}$, $x^\nu \notin Y_\alpha$, hence, $\partial C^*(x^\nu, \alpha)/\partial x < 0$. By continuity, $\lim_{\nu\to\infty} C^*(x^\nu,\alpha)/\partial x = \partial C^*(\underline{x}, \alpha)/\partial x \leqq 0$. Hence, $\partial C^*(\underline{x}, \alpha)/\partial x = 0$.

We shall conclude from the above for the condition;

Proposition 4: In order that an interior location is chosen by the firm as its production location, it suffices to show that the marginal cost $\partial C^*(., \alpha)/\partial x$ is strictly decreasing in x at x = 0, and, it is strictly increasing in x at $x = x(\alpha)$, i.e. for each $\alpha \in A$,

$$\partial C^*(0, \alpha)/\partial x < 0, \text{ and } \partial C^*(x(\alpha), \alpha) > 0.$$

Here, the chosen location is strictly interior of the interval $X(\alpha)$, i.e. $0 < \bar{x}(\alpha) < x(\alpha)$, and $\partial C^*(\bar{x}(\alpha), \alpha)/\partial x = 0$.

Take $\underline{x} = \bar{x}(\alpha)$ in the previous argument for a direct proof. There may be more than one interior points, at which the derivative $\partial C^*(.)/\partial x = 0$, but, we can say that the firm can choose an interior point at which the cost $C^*(.)$ takes its minimum. Also see that if the cost $C^*(.)$ is convex does not matter in the space X x A, nor in X alone.

For each $x \in X$, apply Proposition 4 to $C^*(x, .)$ with respect to A, then, we shall have without any further condition;

Proposition 5 (Khalili, Mathur and Bodenhorn 1974): The cost function takes a minimum in the interior of A, hence, in the ineterior of X x A, for each x in the interior of X, if distance x is fixed. This holds valid as it is, irrespective of whether $\angle M_2OM_3 = 60°$ or not.

For each $x \in X$, $y(x, .)$ and $z(x, .)$ vary continuously with direction angle $\alpha \in A$, hence, the cost function $C^*(x, .)$ takes its minimum and maximum somewhere on A. We may only see that, since,

$$(22) \qquad \frac{\partial C^*(x, \alpha)}{\partial \alpha} = \sum_{i=2}^{3} \frac{\partial P_i(x, \alpha)}{\partial \alpha} M_i(x, \alpha),$$

$$(23) \qquad \frac{\partial C^*(x, 0)}{\partial \alpha} = -P_3'(z) \frac{ax \sin 60°}{z(x, 0)} M_3(x, 0) < 0,$$

(24) $$\frac{\partial C^*(x,\ 60°)}{\partial \alpha} = P_2'(y)\,\frac{ax\ \sin 60°}{y(x,\ 60°)}\,M_2(x,\ 60°) > 0,$$

where observe axsin $60°/u_i$, i =2,3 are symmetric, but not $P_i'(u)M_i(x,\alpha_i)$, and,

(25) $$\frac{\partial u(x,\ \alpha_i)}{\partial \alpha} = \frac{ax\ \sin\alpha_i}{u(x,\ \alpha_i)}\,\frac{\partial \alpha_i}{\partial \alpha}, \qquad \alpha_2 = \alpha, \quad \alpha_3 = 60° - \alpha,$$

(26) $$\frac{\partial P_i(x,\ \alpha)}{\partial \alpha} = P_i'(u)\,\frac{\partial u(x,\ \alpha_i)}{\partial \alpha}, \qquad i = 2,\ 3.$$

Proposition 5 is an important result, and gives

Proposition 6 : 1 The cost function C*(x, .), for each x e X, can never be concave on A, hence, never be concave with respect to the space X x A.

2 The cost function C*(x, .), for each x e X, can never be monotone on A, hence, never be monotone with respect to the space X x A.

Since we have Proposition 5, it follows from Proposition 3 applied to C*(x, .), for each x, on A that the cost function is unable to be concave nor monotone on A, hence, nor, with respect to X x A; see Remark 0.

Thus, we summarize this in a remark:

Remark 3 : Each condition, which appears in Proposition 1-2, fails to hold, except in a linear space of transportation where α is always taken as zero so that the site M_2 and M_3 can be regarded as identical.

Also, we can have; in view of (22) for a fixed x, and (27) for $\alpha = 30°$,

Lemma 7 : The distance function d(x, .) for each $x \in X$ is convex on A, and takes its minimum at $\alpha = 30°$.

Finally, combined with Proposition 5, Proposition 4 may lead to a weaker sufficient condition for an interior location;

Proposition 7: In order that an interior point of location be chosen by the firm, it suffices to show that the marginal cost $\partial C^*(., \bar{\alpha})/ \partial x$ is strictly decreasing in x at x = 0, and, it is strictly increasing in x at $x = x(\bar{\alpha})$, for $\bar{\alpha}$ e A, at which $\partial C^*(x, \bar{\alpha})/ \partial \alpha = 0$ for x e X.

To prove this, first use Proposition 5 to get the fact that there exists $\bar{\alpha}$ e A, such that $\bar{\alpha} = \bar{\alpha}(x)$, $\partial C^*(x, \bar{\alpha}(x))/\partial \alpha = 0$, for each x e X. Then, if the condition of Proposition 7 is met, then, it follows from the proof of Proposition 4 that $\partial C^*(x(\bar{\alpha}), \bar{\alpha})/ \partial x = 0$.

5.2 A Theorem for a Weak Interior Location

We shall establish here a theorem that a location, which is interior or intermediate among markets, is optimal to the firm. We shall say this case weak interior. Figure 3 may help to see easily a proof of it.

Figure 3

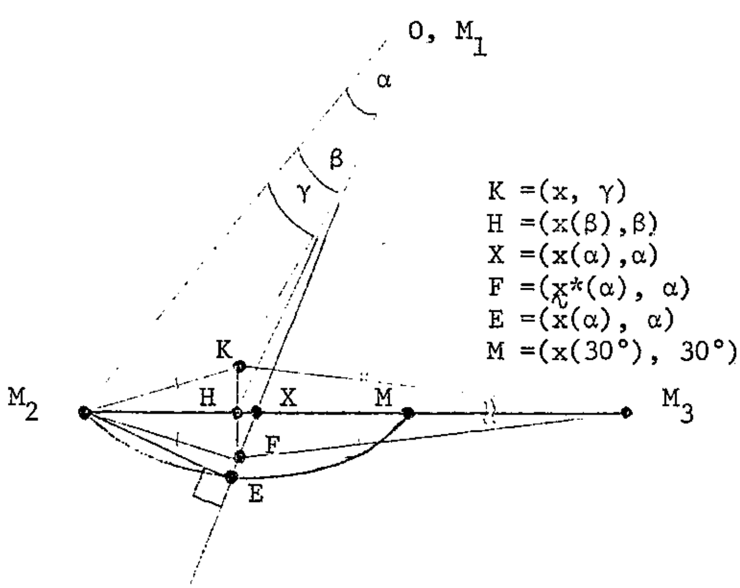


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Figure 3 b

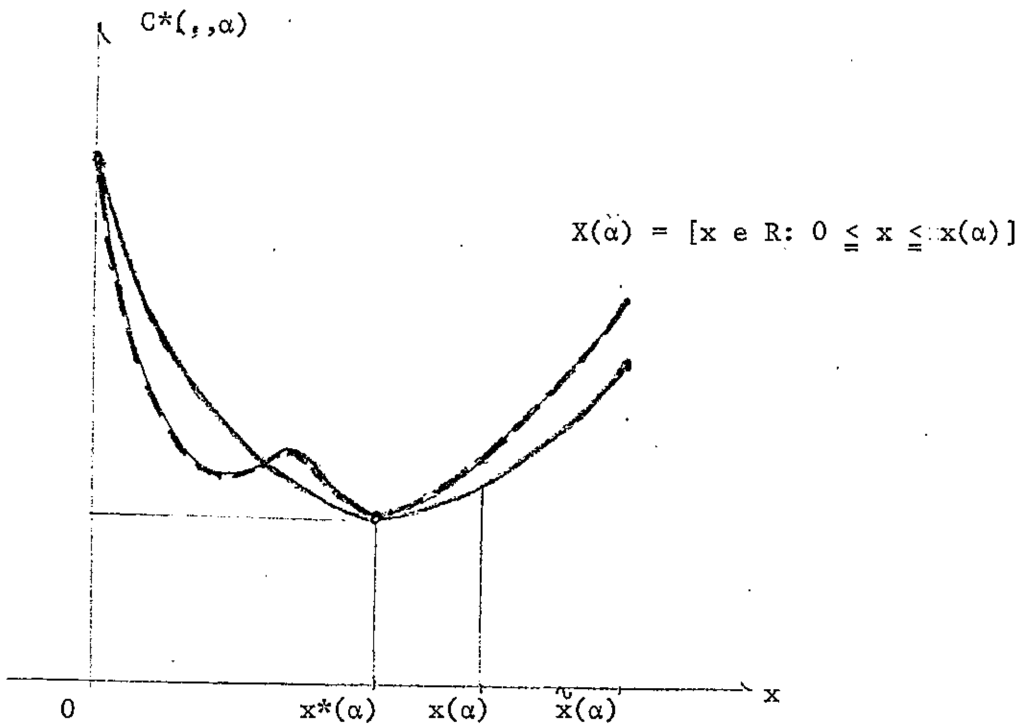
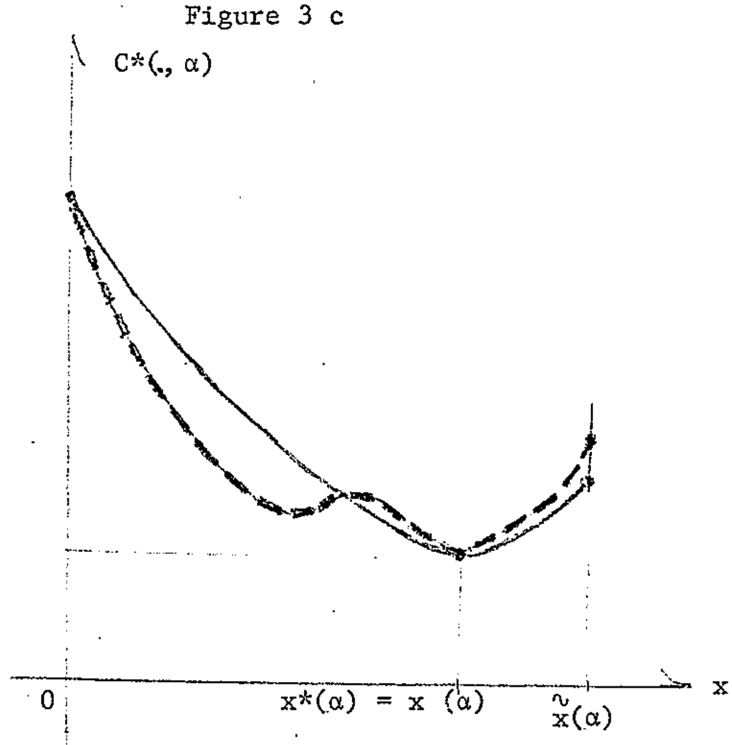


Figure 3 c



Theorem 1 (Weak Interior Location): The cost function $C^*(.,\ .)$ takes its minimum at a weak interior point (x^*, α^*) of the triangle space $X \times A$, hence, an interior or intermediate location is chosen by the firm, provided that

(i-1) $\partial C^*(0, \alpha)/\partial x < 0$, or, equivalently,

(27-1) $R_0'(0)\, Q + P_1'(0)M_1(0, \alpha) < \{P_2'(a)\cos \alpha\}M_2(0, \alpha) + \{P_3'(a)\cos(60°-\alpha)\}M_3(0, \alpha)$,

for each $\alpha \in A$,

(i-2) $\partial C^*(a, 0)/\partial x > 0$, that is,

(27-2) $R_0'(a)\, Q + P_1'(a)M_1(a, 0) > P_2'(0)M_2(a, 0) - \{P_3'(a)/2\}M_3(a, 0)$

and

(i-3) $\partial C^*(a, 60°)/\partial x > 0$, that is,

(27-3) $R_0'(a)\, Q + P_1'(a)M_1(a, 60°) > \{-P_2'(a)/2\}M_2(a, 60°) + P_3'(0)M_3(a, 60°)$.

Theorem 1 is in fact a general result. The cost function $C^*(.)$ is not restricted at all except at the three extreme points by the ad. hoc. conditions (i-j) (27-j) j =1,2,3. These may look stringent but not at all for an interior location. In view of (10), (12)-(15) with that $3/2 \leqq \eta(\alpha) \leqq \sqrt{3}$ (from Table 2 in Appendix A), where $\eta(\alpha) = \cos \alpha + \cos(60°-\alpha)$, it is natural that many interesting cases should satisfy the conditions. In fact, the opposite case, in which the inequality sign is reversed in (27-j), j = 1,2,3, respectively, for each α, gives a counterpart of this theorem; see Theorem 2 (Extreme Point of Location).

An interpretation for (27-1) is: an aggregated sum, weighted with Q, M_i, i =1,2,3, of the changes in the transport (C.I.F.) prices is negative at the site of a (output) market; that is, the cost is decreasing there.

Similar interpretations can be made for the rest of conditions. If, in (27-i) i =2,3, $P_i'(0)$ is so large that the inequality sign can be reversed, then, the extreme point $M_i(a, 0)$ would be chosen. We shall see this later.

Proof: From Proposition 7, we have only to show, at α, such that, for an $x \in X$, $\partial C^*(x, \alpha(x))/\partial\alpha = 0$, $\partial C^*(x+\varepsilon, \alpha)/\partial x \geqq 0$ as $\varepsilon \to 0^+$.

Observe first that $\partial C^*(\tilde{x}(\alpha), \alpha)/\partial x > 0$ (= 0, if $R_0'(x) = P_1'(x) = 0$ and $\alpha = 30°$), where $\tilde{x}(\alpha) = a \cos\alpha$, for each $\alpha \in A$, such that $0 \leqq \alpha \leqq 30°$. Thus, $C^*(., \alpha)$ takes its minimum somewhere in the closed interval $X(\alpha) = [0, \tilde{x}(\alpha)]$. Let $(x^*(\alpha), \alpha)$ be the point at which $\partial C^*(x^*(\alpha), \alpha)/\partial x = 0$ (and $\partial C^*(x^*(\alpha), \alpha(x^*))/\partial\alpha = 0$ if $\alpha = \alpha(x^*)$).

Suppose $x^*(\alpha)$ is in the open interval $(x(\alpha), \tilde{x}(\alpha))$. Then, there are (x, γ)'s, such that $(x, \gamma) \in X \times A$, $x^*(\alpha) \geqq x$, $y(x^*(\alpha), \alpha) \geqq y(x, \gamma)$, and $z(x^*(\alpha), \alpha) \geqq z(x, \gamma)$. Hence,

$$R_0(x^*(\alpha)) \geqq r_0(x^*(\alpha))x \geqq r_0(x)\, x, \quad P_1(x^*(\alpha)) \geqq \bar{P}_1 + r_1(x^*(\alpha))x$$
$$\geqq \bar{P}_1 + r_1(x)\, x, \quad P_i(x^*(\alpha), \alpha) \geqq \bar{P}_i + r_i(x^*(\alpha), \alpha)\, x \geqq \bar{P}_i + r_i(x, \gamma)\, x, \; i=2,3.$$

Define a cost function $C(x, \gamma)$ on such locations (x, γ)'s, to be;

$$C(x, \gamma) = R_0(x)\, Q + P_1(x)\, M_1(x^*(\alpha), \alpha) + \sum_{i=2}^{3} P_i(x, \gamma)\, M_i(x^*(\alpha), \alpha).$$

Then, $C^*(x^*(\alpha), \alpha) \geqq C(x, \gamma)$ with equality only when $x^*(\alpha) = x(\alpha) = x(\beta)$. This inequality holds for every such (x, γ) including $(x(\beta), \beta)$ on the line segment $\overline{M_2M_3}$. Thus, $(x^*(\alpha), \alpha)$ can not be outside of the space $X \times A$, if it minimizes $C^*(., .)$. Let α^* be such that $\partial C^*(x^*(\alpha^*), \alpha^*)/\partial\alpha = 0$.

Such α^* exists in the interior A° by Proposition 5. For this α^*, it must hold that $\partial C^*(x^*+\varepsilon, \alpha^*)/\partial x \geqq 0$. That is, the location $(x^*(\alpha), \alpha)$, at which the cost $C^*(., .)$ takes its minmimum, must be in the interior or on the line $\overline{M_2M_3}$ in the space. Observe that location $(x(0), 0)$, at which $\lim_{x<a} \partial y(x, 0)/\partial x = -1$, and $\lim_{x>a} \partial y(x, 0)/\partial x = 1$, is excluded by (27-2), where note $\lim_{x>a} \partial C^*(x, 0)/\partial x = R_0'(a)Q + P_1'(a)M_1(a, 0) + P_2'(0)M_2(a, 0) + \{P_3'(a)/2\}M_3(a, 0) > \lim_{x<a} \partial C^*(x, 0)/\partial x = \partial C^*(a, 0)/\partial x$ given by (i-2). Similarly for the case in which $30^\circ \leqq \alpha \leqq 60^\circ$ with (27-3).

Remark 4:1 In the proof, if further for each α_i, $0 \leqq \alpha_i \leqq 30^\circ$,

$$\begin{aligned}\partial C^*(x(\alpha_i), \alpha_i)/\partial x &= R_0'(x(\alpha_i))Q + P_1'(x(\alpha_i))M_1(x(\alpha_i), \alpha_i) \\ &+ (-1)^{i+1}P_2'(y(\alpha_i))|\partial y(\alpha_i)/\partial x|\, M_2(x(\alpha_i), \alpha_i) \\ &+ (-1)^{i}P_3'(z(\alpha_i))|\partial z(\alpha_i)/\partial x|\, M_3(x(\alpha_i), \alpha_i) > 0, \quad i = 2,3, \end{aligned} \tag{27-4}$$

then, the location will be chosen strictly in the interior of the space, where $x(\alpha_i) = (a\sqrt{3}/2)/(\sin(60^\circ + \alpha_i)$, $y(\alpha_i) = y(x(\alpha_i), \alpha_i)$, $z(\alpha_i) = z(x(\alpha_i), \alpha_i)$, $\frac{1}{a}|\partial y(\alpha_i)/\partial x| = (-1)^{i+1}\phi(\alpha_i)/y(\alpha_i) > 0$, $\phi(\alpha_i) = x(\alpha_i)/a - \cos \alpha_i$, $\frac{1}{a}|\partial z(\alpha_i)/\partial x| = (-1)^{i}\psi(\alpha_i)/z(\alpha_i) > 0$, $\psi(\alpha_i) = x(\alpha_i)/a - \cos(60^\circ - \alpha_i)$ and $\partial y(\alpha_i)/\partial x + \partial z(\alpha_i)/\partial x \geqq 0$. This condition will be satisfied if, $i = 2, 3$,

$$\begin{aligned}& R_0'(x(\alpha_i))Q + P_1'(x(\alpha_i))M_1(x(\alpha_i), \alpha_i) \\ &\quad - \max\{P_2'(y(\alpha_i))M_2(x(\alpha_i), \alpha_i),\ P_3'(z(\alpha_i))M_3(x(\alpha_i), \alpha_i)\} \\ &\quad + \frac{1}{2}\min\{P_2'(y(\alpha_i))M_2(x(\alpha_i), \alpha_i),\ P_3'(z(\alpha_i))M_3(x(\alpha_i), \alpha_i)\} \\ &\quad > 0, \quad \alpha_2 = \alpha, \ \alpha_3 = 60^\circ - \alpha. \end{aligned} \tag{27-5}$$

Taking $\alpha = 0$, we have (27-2) for $i = 2$, and (27-3) for $i = 3$.

We shall examine this condition in case of Leontief technology; see Section 6.

Remark 4:2 In the proof of the theorem, observe; there is no ordering among the costs $C(x, \gamma)$'s for such inside points (x, γ)'s, whereas, the cost $C^*(x^*(\alpha), \alpha)$ outside is always larger than the cost $C(x, \gamma)$ inside. Note also that, $C(x, \gamma) \geqq C^*(x, \gamma)$ for each such (x, γ), and $C^*(x, \gamma)$ and $C^*(x(\gamma), \gamma)$ can not be compared yet with each other.

Remark 5: This condition will become, however, so much more hardly established, as (relatively) more input markets for indispensable factors, $M_4, M_5, \ldots,$ are spacially agglomerated at the output market site. In fact, $R_0'(0)Q + P_1'(0)M_1(0, \alpha) + \sum_{j \geqq 4}^{s} P_j'(0)M_j(0, \alpha)$ will become larger definitely than $P_2'(0, \alpha)\cos\alpha\, M_2(0, \alpha) + P_3'(0, \alpha)\cos(60°-\alpha)M_3(0, \alpha)$, if the number s becomes larger enough. Then, the exclusion argument will be easier to hold, irrespective of whether technology is of a Leontief type or of a neoclassical type.

The next remark will deserve of attention:

Remark 6: The cost function $C^*(., \alpha^*)$ can not be concave with respect to X, that is, $\partial^2 C^*(., \alpha^*)/\partial^2 x > 0$, for some x e X.

If for all x e X it is negative or zero, then, $\partial C^*(., \alpha^*)/\partial x$ is negative hence, contradicting to $\partial C^*(x^*+\varepsilon, \alpha^*)/\partial x \geqq 0$. Hence, it can not be true that $\partial^2 C^*(., \alpha)/\partial x^2 \leqq 0$, for all x e X, for each α e A.

Remark 6 also supports Remark 3.

6. A Leontief Type of Technology and the Location Problem

In order to disclose specific characteristics of transport cost and range structure, which would let a firm choose an interior, intermediate, or extreme point of location in the space. we shall first presume that the given production function is of a Leontief type. This excludes substitutions among inputs (factors of production), hence, negative substitution effects (with respect to price hence distance). That is, $S_x(x, \alpha) = 0$, and $S_\alpha(x, \alpha) = 0$.

The technology employed by a firm is given in a fixed vector form called an input vector $a_1 = (a_{11}, a_{21}, a_{31}) \geqq 0$. Take $Q = 1$. We shall examine throughout this section how the unit cost function $C^*(.)$ varies on X for each $\alpha \in A$. To see the fundamental feature of this model, we presume that

$$(28) \qquad R_0''(x) = P_1''(x) = P_i''(u) = 0, \quad i = 2, 3, \quad u = u(x, \alpha_i).$$

Then, for positive (non-negative) constants c_0, c_1, c_2, and c_3,

$$(29) \quad C^*(x, \alpha) = c_0 x + \{\bar{P}_1 + c_1 x\} a_{11} + \{\bar{P}_2 + c_2 y(x, \alpha)\} a_{21} + \{\bar{P}_3 + c_3 z(x,\alpha)\} a_{31},$$

which is a nonlinear cost function with respect to location x and α.

$$(30) \qquad \frac{\partial C^*(x, \alpha)}{\partial x} = c_0 + c_1 a_{11} + c_2 a_{21} \frac{\partial\, y(x, \alpha)}{\partial\, x} + c_3 a_{31} \frac{\partial\, z(x, \alpha)}{\partial\, x},$$

$$(31) \qquad \frac{\partial^2 C^*(x, \alpha)}{\partial x^2} = c_2 a_{21} \frac{\partial^2 y(x, \alpha)}{\partial x^2} + c_3 a_{31} \frac{\partial^2 z(x, \alpha)}{\partial x^2} > 0.$$

The last inequality (31) proves that $C^*(.,)$ is strictly convex on X for each $\alpha \in A$, hence, it follows from Proposition 4, 5 & 7 that, if

(32) $$\frac{\partial C^*(0, \alpha)}{\partial x} < 0, \quad \text{and} \quad \frac{\partial C^*(x(\alpha), \alpha)}{\partial x} > 0,$$

then, an interior point is optimal, and the location is taken at that point, which is unique in the triangle space, at which the partial derivative of the cost C* with respect to distance x vanishes. Otherwise, an extreme point of location is chosen from monotonicity. We shall raise three examples, the first two of which may serve to see how an extreme or interior point of location is taken as optimal by the properties of the transport distance function $l(.) + d(., \alpha)$. The third one will serve to show even in case of the convexity of the cost function C*(.) that an extreme point is optimal, due to the ad. hoc. transport cost structure assumed. Corollaries to Theorem 1 may apply to the second case (Case 4), while, Theorem 2, which will be established later, may apply to the rest of examples; Case 3 and 5.

6.1 Special Cases in which No Substitution Prevails

Case 3: Suppose further that $c_0 = c_i a_{i1}$, i = 1, 2, 3, as in that transport costs of a unit of output Q, and a_{i1} unit of input factor, i = 1, 2, 3, are adjusted so that they all may be taken as the same.

Then,

(33) $$\frac{\partial C^*(x, \alpha)}{\partial x} = c_0\{1 + \frac{\partial d(x, \alpha)}{\partial x}\} > 0$$

where the positivity comes from Lemma 4:1.

Thus, the location should be chosen at output market 0.

Case 4: Alternatively, suppose further that $c_0 = c_i a_{i1}$, $i = 2, 3$, and $R_1(x)$ is constant over x, or $a_{11} = 0$. This may serve as a typical example of a triangle case. Then, the condition (27) is met and at least one point is optimal and weakly interior. Further, we have

$$\frac{\partial C^*(x, \alpha)}{\partial x} = c_0 \frac{\partial d(x, \alpha)}{\partial x}. \tag{34}$$

Thus, the cost function $C^*(.)$ takes its minimum on X for each $\alpha \in A$, just as the distance function $d(.)$ does on X. From Lemma 6, it follows that the gravity point $((\sqrt{3}/3)a, 30°)$ is uniquely optimal.

We can generally state that there is a uniformly convergent sequence of continuous cost functions $C^{*n}(.)$ for which c_0, c_i^n, $i = 1, 2, 3$, are involved, such that for each $\varepsilon > 0$ and for some number $N(\varepsilon)$,

$$\left| \frac{\partial C^{*n}(x, \alpha)}{\partial x} - \{1 + \frac{\partial d(x, \alpha)}{\partial x}\} c_0 \right| < \varepsilon, \quad n > N(\varepsilon), \tag{35}$$

where the limit function is $c_0(x + d(x, \alpha)) + \bar{c}$, or,

$$\left| \frac{\partial C^{*n}(x, \alpha)}{\partial x} - \frac{\partial d(x, \alpha)}{\partial x} c_0 \right| < \varepsilon, \quad n > N(\varepsilon), \tag{36}$$

where the limit function is simply $c_0 d(x, \alpha) + \bar{c}$, $\bar{c}$ is arbitrary. Thus, Case 3 and 4 may be said to be the generic cases in which typical features are explicitly visualized. There is another interesting case.

Let $\kappa(\alpha) = c_2a_{21}\cos\alpha + c_3a_{31}\cos(60°-\alpha)$. Then, it follows that $\kappa(.)$ is concave on A and $\partial\kappa(0)/\partial\alpha = \sqrt{3}c_3a_{31}/2$, $\partial\kappa(60)/\partial\alpha = -\sqrt{3}c_2a_{21}/2$, hence, from Proposition 4, there exists a unique point $\bar{\alpha}$ such that $c_2a_{21}/c_3a_{31} = \sin(60°-\bar{\alpha})/\sin\bar{\alpha}$, provided that $0 < c_2a_{21}/c_3a_{31} < \infty$.

In order to see that in case of no substitution prevailing a certain cost structure imposed will make the firm choose an extreme point as an optimal location, we shall have;

Case 5: Suppose $c_0 + c_1a_{11} \geqq \kappa(\bar{\alpha})$ for $\kappa(\bar{\alpha})$ thus defined. Then, $c_0 + c_1a_{11} + c_3a_{31}\,\partial y(0,\alpha)/\partial x + c_3a_{31}\,\partial z(0,\alpha)/\partial x \geqq 0$, with equality when $\alpha = 0$, that is, $\partial C^*(0,\alpha)/\partial x \geqq 0$. This in turn implies, with the convexity of C^* on X for each α, the location is at output site 0. In case at each site only one market is distributed, where a_{11} can be regarded as zero, the argument holds if $c_0 \geqq \kappa(\bar{\alpha})$.

Suppose $c_0 + c_1a_{11} = c_2a_{21} + c_3a_{31}$, then, the condition in Case 5 is satisfied. By this, an intuitive observation is justified so that the production location should be taken at a site, at which many markets are spacially agglomerated; see Remark 5.

Case 3 and 5 will be generalized in a theorem; Theorem 2, whereas, Case 4 will be generalized in a corollary to Theorem 1.

6.2 Corolaries for an Interior Location

We have two corollaries to Theorem 1, associated with the Leontief technology.

Corollary 1 (Interior Location): Let the production function be given of a Leontief type. Then, the cost function $C^*(.,.)$ takes its minimum at an interior point (x^*, α^*) of the space $X \times A$, if Condition (i-1,2,3) or equivalently (27-1,2,3) is satisfied, and transport cost functions are concave.

Note that an interior point of location is chosen here.

Proof: Apply Theorem 1 to have an intermediate location first. Then, we like to show $\partial C^*(x(\alpha), \alpha)/\partial x > 0$ for each α such that $0 \leqq \alpha \leqq 30°$.

$$\partial C^*(x(\alpha), \alpha)/\partial x = R_0'(x(\alpha)) + P_1'(x(\alpha))\, a_{11} + P_2'(y(\alpha))\, a_{21}\, \partial y(\alpha)/\partial x + P_3'(z(\alpha))\, a_{31}\, \partial z(\alpha)/\partial x.$$

From Remark 4:1, $\partial y(\alpha)/\partial x + \partial z(\alpha)/\partial x \geqq 0$, and also in view of (10), $|\partial y(\alpha)/\partial x| \leqq 1$, $|z(\alpha)/\partial x| \leqq 1$. Hence, Condition (27-2) implies;

$$\begin{aligned} 0 < \partial C^*(x(0), 0)/\partial x &= R_0'(x(0)) + P_1'(x(0))\, a_{11} - P_2'(0)\, a_{21} + P_3'(z(0))/2\; a_{31} \\ &< R_0'(x(0)) + P_1'(x(0))\, a_{11} - P_2'(0)\, a_{21} + P_3'(z(0))\, a_{31} \\ &< R_0'(x(0)) + P_1'(x(0))\, a_{11} + \{P_2'(0)\, a_{21} - P_3'(z(0))\, a_{31}\}\, \partial y(\alpha)/\partial x \\ &\leqq \partial C^*(x(\alpha), \alpha)/\partial x, \text{ if } R_0''(x) = P_1''(x) = P_i''(u) = 0,\ i = 2, 3. \end{aligned}$$

Suppose transport cost functions are concave so that $R_0''(x) \leqq 0$, $P_1''(x) \leqq 0$, $P_i''(u) \leqq 0$, $i = 2, 3$. Then, since,

$$\partial C^*(x(\alpha), \alpha)/\partial x \quad > \quad R_0'(x(\alpha)) + P_1'(x(\alpha))a_{11} - P_2'(y(\alpha))a_{21} + P_3'(z(\alpha))a_{31},$$

and the right hand side (denoted by $\Pi(\alpha)$) is increasing in α;

$$\partial\Pi(\alpha)/\partial\alpha = \{R_0''(x(\alpha)) + P_1''(x(\alpha))a_{11}\}\partial x(\alpha)/\partial\alpha$$

$$- P_2''(y(\alpha))a_{21}\ \partial y(\alpha)/\partial\alpha + P_3''(z(\alpha))a_{31}\ \partial z(\alpha)/\partial\alpha > 0,$$

it follows that $\Pi(\alpha) > 0$ if $\Pi(0) > 0$, which is established if, for $a = x(0)$,

$$\partial C^*(x(0), 0)/\partial x = R_0'(a) + P_1'(a)a_{11} - P_2'(0)a_{21} + (P_3'(a)/2)a_{31} = \Pi(0) > 0, (27\text{-}2).$$

Thus, $\partial C^*(x(\alpha), \alpha)/\partial x \geqq \Pi(\alpha) > 0$, for each such α. Similarly for α such that $30° \leqq \alpha \leqq 60°$. Hence, the chosen location must be interior.

The next one concerns itself with stronger conditions which will make the results stronger. See Remark 7. With the help of;

<u>Lemma 8</u>: <u>Suppose that transport cost functions are convex in ranges, i.e.</u> $R_0''(x) \geqq 0$, $P_1''(x) \geqq 0$, and $P_i''(u(x, \alpha_i))$ $i = 2,3$, $\geqq 0$. <u>Then, the cost function $C^*(., \alpha)$ is convex on $X(\alpha)$ for each α e A.</u>

(In view of (9) and (5), where $S_x(x, \alpha) = 0$, the proof is obvious.)

We shall prove a stronger result;

<u>Corollary 2 (Strong Interior Location)</u>: <u>Let the production function be given of a Leontief type. Then, the cost function $C^*(., .)$ takes its minimum at a unique interior point (x^*, α^*) of the space X x A, provided that (i -1) $\partial C^*(0, \alpha)/\partial x < 0$, (equivalently, Condition(27-1) is met for each α e A)</u>

<u>and</u> (27-6) $c_0 + c_1 a_{11} \geqq \max(c_2 a_{21}, c_3 a_{31})$; c_0, c_i, $i = 1,2,3$, constant;

or, (27-7) $R_0'(a/2) + R_1'(a/2)a_{11} \geqq \max\{P_2'(a/2)a_{21}, P_3'(a/2)a_{31}\}$ if $R_0''(x) \geqq 0$, $P_1''(x) \geqq 0$, and $P_i''(u) \geqq 0$, $i = 2,3$.

Condition (27-1) is here rewritten as

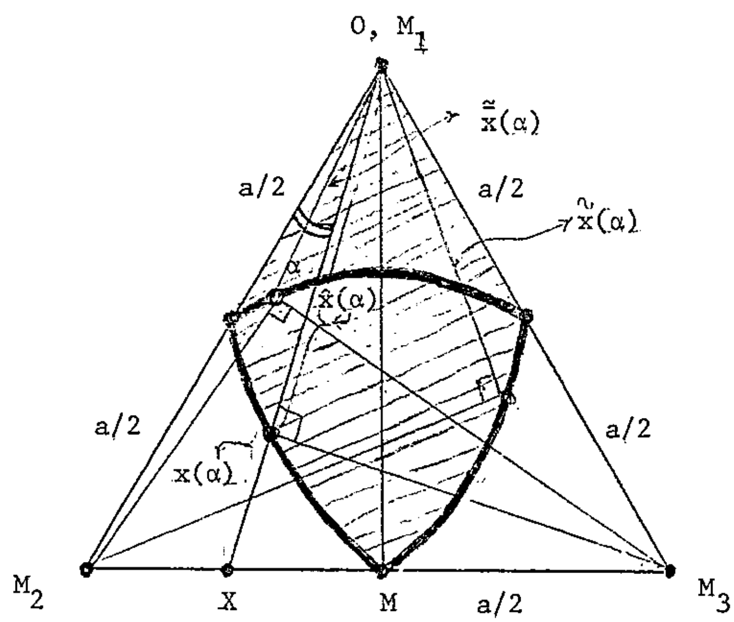
$$(27)' \quad c_0 + c_1 a_{11} < c_2 a_{21} \cos\alpha + c_3 a_{31} \cos(60°-\alpha)$$

where the right hand side is $\kappa(\alpha)$ defined before and takes its minimum at

$\alpha=0°$ or $60°$. We assume $\max(c_2a_{21}, c_3a_{31}) = c_2a_{21}$ without loss of generality. The condition (ii) and (27)' imply $\kappa(\alpha) > \max(c_2a_{21}, c_3a_{31}) \geqq c_2a_{21}$, but, $\kappa(\alpha) \geqq \kappa(0°) = c_2a_{21} + c_3a_{31}/2 > c_2a_{21}$. Thus, the two conditions are compatible with each other. Similarly for the case $30° \leqq \alpha \leqq 60°$.

An economic interpretation for the set of condition (27)' and (27-4)is; an aggregated sum of transport unit cost per a unit of distance x for output

Figure 4



and that for the first input is, at the site of output market, less than the sum of those for the second and third inputs, but larger than any of those evaluated everywhere for the second and third inputs. A similar but more restrictive interpretation can be made for the condition (iii).

Proof: Apply Theorem 1 to obtain the weak intermediate location. Then, we have only to show this is in the interior. To this end, we must show $\partial C^*(x(\alpha^*), \alpha^*)/\partial x > 0$. First, we see, for $\hat{x}(\alpha) = a\cos(60°-\alpha)$, $\hat{y} = \hat{y}(x(\alpha), \alpha)$, $\partial y(\hat{x}(\alpha), \alpha)/\partial x = a(\cos(60°-\alpha)-\cos\alpha)/y(\hat{x}(\alpha), \alpha) = a\sin(\alpha - 30°)/y(\hat{x}(\alpha),\alpha)$ $\leqq 0$, and $\partial z(\hat{x}(\alpha), \alpha)/\partial x = 0$ for α such that $0 \leqq \alpha \leqq 30°$. Since, for this α,

$$\frac{\partial}{\partial\alpha}\{\frac{1}{\hat{y}}\sin(\alpha - 30°)\} = \frac{1}{\hat{y}}\cos(\alpha - 30°) - \frac{\sin(\alpha-30°)}{\hat{y}}\frac{xa\sin\alpha}{\hat{y}} > 0.$$

This implies, in (30), and under the condition (ii), that $\partial C^*(\hat{x}(\alpha), \alpha)/\partial x \geqq 0$, with equality only when $\alpha = 0°$, for such α. By the strict convexity of $C^*(., \alpha)$ with respect to X for this α, it follows that $\partial C^*(x, \alpha)/\partial x > 0$ for each x such that $\hat{x}(\alpha) \leqq x$. hence, $\partial C^*(x(\alpha), \alpha)/\partial x > 0$.

With the condition (iii), in stead of (ii), we have

$$\partial C^*(\hat{x}(\alpha), \alpha)/\partial x = R_0'(\hat{x}(\alpha)) + P_1'(\hat{x}(\alpha))a_{11} + \{P_2'(\hat{x}(\alpha), \alpha)\partial y(\hat{x}(\alpha), \alpha)/\partial x\}\, a_{21}.$$

Hence, $\partial C^*(\hat{x}(0), 0)/\partial x = R_0'(a/2) + P_1'(a/2)a_{11} - P_2'((a/2), 0)\, a_{21} \geqq 0$.
It also follows from Lemma 8 that $\partial C^*(x(\alpha), \alpha)/\partial x > 0$ if $\partial C^*(\hat{x}(\alpha), \alpha)/\partial x > 0$. Thus, we shall show $\partial C^*(\hat{x}(\alpha), \alpha)/\partial x > 0$, for which it is sufficient that

$$\partial^2 C^*(\hat{x}(\alpha), \alpha)/\partial x\partial\alpha = R_0''(\hat{x}(\alpha))\partial\hat{x}(\alpha)/\partial\alpha + \{P_1''(\hat{x}(\alpha))\,\partial\hat{x}(\alpha)/\partial\alpha\}a_{11}$$
$$+ P_2''(\hat{x}(\alpha), \alpha)(\partial y(\hat{x}(\alpha), \alpha)/\partial\alpha)^2 a_{21} + \{P_2'(\hat{x}(\alpha), \alpha)\,\partial^2 y(\hat{x}(\alpha), \alpha)/\partial x\partial\alpha\}\, a_{21} > 0.$$

The last inequality comes from $\partial\hat{x}(\alpha)/\partial\alpha = a\sin(60°-\alpha) > 0$, $\partial y(\hat{x}(\alpha), \alpha)/\partial x = a\sin(\alpha-30°)/y(\hat{x}(\alpha),\alpha) \leqq 0$, and $\partial^2 y(\hat{x}(\alpha), \alpha)/\partial x\partial\alpha > 0$ above.

The uniqueness follows from the convexity with respect to X and A.

In the Leontief case, suppose transport functions are linear so that $c_0 = R_0'(x)$, $c_1 = P_1'(x)$, and $c_i = P_i'(u)$ $i = 2,3$. Then, Condition (27-6) implies Condition (27-2 and 3) and hence Corollary 1 holds but not vice versa.

Remark 7: The conclusion of Corollary 2 is stronger in the sense that the location is chosen at a point inside the shaded area of the space; see Figure 4 whereas, in Corollary 1, it is somewhere in the whole interior.

Perhaps the strongest result can be obtained, if we assume, as Corollary 3, instead of (i) and (ii) or (iii), that $\partial C^*(\hat{x}(\alpha), \alpha)/\partial x > 0$, $\partial C^*(\tilde{x}(\alpha), \alpha)/\partial x > 0$, and $\partial C^*(\tilde{\tilde{x}}(\alpha), \alpha)/\partial x < 0$, except at $\alpha = 0°$, or $60°$. Then, the location will be chosen at a point inside the area surrounded by the solid line; see Figure 4. Here $\tilde{x}(\alpha)$ is evaluated at α such that $30° \leqq \alpha \leqq 60°$, $\tilde{\tilde{x}}(\alpha) = \frac{a}{2}\{ \eta(\alpha) - \sqrt{\eta^2(\alpha)-2}\}$, and is evaluated at α such that $0° \leqq \alpha \leqq 30°$. Corollary 2 satisfies the first two conditions, but does not necessarily satisfy the third one. Case 4 is a special case in which all the conditions are satisfied and this strongest result is obtained.

Observe also that the optimal location is independent of the level of output. This property of the optimal location will be considered later with the substitutive production technology. See Proposition 8.

7. A General Technology and the Economic Location Problem

Suppose, in Section 6, that a production function f defined on an input factor space is substitutive among factors as well as constant returns to scale, as in case it is neoclassical. Then, the average cost $C^*(.)/Q$ and marginal cost $\partial C^*(.)/\partial Q$ are equal, constant for all output Q, for each $(x, \alpha) \in X \times A$. We may take $Q = 1$ for each $(x, \alpha) \in X \times A$. In stead of a_{i1}, $i = 1, 2, 3$, in equation (29), $m_i(x, \alpha)$, $i = 1, 2, 3$, may be taken here. The difference between them is of course that $m_i(x, \alpha)$ varies with (x, α) on $X \times A$, but, a_{i1}, constant for each $i = 1,2,3$. Including this constant returns to scale or a homothetic production as special cases, we shall examine how the substitution effect produces effects on the firm's locational decision making. The substitution effects on the cost function C* does not vanish so that $S_x(x, \alpha) \neq 0$, and $S_\alpha(x, \alpha) \neq 0$. If production function f is neo-classical, then, the effects take non-positive value. If it is of a positive semi-definite inverse of its bordered Hessian everywhere, then, they are non-negative. The latter case is not neoclassical, but worth of attention especially for analysis of imperfect competitive production.

First, we shall point out the properties of an optimal location that a class of homothetic production functions may yield.

Find an optimal point (x^*, α^*) and fix it, at which the cost function C* takes its minimum in the space. Then, the unit cost $C^*(x^*, \alpha^*)/Q$ is independent of the level of output Q, if and only if the optimal location (x^*, α^*) is invariant with respect to the level of output Q. The former is equivalent to that $M_1(x^*, \alpha^*)/Q$, $M_i(x^*, \alpha^*)/Q$, i =2,3, are invariant with respect to the level of output Q, which is also equivalent to that f is linearly homogenious in inputs M_1, M_i, i =2,3.

In view of (22), Suppose for a fixed x, $\partial C^*(x, \alpha^*)/\partial\alpha = 0$, $\alpha^* \in A$. That is, $\{\sin\alpha^*/y(x,\alpha^*)\}/\{\sin(60°-\alpha^*)/z(x, \alpha^*)\} = \{P_3'(z(x, \alpha^*)/P_2'(y(x,\alpha^*)\}\{M_3(x,\alpha^*)/Q\}/\{M_2(x, \alpha^*)/Q\}$. Then, in the right hand side $\{M_3(x,\alpha^*)/Q\}/\{M_2(x, \alpha^*)/Q\}$ is invariant with respect to the level output Q if and only if production function f is homothetic hence the expansion path is linear and a ray through the origin in the input-output space.

Suppose there is no need for the firm to pay the cost of transporting output to the output market, as in case buyers pay for the costs. Then, in view of (4), $R_0'(x) = 0$ for each x, and an optimal point $(x^*_\ast \alpha^*)$ for a level of output satisfies; $\partial C^*(x^*_\ast, \alpha^*)/\partial x = 0 \leftrightarrow$

$$-P_1'(x) = \Sigma_{i=2}^{3} \{\partial P_i(x^*, \alpha^*)/\partial x\}\{M_i(x^*_\ast \alpha^*)/Q\}/\{M_1(x^*_\ast \alpha^*)/Q\} .$$

For the point $(x^*_\ast \alpha^*)$ to remain to be optimal for all levels of output it is necessary and sufficient that $\{M_i(x^*_\ast \alpha^*)/Q\}/\{M_1(x^*_\ast \alpha^*)/Q\}$ i =2,3 are constant over any levels of output Q. These are easily extended to many inputs, by taking the number 3 to any integer n.

We may thus have a <u>single</u> optimal location:

<u>Proposition 8:1 (Bradfield 1971, Khalili, Mathur and Bodenhorn 1974)</u>
<u>An equivalent condition for a location $(x^*_\ast \alpha^*)$ in the space to remain optimal for any level of ooutput is that production is governed by constant returns to scale, that is, f is homogeneous</u>.

8:2 (Khalili, Mathur and Bodenhorn 1974) An equivalent condition for a location (x, α^*) for each fixed x to remain optimal for any levels of ouput is that production is governed by homotheticity with respect to scale, that is, f is homothetic.

8:3 An equivalent condition for a location (x^*, α^*) to remain optimal for any level of output is that production is governed by homotheticity with respect to scale, provided that there is no need for the firm to pay the cost of transporting output to the output market.

As Moses pointed out, the optimal location, in general, varies with the level of output and there is a different optimal location for every level of output. However, almost all the production functions used in empirical economic analysis are homothetic. The restriction of the above results to homotheticity or linear homogeneity is not so limiting on their application.

The substitution property is the property with respect to prices $P_i(x, \alpha)$, i =1, 2, 3, hence, to location variable (x, α), and provides a decreasing effect in the marginal cost $\partial C^*(x, \alpha)/\partial v$, $v = x, \alpha$. If this factor is so dominant with respect, for example, to x that $C^*(., \alpha)$ is concave with respect to $X(\alpha)$ at x for an α e A, then, it takes minimum value at an boundary point of $X(\alpha)$. Thus, in general, the neoclassical factor reinforces any concavity of $C^*(.)$ which may appear. On the other hand, the convexity property of range $y(., \alpha)$, $z(., \alpha)$ and of the distance function $d(.)$ on $X(\alpha)$ will provide an increasing effect. The convexity property will make the cost function $C^*(.)$ convex if it is dominant in (5) so that the sign can be determined to be positive. Proposition 6:1-2 implies the cost function can not be monotone nor concave on A, nor it is concave on $X(\alpha)$ for each e A, if Condition (27) is satisfied; see Remark 6. Thus, the case in which $C^*(.)$ is concave on X, must be examined without this condition.

7.1 We may add in this general case a corollary to Theorem 1:

Corollary 4 (Strong Interior Location): Let the transport distance factor is so dominant that the cost function $C^*(., \alpha)$ for each $\alpha \in A$, is strictly convex on X. Then, the cost function $C^*(., .)$ takes its minimum at an interior point (x^*, α^*) of the space X x A, provided that, in addition to (i-1),

(27-8)
$$R_0'(\hat{x}(\alpha_i))Q + P_1'(\hat{x}(\alpha_i))M_1(\hat{x}(\alpha_i), \alpha_i) > P_i'(\hat{x}(\alpha_i), \alpha_i)|\partial u(\hat{x}(\alpha_i), \alpha_i)/\partial x|M_i(\hat{x}(\alpha_i), \alpha_i),$$

where $i = 2,3$, $\alpha_2 = \alpha$, $\alpha_3 = 60° - \alpha$, $0 \leqq \alpha_i \leqq 30°$, and $\hat{x}(\alpha_i) = a\cos(60° - \alpha_i)$.

Proof: See and follow the latter half in the proof of Corollary 2.

Remark 8: Condition (27-8) is not so strong as it looks. This is satisfied if it holds that,

(27-9) $$R_0'(\hat{x}(\alpha_i))Q + P_1'(\hat{x}(\alpha_i)M_i(\hat{x}(\alpha_i), \alpha_i) \geqq P_i'(\hat{x}(\alpha_i), \alpha_i)M_i(\hat{x}(\alpha_i), \alpha_i), \quad i = 2,3,$$

$0 \leqq \alpha_i \leqq 30°$.

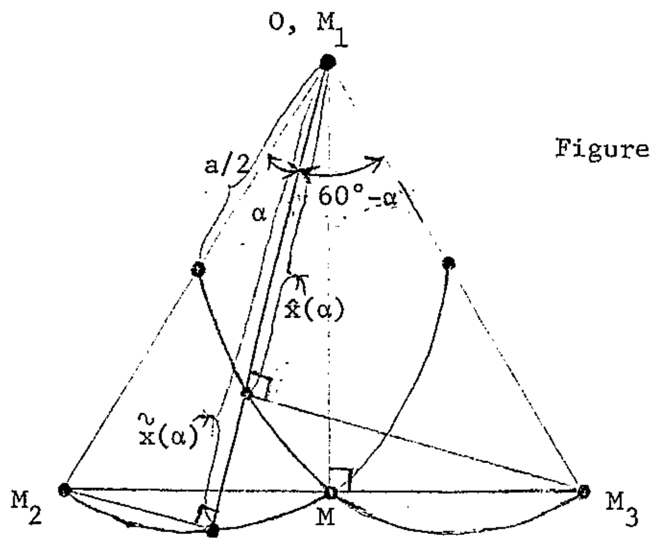


Figure 5

7.2 A Theorem for Exclusion of an Intermediate Location.

In what follows, we wish to show that, with Condition (v), (vi), (viii) or (ix), the convexity or concavity of C^* implies the monotonicity of C^* on $X(\alpha)$ for each α, hence, with the fact that an optimal point is inside the space, implies that an extreme point is optimal. With Condition (vii), the concavity of the cost function C^*, which is not monotone, implies that one of the three sites is optimal.

As referred before, we can presume the concavity of cost function C^* if condition (i) is not satisfied; see Remark 5 and 6.

Theorem 2 ((Weak) Exclusion): The cost function $C^*(.,.)$ takes its minimum at a market site, hence, the market site is chosen by the firm, provided that (iv) $\partial C^*(0, \alpha)/\partial x \geqq 0$, or, equivalently,

(37) $R_0'(0)\,Q + P_1'(0)M_1(0, \alpha) \geqq \{P_2'(a)\cos\alpha\}M_2(0, \alpha) + P_3'(a)\cos(60°-\alpha)M_3(0, \alpha)$,

and (v) it is convex on (and is not constant over) X, for each $\alpha \in A$

or, (vi) $C^*(., \alpha)$ is concave so that $\partial^2 C^*(x, \alpha)/\partial x^2 \leqq 0$ for each $x \in [0, \tilde{x}(\alpha)]$,

or, (vii) $C^*(., \alpha)$ is concave so that $\partial^2 C^*(x, \alpha)/\partial x^2 \leqq 0$ for each $x \in [0, \bar{x}(\alpha)]$

and $\partial C^*(x(0), 0)/\partial x < 0$, Alternatively, the conclusion holds, provided that, in addition to (i-1) (viii) $\partial^2 C^*(., \alpha)/\partial x^2 < 0$ (concavity) on X, or (ix) $C^*(.,\alpha)$ is convex so $\partial^2 C^*(x,\alpha)/\partial x^2 \geqq 0$ on X for each α, and $\partial C^*(x(\alpha),\alpha)/\partial x < 0$.

Proof: With the condition (iv);(37), the convexity of C^* on X implies $\partial C^*(x, \alpha)/\partial x > 0$ for each α e A. Apply Proposition 2:1 and 3 here. In stead of (v), with (vi), in addition to (iv), the concavity of C^* on $[0, x(\alpha)]$ implies $\partial C^*(x, \alpha)/\partial x > \partial C^*(x(\alpha), \alpha)/\partial x > 0$. Proposition 2:1 and 3 may apply here and the output market $(0,\alpha)$ is chosen.

Suppose (37) and (vii) hold. Then, from Proposition 1 and 3, the point (x^*, α), at which $C^*(x^*, \alpha) \leqq C^*(x, \alpha)$ for every x e X, for each α. Since $\partial C^*(x(0), 0)/\partial x < 0$ by assumption, there exists a neighborhood $N_\varepsilon(0)$ of 0° e A, such that α e $N_\varepsilon(0)$ A, $\partial C^*(x(\alpha), \alpha)/\partial x < 0$. For all α's in the neighborhood of 0° this and $\partial C^*(\tilde{x}(\alpha), \alpha)/\partial x > 0$, imply there is an x^* e $(x(\alpha), \tilde{x}(\alpha))$, such that $\partial C^*(x^*(\alpha), \alpha)/\partial x = 0$ and $C^*(x^*(\alpha), \alpha) \leqq C^*(x, \alpha)$ over the open interval. Then, there exists a point $(x(\beta), \beta)$ for a β e A such that $C(x(\beta), \beta) \leqq C^*(x^*(\alpha), \alpha)$. Since $C^*(x(\beta), \beta) \leqq C(x(\beta), \beta) < C^*(x(\alpha), \alpha)$,$(x(\alpha), \alpha))$ can not be the location. For this β e A such that $\beta < \alpha$, there exists an x^* e $(x(\beta), \tilde{x}(\beta))$ at which $C^*(x^*(\beta), \beta) \leqq C^*(x, \beta)$ over the interval again. Repeatedly, we could find a γ e A, such that $C(x(\gamma), \gamma) < C^*(x(\beta), \beta)$, $\gamma < \beta$. Thus,

$C^*(x(\gamma), \gamma) < C^*(x(\beta), \beta) < C^*(x(\alpha), \alpha)$, $0 < \gamma < \beta < \alpha < 30°$.

By continuity of $C^*(., .)$ on X x A, $C^*(x(0), 0)$ is the lower boundary of such $C^*(x(\alpha), \alpha)$ on $N_\varepsilon(0)$ A.

For any α, β such that $\alpha, \beta \notin N_\varepsilon(0)$, and $\alpha > \beta$, suppose $\partial C^*(x(\alpha),\alpha)/\partial x < 0$, $\partial C^*(x(\beta), \beta)/\partial x > 0$. Then, by continuity, there is γ e A such that $\beta<\gamma<\alpha$, $\partial C^*(x(\gamma), \gamma)/\partial x = 0$, and without loss of generality it comes out; (a), from the above argument, $(C^*(0, \gamma) \leq) C^*(x(\gamma'), \gamma') \leqq C^*(x(\alpha), \alpha)$, $\gamma \leqq \gamma' \leqq \alpha$, (b) $C^*(0, \gamma') \leqq C^*(x(\gamma'), \gamma') \leqq C^*(x(\alpha), \alpha)$ for γ' such that $\gamma<\gamma'<\alpha$ and $\partial C^*(x(\gamma'), \gamma')/\partial x > 0$, or (c) $C^*(x(0), 0) \leqq C^*(x(\alpha), \alpha)$ γ' e $N_\varepsilon(0)$. (in (c), if $\partial C^*(x(\gamma'), \gamma')/\partial x < 0$ and $\partial C^*(x(\gamma''), \gamma'')/\partial x > 0$, $\gamma' > \gamma''$, then, (a), (b), or (c) follows again, hence, the repeated process may apply until (c) alone follows.)

For (viii), $\partial C^*(0, \alpha)/\partial x < 0$ implies $\partial C^*(x, \alpha)/\partial x < 0$ on X for each α, so that from Proposition 2, 3 and from the above argument for (vii), the site M_2 or M_3 is chosen. (ix) implies $\partial C^*(x, \alpha)/\partial x < 0$, x e X for each α, and from Proposition 2,3 and from the above argument for (vii), M_2, or, M_3 is chosen as the optimal location. Similarly for the case in which $30° < \alpha \leqq 60°$. At 30°, $\partial C^*(x(30°), 30°)/\partial x > 0$, so that (0, 30°) is the location. Hence, the three market site O(0, 0), M_2(a, 0) amd M_3(a, 60°) are eligible for the optimal location.

Remark 9:1 That $C^*(.)$ is monotone on X, irrespective of whether it is concave or convex, will imply an optimal extreme point of location.

Remark 9:2 Case 3 and 5 satisfy Condition (37) and (v) in Theorem 2.

It is worthy to point out a sufficient condition for the concavity of C^* to hold in this general case. The proof of this is trivial, and omitted. Let us call the effect of a change in the distance x (range from output market) on the input demand for i th factor; for each $i = 1, 2, 3$, and for a fixed α,

$$\{\partial M_i(x, \alpha)/\partial x\}/\{M_i(x, \alpha)/x\}$$

the distance elasticity of i th factor substitution, and let us call the effect of a change in x on the rate of change in marginal transport cost;

$$x\{\partial^2 P_i(x, \alpha)/\partial x^2\}/\{\partial P_i(x, \alpha)/\partial x\},$$

the distance elasticity of marginal transport cost for i th input.

Remark 10: Suppose that transport cost and input demand vary conversely for each input factor, i.e. $\{\partial P_i(x, \alpha)/\partial x\}\ \partial M_i(x, \alpha)/\partial x < 0$. Suppose also that the distance elasticity of i th factor substitution is larger in the "absolute value" sense than the distance elasticity of marginal transport cost for i th factor. Then, the concavity of transport cost function for output, $R_0''(.) \leq 0$, will imply the concavity of the cost function $C^*(., \alpha)$ on $X(\alpha)$ for a fixed $\alpha \in A$.

This condition is easy to understand, and intuitively true it is, a condition that guarantees the neo-classical factor dominant.

8. Concluding Discussion

The optimal choice by an economic agent of a facility location and the design of the facility has been treated simultaneously and solved in terms of a competitive firm. This firm may be taken as a supplier who is not a perfect competitive. The firm finds the optimal location as well as the optimal quantity of each input for production.

We have applied the mathematical method of a constrained extremum problem with a certain class of spatial perturbations of the objective (total cost) function. It has th same objective function and yet has a different Lagrangian function and its Kuhn-Tucker coefficient, which continuously corresponds to each of perturbation variables.

Alternatively, we may take a cost-minimizing consumer or worker as the agent who is looking for a house to live in, for example. In terms of the formulation discussed above, one has only to regard the set of inputs and output as the set of consumption goods or commodities, production function as utility function or labor reproduction function, etc.. In the analysis throughout the preceding sections, concerning whether an extreme or interior point is optimal in the triangle, we have so far witnessed; In general:

(i) There exist two contradicting determining factors in the firm's decision making, spatial and technological: One is the transport distance factor, that will normally make an interior point optimal if it is dominant over the other, as in case of no substitution among inputs allowed in production. The other is the neoclassical (substitution) factor, which will make an extreme point (one of the vertices) optimal if it is dominant over the other as in case of the linear space of transportation.

(ii) However, the concavity property, due to the neoclassical factor, of the cost function is hardly established for every location angle and easily subject to the ad hoc assumptions on transport cost structure; see Theorem 1, 2 and Remark 6. The strict convexity of the distance function may imply the monotonicity, hence, imply an optimal extreme point with the ad hoc assumptions. But, the other property that the transport distance takes its minimum value in the interior of the space, is robust in making an interior location optimal; see Theorem 1, Corollary 1,2,3 and Case 3,4,and 5.

(iii) That transport cost functions in ranges are concave or convex seems to be ad hoc presumptions. That the total cost function is monotonely increasing or decreasing at the vertices of the triangle, also seems to be ad hoc assumptions. Both depend on the assumptions (A-j) j =1,2,3 on transport cost structure with respect to ranges from the firm's location to the three vertices. Each plays, however, a critical role in the firm's decision making.

To be more specific:

(iv) Case 1 and 2 are in an essetially linear space and the distance factor is eliminated. Remark 10 concerns itself with a dominant neoclassical factor in the triangle case. The concavity of the cost function attributes only to this factor, if transport cost functions are not concave.

However, Case 3 & 5 are very simple cases, in which production is of a Leontief type, hence, the neoclassical substitution disappear, and, neverthless, a market site is optimal. In Case 3, the optimal location is taken at the site where two markets are spacially agglomerated. Case 5 is a more specific example, in which, from the view point of minimizing transport cost, with a market, the other markets compare so favorable that the location is chosen at the site of the market so as to save the unfavourable transport cost. Case 4 is an typical example, in which no markets are spacially agglomerated so that at each site located is only one market, exemplifies an interior point chosen as optimal, as if the transport distance attains its minimum at the interior point. Case 1,2,3, and 5 are generalized in Theorem 2, while, Case 4 is generalized in Theorem 1, Corollary 1,2,3 and 4.

Finally, we also have observed in the general framework;

(v) in the three different cases, a point of location would be optimal at any level of output if and only if production is governed by homotheticity or linear homogeneity; see Proposition 8.

Thus, the global investigation of the basic properties of the optimal location with inputs have drawn, extended the contribution of the various authors referred to above, and reinforced the foundations of the economic theory of location, which unifies traditional location analysis and production (utility) analysis, each having been abstracting from other.

Appendix A

Table 1 ; $d(x, \alpha)$, $\alpha \leqq 30°$.

α \ x	0	$a/2$	$a/\sqrt{3}$	$\bar{x}(\alpha)$	$x(\alpha)$	$(\sqrt{3}/2)a$	a
0	$2a$	$(7/4)a$	+		+	+	$2a$
30°	$2a$	$(\frac{1}{2}+\sqrt{5-2\sqrt{3}})a$	$\sqrt{3}\ a$		$(1+\sqrt{3}/2)a$		
α	$2a$	$\frac{1}{2}a + \mu(\alpha)a$	+	+	$(1+x(\alpha))a$		

where $\mu(\alpha) = 1/2 + \sqrt{5/4 - \cos\alpha} + \sqrt{5/4 - \cos(60°-\alpha)}$, and

$x(\alpha) = \{(\sqrt{3}/2)/\sin(60°+\alpha)\}a.$

Table 2; $\dfrac{\partial d(x, \alpha)}{\partial\ x}$, $\alpha \leqq 30°$

α \ x	0	$a/2$	$a/\sqrt{3}$	$\bar{x}(\alpha)$	$x(\alpha)$	$\sqrt{3}/2)a$	a
0	$-1/2$	0	+		+	+	$1/2$
30°	$-(\sqrt{3} - 1)$	−	0		+	1	
α	$-\eta(\alpha)+1$	−	−	0	$1 + \sigma(\alpha)$		

where $\eta(\alpha) = \cos\alpha + \cos(60° - \alpha)$, $3/2 \leqq \cos\alpha + \cos(60° - \alpha) \leqq \sqrt{3}$,

$x(\alpha) - y\,\eta(\alpha) \geqq 0$ if $\alpha \leqq 30°$ with equality when $\alpha = 30°$, and

$\sigma(\alpha) = (1/y(a-y))\{x(\alpha) - y\eta(\alpha)\}$, $\dfrac{\sqrt{3}}{2}a \leqq x(\alpha) \leqq a.$

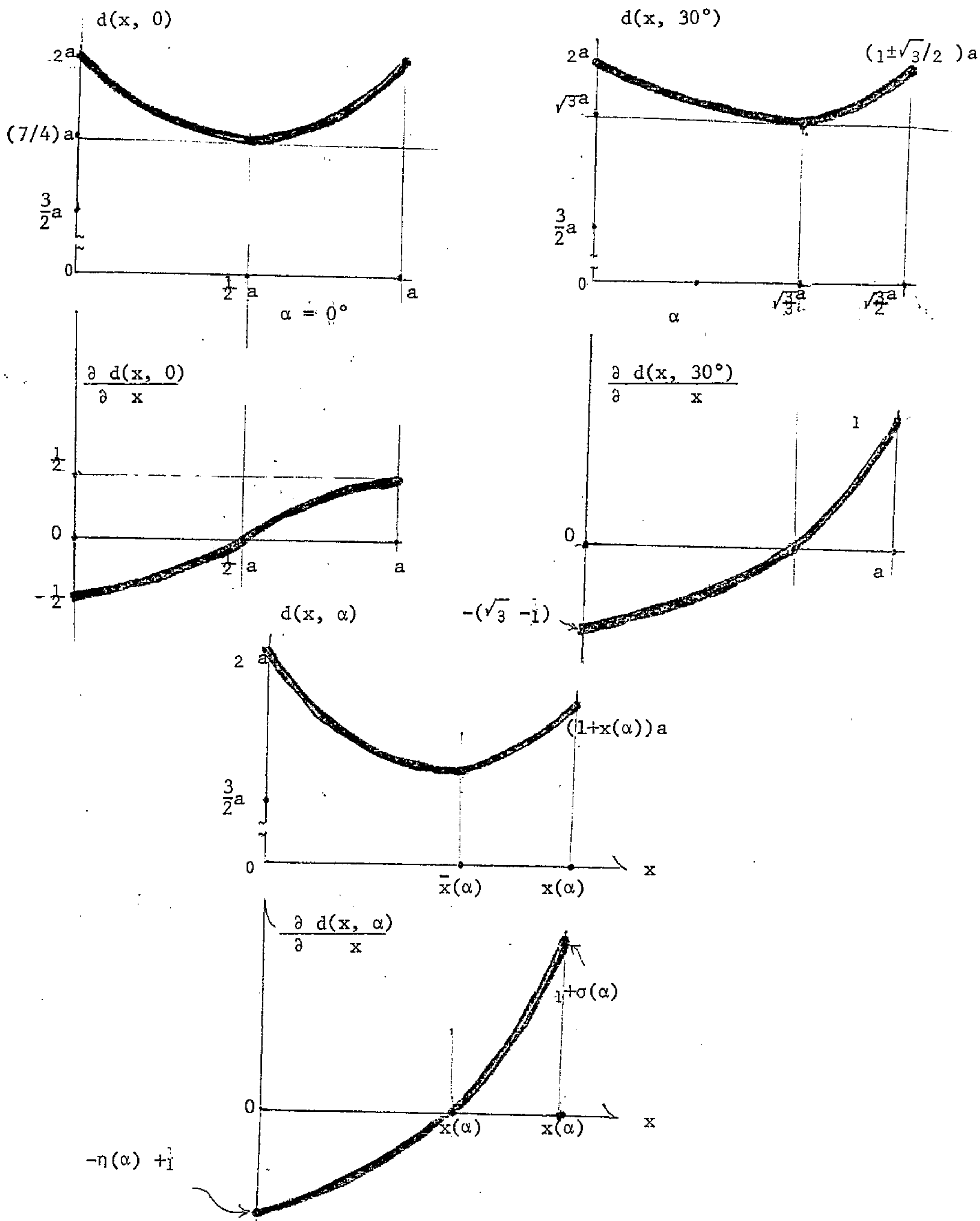

d(x, 0)
2a
(7/4)a
3/2a
0
1/2 a
a
α = 0°
d(x, 30°)
2a
(1±√3/2)a
√3a
3/2a
0
√3/3a
√3/2a
α
∂ d(x, 0) / ∂ x
1/2
0
1/2 a
a
-1/2
∂ d(x, 30°) / ∂ x
1
0
a
-(√3 -1)
d(x, α)
2 a
(1+x(α))a
3/2a
0
x̄(α)
x(α)
x
∂ d(x, α) / ∂ x
1+σ(α)
0
x(α)
x(α)
x
-η(α) +1

Figure 6

Appendix B

Next, we like to know where $d(\bar{x}(.), .)$ achieves its minimum distance on A. To this end, what we must know is the values of $d(\bar{x}(.), .)$, and the signs of $\frac{\partial d(\bar{x}(.), .)}{\partial \alpha}$, at $\alpha = 0°$, and $\alpha = 30°$ etc.,

We have $y(\bar{x}(\alpha), \alpha) = \bar{x}(\alpha)$, and

(22) $$\frac{\partial^2 d(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} = \frac{\partial^2 y(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} + \frac{\partial^2 z(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} > 0,$$

since , $0° \leqq \alpha \leqq 30°$,

(23) $$0 \leqq \frac{1}{y} \frac{\partial y(\bar{x}(\alpha), \alpha)}{\partial \alpha} = \frac{a}{\bar{x}(\alpha)} \sin \alpha \leqq 1$$

$$\sqrt{3} \leqq \tan^{-1} \alpha \quad \text{and,}$$

(24) $$\sqrt{3}^{\,-1} \leqq -\frac{1}{z} \frac{\partial z(\bar{x}(\alpha), \alpha)}{\partial \alpha} = \frac{a\bar{x}(\alpha)}{z(x(\alpha), \alpha)^2} \sin(60° - \alpha) \leqq \frac{3}{2},$$

where let $y = y(\bar{x}(\alpha), \alpha)$ and $z = z(\bar{x}(\alpha), \alpha))$,

(25) $$\frac{\partial^2 y(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} = \frac{\partial y}{\partial \alpha} (\tan^{-1} \alpha - \frac{1}{y} \frac{\partial y}{\partial \alpha})$$

(26) $$\frac{\partial^2 z(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} = -\frac{\partial z}{\partial \alpha} (\tan^{-1}(60° - \alpha) - \frac{1}{z} \frac{\partial z}{\partial \alpha}).$$

We also have,

(27) $$\frac{\partial d(\bar{x}(0), 0)}{\partial \alpha} = -\frac{1}{2} a, \qquad \frac{\partial d(\bar{x}(30°), 30°)}{\partial \alpha} = 0.$$

Now , we can illustrate how the distance $d(\bar{x}(.), ,)$ varies with direction angle α on A, as follows

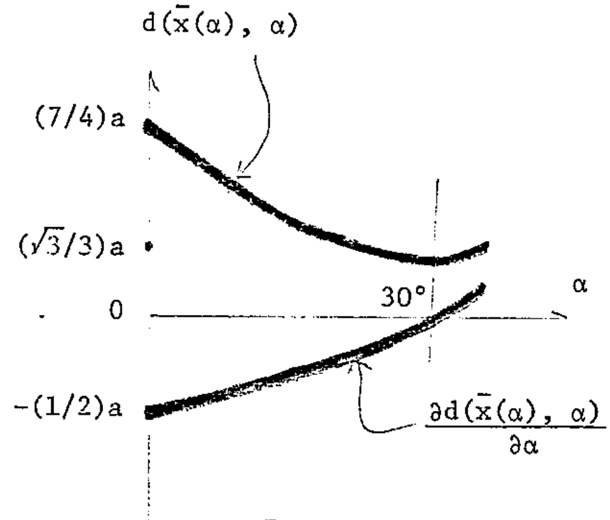


Figure 7

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